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QUANTUM STOCHASTIC CALCULUS WITH MAXIMAL OPERATOR DOMAINS[1]

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Dedicated to Robin Hudson in the year of his 60th birthday

Quantum stochastic calculus is extended in a new formulation in which its stochastic integrals achieve their natural and maximal domains. Operator adaptedness, conditional expectations and stochastic integrals are all defined simply in terms of the orthogonal projections of the time filtration of Fock space, together with sections of the adapted gradient operator. Free from exponential vector domains, our stochastic integrals may be satisfactorily composed yielding quantum Itô formulas for operator products as sums of stochastic integrals. The calculus has seen two reformulations since its discovery—one closely related to classical Itô calculus; the other to noncausal stochastic analysis and Malliavin calculus. Our theory extends both of these approaches and may be viewed as a synthesis of the two. The main application given here is existence and uniqueness for the Attal–Meyer equations for implicit definition of quantum stochastic integrals.

0. Introduction. Quantum stochastic calculus is now a well-established noncommutative extension of classical Itô calculus [10, 32, 37]. There are other such extensions, notably Itô–Clifford theory [7, 40], fermionic [1] and quasi-free [8, 26] stochastic calculi and the calculus based on free independence [11, 25]. However, to date, the most developed [2, 5, 9, 16, 17, 19, 22, 27, 29, 30, 34, 33, 38, 43] is the bosonic theory originated by Hudson and Parthasarathy [23]. Moreover, fermionic theory has been incorporated into the bosonic by means of a continuous Jordan–Wigner transformation which has a simple quantum stochastic description [24].

As originally set down the homogeneity of exponential vectors with respect to the continuous tensor product structure of (symmetric) Fock space was key to the development of the calculus. Quantum stochastic (QS) integrals were constructed for time-indexed families of operators which are defined on exponential vectors and satisfy an adaptedness property which is itself nicely described in terms of these vectors. The QS integrals were thereby defined on such exponential domains, too. Their composition as operators is thus inadmissible, strictly speaking, within this *exponential vector formulation*, since they typically do not leave the linear

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span of exponential vectors invariant. Only inner products of QS integrals acting on such vectors may be formed. Perhaps, surprisingly, this limitation has not been felt until recently—a rich stock of QS processes has been constructed through an effective theory of QS differential equations. This limitation does make itself felt when one is interested in algebraic questions such as the structure of the collection of bounded operator-valued quantum semimartingales [2].

One way in which QS calculus has been extended beyond exponential domains is by means of the Hitsuda–Skorohod integral of anticipative processes [21, 41] and the related gradient operator of Malliavin calculus [18, 35]. In this *noncausal formulation* [9, 27] the action of each QS integral is defined explicitly on Fock space vectors, and the essential quantum Itô formula (in inner product form) is seen in terms of the *Skorohod isometry*. Neither exponential domains nor adaptedness of the operator-valued integrands are required. Set against these advantages, the domains of both the annihilation and number integrals in this approach are still restricted—this time to parts of the domain of the square root of the number operator—even when the resulting operator is bounded. This domain limitation again artificially limits operator composition.

A second way in which the scope and domain of QS calculus has been extended is by means of an abstract Itô calculus on Fock space [5]. Specifically, in this approach the fact that all vectors of the Fock space admit an abstract predictable representation, $f = \mathbb{E}[f] + \int_0^\infty \xi_s(f)\,d\chi_s$, is exploited to obtain a formula for the action of QS integrals which makes good sense for nonexponential vectors. For example, if X_t is the creation integral $\int_0^t H_s\,dA_s^\dagger$ and $f_t = \int_0^t \xi_s(f)\,d\chi_s$, then $X_t f_t = \int_0^t (X_s\xi_s(f) + H_s f_s)\,d\chi_s$. This leads to a definition of QS integrals which agrees with the original one when restricted to exponential vectors. In this *Itô calculus formulation* operator composition of QS integrals is admitted. Under some conditions the domain of these QS integrals may be the whole of Fock space—a fact which plays an important role in the theory of quantum semimartingale algebras and the theory of quantum square and angle brackets [2, 43]. The main disadvantage of this formulation is that the QS integrals are only defined *implicitly*. In fact, the definitions amount to a system of abstract stochastic differential equations and, up to now, the general existence and uniqueness of solutions for these equations were not known. Moreover, the maximal operator domains of these QS integrals have been far from clear.

The purpose of this work is to unify and extend both of the above approaches. We give definitions which provide the action of QS integrals explicitly; introduce no unnatural domain limitations; settle the existence and uniqueness question for the systems of stochastic differential equations arising in the Itô calculus approach; and permit operator composition of QS integrals, governed by a quantum Itô product formula. Moreover, we demonstrate maximality of operator domains for these QS integrals.

The main idea is to base the calculus on a finely tuned definition of operator adaptedness, exploiting an *adapted gradient operator* inspired by classical stochastic analysis. On exponential vectors the usual definition is recovered; however, the new definition frees us from any prescribed domains and imposes no extraneous domain constraints. For example, the domain of a t-adapted operator can now be all of Fock space when it is bounded and need no longer be an algebraic tensor product when it is unbounded. The new definition provides a clearer picture of the relationship between quantum and classical Itô calculus; it also leads to a definition of conditional expectation for Fock space operators which enjoys all the algebraic properties one could hope for, given the vagaries of unbounded operators.

The refinement of operator adaptedness is also the point of departure for the new definitions of QS integrals given here. In particular, the gradient operator, used in the noncausal formulation of QS calculus, is replaced by the adapted gradient. This overcomes the unnatural domain constraint imposed in the noncausal approach while maintaining explicitness of the action of QS integrals. The connection with the Itô calculus approach is then seen through commutation relations between the Skorohod and time integrals and the adapted gradient operator, and a recursion formula enjoyed by the QS integrals.

A brief preliminary account of this work has appeared in [4].

1. Notation and conventions. The collection of subsets of $\mathbb{R}_+$ having finite cardinality:

$$\{\sigma \subset \mathbb{R}_+ : \#\sigma < \infty\}$$

will be referred to as the *finite power set* of $\mathbb{R}_+$ and denoted Γ, with $\Gamma^{(n)}$ denoting the collection of n-element subsets. For $n \geq 1$, Lebesgue measure induces a measure on $\Gamma^{(n)}$ through the bijection $\mathbf{s} \mapsto \{s_1, \dots, s_n\}$ from $\{\mathbf{s} \in \mathbb{R}_+^n : s_1 < \cdots < s_n\}$ to $\Gamma^{(n)}$. By letting $\varnothing \in \Gamma^{(0)}$ be an atom of measure 1, we arrive at a σ-finite measure on $\bigcup_{n\geq 0} \Gamma^{(n)} = \Gamma$ called the *symmetric measure* of Lebesgue measure on $\mathbb{R}_+$ [20].

Measurable for Hilbert space–valued maps means here strongly measurable, and *integrable* means Bochner integrable; since all Hilbert spaces appearing will be separable, weak measurability implies measurability [14]. Fixing a complex separable Hilbert space $\mathfrak{h}$, *Guichardet–Fock space* (or simply Fock space) is the Hilbert space tensor product $\mathfrak{h} \otimes L^2(\Gamma)$, which we identify with the space of (classes of) square-integrable functions $L^2(\Gamma; \mathfrak{h})$, and is denoted $\mathcal{F}$. Elements of Γ will always be denoted by lowercase Greek letters such as α, β, σ, τ and ω, and these will be used exclusively for this purpose. With this convention we write simply $\int f(\sigma)\, d\sigma$ to denote the integral of a Hilbert space–valued function f over Γ with respect to the symmetric measure of Lebesgue measure on $\mathbb{R}_+$. Similarly, $\int \varphi(s)\, ds$ will always denote the integral of a function φ over $\mathbb{R}_+$ with respect to Lebesgue measure. The following elementary identity is fundamental—a proof may be found in [28].

LEMMA 1.1 (Integral–sum lemma). *Let g be a nonnegative measurable function $\Gamma \times \Gamma \to \mathbb{R}$ (or a Bochner-integrable function $\Gamma \times \Gamma \to \mathfrak{h}$) and let G be the function defined by $G(\sigma) = \sum_{\alpha \subset \sigma} g(\alpha, \sigma \setminus \alpha)$. Then G is nonnegative and measurable (resp. integrable) and*

$$\int G(\sigma)\, d\sigma = \iint g(\alpha, \beta)\, d\alpha\, d\beta.$$

The following subspaces of Guichardet–Fock space are useful:

$$\mathcal{K}^{(a)} := \operatorname{Dom} a^N, \qquad \mathcal{K} := \bigcap_{a \geq 1} \mathcal{K}^{(a)}, \tag{1.1a}$$

$$\mathcal{F}_{\text{fin}} := \left\{ f \in \mathcal{F} : \operatorname{supp} f \subset \bigcup_{n \leq m} \Gamma^{(n)} \text{ for some } m \right\}, \tag{1.1b}$$

$$\mathcal{E}(S) := \operatorname{Lin}\{\varepsilon(\varphi) : \varphi \in S\}, \tag{1.1c}$$

where S is a subset of $L^2(\mathbb{R}_+)$. Here Dom denotes the domain of a Hilbert space operator; N is the *number operator*, $Nf(\sigma) = \#\sigma f(\sigma)$, with maximal domain; and a^N is defined through the functional calculus. Also, $\varepsilon(\varphi)$ denotes the *exponential* vector of the test function φ [37] which in Guichardet–Fock space is the measure equivalence class of the function

$$\sigma \mapsto \prod_{s \in \sigma} \varphi(s).$$

Following is a list of set-theoretic notation and measure-theoretic conventions that we shall adopt throughout. Let $s, t \in \mathbb{R}_+$ and let $\omega, \sigma, \tau \in \Gamma$. Then

$\omega_{t)} := \omega \cap [0, t[, \qquad \omega_{[t} := \omega \cap [t, \infty[, \qquad$ and so on;

$\vee\sigma := \max\{s : s \in \sigma\}, \qquad \sigma_- := \sigma \setminus \{\vee\sigma\}, \qquad \wedge\sigma := \min\{s : s \in \sigma\};$

$\omega \cup s := \omega \cup \{s\}, \qquad \sigma \setminus s := \sigma \setminus \{s\};$

"$\sigma < \tau$" means $s < t\ \forall s \in \sigma, t \in \tau$;

$\Gamma_s := \{\omega \in \Gamma : \omega \subset [0, s[\}, \qquad \Gamma^s := \{\omega \in \Gamma : \omega \subset [s, \infty[\};$

"a.a. $\tau > s$" means almost all $\tau \in \Gamma^s$ (here s is fixed), whereas

"a.a. $(\tau > s)$" means almost all elements of $\{(\tau, s) \in \Gamma \times \mathbb{R}_+ : \tau > s\}$;

$\mathcal{F}_s := \mathfrak{h} \otimes L^2(\Gamma_s), \qquad \mathcal{F}^s = L^2(\Gamma^s).$

Guichardet–Fock space enjoys a *continuous tensor product* structure: for each $s \geq 0$ the map

$$f \otimes g \mapsto \big(\omega \mapsto f(\omega_{s)}) g(\omega_{(s})\big) \tag{1.2}$$

extends uniquely to an isometric isomorphism $\mathcal{F}_s \otimes \mathcal{F}^s \to \mathcal{F}$. This structure is elegantly carried by the exponential vectors, being determined by the following restriction of (1.2):

$$v\varepsilon(\varphi_{[0,s[}) \otimes \varepsilon(\varphi_{[s,\infty[}) \mapsto v\varepsilon(\varphi),$$

where, under the natural isometry $\mathcal{F}_s \to \mathcal{F}_s \otimes \mathcal{F}^s$ given by $f \mapsto f \otimes \delta_\varnothing$, with $\delta_\varnothing := \varepsilon(0)$, $\mathcal{F}_s$ is first viewed as a *subspace* of $\mathcal{F}$. The notation here is $\varphi_{[a,b[} := \varphi \mathbf{1}_{[a,b[}$, and $\mathbf{1}$ denotes the indicator function.

An important consequence of the integral–sum lemma is the following identity:

$$\iint f(\sigma \cup t)\, d\sigma\, dt = \int \#\omega f(\omega)\, d\omega, \tag{1.3}$$

which is valid for nonnegative measurable functions $\Gamma \to \mathbb{R}$ and for measurable functions $\Gamma \to \mathfrak{h}$ for which either/both sides are defined.

We use the following notation for algebraic tensor products: for subspaces U and V of Hilbert spaces H and K, $U \odot V \subset \mathsf{H} \otimes \mathsf{K}$ is the subspace $\mathrm{Lin}\{u \otimes v : u \in U, v \in V\}$, and for operators R on H and S on K, $R \odot S$ denotes the operator on $\mathsf{H} \otimes \mathsf{K}$ with domain $\mathrm{Dom}\, R \odot \mathrm{Dom}\, S$ satisfying $(R \odot S)(x \otimes y) = Rx \otimes Sy$. Finally, a pair of Hilbert space operators satisfying

$$\langle Ru, v\rangle = \langle u, Sv\rangle, \qquad u \in \mathrm{Dom}\, R,\ v \in \mathrm{Dom}\, S, \tag{1.4}$$

will be called an *adjoint pair*. When $\mathrm{Dom}\, R$ is dense this amounts to the condition $R^* \supset S$.

2. Itô calculus in Fock space. Our aim in this section is twofold. First, we construct part of the Itô calculus on Fock space, describing familiar probabilistic concepts in this unfamiliar language while emphasizing the universality of Fock space. Second, we develop relationships between the components of this calculus (derivative, projection and integrals). These will be applied later, once we have introduced noncommuative processes. The section ends with a discussion of the connection with classical Itô calculus through probabilistic interpretations of the objects introduced.

2.1. *Integration.* The measurable structure on $\Gamma \times \mathbb{R}_+$ is the completed product measure of the Guichardet measure on Γ and the Lebesgue measure on $\mathbb{R}_+$. We need a spectrum of integrability conditions for a Hilbert space–valued map $x : \Gamma \times \mathbb{R}_+ \to \mathfrak{h}$. We write $x_s(\omega)$ for $x(\omega, s)$. Then:

- x is *time integrable* if, for a.a. ω, the map $x.(\omega)$ is integrable $\mathbb{R}_+ \to \mathfrak{h}$ and the following a.e. defined map is square integrable:

$$\mathcal{L}(x) : \omega \to \int x_s(\omega)\, ds.$$

- x is *absolutely time integrable* if x is measurable and the map $(\omega, s) \to \|x_s(\omega)\|$ is time integrable.
- x is *Bochner integrable* if, for a.a. s, the map x_s is square integrable $\Gamma \to \mathfrak{h}$ and $s \mapsto [x_s]$ is integrable $\mathbb{R}_+ \to \mathcal{F}$.
- x is *Skorohod integrable* if the following map is square integrable $\Gamma \to \mathfrak{h}$:

$$\mathcal{S}(x) : \sigma \mapsto \sum_{s \in \sigma} x_s(\sigma \setminus s).$$

- x *belongs to* $\operatorname{Dom} \mathcal{S}$ if x is square integable $\Gamma \times \mathbb{R}_+ \to \mathfrak{h}$ and x is Skorohod integrable.
- x is *absolutely Skorohod integrable* if x is measurable and the map $(\omega, s) \mapsto \|x_s(\omega)\|$ is Skorohod integrable.

$\mathcal{L}(x)$ and $\mathcal{S}(x)$ are called the *time integral of* x and the *Skorohod integral of* x, respectively. We emphasize here that, for the definitions of both time integrability and Skorohod integrability, we assume *neither* the square integrability of each x_s *nor* the (joint) measurability of x. Note, however, that if x and x' are maps $\Gamma \times \mathbb{R}_+ \to \mathfrak{h}$ which agree a.e., then x' is time integrable if and only if x is, in which case $\mathcal{L}(x') = \mathcal{L}(x)$, and similarly for the Skorohod integral. Therefore, although $\mathcal{L}(x)$ and $\mathcal{S}(x)$ are defined pointwise, we view both $\mathcal{L}$ and $\mathcal{S}$ as mappings from measure equivalence classes into $\mathcal{F}$. The definition of Bochner integrability is the standard one, rephrased here for easy comparison with the pointwise integrability conditions. The space $\operatorname{Dom} \mathcal{S}$ is merely the domain of the Skorohod integral when it is viewed as an unbounded Hilbert space operator $L^2(\Gamma \times \mathbb{R}_+; \mathfrak{h}) \to \mathcal{F}$.

PROPOSITION 2.1. *Let x be a map $\Gamma \times \mathbb{R}_+ \to \mathfrak{h}$.*

(ai) *If x is square integrable, then x is locally Bochner integrable and*

$$\int_0^t \|x_s\| \, ds \leq \sqrt{t} \Big(\int_0^t \int \|x_s(\omega)\|^2 \, d\omega \, ds \Big)^{1/2}.$$

(aii) *If x is Bochner integrable, then x is absolutely time integrable and*

$$\|\mathcal{L}(x)\| \leq \int \|x_s\| \, ds.$$

(bi) *If x is measurable, then*

$$\iiint \|x_s(\omega \cup t)\| \, \|x_t(\omega \cup s)\| \, d\omega \, dt \, ds \leq \iint \#\omega \|x_s(\omega)\|^2 \, d\omega \, ds.$$

(bii) *If x is square integrable and the function $(\omega, s, t) \to \langle x_s(\omega \cup t), x_t(\omega \cup s) \rangle$ is integrable, then $x \in \operatorname{Dom} \mathcal{S}$ and*

$$\|\mathcal{S}(x)\|^2 = \int \|x_s\|^2 \, ds + \iiint \langle x_s(\omega \cup t), x_t(\omega \cup s) \rangle \, d\omega \, dt \, ds. \tag{2.1}$$

(biii) *If x is absolutely Skorohod integrable, then x is square integrable and the function $(\omega, s, t) \mapsto \|x_s(\omega \cup t)\| \|x_t(\omega \cup s)\|$ is integrable.*

Identity (2.1) will be referred to as *Skorohod isometry*.

PROOF OF PROPOSITION 2.1. (ai) follows from the Cauchy–Schwarz inequality in $L^2(\mathbb{R}_+)$. If x is Bochner integrable, then, by standard vector integration theory [14], it is jointly measurable. Moreover, by a continuous version of Minkowski's inequality [42],

$$\left\{\int \left[\int \|x(\omega, s)\| \, ds\right]^2 d\omega\right\}^{1/2} \leq \int \left\{\int \|x(\omega, s)\|^2 \, d\omega\right\}^{1/2} ds = \int \|x_s\| \, ds,$$

which establishes (aii). (bi)–(biii) follow from straightforward applications of the integral–sum lemma—see [27] for further details. $\square$

Let x be a map $\Gamma \times \mathbb{R}_+ \to \mathfrak{h}$. Then x is *adapted* if

$$x_s(\omega) = 0 \qquad \text{for } \omega \not\subset \Gamma_s, \tag{2.2}$$

and x is *Itô integrable* if x is adapted and the map

$$\mathcal{I}(x) : \Gamma \to \mathfrak{h}, \qquad \sigma \mapsto \begin{cases} 0, & \text{if } \sigma = \varnothing, \\ x_{\vee\sigma}(\sigma_-), & \text{otherwise}, \end{cases}$$

is square integrable. Note $\mathcal{I}(x)$ is that called the *Itô integral of x*. Like $\mathcal{L}(x)$ and $\mathcal{S}(x)$, it will be viewed as an element of $\mathcal{F}$. As with time and Skorohod integrals, Itô integrability depends only on the measure equivalence class of x, and the Itô integral lifts to a mapping from measure equivalence classes into $\mathcal{F}$. In contrast to time integrals and Skorohod integrals, Itô-integrable maps are necessarily measurable.

A *Fock vector process* is a family $x = (x_s)_{s \geq 0}$ in $\mathcal{F}$. It is *adapted* if $x_s \in \mathcal{F}_s$ for each s, and *measurable* if the map $s \mapsto x_s$ is measurable $\mathbb{R}_+ \to \mathcal{F}$. For a measurable vector process x, there is a measurable map $\widetilde{x} : \Gamma \times \mathbb{R}_+ \to \mathfrak{h}$ such that $\widetilde{x}_s(\cdot)$ is a version of x_s for each s. If x is adapted, then $\widetilde{x}$ may be chosen to be adapted in the sense of (2.2). The measure equivalence class of $\widetilde{x}$ is unique, and we shall therefore abuse notation by using x for the map as well as for the process.

Using the integral–sum lemma, the following is a straightforward consequence of our definitions.

PROPOSITION 2.2. *Let x be an adapted map $\Gamma \times \mathbb{R}_+ \to \mathfrak{h}$. Then the following are equivalent*:

(a) *x is Itô integrable*;
(b) *x is Skorohod integrable*;
(c) *x is square integrable.*

Moreover, in any of these cases we have $\mathcal{I}(x) = \mathcal{S}(x)$ *and*

$$\|\mathcal{I}(x)\|^2 = \int \|x_s\|^2 \, ds. \tag{2.3}$$

We shall refer to identity (2.3) as *Itô isometry*. Comparison with (2.1) shows that Skorohod isometry extends Itô isometry beyond adapted maps. Another way of expressing Itô integrability is in terms of the set

$$\Gamma_{\text{ad}} := \{(\omega, s) \in \Gamma \times \mathbb{R}_+ : \omega < s\}.$$

The collection of equivalence classes of Itô-integrable maps may be identified with $L^2(\Gamma_{\text{ad}}; \mathfrak{h})$. The *adapted projection* on $L^2(\Gamma \times \mathbb{R}_+; \mathfrak{h})$ is the orthogonal projection onto the closed subspace $L^2(\Gamma_{\text{ad}}; \mathfrak{h})$:

$$P_{\text{ad}} x : (\omega, s) \mapsto \mathbf{1}_{\{\omega < s\}} x_s(\omega).$$

In the discussion of the Itô calculus approach to QS calculus, we shall use the notation $\int_0^\infty x_s \, d\chi_s$ for $\mathcal{I}(x)$, in recognition of the fact that it may be viewed as an integral in $\mathcal{F}$ with respect to the continuous path $(\chi_s)_{s \geq 0}$, where χ_s is (the measure equivalence class of) the indicator function of $\Gamma_s^{(1)}$ [3].

Letting $\mathcal{R}$ stand for any of the integrals $\mathcal{L}$, $\mathcal{S}$ or $\mathcal{I}$, we write for $a \leq b \in [0, \infty]$,

$$\mathcal{R}_a^b(x) = \mathcal{R}\big(\mathbf{1}_{[a,b[}(\cdot) x_.\big).$$

Note that the Skorohod integrability of $\mathbf{1}_{[a,b[}(\cdot)x_.$ *is not implied by* the Skorohod integrability of x and the same goes for time integrability. This is an essential feature of these integrals [6].

2.2. *Differentiation and projection.* For a vector space–valued map $f : \Gamma \to V$, let ∇f and Df be the maps $\Gamma \times \mathbb{R}_+ \to V$ given by

$$\nabla f(\omega, s) = f(\omega \cup s), \qquad Df(\omega, s) = \mathbf{1}_{\{\omega < s\}} f(\omega \cup s).$$

The operators ∇ and D *commute with* the norm in $\mathfrak{h}$ in the sense that if $k = \|f(\cdot)\|_{\mathfrak{h}}$ where $f : \Gamma \to \mathfrak{h}$, then $\nabla k(\omega, s) = \|\nabla f(\omega, s)\|$, and likewise for D. Straightforward application of the integral–sum lemma [cf. (1.3)] gives the following result.

PROPOSITION 2.3. *Let* $f : \Gamma \to \mathfrak{h}$ *be measurable. Then* ∇f *and* Df *are measurable and satisfy*

$$\iint \|\nabla f(\omega, s)\|^2 \, d\omega \, ds = \int \#\sigma \, \|f(\sigma)\|^2 \, d\sigma, \tag{2.4a}$$

$$\iint \|Df(\omega, s)\|^2 \, d\omega \, ds = \int \|f(\sigma)\|^2 \, d\sigma - \|f(\emptyset)\|^2. \tag{2.4b}$$

It follows that we may view ∇ and D as (measure equivalence) class mappings. When $f \in \mathcal{F}$, we call ∇f and Df the *stochastic gradient of* f and the *adapted gradient of* f, respectively. Moreover, we write $\operatorname{Dom}\nabla$ for the domain of the stochastic gradient viewed as an unbounded Hilbert space operator (cf. $\operatorname{Dom}\mathcal{S}$):

$$\operatorname{Dom}\nabla := \{f \in \mathcal{F} : \nabla f \in L^2(\Gamma \times \mathbb{R}_+; \mathfrak{h})\}.$$

For $\sigma \in \Gamma$, $s \in \mathbb{R}_+$ and a vector space–valued map $f : \Gamma \to V$, let $\nabla_\sigma f$, $D_\sigma f$ and $P_s f$ be the maps $\Gamma \to V$ given by

$$\nabla_\sigma f(\omega) = f(\omega \cup \sigma), \qquad D_\sigma f(\omega) = \mathbf{1}_{\{\omega < \sigma\}} f(\omega \cup \sigma), \qquad P_s f = \mathbf{1}_{\Gamma_s} f.$$

Thus, writing $D_s f$ for $D_{\{s\}} f$, we have

$$D_s f = Df(\cdot, s), \qquad D_\varnothing f = f$$

and

$$D_\sigma f = D_{s_1} \cdots D_{s_n} f \qquad \text{if } \sigma = \{s_1 < \cdots < s_n\}.$$

The following algebraic relations are evident for $s < t$:

$$P_0 f = f(\varnothing)\delta_\varnothing, \qquad P_s P_t f = P_t P_s f = P_s f, \tag{2.5a}$$

$$D_t D_s f = D_t P_s f = 0, \tag{2.5b}$$

$$D_s P_t f = P_t D_s f = D_s f, \tag{2.5c}$$

as is the *reproducing relation* $D_\tau f(\sigma) = f(\sigma \cup \tau)$ for $\sigma < \tau$, with special cases:

$$f(\omega) = (D_\omega f)(\varnothing) = D_{\vee\omega} f(\omega_-) \qquad \text{if } \omega \neq \varnothing. \tag{2.5d}$$

2.3. *Integro-differential and adjoint relations.* First, we relate Skorohod integration with stochastic differentiation and give the adapted counterpart.

PROPOSITION 2.4. *Let $f \in \mathcal{F}$ and let $x : \Gamma \times \mathbb{R}_+ \to \mathfrak{h}$ be Skorohod integrable*:

(a) *If the map $(\omega, s) \mapsto \langle x_s(\omega), f(\omega \cup s)\rangle$ is integrable, then*

$$\langle \mathcal{S}(x), f\rangle = \iint \langle x_s(\omega), \nabla_s f(\omega)\rangle \, d\omega \, ds.$$

(b) *If x is Itô integrable, then*

$$\langle \mathcal{I}(x), f\rangle = \int \langle x_s, D_s f\rangle \, ds.$$

PROOF. More straightforward applications of the integral–sum lemma. $\square$

Next, we summarize the Hilbert space properties of the stochastic and adapted gradients and the Skorohod and Itô integrals. For further details, see [3] and [27].

THEOREM 2.5. *As Hilbert space operators* $\mathcal{S}$, ∇, D *and* $\mathfrak{I}$ *enjoy the following properties*:

(a) $(\mathcal{S}, \operatorname{Dom}\mathcal{S})$ *and* $(\nabla, \operatorname{Dom}\nabla)$ *are closed, densely defined operators.*
(b) $\mathcal{S}^* = \nabla$ (*and* $\nabla^* = \mathcal{S}$).
(c) $\operatorname{Dom}\mathcal{S} \supset \operatorname{Dom}\sqrt{N \otimes I}$; $\operatorname{Dom}\nabla = \operatorname{Dom}\sqrt{N}$.
(d) *The Itô integral is an isometric operator* $L^2(\Gamma_{\mathrm{ad}}; \mathfrak{h}) \to \mathcal{F}$ *with final space* $[\delta_\varnothing]^\perp$, *whose adjoint is the adapted gradient* D:

$$D\mathfrak{I} = I,$$

that is, for all $x \in L^2(\Gamma_{\mathrm{ad}}; \mathfrak{h})$,

$$D_t \int_0^\infty x_s \, d\chi_s = x_t \qquad a.a.\ t \in \mathbb{R}_+;$$

$$\operatorname{Ker} D = (\operatorname{Im}\mathfrak{I})^\perp = \mathbb{C}\delta_\varnothing; \qquad \mathfrak{I}^* = D;$$

$$\mathfrak{I}D = P_0^\perp,$$

that is, for all $f \in \mathcal{F}$,

$$f = P_0 f + \int_0^\infty D_s f \, d\chi_s.$$

(e) *The Skorohod integral is an extension of the Itô integral*: $\mathfrak{I} = \mathcal{S}|_{L^2(\Gamma_{\mathrm{ad}}; \mathfrak{h})}$.
(f) *The adapted gradient is the closure of the product of the adapted projection and the stochastic gradient*: $D = \overline{P_{\mathrm{ad}}\nabla}$.

2.4. *Almost everywhere defined operators.* Our philosophy in this paper is to treat the maps D_s like operators on $\mathcal{F}$, exploiting the fact that D is a *bounded operator* on $\mathcal{F}$ so that, unlike ∇, it is defined on the whole of $\mathcal{F}$. With each $f \in \mathcal{F}$, $D_s f$ is a well-defined element of $\mathcal{F}$ for almost every s. Of course, the null set depends on f, and for this reason D_s is not an operator on $\mathcal{F}$ in the usual sense—we shall speak of *almost everywhere defined operators* on $\mathcal{F}$. We take this viewpoint in order to exploit the relations (2.5a–d). On measure equivalence classes of maps such as elements of $\mathcal{F}$, these translate to the a.e. relations and the a.e. *reproducing property*

(2.6a) $\quad D_t D_s f = D_t P_s f = 0, \qquad D_s P_t f = P_t D_s f = D_s f \qquad$ for a.a. $(s < t)$,

(2.6b) $\quad f(\omega) = (P_s D_{\omega_{(s}} f)(\omega_{s)}) \qquad$ for all s and a.a. ω.

2.5. *Commutation relations.* In this section we describe the effect of the operators P_t and the a.e. defined operators D_t on Skorohod and time integrals. The relations we obtain will be applied to QS integrals in Section 5.2. Note the a.e. properties

$$f \in \mathcal{F} \Rightarrow P_s f, D_s f \in \mathcal{F}_s, \qquad f \in \mathcal{F}_s, s < t \Rightarrow D_t f = 0.$$

PROPOSITION 2.6. *Let $x:\Gamma\times\mathbb{R}_+\to\mathfrak{h}$ be measurable. If $P_t x_t$ is square integrable $\Gamma\to\mathfrak{h}$ for almost every $t\geq 0$, then the following are equivalent*:

(a) *x is Skorohod integrable.*

(b) *$s\mapsto \mathbf{1}_{\{s<t\}}D_t x_s$ is Skorohod integrable and the map $t\mapsto \mathcal{S}_0^t(D_t x.)+P_t x_t$ is Itô integrable.*

In this case we have, for a.a. t,

$$D_t\mathcal{S}(x)=\mathcal{S}_0^t(D_t x.)+P_t x_t. \tag{2.7}$$

PROOF. In view of the identity

$$\mathbf{1}_{\{\sigma<t\}}\mathcal{S}(x)(\sigma\cup t)=\sum_{s\in\sigma}\mathbf{1}_{\{\sigma<t\}}\mathbf{1}_{[0,t[}(s)x_s\big((\sigma\setminus s)\cup t\big)+\mathbf{1}_{\{\sigma<t\}}x_t(\sigma),$$

we have

$$D_t\mathcal{S}(x)(\sigma)=\mathcal{S}\big(\mathbf{1}_{[0,t[}(\cdot)D_t x.\big)(\sigma)+(P_t x_t)(\sigma). \tag{2.8}$$

If x is Skorohod integrable, then, since $P_t x_t$ is square integrable, $\mathbf{1}_{[0,t[}(\cdot)D_t x.$ is Skorohod integrable and (2.7) holds for a.a. t; moreover, the a.e. defined map $(\sigma,t)\mapsto \mathcal{S}_0^t(D_t x.)(\sigma)+(P_t x_t)(\sigma)$ is adapted and square integrable and thus Itô integrable. Conversely, if x satisfies (b), then, since x is measurable and

$$\int\|\mathcal{S}(x)(\sigma)\|^2\,d\sigma=\iint\|D_s\mathcal{S}(x)(\omega)\|^2\,d\omega\,ds$$

by (2.4b), x is Skorohod integrable by (2.8). □

PROPOSITION 2.7. *Let $x:\Gamma\times\mathbb{R}_+\to\mathfrak{h}$ be measurable. If $x.(\varnothing)$ is integrable, then the following are equivalent*:

(a) *x is time integrable.*

(b) *$s\mapsto D_t x_s$ is time integrable for a.a. t and the map $t\mapsto\mathcal{L}(D_t x.)$ is square integrable.*

In this case we have the a.e. identity

$$D_t\mathcal{L}(x)=\mathcal{L}(D_t x.). \tag{2.9}$$

PROOF. Let x be time integrable. Then, for a.a. (ω,t), the map $s\mapsto \mathbf{1}_{\{\omega<t\}}x_s(\omega\cup t)$ is integrable and so, for a.a. t,

$$\begin{aligned}D_t\mathcal{L}(x)(\omega)&=\mathbf{1}_{\{\omega<t\}}\int x_s(\omega\cup t)\,ds\\&=\int\mathbf{1}_{\{\omega<t\}}x_s(\omega\cup t)\,ds=\int(D_t x_s)(\omega)\,ds\end{aligned}$$

for a.a. ω. Hence, for a.a. t, $D_t x.$ is time integrable and (2.9) holds—in particular, the map $t\mapsto\mathcal{L}(D_t x.)$ is square integrable.

Conversely, if (b) holds, then the map $x.(\omega)$ is either $x.(\varnothing)$ or $(D_{\vee\omega}x.)(\omega_-)$ and so is integrable for a.a. ω. Moreover, the map $\alpha \in \Gamma \setminus \{\varnothing\} \mapsto \int x_s(\alpha)\, ds$ is the composition of the measure isomorphism $\alpha \mapsto (\alpha_-, \vee\alpha)$ from $\Gamma \setminus \{\varnothing\}$ into Γ_{ad} and the square-integrable map $(\omega, t) \mapsto \int (D_t x_s)(\omega)\, ds$. Hence, it is square integrable, so that x is time integrable. $\square$

The proof of the following is now straightforward.

PROPOSITION 2.8. *Let x be a measurable map $\Gamma \times \mathbb{R}_+ \to \mathfrak{h}$ and let $t \geq 0$.*

(a) *If x is time integrable, then $P_t x.$ is time integrable and*

$$\mathcal{L}(P_t x.) = P_t \mathcal{L}(x).$$

(b) *If x is Skorohod integrable, then $\mathbf{1}_{[0,t[} P_t x.$ is Skorohod integrable and*

$$\mathcal{S}_0^t(P_t x.) = P_t \mathcal{S}(x).$$

Moreover, if also $\mathbf{1}_{[t,\infty[}(\cdot) P_t x.$ is Itô integrable, then $P_t x.$ is Skorohod integrable and

$$\mathcal{S}(P_t x.) = P_t \mathcal{S}(x) + \mathfrak{I}_t^\infty(P_t x.).$$

REMARK. Each of the supplementary conditions in Propositions 2.6, 2.7 and 2.8—namely, square integrability of $\mathbf{1}_{\Gamma_t} x_t$ for a.a. t, integrability of $x.(\varnothing)$ and Itô integrability of $\mathbf{1}_{[t,\infty[}(\cdot) P_t x.$—is a condition on the $\mathbb{R}_+$-valued map $(\omega, s) \mapsto \|x_s(\omega)\|$. In view of the fact that P_t and D_t commute with the norm $\|\cdot\|_{\mathfrak{h}}$ (see the remark following the definitions of ∇ and D), each of these results also holds if time and Skorohod integrability are replaced by absolute time and absolute Skorohod integrability, respectively.

2.6. *Probabilistic interpretations.* In this section we describe explicitly the connection between the objects we have introduced in Fock space (P_{ad}, D, ∇, $\mathcal{S}$, $\mathfrak{I}$) and their classical probabilistic counterparts. While formally independent of the rest of the paper, the ideas here underlie the whole work.

A *probabilistic interpretation* of Fock space is provided by a quintuple of the form $(\Omega, \mathfrak{F}, (\mathfrak{F}_t)_{t\geq 0}, \mathbb{P}, m)$ in which $(\Omega, \mathfrak{F}, (\mathfrak{F}_t)_{t\geq 0}, \mathbb{P})$ is the canonical filtered space of $m = (m_t)_{t\geq 0}$ and m is a *normal martingale*—that is, a martingale for which $(m_t^2 - t)_{t\geq 0}$ is also a martingale—which has the chaotic representation property. Examples of such martingales include Brownian motion, the compensated Poisson process and some of the Azema martingales [15]. The chaotic representation of random variables leads to a natural isomorphism Ψ between $\mathcal{F}$ and $\mathfrak{X} := L^2(\Omega, \mathfrak{F}, \mathbb{P}; \mathfrak{h})$, which may suggestively be expressed as $f \mapsto \int f(\sigma)\, dm_\sigma$ [31]. Each of the operations P_t, D_t, ∇_t, $\mathcal{S}$, $\mathfrak{I}$ and $\mathcal{L}$ has interpretations on $\mathfrak{X}$ as well-known probabilistic operations. For this point of view in the Poisson case, see [36].

The orthogonal projection P_t is $\Psi^{-1} \circ \mathbb{E}[\,\cdot\,|\mathfrak{F}_t] \circ \Psi$, and $\mathcal{F}_t = \Psi^{-1}(\mathcal{X}_t)$, where $\mathcal{X}_t = L^2(\Omega, \mathfrak{F}_t, \mathbb{P}; \mathfrak{h})$. In particular, a square-integrable classical stochastic process in $\mathcal{X}$ is adapted if and only if its image under Ψ^{-1} is adapted in the sense of Section 2.1. Since the martingale m has the chaotic representation property, it also possesses the predictable representation property. Any random variable f in $\mathcal{X}$ may therefore be expressed as $f = \mathbb{E}[f] + \int_0^\infty \xi_t(f)\, dm_t$ for some predictable process $(\xi_t(f))_{t\geq 0}$ in $\mathcal{X}$. Viewing $(\xi_t)_{t\geq 0}$ as a family of a.e. defined operators on $\mathcal{X}$, we see it as a probabilistic interpretation of $(D_t)_{t\geq 0} : D_t = \Psi^{-1} \circ \xi_t \circ \Psi$. Similarly, ∇ corresponds precisely to the gradient operator in Malliavin calculus, and ∇_t corresponds to the stochastic derivative, along the element $\chi : s \mapsto s \wedge t$ of the Cameron–Martin space, on $\mathcal{X}$ [35]. By Theorem 2.5(b) $\mathcal{S}$ is the adjoint of ∇ and therefore [18] corresponds to the Hitsuda–Skorohod integral with respect to the process m. It also follows from Theorem 2.5(e) that $\mathcal{I}$, being the restriction of $\mathcal{S}$ to adapted Fock vector processes, corresponds to the Itô integral with respect to m. Theorem 2.5(d) includes an expression of the predictable respresentation property of m, and the isometry of Itô integration with respect to m, on $\mathcal{F}$. Finally, 2.5(f) implies that $D_t = P_t \nabla_t$ (in the sense of a.e. defined operators), which corresponds to Clark's formula [12, 13].

Thus, each of the operations introduced in Sections 2.1–2.4 corresponds to well-known operations of classical stochastic analysis once Fock space is interpreted as the *chaotic space* of some normal martingale. In fact, one should rather think the other way around. Probabilistic operations such as Skorohod integration, stochastic differentiation, predictable representation and so on may be expressed merely in terms of the chaotic expansion of random variables. They use no specific property of the particular martingale beyond chaotic representation and the form of Itô isometry. The normality of the martingale implies that its angle bracket $\langle m, m\rangle_t$ equals t, and so the fomula for Itô isometry remains the same for each such martingale. Fock space is thereby seen as an abstract chaos space which encodes the chaotic representation property and the Itô isometry formula of normal martingales and which carries simple intrinsic operations which perform the L^2-stochastic calculus of the martingale.

3. Operator adaptedness. In this section three natural candidates for *time-s-adaptedness* for a (possibly unbounded) Fock space operator are shown to be equivalent, and, using the a.e. reproducing property, a very useful further equivalent condition is found. We take these as our definition and verify that they generalize the obvious definition on exponential domains. Under the new definition composition of time-s-adapted operators yields a time-s-adapted operator. We note that all domains previously used in QS calculus are adapted subspaces in the sense defined below. Recall the a.e. reproducing relation (2.6b).

3.1. *Definitions and basic properties.* A subspace V of $\mathcal{F}$ is called *s-adapted* if, for any f in V:

(i) $P_s f \in V$;
(ii) $D_t f \in V$ for a.a. $t > s$.

Clearly, any intersection of s-adapted subspaces is s-adapted. A subspace V of $\mathcal{F}$ is called *adapted* if it is s-adapted for every s; these will be referred to as *Fock-adapted spaces*.

In the proof of the following result, repeated use is made of the a.e. relations (2.6a, b), as well as the identity

$$(P_t - P_s)f = \mathcal{I}_s^t(D.f), \qquad s < t \le \infty,$$

in which P_∞ is the identity operator.

THEOREM 3.1. *Let C be an operator on $\mathcal{F}$ with s-adapted domain $\mathcal{D}$. Then the following are equivalent*:

(a) *For all $f \in \mathcal{D}$*, (i) $P_s Cf = P_s C P_s f$ *and* (ii) $D_t Cf = C D_t f$ *for a.a.* $t \ge s$.

(b) *For all $f \in \mathcal{D}$*, (i) $C P_s f = P_s C P_s f$ *and* (ii) $C D_t f \in \mathcal{F}_t$ *for a.a.* $t \ge s$, $(C D_t f)_{t \ge s}$ *defines an Itô-integrable process and*

$$C(f - P_s f) = \mathcal{I}_s^\infty(CD.f). \tag{3.1}$$

(c) *For all $f \in \mathcal{D}$*, (i) $P_s Cf = C P_s f$ *and* (ii) $D_t Cf = C D_t f$ *for a.a.* $t > s$.

(d) *For all $f \in \mathcal{D}$*,

$$Cf(\omega) = (C P_s D_{\omega_{(s}} f)(\omega_{s)}) \qquad \textit{for a.a. } \omega.$$

PROOF. Obviously, (c) $\Rightarrow$ (a) and (a) + (b) $\Rightarrow$ (c). It therefore suffices to establish (a) $\Leftrightarrow$ (b) and (c) $\Leftrightarrow$ (d). Suppose that (a) holds and let $f \in \mathcal{D}$. Then, by (aii), $C D_t f \in \mathcal{F}_t$ for a.a. $t > s$ and $(C D_t f)_{t \ge s}$ is Itô integrable with

$$\mathcal{I}_s^\infty(CD.f) = \mathcal{I}_s^\infty(D.Cf) = Cf - P_s Cf. \tag{3.2}$$

Applying this identity to $P_s f$ gives

$$0 = C P_s f - P_s C P_s f,$$

and so (bi) holds and also, by (ai), $P_s Cf = C P_s f$. Putting this back into (3.2) gives (bii); hence, (b) holds.

Suppose that (b) holds and let $f \in \mathcal{D}$. Then, applying (bii) then (bi), for a.a. $t > s$,

$$D_t Cf = D_t C P_s f + D_t \mathcal{I}_s^\infty(CD.f) = D_t P_s C P_s f + C D_t f = C D_t f,$$

so (aii) holds. Applying (3.1),

$$P_s C P_s^\perp f = P_s \mathcal{I}_s^\infty(CD.f) = 0,$$

since $\mathcal{I}_s^\infty(CD.f)$ is orthogonal to $\mathcal{F}_s$, so (ai) holds, too.

Suppose that (c) holds and let $f \in \mathcal{D}$. Then iterating (cii) shows that $P_s D_\beta C f = C P_s D_\beta f$ for a.a. $\beta > s$. Therefore, using the a.e. reproducing property,

$$Cf(\omega) = (P_s D_{\omega_{(s}} Cf)(\omega_{s)}) = (C P_s D_{\omega_{(s}} f)(\omega_{s)})$$

for a.a. ω, so (d) holds.

Suppose, finally, that (d) holds and let $f \in \mathcal{D}$. Then, applying (d) first to $P_s f$ and last to f,

$$\begin{aligned} C P_s f(\omega) &= (C P_s D_{\omega_{(s}} P_s f)(\omega_{s)}) = \mathbf{1}_{\{\omega < s\}} (C P_s f)(\omega_{s)}) \\ &= \mathbf{1}_{\{\omega < s\}} (C P_s f)(\omega) = \mathbf{1}_{\{\omega < s\}} (C f)(\omega) = P_s C f(\omega) \end{aligned}$$

for a.a. ω, so $C P_s f = P_s C f$. Also, applying (d) to f and then to $D_t f$,

$$\begin{aligned} (D_t C f)(\omega) &= \mathbf{1}_{\{\omega < t\}} C f(\omega \cup t) = \mathbf{1}_{\{\omega < t\}} (C P_s D_{\omega_{(s}} D_t f)(\omega_{s)}) \\ &= (C P_s D_{\omega_{(s}} D_t f)(\omega_{s)}) = (C D_t f)(\omega) \end{aligned}$$

for a.a. (ω, t) with $t > s$, so $D_t C f = C D_t f$ for a.a. $t > s$. Hence, (c) holds. $\square$

An operator C on $\mathcal{F}$ is called *s-adapted* if it has an s-adapted domain on which it satisfies any/all of the equivalent conditions of the above theorem. We call (a), (b), (c) and (d), respectively, the *differential, integral, commuting* and *projective* definitions of adaptedness.

REMARKS. (o) Notice how the projective definition builds on the a.e. reproducing relation (2.6b).

(i) If $g \in \mathcal{F}_s$, $h \in \mathcal{F}^s$ and $t > s$, then $P_s(g \otimes h) = h(\varnothing) g \otimes \delta_\varnothing$ and $D_t(g \otimes h) = g \otimes D_t h$. It follows that, for any operator $\widetilde{C}$ on $\mathcal{F}_s$ with domain $\widetilde{V}$, the operator $\widetilde{C} \odot I$ (having domain $\widetilde{V} \odot \mathcal{F}^s$) is s-adapted.

(ii) From the projective definition of adaptedness, one sees that an s-adapted operator satisfies

$$C P_t f = P_t C f \qquad \text{for } t > s \text{ and } f \in \operatorname{Dom} C \cap \operatorname{Dom}(C P_t),$$

and so is t-adapted, provided only that $\operatorname{Dom} C$ is t-adapted.

(iii) Also, from the projective definition, it follows that if s-adapted operators C and C' agree on $\mathcal{F}_s \cap \operatorname{Dom} C \cap \operatorname{Dom} C'$, then they agree on their common domain $\operatorname{Dom} C \cap \operatorname{Dom} C'$.

(iv) Given an operator $\widetilde{C}$ on $\mathcal{F}_s$ with domain $\mathcal{F}_s \cap V$, where V is an s-adapted subspace, $Cf(\tau) = (\widetilde{C} P_s D_{\tau_{(s}} f)(\tau_{s)})$ defines an s-adapted operator C on V extending $\widetilde{C}$, called the *s-adapated extension* of $\widetilde{C}$ to V.

The following property of s-adapted operators follows easily from the a.e. reproducing property and the integral–sum lemma.

PROPOSITION 3.2. *Let C be an s-adapted operator on $\mathcal{F}$. Then, for all $f \in \mathcal{F}$ and $g \in \operatorname{Dom} C$,*

$$\langle f, Cg\rangle = \int_{\{\beta > s\}} \langle P_s D_\beta f, C P_s D_\beta g\rangle \, d\beta.$$

Let $\mathcal{A}_s$ denote the collection of s-adapted operators on $\mathcal{F}$.

PROPOSITION 3.3. *$\mathcal{A}_s$ is closed under operator products, sums and scalar multiples.*

PROOF. Let C and C' be s-adapted operators on $\mathcal{F}$ and let $f \in \operatorname{Dom}(CC')$. Then $P_s f$ and $D_t f$ lie in $\operatorname{Dom}(C')$, $C' P_s f = P_s C' f$ and $C' D_t f = D_t C' f$ for a.a. $t > s$, since $f \in \operatorname{Dom}(C')$ and C' is s-adapted. But $C' f \in \operatorname{Dom} C$ and C is s-adapted, so $P_s C' f$ and $D_t C' f$ lie in $\operatorname{Dom} C$, $C(P_s C' f) = P_s C C' f$ and $C(D_t C' f) = D_t C C' f$ for a.a. $t > s$. This shows that $\mathcal{A}_s$ is closed under operator multiplication. Since an intersection of s-adapted subspaces is s-adapted, $\mathcal{A}_s$ is also closed under addition. It is obviously closed under scalar multiplication. $\square$

$\mathcal{A}_s$ fails to be an associative algebra in the same sense in which the collection of all unbounded operators on $\mathcal{F}$ does; namely, an element C whose domain is not all of $\mathcal{F}$ fails to have an additive inverse, and scalar multiplication by 0 yields not the zero operator, but its restriction to $\operatorname{Dom} C$.

The adjoint of a densely defined s-adapted operator C may fail to be s-adapted as it stands. However, we shall see in the next section (Corollary 4.4) that *conditioning* an operator which is adjoint to C yields an s-adapted operator adjoint to C.

The next result addresses the question of when an s-adapted operator can pass under an Itô integral.

PROPOSITION 3.4. *Let C be an s-adapted operator and let $x : [a, b[\to \mathcal{F}$ be an Itô-integrable* $\operatorname{Dom} C$*-valued vector process, where $s \le a < b \le \infty$. If $\mathcal{I}_a^b(x) \in \operatorname{Dom} C$, then the adapted Fock vector process $\mathbf{1}_{[a,b[}(\cdot) C x.$ is also Itô integrable, and*

$$\mathcal{I}_a^b(Cx.) = C\mathcal{I}_a^b(x). \tag{3.3}$$

PROOF. Set $g = \mathcal{I}_a^b(x)$. Then $D_t g = x_t$ for a.a. $t \in [a, b[$ and, by the s-adaptedness of C,

$$Cx_t = CD_t g = D_t C g \qquad \text{for a.a. } t \in [a, b[.$$

Hence, $\mathbf{1}_{[a,b[}(\cdot) C x.$ is Itô integrable and

$$\mathcal{I}_a^b(Cx.) = \mathcal{I}_a^b(D.Cg) = (P_b - P_a)Cg = C(P_b - P_a)g.$$

Since $P_b g = g$ and $P_a g = 0$, (3.3) follows. $\square$

We shall see later [Proposition 4.1(v)] that the condition $\mathcal{I}_a^b(x) \in \operatorname{Dom} C$ is automatically satisfied for s-adapted operators which have their natural s-adapted domains.

3.2. *Examples and comparisons.* Recall that in the original (exponential vector) formulation of QS calculus, all processes are defined on a domain of the form $V_0 \odot \mathcal{E}(S)$, where V_0 is a dense subspace of $\mathfrak{h}$ and S is an *admissible subset* of $L^2(\mathbb{R}_+)$, that is, a subset for which $\mathcal{E}(S)$ is dense in $\Gamma(L^2(\mathbb{R}_+))$ and $\varphi_{[0,s[} \in S$ whenever $\varphi \in S$ and $s \geq 0$ [23]. Since, for all s and a.a. t,

$$P_s v\varepsilon(\varphi) = v\varepsilon(\varphi_{[0,s[}) \quad \text{and} \quad D_t v\varepsilon(\varphi) = \varphi(t) v\varepsilon(\varphi_{[0,t[}), \tag{3.4}$$

such domains are adapted in our sense. Note the a.e. identity

$$D_\tau v\varepsilon(\varphi) = v\varepsilon(\varphi_{[0,t[})\varepsilon(\varphi_{[t,\infty[})(\tau) \qquad \text{where } t = \wedge\tau.$$

Commonly used admissible subsets are dense subspaces of $L^2(\mathbb{R}_+)$ such as $(L^2 \cap L^\infty_{\mathrm{loc}})(\mathbb{R}_+)$ [23], the set $\{\varphi \in (L^2 \cap L^\infty)(\mathbb{R}_+) : \|\varphi\|_2 \leq 1 \text{ and } \|\varphi\|_\infty \leq 1\}$ [17] and the set $\{\mathbf{1}_B : B \text{ is a finite union of bounded intervals}\}$. The admissibility of this last set was established in [39].

PROPOSITION 3.5. *Let C be an operator on $\mathcal{F}$ with domain of the form $V_0 \odot \mathcal{E}(S)$, where V_0 is a dense subspace of $\mathfrak{h}$ and S is admissible. Then C is s-adapted if and only if, for all $v \in V_0$ and $\varphi \in S$*:

(i) $Cv\varepsilon(\varphi_{[0,s[}) \in \mathcal{F}_s$;
(ii) $Cv\varepsilon(\varphi) = Cv\varepsilon(\varphi_{[0,s[}) \otimes \varepsilon(\varphi_{[s,\infty[})$.

PROOF. First note that if C is s-adapted, then (i) holds. Suppose therefore that C satisfies (i) and let $v \in V_0$ and $\varphi \in S$. Then [by (3.4)], for a.a. $\omega > s$, $P_s D_\omega v\varepsilon(\varphi) = v\varepsilon(\varphi_{[0,s[})\varepsilon(\varphi_{[s,\infty[})(\omega)$ and so, for a.a. ω,

$$\begin{aligned}(CP_s D_{\omega_{(s}} v\varepsilon(\varphi))(\omega_{s)}) &= Cv\varepsilon(\varphi_{[0,s[})(\omega_{s)})\varepsilon(\varphi_{[s,\infty[})(\omega_{(s}) \\ &= (Cv\varepsilon(\varphi_{[0,s[}) \otimes \varepsilon(\varphi_{[s,\infty[}))(\omega).\end{aligned}$$

Appealing to the projective definition, we see that C is s-adapted if and only if (ii) holds. $\square$

Thus, the new notion of adaptedness for Fock space operators extends the original definition beyond exponential domains.

All the domains commonly used in QS calculus are adapted. Recall the spaces defined in (1.1a–c).

- $\mathcal{F}$ itself is obviously adapted. This is the proper domain for bounded operator-valued processes.

- Each $\mathcal{K}^{(a)}$ is adapted since P_s leaves $\mathrm{Dom}(a^N)$ invariant and $\int \|a^N D_t f\|^2\, dt = a^{-2}\|Da^N f\|^2$. Thus, $\mathcal{K}$ is adapted, too. This is a natural domain for both noncausal QS calculus and for integral–sum kernel operators on $\mathcal{F}$.
- $\mathcal{F}_{\text{fin}}$ is obviously adapted since if $f \in \mathcal{F}$ has support in $\Gamma^{(n)}$, then $P_s f$ and $D_t f$ have support in $\Gamma^{(n)}$ and $\Gamma^{(n-1)}$, respectively.
- For any subspace M of $L^2(\mathbb{R}_+)$ satisfying $\varphi_{[0,t[} \in M$ whenever $\varphi \in M$ and $t > 0$, the symmetric tensor algebra $\mathrm{Lin}\{\otimes^{(n)}\varphi : \varphi \in M, n \geq 0\}$ is adapted since $P_s \otimes^{(n)} \varphi = \otimes^{(n)}\varphi_{[0,s[}$ and $D_t \otimes^{(n)} \varphi = \varphi(t) \otimes^{(n-1)} \varphi_{[0,t[}$.
- The original domain used by Maassen for expressing QS integrals as integral–sum kernel operators [30]:

$$\{f \in \mathcal{F} : \operatorname{supp} f \subset \Gamma_T \text{ and } \|f(\omega)\| \leq CK^{\#\omega} \text{ for some } T, C \text{ and } K\},$$

  is adapted since both the support and boundedness properties are clearly invariant under P_s and D_t; for example, $\|D_t f(\omega)\| \leq C'K^{\#\omega}$, where $C' = CK$.

4. Conditional expectation and operator processes. The projective definition of s-adaptedness leads to a natural way of defining conditional expectation for Fock space operators. When applied to any operator it yields an s-adapted operator; when applied to an operator which is already s-adapted, it yields an extension of the operator to a natural domain for the purposes of QS calculus; and when applied to bounded operators, it gives the usual result. In this section the basic classes of Fock operator processes are introduced: adapted, measurable and continuous processes, and martingales, and their stability is discussed.

4.1. *Conditioned spaces.* The idea is to construct the domain of the conditioned operator so that it is maximal given the domain constraint of the unconditioned operator. Thus, for any subspace V of $\mathcal{F}$, its *time-s conditioned space* is defined by

$$\mathbb{D}_s[V] := \{f \in \mathcal{F} : P_s D_\tau f \in V \text{ for a.a. } \tau > s\}.$$

Clearly, $\mathbb{D}_s[V]$ is an s-adapted subspace. A list of additional properties enjoyed by this construction follows.

PROPOSITION 4.1. *Let V and V' be subspaces of $\mathcal{F}$ and let $s \geq 0$.*

(o) $\mathbb{D}_s[\mathcal{F}] = \mathcal{F}$, $\mathbb{D}_s[V \cap V'] = \mathbb{D}_s[V] \cap \mathbb{D}_s[V']$.
(i) $\mathbb{D}_s[V]$ *is t-adapted for all $t \geq s$.*
(ii) *If V is s-adapted, then* $\mathbb{D}_s[V] \supset V$.
(iii) $\mathbb{D}_t[\mathbb{D}_s[V]] = \mathbb{D}_s[\mathbb{D}_t[V]] = \mathbb{D}_s[V]$ *for all $t \geq s$.*
(iv) $\mathbb{D}_s[V] \supset (V \cap \mathcal{F}_s) \odot \mathcal{F}^s$.
(v) *Let $b > a \geq s$ and suppose that $(x_t)_{t\in[a,b[}$ is a $\mathbb{D}_s[V]$-valued Itô-integrable vector process, then $\mathfrak{I}_a^b(x) \in \mathbb{D}_s[V]$.*

PROOF. These are routine verifications. For example, in (v), $\mathcal{I}_a^b(x) \in \mathcal{F}_b \ominus \mathcal{F}_a$; it follows that $P_s D_\tau \mathcal{I}_a^b(x) = P_s D_{\tau\_} x_{\vee\tau}$ if $\tau \in \Gamma_b \setminus \Gamma_a$, and is 0 otherwise. □

Thus, the map $\mathbb{D}_s$ manufactures an s-adapted subspace from any subspace V which moreover contains V if V is already s-adapted. (iii) is a tower property of the maps, and (v) is a technical property which will be useful later [in the proof of Theorem 4.3(v)].

4.2. *Conditioned operators.* We come now to a central definition of our approach. Let C be an operator on $\mathcal{F}$ with domain V. Taking our cue from the projective definition for adaptedness, we define an operator $\mathbb{E}_s[C]$ on $\mathcal{F}$ by the a.e. prescription

$$(\mathbb{E}_s[C]f)(\omega) = (CP_s D_{\omega_{(s}} f)(\omega_{s)}),$$

with domain

$$\{f \in \mathbb{D}_s[V] : \tau \mapsto \mathbf{1}_{\{\tau > s\}} P_s C P_s D_\tau f \text{ is square integrable } \Gamma \to \mathcal{F}\}.$$

Using the integral–sum lemma, it is easily verified that $\mathbb{E}_s[C]f \in \mathcal{F}$ and that the operator $\mathbb{E}_s[C]$ is s-adapted. The next result therefore includes an extension of Proposition 3.2.

PROPOSITION 4.2. *Let C be an operator on $\mathcal{F}$ and let $s \geq 0$.*

(a) $\mathbb{E}_s[C] = \mathbb{E}_s[P_s C P_s]$.
(b) *If $g \in \operatorname{Dom} \mathbb{E}_s[C]$, then* $P_s \mathbb{E}_s[C] g = P_s \mathbb{E}_s[C] P_s g = P_s C P_s g$.
(c) *If $g \in \operatorname{Dom} \mathbb{E}_s[C]$ and $f \in \mathcal{F}$, then*

$$\langle f, \mathbb{E}_s[C] g \rangle = \int_{\{\beta > s\}} \langle P_s D_\beta f, C P_s D_\beta g \rangle \, d\beta.$$

(d) *If $(C, C^\dagger)$ is an adjoint pair of operators on $\mathcal{F}$* [*see* (1.4)], *then* $(\mathbb{E}_s[C], \mathbb{E}_s[C^\dagger])$ *is also an adjoint pair.*
(e) *If C and $\mathbb{E}_s[C]$ are densely defined, then* $\mathbb{E}_s[C]^* \supset \mathbb{E}_s[C^*]$.

PROOF. Parts (a) and (b) are immediate consequences of the definition. Part (c) follows from (b) and Proposition 3.2. Part (d) follows from (c), and (e) from (d). □

Notice that if C is s-adapted then the subspaces $\mathcal{F}_s \cap \operatorname{Dom} C$ and $\mathcal{F}_s \cap \operatorname{Dom} \mathbb{E}_s[C]$ coincide, and $\mathbb{E}_s[C]g = Cg$ for g in this subspace. It follows that

$$\mathbb{E}_s[C] P_s f = C P_s f \quad \text{and} \quad \mathbb{E}_s[C] D_s f = C D_s f,$$

whenever $P_s f$ (resp. $D_s f$) belongs to $\operatorname{Dom} C$. What follows is a list of the basic properties of time-s conditional expectation. A refinement of (d) of the above proposition is included.

THEOREM 4.3. *Let C and C' be operators on $\mathcal{F}$, let $\lambda \in \mathbb{C}$ and let $s \geq 0$.*

(o) $\mathbb{E}_s[I] = I$, $\mathbb{E}_s[C + \lambda C'] \supset \mathbb{E}_s[C] + \lambda \mathbb{E}_s[C']$.

(i) $\mathbb{E}_s[C]$ *is t-adapted for every* $t \geq s$.

(ii) C *is s-adapted if and only if* $\operatorname{Dom} C$ *is s-adapted and* $C \subset \mathbb{E}_s[C]$.

(iii) $\mathbb{E}_s[\mathbb{E}_t[C]] = \mathbb{E}_s[C] \subset \mathbb{E}_t[\mathbb{E}_s[C]]$ *for all* $t \geq s$.

(iv) $\mathbb{E}_s[C] \supset \widetilde{C} \odot I$, *where* $\widetilde{C} = P_s C|_{\mathcal{F}_s \cap \operatorname{Dom} C}$ *and* $I = I_{\mathcal{F}^s}$.

(v) *If* C, $\mathbb{E}_s[C]$ *and* $\mathbb{E}_s[C]^*$ *are all densely defined, then* $\mathbb{E}_s[C]^*$ *is s-adapted and* $\mathbb{E}_s[C]^* \supset \mathbb{E}_s[C^*]$.

(vi) *If C is bounded* (*with domain* $\mathcal{F}$), *then* $\mathbb{E}_s[C]$ *is bounded* (*with domain* $\mathcal{F}$), *too, and has norm at most* $\|C\|$.

(vii) *If C is nonnegative, then so is* $\mathbb{E}_s[C]$.

(viii) *If S is an s-adapted operator, then* $\mathbb{E}_s[CS] \supset \mathbb{E}_s[C]S$.

(ix) *If B is a bounded s-adapted operator with domain* $\mathcal{F}$, *then* $\mathbb{E}_s[BC] \supset B\mathbb{E}_s[C]$.

(x) *If* $C = C_1 \odot C_2$, *where* C_1 *is an operator on* $\mathcal{F}_s$ *and* C_2 *is an operator on* $\mathcal{F}^s$ *whose domain includes* $\delta_\varnothing$, *then* $\mathbb{E}_s[C] \supset \langle \delta_\varnothing, C_2 \delta_\varnothing \rangle C_1 \odot I_{\mathcal{F}^s}$.

PROOF. Most of these properties follow from straightforward applications of the a.e. relations (2.6a, b), the integral–sum lemma and Propositions 4.1 and 4.2 to the definitions. For example, (i) follows from (i) of Proposition 4.1 and remark (ii) following Theorem 3.1. Parts (iii) and (v) are a little more delicate.

(iii) The inclusion follows from (i) and (ii)—it is the equality that still needs proof. For $f \in \mathcal{F}$, f belongs to $\operatorname{Dom} \mathbb{E}_s[\mathbb{E}_t[C]]$ if and only if $P_s D_\alpha f \in \operatorname{Dom} \mathbb{E}_t[C]$ for a.a. $\alpha > s$, and the map $\alpha \mapsto \mathbf{1}_{\{\alpha > s\}} P_s \mathbb{E}_t[C] P_s D_\alpha f$ is square integrable. By Proposition 4.2 these hold if and only if:

(a) $P_t D_\beta P_s D_\alpha f \in \operatorname{Dom} C$ for a.a. $\beta > t$ and a.a. $\alpha > s$;

(b) $\beta \mapsto \mathbf{1}_{\{\beta > t\}} P_t C P_t D_\beta P_s D_\alpha f$ is square integrable for a.a. $\alpha > s$;

(c) $\alpha \mapsto \mathbf{1}_{\{\alpha > s\}} P_s C P_s D_\alpha f$ is square integrable.

But since $t \geq s$, $P_t D_\beta P_s D_\alpha f = \delta_\varnothing(\beta) P_s D_\alpha f$ for a.a. β and (b) is vacuous, so $f \in \operatorname{Dom} \mathbb{E}_s[\mathbb{E}_t[C]]$ if and only if $f \in \operatorname{Dom} \mathbb{E}_s[C]$. Moreover,

$$\mathbb{E}_s\big[\mathbb{E}_t[C]\big] f(\omega) = (\mathbb{E}_t[C] P_s D_{\omega_{(s}}) f(\omega_{s)}) = (C P_s D_{\omega_{(s}}) f(\omega_{s)}) = \mathbb{E}_s[C] f(\omega)$$

for a.a. ω. Hence, $\mathbb{E}_s[\mathbb{E}_t[C]] = \mathbb{E}_s[C]$.

(v) In view of Proposition 4.2(e), all that remains to be proved is that $\mathbb{E}_s[C]^*$ is s-adapted under the assumption that, along with C and $\mathbb{E}_s[C]$, it is densely defined. Thus, set $V = \operatorname{Dom} \mathbb{E}_s[C]$ and $V^* = \operatorname{Dom} \mathbb{E}_s[C]^*$ and let $f \in V^*$ and $g \in V$. Then, by Proposition 4.2(a),

$$\langle P_s f, \mathbb{E}_s[C] g \rangle = \langle f, \mathbb{E}_s[C] P_s g \rangle = \langle P_s \mathbb{E}_s[C]^* f, g \rangle.$$

Thus, $P_s f \in V^*$ and $\mathbb{E}_s[C]^* P_s f = P_s \mathbb{E}_s[C]^* f$. Next, note that, for any $k, h \in \mathcal{F}$, $P.k$ is locally Itô integrable, and $\langle D.k, h \rangle = \langle D.k, P.h \rangle$ is locally integrable with

$\int_a^b \langle D_t k, h\rangle\, dt = \langle k, \mathfrak{I}_a^b(P.h)\rangle$. Using this together with Propositions 3.4 and 4.1(v) and the fact that $P_t \mathbb{E}_s[C] f = \mathbb{E}_s[C] P_t f$ for $t \geq s$, we obtain

$$\int_s^b \langle D_t f, \mathbb{E}_s[C] g\rangle\, dt = \langle f, \mathfrak{I}_s^b(\mathbb{E}_s[C]P.g)\rangle = \int_s^b \langle D_t \mathbb{E}_s[C]^* f, g\rangle\, dt$$

for $b > s$. Since b is arbitrary, there is a Lebesgue null set N_g of $[s, \infty[$ such that $\langle D_t f, \mathbb{E}_s[C] g\rangle = \langle D_t \mathbb{E}_s[C]^* f, g\rangle$ for $t \notin N_g$. Letting g run through a countable family in V, whose linear span is a core for the closure of $\mathbb{E}_s[C]$, we see that, for a.a. $t > s$, $D_t f \in V^*$ and $\mathbb{E}_s[C]^* D_t f = D_t \mathbb{E}_s[C]^* f$. Hence, $\mathbb{E}_s[C]^*$ is s-adapted. □

By (iv) and (vi) our definition extends the usual definition for bounded (everywhere defined) operators C:

$$\mathbb{E}_s[C] = \widetilde{C} \otimes I \qquad \text{where } \widetilde{C} = P_s C|_{\mathcal{F}_s} \text{ and } I = I_{\mathcal{F}^s}.$$

In view of property (ii) above, we say that an operator C is *maximally s-adapted* if $\mathbb{E}_s[C] = C$.

Property (v) expresses the sense in which conditional expectation commutes with the adjoint operation. When applied to already s-adapted operators, or maximally s-adapted operators, it gives us the following useful result.

COROLLARY 4.4. *Let C be an operator on $\mathcal{F}$ which is densely defined and s-adapted. If $\mathbb{E}_s[C]$ is closable, then*:

(a) $\mathbb{E}_s[C^*] = \mathbb{E}_s[C]^*$;
(b) $\mathbb{E}_s[\overline{C}] = \overline{\mathbb{E}_s[C]}$ *provided that C^* is s-adapted.*

In particular, the operators $\mathbb{E}_s[C]^*$ and $\overline{\mathbb{E}_s[C]}$ are maximally s-adapted.

PROOF. By (v) and (ii), $\mathbb{E}_s[\mathbb{E}_s[C]^*] \supset \mathbb{E}_s[C]^* \supset \mathbb{E}_s[C^*]$. But $C \subset \mathbb{E}_s[C]$, so $C^* \supset \mathbb{E}_s[C]^*$ and therefore $\mathbb{E}_s[C^*] \supset \mathbb{E}_s[\mathbb{E}_s[C]^*]$. Combining these, we obtain $\mathbb{E}_s[C^*] \supset \mathbb{E}_s[C]^* \supset \mathbb{E}_s[C^*]$, which gives (a). Now (b) follows easily by applying (a) to C^*. □

Thus, if C is densely defined, closable and maximally s-adapted, then both C^* and $\overline{C}$ are maximally s-adapted, too. Let $\mathcal{A}_s^\ddagger$ denote the collection of closed densely defined and maximally s-adapted operators on $\mathcal{F}$. The next result is complementary to Proposition 3.3.

PROPOSITION 4.5. *The collection $\mathcal{A}_s^\ddagger$ is closed under the Hilbert space adjoint operation and contains $B(\mathcal{F}) \cap \mathcal{A}_s = B(\mathcal{F}_s) \otimes I_{\mathcal{F}^s}$.*

PROOF. The first part is contained in Corollary 4.4. Let $C \in B(\mathcal{F}) \cap \mathcal{A}_s$ and write I for $I_{\mathcal{F}^s}$. Then, for $u \in \mathcal{F}_s$, $C(u \otimes \delta_\varnothing) = CP_s(u \otimes \delta_\varnothing) = P_sC(u \otimes \delta_\varnothing) = u' \otimes \delta_\varnothing$ for some $u' \in \mathcal{F}_s$. Therefore, for $v \in \mathcal{F}^s$, $C(u \otimes v)(\sigma) = CP_sD_{\sigma_{(s}}(u \otimes v)(\sigma_{s)}) = v(\sigma_{(s})C(u \otimes \delta_\varnothing)(\sigma_{s)}) = (u' \otimes v)(\sigma)$. It follows that $C = \widetilde{C} \otimes I$ for an operator $\widetilde{C}$ in $B(\mathcal{F}_s)$. Conversely, if $\widetilde{C} \in B(\mathcal{F}_s)$, then $\widetilde{C} \odot I$ is s-adapted by (x) of Theorem 4.3, and $\widetilde{C} \otimes I = \mathbb{E}_s[\widetilde{C} \odot I]$ by the corollary, so $\widetilde{C} \otimes I$ is s-adapted. □

4.3. *Fock operator processes.* Let H be a *Fock operator process*, that is, a family $(H_s)_{s \geq 0}$ of operators on $\mathcal{F}$. The *process domain* of H, denoted $\mathbb{P}\,\mathrm{Dom}\, H$, is the intersection of the domains of its constituent operators; H is *measurable*, or *continuous*, if, for each $f \in \mathbb{P}\,\mathrm{Dom}\, H$, the map $s \mapsto H_s f$ is measurable (resp. continuous); and H is *adapted*, or *bounded*, if each H_s is s-adapted (resp. bounded). To H we associate a process $\widehat{H}$ by

$$\widehat{H}_s = \mathbb{E}_s[H_s], \qquad s \geq 0. \tag{4.1}$$

Thus, $\widehat{H}$ is an adapted Fock operator process and, when H itself is adapted, $\widehat{H}_s \supset H_s$ for each s. Therefore, when applied to adapted processes, this procedure systematically extends the domain of the process so that it becomes *maximally adapted*: $\mathbb{E}_s[\widehat{H}_s] = \widehat{H}_s$. This is helpful for dealing with unbounded operator-valued processes—in particular, for providing a robust definition of a martingale. A process H will be called a *Fock operator martingale* if it is adapted and satisfies

$$\mathbb{E}_s[\widehat{H}_t] \subset \widehat{H}_s, \qquad t \geq s.$$

By the tower property of conditional expectations [Theorem 4.3(iii)], this may be written in the equivalent form $\mathbb{E}_s[H_t] \subset \mathbb{E}_s[H_s]$. If H is a martingale and satisfies

$$\mathbb{E}_s[H_\infty] \subset \widehat{H}_s, \qquad s \geq 0,$$

for some operator H_∞, then H is said to be *complete with closure* H_∞. For any operator C on $\mathcal{F}$, $(\mathbb{E}_s[C])_{s \geq 0}$ defines a complete martingale with closure C—such martingales are called *exact*. Note that closures are nonunique (every martingale has a truly trivial closure!).

A pair of Fock operator processes $(H, H^\dagger)$ is called an *adjoint pair* of processes if each $(H_s, H_s^\dagger)$ is an adjoint pair of operators [see (1.4)]. As we have already remarked, adaptedness of H does not automatically entail adaptedness of $H^\dagger$ when the process is unbounded. On the other hand, whenever the process domain of H is dense, Bessel's equality using an orthonormal basis drawn from $\mathbb{P}\,\mathrm{Dom}\, H$, shows that $H^\dagger$ is measurable if H is.

Let $\mathcal{A}$ denote the collection of adapted Fock operator processes, let $\mathcal{A}^\ddagger = \{H \in \mathcal{A} : H_s \in \mathcal{A}_s^\ddagger \text{ for each } s\}$, let $\mathcal{A}^b = \{H \in \mathcal{A} : H \text{ is bounded, with process domain } \mathcal{F}\}$ and let $\mathcal{M}$ denote the collection of Fock operator martingales. Propositions 3.3 and 4.5 give the following result.

PROPOSITION 4.6. *$\mathcal{A}$ is closed under sums, products and scalar multiples; $\mathcal{A}^\ddagger$ is closed under adjoints; and $\mathcal{A}^b$ is a unital *-algebra contained in $\mathcal{A}^\ddagger$.*

Due to the unavoidable inclusion relations involved in the definition of martingales, there is a dirth of algebraic properties of $\mathcal{M}$. However, the sum of two exact martingales is a complete martingale, and the collection of bounded operator-valued martingales forms a linear space closed under adjoints. Moreover, the following *-subalgebra of $\mathcal{A}^b$ has been investigated in [2]:

$$\begin{aligned}\mathcal{A}^r = \{H \in \mathcal{A}^b : \exists \text{ Radon measure } \mu \text{ such that } \forall\, t > s > 0 \text{ and } f \in \mathcal{F}_s \\ \text{s.t. } \|f\| = 1, \|(H_t - H_s)f\|^2 + \|(H_t^* - H_s^*)f\|^2 \\ + \|(P_s H_t - H_s)f\| \le \mu([s, t[)\},\end{aligned}$$

where, following [38], its elements have been called *regular semimartingales*; they are shown to be expressible as sums of QS integrals of processes in $\mathcal{A}^b$; moreover, the resulting integrands are characterized.

5. QS integrals. In this section we introduce new definitions of stochastic integrals, with respect to the basic processes of QS calculus, for adapted Fock operator processes. The technical core is Section 5.2 on commutation relations between the noncommutative stochastic integrals and sections of the adapted gradient operator. It is also verified that the integrals produce martingales and that the martingales are complete.

5.1. *Definitions.* Let H be an adapted Fock operator process. Allowing Q to stand for either P or D, let $V^Q(H)$ denote the subspace of $\mathcal{F}$ consisting of those vectors f for which:

- $Q_s D_\tau f \in \operatorname{Dom} H_s$ for a.a. $(s < \tau)$, and
- the a.e. defined $\mathcal{F}$-valued map $(s, \tau) \mapsto \mathbf{1}_{\{\tau > s\}} H_s Q_s D_\tau f$ is measurable.

For each f in $V^Q(H)$ there is a measurable $\mathfrak{h}$-valued map, written $(\omega, s) \mapsto H_s^Q f(\omega)$, such that $\mathbf{1}_{\Gamma_s}(\cdot) H_s^Q f(\cdot \cup \tau)$ is a representative of $H_s Q_s D_\tau f$ for a.a. $(s < \tau)$. This map is uniquely defined up to a set of measure 0, and satisfies the a.e. identity

$$H_s^Q f(\omega) = (H_s Q_s D_{\omega_{(s}} f)(\omega_{s)}).$$

We emphasize here that, for each $s \ge 0$, while $H_s^Q f$ is a measurable map $\Gamma \to \mathfrak{h}$, it need not be square integrable—in other words, in general, $[H_s^Q f] \notin \mathcal{F}$. Thus, $H_\cdot^Q f$ should *not* be thought of as a Fock vector process—in general, it is not. However, the following result describes subspaces of vectors f for which the maps $H_\cdot^Q f$ simplify, and it also gives conditions on H for the spaces $V^Q(H)$ to have a simple description. Recall that the domain of the stochastic gradient ∇

coincides with that of $\sqrt{N}$. Along with $V^P(H)$ and $V^D(H)$ we associate to H two additional subspaces of $\mathcal{F}$:

$$V(H) := \{f \in \mathbb{P}\,\mathrm{Dom}\,H : s \mapsto H_s f \text{ is measurable}\},$$

$$V^{\nabla}(H) := \{f \in \mathrm{Dom}\,\nabla : \nabla_s f \in \mathrm{Dom}\,H_s \text{ for a.a. } s;\ s \mapsto H_s \nabla_s f \text{ is measurable}\}.$$

Here H is a Fock operator process which is not assumed to be adapted.

PROPOSITION 5.1. *Let H be a Fock operator process.*

(a) *Suppose that H is adapted.*

(i) *If $f \in V(H)$, then $f \in V^P(H)$ and $[H_s^P f] = H_s f$ for a.a. s.*
(ii) *If $f \in V^{\nabla}(H)$, then $f \in V^D(H)$ and $[H_s^D f] = H_s \nabla_s f$ for a.a. s.*

(b) *Suppose that H is measurable and adapted. Then $V^P(H)$ contains $\mathbb{P}\,\mathrm{Dom}\,H$.*

(c) *Suppose that H has an adjoint process $H^{\dagger}$ which is measurable and has dense process domain. Then*:

(i) $V^{\nabla}(H) = \{f \in \mathrm{Dom}\sqrt{N} : \nabla_s f \in \mathrm{Dom}\,H_s$ *for a.a.* $s\}$.
(ii) *If H is also adapted*, *then*

$$V^Q(H) = \{f \in \mathcal{F} : Q_s D_\tau f \in \mathrm{Dom}\,H_s \text{ for a.a. } (s < \tau)\}.$$

(d) *Suppose that H is measurable and bounded*, *with process domain $\mathcal{F}$. Then*:

(i) $V^{\nabla}(H) = \mathrm{Dom}\sqrt{N}$.
(ii) *If H is also adapted*, *then* $V^D(H) = V^P(H) = \mathcal{F}$.

PROOF. (ai) This follows easily from the a.e. reproducing property.

(aii) Let $f \in V^{\nabla}(H)$. Then $f \in \mathrm{Dom}\sqrt{N}$ for a.a. $(s < \tau)$ $P_s D_\tau \nabla_s f = P_s \nabla_s D_\tau f = D_s D_\tau f$ and for a.a. s, $\nabla_s f \in \mathrm{Dom}\,H_s$. Hence, if H is adapted, $D_s D_\tau f \in \mathrm{Dom}\,H_s$ for a.a. $(s < \tau)$, and $\mathbf{1}_{\{\tau > s\}} H_s D_s D_\tau f = \mathbf{1}_{\{\tau > s\}} P_s D_\tau H_s \nabla_s f$, which is a measurable function of (s, τ). Hence, $f \in V^D(H)$ and, for a.a. (s, ω), $H_s^D f(\omega) = H_s \nabla_s f(\omega)$ by the a.e. reproducing property.

(b) This is immediate.

(c) Let H satisfy the condition of (c) and let (e_n) be an orthonormal basis for $\mathcal{F}$ selected from $\mathbb{P}\,\mathrm{Dom}\,H^{\dagger}$.

(i) If $f \in \mathrm{Dom}\sqrt{N}$ and $\nabla_s f \in \mathrm{Dom}\,H_s$ for a.a. s, then, by Bessel's equality, $H_s \nabla_s f = \sum_n \langle H_s^{\dagger} e_n, \nabla_s f\rangle e_n$ for a.a. s. But this is manifestly a measurable function of s; therefore, $f \in V^{\nabla}(H)$.

(ii) If H is adapted and $f \in \mathcal{F}$ satisfies $Q_s D_\tau f \in \mathrm{Dom}\,H_s$ for a.a. $(s < \tau)$, then, by another application of Bessel's equality, for a.a. $(s < \tau)$,

$$\mathbf{1}_{\{\tau > s\}} H_s Q_s D_\tau f = \mathbf{1}_{\{\tau > s\}} \sum_n \langle H_s^{\dagger} e_n, Q_s D_\tau f\rangle e_n,$$

which is a measurable function of (s, τ). Thus, $f \in V^Q(H)$.

(d) This is a special case of (c). $\square$

The *creation, number, annihilation and time integrals* of an adapted Fock operator process H are given, respectively, by the (a.e. defined) actions:

$$A^{\dagger}(H)f = \mathcal{S}(H^P f) : \omega \mapsto \sum_{s \in \omega} (H_s P_s D_{\omega_{(s}} f)(\omega_{s)}),$$

$$N(H)f = \mathcal{S}(H^D f) : \omega \mapsto \sum_{s \in \omega} (H_s D_s D_{\omega_{(s}} f)(\omega_{s)}),$$

$$A(H)f = \mathcal{L}(H^D f) : \omega \mapsto \int (H_s D_s D_{\omega_{(s}} f)(\omega_{s)})\, ds,$$

$$T(H)f = \mathcal{L}(H^P f) : \omega \mapsto \int (H_s P_s D_{\omega_{(s}} f)(\omega_{s)})\, ds,$$

with the following natural domains:

$$\operatorname{Dom} A^{\dagger}(H) = \{f \in V^P(H) : H^P f \text{ is Skorohod integrable}\},$$
$$\operatorname{Dom} N(H) = \{f \in V^D(H) : H^D f \text{ is Skorohod integrable}\},$$
$$\operatorname{Dom} A(H) = \{f \in V^D(H) : H^D f \text{ is time integrable}\},$$
$$\operatorname{Dom} T(H) = \{f \in V^P(H) : H^P f \text{ is time integrable}\}.$$

Recall (4.1) defining the extension of an adapted Fock operator process H to its maximally adapted form $\widehat{H}$. From the remarks following Proposition 4.2, it follows that $V^Q(H) = V^Q(\widehat{H})$ and $\widehat{H}^Q f = H^Q f$ for $f \in V^Q(H)$ and $Q = P$ or D. Therefore, each of the QS integrals is unaffected by allowing the integrand to achieve its maximally adapted form:

$$\Lambda(H) = \Lambda(\widehat{H}),$$

where here, and from now on, Λ stands for a generic QS integrator.

The following linear relations are clear from the definitions:

$$V^Q(H+K) \supset V^Q(H) \cap V^Q(K), \qquad V^Q(\lambda H) = V^Q(H), \qquad V^Q(0) = \mathcal{F},$$
$$\Lambda(H+K) \supset \Lambda(H) + \Lambda(K), \qquad \Lambda(\lambda H) = \lambda \Lambda(H), \qquad \Lambda(0) = 0,$$

where H and K are adapted Fock operator processes and $\lambda \in \mathbb{C} \setminus \{0\}$. Multiplicative relations between the QS integrals constitute the quantum Itô product formulas, to be described in the final section.

Notice that each of the QS integrals is associated with either adapted differentiation or projection and with either Skorohod or time integration. It will considerably simplify the development of the basic theory if we forge a unified notation to describe the integrals. Thus, to each QS integrator Λ, we associate $\mathcal{R}^{\Lambda} \in \{\mathcal{S}, \mathcal{L}\}$ as well as $R^{\Lambda} \in \{P, D\}$ and $Q^{\Lambda} \in \{P, D\}$ as follows: for $\Lambda = A^{\dagger}, N, A$ or T, the triple $(\mathcal{R}, R, Q)$, respectively, equals

$$(5.1) \qquad (\mathcal{S}, D, P), \qquad (\mathcal{S}, D, D), \qquad (\mathcal{L}, P, D) \quad \text{or} \quad (\mathcal{L}, P, P).$$

Thus, Λ is determined by the pair $(\mathcal{R}^\Lambda, Q^\Lambda)$, R^Λ is determined by $\mathcal{R}^\Lambda$ and vice versa. The definitions of the four QS integrals are thereby unified:

$$\begin{aligned}\operatorname{Dom}\Lambda(H) &= \{f \in V^Q(H) : H^Q f \text{ is } \mathcal{R}\text{-integrable}\},\\ \Lambda(H)f &= \mathcal{R}(H^Q f),\end{aligned}$$

where $Q = Q^\Lambda$ and $\mathcal{R} = \mathcal{R}^\Lambda$. The notation R^Λ will come into its own when the fundamental formulas are extended (Section 6.3).

One further notation—each QS integrator has an adjoint integrator:

$$(A^\dagger)^\dagger = A, \qquad N^\dagger = N, \qquad T^\dagger = T.$$

Having found a compact expression for the integrals, let us unravel somewhat, to get a better view of their workings. Let H be an adapted Fock operator process and let $f \in V^Q(H)$. Then, for $\tau = \{t_1 < \cdots < t_n\}$, the following a.e. relation holds:

$$H_s^Q f(\tau) = \sum_{k=0}^{n} \mathbf{1}_{[t_k, t_{k+1}[}(s)\big(H_s Q_s D_{t_{k+1}} \cdots D_{t_n} f\big)(t_1, \ldots, t_k),$$

where $t_0 = 0$ and $t_{n+1} = \infty$. Thus,

$$\mathcal{S}(H^Q f)(\tau) = \sum_{k=1}^{n} \big(H_{t_k} Q_{t_k} D_{t_{k+1}} \cdots D_{t_n} f\big)(t_1, \ldots, t_{k-1}), \tag{5.2a}$$

and, if $s \mapsto H_s^Q f(\tau)$ is integrable,

$$\mathcal{L}(H^Q f)(\tau) = \sum_{k=0}^{n} \int_{t_k}^{t_{k+1}} \big(H_s Q_s D_{t_{k+1}} \cdots D_{t_n} f\big)(t_1, \ldots, t_k). \tag{5.2b}$$

Therefore, (5.2a) and (5.2b) are a.e. expressions for $\Lambda(H)f(\tau)$ when $f \in \operatorname{Dom}\Lambda(H)$, for $\Lambda = A^\dagger$ or N, respectively, A or T.

The following identities are easily established.

LEMMA 5.2. *Let H be an adapted Fock operator process and let $f \in V^Q(H)$, where $Q = P$ or D. Then the following relations hold* (*a.e.*):

$$D_t f, P_t f \in V^Q\big(\mathbf{1}_{[0,t[} H\big), \tag{5.3a}$$

$$D_t H_s^Q f = \mathbf{1}_{[0,t[}(s) H_s^Q D_t f + D_t H_s Q_s f, \tag{5.3b}$$

$$P_t H_s^Q f = \mathbf{1}_{[0,t[}(s) H_s^Q P_t f + \mathbf{1}_{[t,\infty[}(s) P_t H_s Q_s f, \tag{5.3c}$$

$$P_t H_t^Q f = H_t Q_t f. \tag{5.3d}$$

In particular, $\mathbf{1}_{[0,t[}(s)(D_t H_s^Q f)(\omega)$ gives a version of $\mathbf{1}_{[0,t[}(s)(H_s^Q D_t f)(\omega)$ which is jointly measurable in s, t, ω.

5.2. *Commutation relations.* We next apply the commutation relations obtained in Section 2.5 to QS integrals. This allows us to deduce adaptedness and martingale properties in the next section. It is also the first step toward solving the problems raised by the Itô calculus approach to QS calculus (see Section 7.2).

In the rest of the paper, $\Lambda_t(H)$ will be used to abbreviate $\Lambda(\mathbf{1}_{[0,t[}H)$.

THEOREM 5.3. *Let H be an adapted Fock operator process and let $f \in V^Q(H)$, where $Q = Q^\Lambda$ and $\Lambda = A^\dagger$ or N. Then the following are equivalent*:

(a) $f \in \operatorname{Dom}\Lambda(H)$.

(b) *$D_t f \in \operatorname{Dom}\Lambda_t(H)$ for a.a. t, and $t \mapsto \Lambda_t(H)D_t f + H_t Q_t f$ is Itô integrable.*

When these hold we have the a.e. identity

$$D_t \Lambda(H) f = \Lambda_t(H) D_t f + H_t Q_t f. \tag{5.4}$$

PROOF. In view of (5.3d), Proposition 2.6 applies to $H^Q_{\cdot} f$. If $f \in \operatorname{Dom}\Lambda(H)$, then $H^Q_{\cdot} f$ is Skorohod integrable so $\mathbf{1}_{[0,t[}(\cdot) D_t H^Q_{\cdot} f$ is Skorohod integrable for a.a. t, and

$$D_t \mathcal{S}(H^Q_{\cdot} f) = \mathcal{S}_0^t(D_t H^Q_{\cdot} f) + P_t H^Q_t f,$$

which is square integrable in t. By (5.3a) and (5.3b), f satisfies (b), and the a.e. identity (5.4) holds. Conversely, if f satisfies (b), then, since (5.3b) implies that

$$\Lambda_t(H) D_t f = \mathcal{S}_0^t(H^Q_{\cdot} D_t f) = \mathcal{S}_0^t(D_t H^Q_{\cdot} f),$$

Proposition 2.6 gives the Skorohod integrability of $H^Q_{\cdot} f$—in other words, $f \in \operatorname{Dom}\Lambda(H)$. $\square$

THEOREM 5.4. *Let H be an adapted Fock operator process and let $f \in V^Q(H)$ be such that $(H.Q.f)(\varnothing)$ is integrable, where $Q = Q^\Lambda$ and $\Lambda = A$ or T. Then conditions* (a) *and* (b) *are equivalent*:

(a) (i) $f \in \operatorname{Dom}\Lambda(H)$; (ii) *$H.Q.f$ is time integrable.*

(b) (i) *$D_t f \in \operatorname{Dom}\Lambda_t(H)$ for a.a. t*; (ii) *$D_t H.Q.f$ is time integrable for a.a. t*; (iii) *the maps $t \mapsto \Lambda_t(H) D_t f$ and $t \mapsto \mathcal{L}(D_t H.Q.f)$ are Itô integrable.*

When these hold we have the a.e. identity,

$$D_t \Lambda(H) f = \Lambda_t(H) D_t f + \mathcal{L}(D_t H.Q.f). \tag{5.5}$$

PROOF. In view of (5.3d), $H^Q_{\cdot} f(\varnothing)$ is integrable and so Proposition 2.7 applies. If $f \in \operatorname{Dom}\Lambda(H)$ and $H.Q.f$ is time integrable, then, by Proposition 2.7, both $D_t H^Q_{\cdot} f$ and $D_t H.Q.f$ are time integrable for a.a. t, and

$$\mathcal{L}(D_t H^Q_{\cdot} f) = D_t \mathcal{L}(H^Q_{\cdot} f) = D_t \Lambda(H) f,$$
$$\mathcal{L}(D_t H.Q.f) = D_t \mathcal{L}(H.Q.f),$$

both of which are square integrable in t. By (5.3b), therefore, $D_t f \in \operatorname{Dom}\Lambda_t(H)$, (5.5) holds and both $\Lambda(H)D_t f$ and $\mathcal{L}(D_t H.Q.f)$ are square integrable in t. Thus, f satisfies (b). Conversely, if f satisfies (b), then, by (5.3b), $D_t H^Q_{\boldsymbol{\cdot}} f = \mathbf{1}_{[0,t[}(\cdot)H^Q_{\boldsymbol{\cdot}} D_t f + D_t H.Q.f$, which is time integrable by (bi) and (bii), with time integral $\Lambda_t(H)D_t f + \mathcal{L}(D_t H.Q.f)$, which is Itô integrable by (biii). Hence, using Proposition 2.7 once more, $H^Q_{\boldsymbol{\cdot}} f$ is time integrable—in other words, $f \in \operatorname{Dom}\Lambda(H)$, so that (a) holds. $\square$

PROPOSITION 5.5. *Let H be an adapted Fock operator process and let $t \geq 0$.*

(a) *If $f \in \operatorname{Dom}\Lambda(H)$, where $\Lambda = A^\dagger$ or N, then $P_t f \in \operatorname{Dom}\Lambda_t(H)$ and*

$$\Lambda_t(H)P_t f = P_t\Lambda(H)f.$$

(b) *If $f \in \operatorname{Dom}\Lambda(H)$, where $\Lambda = A$ or T, then z^t is time integrable and*

$$\mathcal{L}(z^t) = P_t\Lambda(H)f,$$

where $z^t = (\mathbf{1}_{[0,t[}(\cdot)H^Q_{\boldsymbol{\cdot}} P_t f + \mathbf{1}_{[t,\infty[}(\cdot)P_t H.Q.f)$ and $Q = Q^\Lambda$.

(c) *If $f \in \mathcal{F}$ and $P_t f \in \operatorname{Dom}\Lambda(H)$, where $\Lambda = A^\dagger, N$ or A, then $P_t f \in \operatorname{Dom}\Lambda_t(H)$ and*

$$\Lambda_t(H)P_t f = P_t\Lambda(H)P_t f.$$

(d) *If $\Lambda = N$ or A, then the subspaces $\mathcal{F}_t \cap \operatorname{Dom}\Lambda(H)$ and $\mathcal{F}_t \cap \operatorname{Dom}\Lambda_t(H)$ coincide and*

$$\Lambda_t(H)P_t f = \Lambda(H)P_t f,$$

whenever $P_t f \in \operatorname{Dom}\Lambda_t(H)$.

PROOF. Each of these commutation relations follows easily from Proposition 2.8 by using (5.3c). $\square$

5.3. *Adaptedness and martingale properties.* The next two results confirm that our definitions synchronize satisfactorily.

PROPOSITION 5.6. *Let H be an adapted Fock operator process and let $t \geq 0$. Then, for each QS integrator Λ, the operator $\Lambda_t(H)$ is u adapted for each $u \geq t$.*

PROOF. Let $f \in \operatorname{Dom}\Lambda_t(H)$ and let $u \geq t$. First, note that, by (5.3a), $P_u f, D_u f \in V^Q(\mathbf{1}_{[0,t[}H)$. Since $\Lambda_u(\mathbf{1}_{[0,t[}H) = \Lambda_t(H)$ and $\mathbf{1}_{[u,\infty[}(\cdot)P_u \times \mathbf{1}_{[0,t[}(\cdot)H.Q.f = 0$, Proposition 5.5 implies that $P_u f \in \operatorname{Dom}\Lambda_t(H)$ and $P_u\Lambda_t(H)f = \Lambda_t(H)P_u f$. Since $\mathbf{1}_{\{u<t\}}H_u Q_u f = 0$, Theorem 5.3 implies that $D_u f \in \operatorname{Dom}\Lambda_t(H)$ and $\Lambda_t(H)D_u f = D_u\Lambda_t(H)f$ for $\Lambda = A^\dagger$ or N. For $\Lambda = A$ or T, (5.3b) implies that $D_u H^Q_s f = H^Q_s D_u f$ for $s < t$. Therefore, by Proposition 2.7, $\mathbf{1}_{[0,t[}H^Q_{\boldsymbol{\cdot}} D_u f$ is time integrable, so that $D_u f \in \operatorname{Dom}\Lambda_t(H)$ and

$$\Lambda_t(H)D_u f = \mathcal{L}_t(H^Q_{\boldsymbol{\cdot}} D_u f) = D_u\mathcal{L}_t(H^Q f) = D_u\Lambda_t(H)f.$$

This completes the proof. $\square$

THEOREM 5.7. *Let H be an adapted Fock operator process. Then, for $\Lambda = A^\dagger$, N or A, $\Lambda.(H)$ is a complete martingale with closure $\Lambda(H)$.*

PROOF. Let $t \geq 0$, let $u \in [t, \infty]$ and let $f \in \mathrm{Dom}(\mathbb{E}_t[\Lambda_u(H)])$. Then $P_t D_\beta f \in \mathrm{Dom}(\Lambda_u(H))$, the map $\beta \mapsto \mathbf{1}_{\{\beta > t\}} P_t \Lambda_u(H) P_t D_\beta f$ is square integrable $\Gamma \to \mathcal{F}$ and $\mathbb{E}_t[\Lambda_u(H)] f(\omega) = (P_t \Lambda_u(H) P_u D_{\omega_{(t}} f)(\omega_{t)})$ for a.a. ω. Thus, by part (c) of Theorem 5.5, $f \in \mathrm{Dom}\, \mathbb{E}_t[\Lambda_t(H)]$ and $\mathbb{E}_t[\Lambda_t(H)] f = \mathbb{E}_t[\Lambda_u(H)] f$. This shows that

$$\mathbb{E}_t[\Lambda_u(H)] \subset \mathbb{E}_t[\Lambda_t(H)], \tag{5.6}$$

and so $\Lambda.(H)$ is a complete martingale, with closure $\Lambda(H)$. $\square$

REMARK. In view of Proposition 5.5(d), we have the following cases of equality in the complete martingale inclusion relations (5.6):

$$\mathbb{E}_t[N_u(H)] = \mathbb{E}_t[N_t(H)], \qquad \mathbb{E}_t[A_u(H)] = \mathbb{E}_t[A_t(H)].$$

In other words, the martingales $\mathbb{E}.[N.(H)]$ and $\mathbb{E}.[A.(H)]$ are exact.

5.4. *Recursion formula.* The commutation relations between adapted gradient and QS integrals lead to recursion formulas for the integrals. This is the second step toward solving the problems raised by the Itô calculus approach.

THEOREM 5.8. *Let H be an adapted Fock operator process and let $f \in \mathrm{Dom}\,\Lambda(H)$. If $H.Q.f$ is $\mathcal{R}$-integrable, then the following is well defined and valid:*

$$\Lambda(H) f = \mathfrak{I}\big(\Lambda.(H) D.f\big) + \mathcal{R}(H.Q.f),$$

where $\mathcal{R} = \mathcal{R}^\Lambda$ and $Q = Q^\Lambda$.

PROOF. Theorem 5.3 (resp. Theorem 5.4) applies. Noting that $H.Q.f$ is adapted, so that if $\mathcal{R} = \mathcal{S}$ then $\mathcal{R}(H.Q.f) = \mathfrak{I}(H.Q.f)$, the result follows by Itô integration of (5.4) [resp. (5.5)]. $\square$

6. Restricted-domain QS integrals. In this section we introduce restricted domains for QS integrals, which lead to good adjoint relations, as well as extensions of the fundamental formulas of QS calculus. These latter are a cornerstone of the calculus [37].

6.1. *Definition and example.* Let H be an adapted Fock operator process. We define the *restricted-domain QS integral* ${}^R\Lambda(H)$ to be the restriction of $\Lambda(H)$ to

$$\mathrm{Dom}\,{}^R\Lambda(H) := \{f \in V^Q(H) : H^Q f \text{ if absolutely } \mathcal{R}\text{-integrable}\},$$

where $Q = Q^\Lambda$ and $\mathcal{R} = \mathcal{R}^\Lambda$ are as given by (5.1), and absolute $\mathcal{R}$-integrability is defined in Section 2.1.

A simplifying feature of restricted-domain QS integrals is the inclusions:

$$\operatorname{Dom} {}^R\Lambda_s(H) \supset \operatorname{Dom} {}^R\Lambda_t(H)$$

for $s \le t$; in particular, $\mathbb{P} \operatorname{Dom} {}^R\Lambda.(H) \subset \operatorname{Dom} {}^R\Lambda(H)$. Another is that the processes $t \mapsto {}^R\Lambda_t(H)$ are continuous (Proposition 6.4).

Before developing the theory, we illustrate the restriction by the example of Fermi field operators as QS integrals.

EXAMPLE. Let J be the unitary process defined by $J_s f(\omega) = (-1)^{\#\omega_{s)}} f(\omega)$. Then

$$J_s^P f(\omega) = (-1)^{\#\omega_{s)}} f(\omega) \quad \text{and} \quad J_s^D f(\omega) = (-1)^{\#\omega_{s)}} f(\omega \cup s).$$

For every f, $J^P f$ is Skorohod integrable and $J^D f$ is time integrable [4],

$$\mathcal{S}_t\big(\|J_\cdot^P f(\cdot)\|_{\mathfrak{h}}\big)(\omega) = \#\omega_{t)} \|f(\omega)\|$$

and

$$\mathcal{L}_t\big(\|J_\cdot^D f(\cdot)\|_{\mathfrak{h}}\big)(\omega) = \int_0^t \|f(\omega \cup s)\| \, ds$$

—neither of which is square integrable in ω in general—so $J^P f$ is not absolutely Skorohod integrable and $J^D f$ is not absolutely time integrable. Thus, whereas $A_t^\dagger(J)$ and $A_t(J)$ both have domain $\mathcal{F}$, and are, in fact, bounded, ${}^R A_t^\dagger(J)$ and ${}^R A_t(J)$ have strictly smaller domains. These operators are Fermi creation and annihilation field operators realized on boson Fock space [24].

LEMMA 6.1. *Let $f \in \operatorname{Dom} {}^R\Lambda(H)$, where H is an adapted Fock operator process and Λ is a QS integrator, and let $Q = Q^\Lambda$ and $\mathcal{R} = \mathcal{R}^\Lambda$. Then the adapted Fock vector process $H.Q.f$ is absolutely $\mathcal{R}$-integrable.*

PROOF. Since $f \in V^Q(H)$, the map $s \mapsto H_s Q_s f = \mathbf{1}_{\Gamma^s}(\varnothing) H_s Q_s D_\varnothing f$ is measurable $\mathbb{R}_+ \to \mathcal{F}$. Since $H_s Q_s f = \mathbf{1}_{\Gamma_s} H_s^Q f$, $H.Q.f$ is absolutely $\mathcal{R}$-integrable. $\square$

REMARK. By Proposition 2.2, since $H.Q.f$ is adapted, it is Itô integrable if $\Lambda = A^\dagger$ or N.

LEMMA 6.2. *Let $X = {}^R\Lambda.(H)$ for an adapted Fock operator process H and QS integrator Λ. If $f \in \mathbb{P} \operatorname{Dom} X$, then*:

(i) $P_t f \in \operatorname{Dom} X_t$ *for all* $t \ge 0$;
(ii) $D_t f \in \operatorname{Dom} X_t$ *for a.a.* t.

PROOF. Let $x(\omega, s) = \|H_s^Q f(\omega)\|$, so that x is $\mathcal{R}$-integrable, and, by (5.3b),

$$\mathbf{1}_{[0,t[}(s) P_t x_s(\omega) = \mathbf{1}_{[0,t[}(s) \|H_s^Q P_t f(\omega)\|$$

so that $P_t f \in \operatorname{Dom} {}^R\Lambda_t(H)$ for each $t \ge 0$. Using Proposition 2.7 and a.e. identity (5.3c) instead, the above argument yields (ii). $\square$

6.2. *Martingale and continuity properties.* We next show that Proposition 5.6 and Theorem 5.7 are also valid for restricted-domain QS integrals.

PROPOSITION 6.3. *Let H be an adapted Fock operator process and let Λ be one of the QS integrators. Then*:

(a) ${}^R\Lambda_t(H)$ *is u-adapted for each $u \geq t$*;

(b) *if $\Lambda = A^\dagger$, N or A, then ${}^R\Lambda.(H)$ is a complete martingale with closure ${}^R\Lambda(H)$.*

PROOF. Let $X = {}^R\Lambda(H)$—both as operator and as process—let $(\mathcal{R}, Q)$ be the pair associated with Λ according to (5.1) and let $t \geq 0$.

(a) In view of Proposition 5.6, it suffices to show that $\operatorname{Dom} X_t$ is a u-adapted subspace for each $u \geq t$. Let $f \in \operatorname{Dom} X_t$. Then the map $k : (\omega, s) \mapsto \mathbf{1}_{[0,t[}(s) \times \|H_s^Q f(\omega)\|$ is $\mathcal{R}$-integrable. By (5.3a)–(5.3c), if $v \geq u \geq t$, then $P_u f$, $D_v f \in V^Q(\mathbf{1}_{[0,t[}H)$,

$$\mathbf{1}_{[0,t]}(s)\|H_s^Q P_u f(\omega)\| = (P_u k_s)(\omega) \leq k_s(\omega), \tag{6.1a}$$

$$\mathbf{1}_{[0,t]}(s)\|H_s^Q D_v f(\omega)\| = (D_v k_s)(\omega) \tag{6.1b}$$

for a.a. (ω, v). By (6.1a), $P_u f \in \operatorname{Dom} X_t$, and by (6.1b), together with Propositions 2.6 and 2.7, $D_v f \in \operatorname{Dom} X_t$ for a.a. v. Thus, $\operatorname{Dom} X_t$ is u-adapted.

(b) By (a), $(X_s)_{s \geq 0}$ is an adapted Fock operator process so that, in view of Theorem 5.7, it suffices to show that $\operatorname{Dom} \mathbb{E}_t[X] \subset \operatorname{Dom} \mathbb{E}_t[X_t]$. Let $f \in \operatorname{Dom} \mathbb{E}_t[X]$, then $P_t D_\tau f \in \operatorname{Dom} X \subset \operatorname{Dom} X_t$, and, since $\Lambda.(H)$ is a complete martingale with closure $\Lambda(H)$,

$$P_t X_t P_t D_\tau f = P_t \Lambda_t(H) P_t D_\tau f = P_t \Lambda(H) P_t D_\tau f = P_t X P_t D_\tau f$$

for a.a. $\tau > t$. Therefore, $\tau \mapsto \mathbf{1}_{\Gamma^t}(\tau) P_t X_t P_t D_\tau f$ is square integrable—in other words, $f \in \operatorname{Dom} \mathbb{E}_t[X_t]$. This gives the required inclusion. $\square$

PROPOSITION 6.4. *Let H be an adapated Fock operator process and let Λ be a QS integrator. Then the process ${}^R\Lambda.(H)$ is continuous.*

PROOF. Let $X = {}^R\Lambda.(H)$, let $\mathcal{R} = \mathcal{R}^{\mathcal{R}}$ and $Q = Q^A$ according to (5.1) and let $f \in \mathbb{P} \operatorname{Dom} X$. Writing k for the map $(\omega, s) \mapsto \|H_s^Q f(\omega)\|$, the following holds pointwise:

$$\|(X_u f - X_r f)(\cdot)\|_{\mathfrak{h}} = \|\mathcal{R}_r^u (H^Q f)(\cdot)\|_{\mathfrak{h}} \leq \mathcal{R}_r^u(k).$$

Thus, if $\Lambda = A^\dagger$ or N,

$$\|X_u f - X_r f\|^2 \leq \int_r^u \int \{k_s(\omega)\}^2 \, d\omega \, ds + \int_r^u \int_r^u \int k_s(\omega \cup t) k_t(\omega \cup s) \, d\omega \, dt \, ds,$$

which is finite by Proposition 2.1. If $\Lambda = A$ or T, then

$$\|X_u f - X_r f\|^2 \leq \int \left\{ \int_r^u k_s(\omega)\, ds \right\}^2 d\omega < \infty.$$

Thus, continuity follows in all four cases by the monotone convergence theorem. □

6.3. *Fundamental formulas and adjoint relations.* Our next result is an extension of the *first fundamental formula* for QS calculus [23, 37] beyond exponential domains.

PROPOSITION 6.5. (a) *Let H be an adapted Fock operator process. If $f \in \operatorname{Dom}{}^R\Lambda(H)$, then, for all $g \in \mathcal{F}$, the map*

$$(s, \beta) \mapsto \mathbf{1}_{\{\beta > s\}} \langle H_s Q_s D_\beta f, R_s D_\beta g \rangle \tag{6.2}$$

is integrable and

$$\iint_{\{\beta > s\}} \langle H_s Q_s D_\beta f, R_s D_\beta g \rangle \, d\beta\, ds = \langle \Lambda(H) f, g \rangle, \tag{6.3}$$

where $Q = Q^\Lambda$ and $R = R^\Lambda$ are as given in (5.1).

(b) *Let $(H, H^\dagger)$ be an adjoint pair of adapted Fock operator processes. If $f \in \operatorname{Dom} \Lambda(H)$, $g \in \operatorname{Dom} A^\dagger(H^\dagger)$ and the map* (6.2) *is integrable, then*

$$\langle \Lambda(H) f, g \rangle = \langle f, \Lambda(H^\dagger) g \rangle. \tag{6.4}$$

PROOF. In case (a) straightforward calculation leads to the estimate

$$\iint_{\{\beta > s\}} |\langle H_s Q_s D_\beta f, R_s D_\beta g \rangle| \, d\beta\, ds \leq \int h(\omega) \|g(\omega)\| \, d\omega,$$

where

$$h(\omega) = \begin{cases} \displaystyle\sum_{s \in \omega} \|(H_s Q_s D_{\omega_{(s}} f(\omega_{s)})\|, & \text{if } \Lambda = A^\dagger \text{ or } N, \\ \displaystyle\int \|(H_s Q_s D_{\omega_{(s}} f)(\omega_{s)})\| \, ds, & \text{if } \Lambda = A \text{ or } T. \end{cases}$$

Similar calculation also reveals the identity (6.3).

(b) If (6.2) is integrable, then

$$\begin{aligned} &\langle (H_s Q_s D_{\omega_{(s}} f)(\omega_{s)}), (R_s D_{\omega_{(s}} g)(\omega_{s)}) \rangle \\ &\qquad = \langle (Q_s D_{\omega_{(s}} f)(\omega_{s)}), (H_s^\dagger R_s D_{\omega_{(s}} f)(\omega_{s)}) \rangle \end{aligned}$$

is an identity of integrable functions of (ω, s), which integrates up to (6.4). □

COROLLARY 6.6. *Let $(H, H^\dagger)$ be an adjoint pair of adapted Fock operator processes.*

(a) *Then* $({}^R\Lambda(H^\dagger), {}^R\Lambda(H))$ *is also an adjoint pair of adapted Fock operator processes*.

(b) *If* ${}^R\Lambda(H)$ *is densely defined*, *then* $({}^R\Lambda(H))^* \supset \Lambda(H^\dagger)$.

The next result is an integration by parts lemma which contains the essential part of one form of the quantum Itô product formula described in the final section. It is an extension of the *second fundamental formula* for QS calculus [23, 37] beyond expontential domains.

THEOREM 6.7. *Let* F *and* G *be adapted Fock operator processes and let* Λ *and* Λ' *be QS integrators*. *If* $f \in \operatorname{Dom} {}^R\Lambda(F)$ *and* $g \in \operatorname{Dom} {}^R\Lambda'(G)$, *then*

$$\begin{aligned} &\langle \Lambda(F)f, \Lambda'(G)g\rangle \\ &= \iint_{\{\beta>t\}} \big[\langle F_t Q_t D_\beta f, Y_t R_t D_\beta g\rangle \\ &\qquad + \langle X_t R'_t D_\beta f, G_t Q'_t D_\beta g\rangle + \epsilon\langle F_t Q_t D_\beta f, G_t Q'_t D_\beta g\rangle\big]\, d\beta\, dt, \end{aligned} \tag{6.5}$$

where $X = {}^R\Lambda_{.}(F)$ *and* $Y = {}^R\Lambda'_{.}(G)$, ϵ *equals* 1 *if* $\{\Lambda, \Lambda'\} \subset \{A^\dagger, N\}$ *and equals* 0 *otherwise*, *and* $R = R^\Lambda$, $Q = Q^\Lambda$, $R' = R^{\Lambda'}$ *and* $Q' = Q^{\Lambda'}$, *according to* (5.1).

PROOF. First, note that $f \in \mathbb{P}\operatorname{Dom} X_.$ and, by Lemma 6.1, $F_.Q_.f$ is Itô integrable if $\Lambda = A^\dagger$ or N and is absolutely time integrable if $\Lambda = A$ or T. Moreover, successive application of Lemma 6.2 gives $D_u f \in \operatorname{Dom} X_u$, so $D_\beta f \in \operatorname{Dom} X_{\wedge\beta}$ and thus $E_t D_\beta f \in \operatorname{Dom} X_t$ for a.a. u, β and $(t < \beta)$, where E is either P or D. Similarly for Y, G, Q' and g. Therefore, since also $f \in V^Q(F)$ and $g \in V^{Q'}(G)$, each of the expressions in the integrand is a.e. well defined.

Now

$$\langle \Lambda(F)f, \Lambda'(G)g\rangle = \big\langle \mathcal{R}(F^Q f), \mathcal{R}'\big(G^{Q'}g\big)\big\rangle,$$

whereas $F^Q f$ is absolutely $\mathcal{R}$-integrable and $G^{Q'}g$ is absolutely $\mathcal{R}'$-integrable, where $\mathcal{R} = \mathcal{R}^\Lambda$ and $\mathcal{R}' = \mathcal{R}^{\Lambda'}$.

CASE (a): $\{\Lambda, \Lambda'\} \subset \{A, T\}$. Then $R = R' = P$, $F^Q f$ and $G^{Q'}g$ are absolutely time integrable, and Fubini's theorem ensures both the integrability of the function $\Psi : (\omega, t, u) \mapsto \langle F_t^Q f(\omega), G_u^{Q'}(\omega)\rangle$ and that its integral is $\langle Xf, Yg\rangle$. Integrating Ψ first over the region $\{t < u\}$ using the u-adaptedness of X_u, the a.e. reproducing property (2.6a) and (5.3b) and (5.3d) gives

$$\begin{aligned} &\iint \big\langle X_u f(\omega), G_u^{Q'} g(\omega)\big\rangle\, du\, d\omega \\ &\quad = \iint \big\langle (X_u P_u D_{\omega_{(u}} f)(\omega_{u)}), \big(P_u G_u^{Q'} D_{\omega_{(u}} g)(\omega_{u)})\big\rangle\, d\omega\, du \\ &\quad = \iint_{\{\beta>t\}} \langle X_t R'_t D_\beta f, G_t Q'_t D_\beta g\rangle\, d\beta\, dt. \end{aligned}$$

The integral of Ψ over the region $\{u < t\}$ may be obtained by the same argument via complex conjugation, and the sum of the two agrees with (6.5).

CASE (b): $\Lambda \in \{A^\dagger, N\}$, $\Lambda' \in \{A, T\}$. Then $R = D$ and $R' = P$; $F^Q f$ is absolutely Skorohod integrable and $G^{Q'} g$ is absolutely time integrable; moreover, Fubini's theorem together with the integral–sum lemma ensures both the integrability of the function

$$\Phi : (\omega, t, u) \mapsto \langle F_t^Q f(\omega), G_u^{Q'}(\omega \cup t)\rangle$$

and that the value of the integral is $\langle Xf, Yg\rangle$. Integrating Φ over the region $\{t < u\}$ and arguing as in Case (a) gives

$$\begin{aligned} &\iint \sum_{t\in\alpha} \mathbf{1}_{[0,u[}(t) \langle F_t^Q f(\alpha \setminus t), G_u^{Q'}(\alpha)\rangle \, d\alpha \, du \\ &\quad = \iint \langle (X_u f)(\alpha), G_u^{Q'}(\alpha)\rangle \, d\alpha \, du \\ &\quad = \iint_{\{\beta > t\}} \langle X_t R_t' D_\beta f, G_t Q_t' D_\beta g\rangle \, d\beta \, dt. \end{aligned}$$

Integrating Φ over the region $\{u < t\}$, we have, since $D_t D_\beta g \in \operatorname{Dom} Y_t$ for a.a. $(t < \beta)$,

$$\begin{aligned} &\iiint \mathbf{1}_{[0,t[}(u) \langle F_t^Q f(\omega), G_u^{Q'} D_t D_{\omega_{(t}} g)(\omega_{t)})\rangle \, d\omega \, du \, dt \\ &\quad = \iint \langle (P_t F_t^Q D_{\omega_{(t}} f)(\omega_{t)}), (Y_t D_t D_{\omega_{(t}} g)(\omega_{t)})\rangle \, d\omega \, dt \\ &\quad = \iint_{\{\beta > t\}} \langle F_t Q_t D_\beta f, Y_t R_t D_\beta g\rangle \, d\beta \, dt. \end{aligned}$$

Therefore, the result holds in this case.

CASE (c): $\Lambda \in \{A, T\}$ and $\Lambda \in \{A^\dagger, N\}$. This is simply the complex conjugate of Case (b).

CASE (d): $\{\Lambda, \Lambda'\} \subset \{A^\dagger, N\}$. Then $\epsilon = 1$, $R = R' = D$, $F^Q f$ and $G^{Q'} g$ are absolutely Skorohod integrable and the Skorohod isometry (2.1) ensures that both of the maps

$$\Phi : (\omega, t) \mapsto \langle F_t^Q f(\omega), G_t^{Q'} g(\omega)\rangle \quad \text{and}$$
$$\Psi : (\omega, t, u) \mapsto \langle F_t^Q f(\omega \cup u), G_u^{Q'} g(\omega \cup t)\rangle$$

are integrable and also that the sum of their integrals is $\langle Xf, Yg\rangle$. By the integral–sum lemma, the integral of Φ is simply

$$\iint_{\{\beta > t\}} \langle F_t Q_t D_\beta f, G_t Q_t' D_\beta g\rangle \, d\beta \, dt.$$

Since $D_u D_\gamma f \in \operatorname{Dom} X_u$ for a.a. $(u < \gamma)$, the integral of Ψ over the region $\{t < u\}$ is

$$\iiint \mathbf{1}_{[0,u[}(t)\big\langle (F_t^Q D_u D_{\omega(u} f)(\omega_{u)}), G_u^{Q'} g(\omega \cup t)\big\rangle \, dt \, d\omega \, du$$

$$= \iint \sum_{t \in \alpha_{u)}} \mathbf{1}_{[0,u[}(t)\big\langle (F_t^Q D_u D_{\alpha(u} f)(\alpha_{u)} \setminus t), \big(G_u^{Q'} D_{\alpha(u} g\big)(\alpha_{u)})\big\rangle \, d\alpha \, du$$

$$= \iint \big\langle (X_u D_u D_{\alpha(u} f)(\alpha_{u)}), \big(P_u G_u^{Q'} D_{\alpha(u} g\big)(\alpha_{u)})\big\rangle \, d\alpha \, du$$

$$= \iint_{\{\beta > t\}} \langle X_t D_t D_\beta f, G_t Q_t' D_\beta g\rangle \, d\beta \, dt.$$

Again, the integral of Ψ over the region $\{u < t\}$ is given by symmetry and yields the first term in (6.5). Thus, the result holds in this final case, too. $\square$

7. Relation to previous approaches. In this section we show that the integrals defined in the last two sections are consistent with previous approaches. Note that the Hudson–Parthasarathy definitions, on exponential domains, are subsumed by each of the noncausal and the Itô calculus formulations.

7.1. *Noncausal approach* ([9, 27]). Let H be a Fock operator process. Recall the notation at the beginning of Section 5. The noncausal QS integrals are defined as follows:

$$^{\mathrm{NC}}A^\dagger(H) f = \mathcal{S}(H.f), \qquad ^{\mathrm{NC}}N(H) f = \mathcal{S}(H.\nabla.f),$$
$$^{\mathrm{NC}}A(H) f = \int H_s \nabla_s f \, ds, \qquad ^{\mathrm{NC}}T(H) f = \int H_s f \, ds,$$

with respective domains

$$\operatorname{Dom} {}^{\mathrm{NC}}A^\dagger(H) = \{f \in V(H) : H.f \in \operatorname{Dom} \mathcal{S}\},$$
$$\operatorname{Dom} {}^{\mathrm{NC}}N(H) = \{f \in V^\nabla(H) : H.\nabla.f \in \operatorname{Dom} \mathcal{S}\},$$
$$\operatorname{Dom} {}^{\mathrm{NC}}A(H) = \{f \in V^\nabla(H) : H.\nabla.f \text{ is integrable}\},$$
$$\operatorname{Dom} {}^{\mathrm{NC}}T(H) = \{f \in V(H) : H.f \text{ is integrable}\}.$$

We show that, when applied to adapted processes, these give restrictions of the QS integrals of this paper.

THEOREM 7.1. *Let H be an adapted Fock operator process. Then the following inclusions hold*:

$$^{\mathrm{NC}}A^\dagger(H) \subset A^\dagger(H), \qquad ^{\mathrm{NC}}N(H) \subset N(H),$$
$$^{\mathrm{NC}}A(H) \subset {}^R A(H), \qquad ^{\mathrm{NC}}T(H) \subset {}^R T(H).$$

PROOF. Since, for a map $x : \Gamma \times \mathbb{R}_+ \to \mathfrak{h}$, x is Skorohod integrable if $x \in \operatorname{Dom} \mathcal{S}$ and x is absolutely time integrable if it is (Bochner) integrable (Proposition 2.1), the result follows immediately from Proposition 5.1. $\square$

7.2. *Itô calculus approach* ([5]). We first recall the treatment in [5] where QS integrals are extended beyond an exponential domain. Let F, G, H and K be four adapted Fock operator processes which are measurable and have common process domain V, where V is a subspace of $\mathcal{F}$ containing $V_0 \odot \mathcal{E}(S)$ for some dense subspace V_0 and admissible subset S and which also satisfy the local integrability (and implied measurability) conditions

$$\int_0^t \{\|F_s P_s f\|^2 + \|G_s D_s f\|^2 + \|H_s D_s f\| + \|K_s P_s f\|\} \, ds < \infty$$

for all $f \in V$ and $t \geq 0$. An adapted Fock operator process X with process domain V is denoted by

$$\int_0^t F_s \, dA_s^\dagger + \int_0^t G_s \, dN_s + \int_0^t H_s \, dA_s + \int_0^t K_s \, ds$$

provided that, for each $f \in V$ and $t \geq 0$:

(i) $D_s f \in \operatorname{Dom} X_s$ for a.a. s, and $s \mapsto \mathbf{1}_{\{s<t\}} X_s D_s f$ is square integrable;

(ii)

$$X_t f = \int_0^\infty X_{s \wedge t} D_s f \, d\chi_s + \int_0^t \{F_s P_s f + G_s D_s f\} \, d\chi_s + \int_0^t \{H_s D_s f + K_s P_s f\} \, ds.$$

Here the alternative notation for Itô integration discussed after Proposition 2.2 is being employed. When V equals $V_0 \odot \mathcal{E}(S)$, this is equivalent to X_t being the corresponding Hudson–Parthasarthy QS integral, and under various conditions the representation (ii) is valid on larger domains V. This is exploited, in particular, in the QS-integral representability of regular semimartingales [2]. However, since the Fock operator process X appears on the right-hand side, (ii) represents a kind of system of Fock space vector-valued stochastic differential equations. In other words, the Fock operator process X is only defined implicitly through (ii). It was not known in general whether this system has a solution, nor whether any solution it might have is unique; moreover, nothing was known about appropriate (maximal) domains for a Fock operator solution process. We shall see that our integrals solve all three of these problems.

For an adapted Fock operator process H, let ${}^r\Lambda_\cdot(H)$ denote the QS process $\Lambda_\cdot(H)$ restricted as follows:

$$\operatorname{Dom} {}^r\Lambda_t(H) := \{f \in \operatorname{Dom} \Lambda_t(H) : \mathbf{1}_{[0,t]}(\cdot) H.Q.D_\beta f \text{ is } \mathcal{R}\text{-integrable for a.a. } \beta > t\}, \tag{7.1}$$

where, as usual, $Q = Q^\Lambda$ and $\mathcal{R} = \mathcal{R}^\Lambda$ are given by (5.1). By the adaptedness of ${}^R\Lambda.(H)$ and Lemma 6.1, the following inclusions hold:

$$ {}^r\Lambda_t(H) \supset {}^R\Lambda_t(H) \qquad \text{for a.a. } t \geq 0. $$

Recall the recursion formula for QS integrals given in Theorem 5.8. Equation (ii) above, for the single QS integral process $X = \Lambda.(H)$, reads

$$ \text{(A-M)} \qquad X_t f = \int_0^\infty X_{t\wedge s} D_s f \, d\chi_s + \int_0^t H_s Q_s f \, dr_s, $$

where $Q = Q^\Lambda$ and $dr_s = d\chi_s$ or ds, respectively, according as $\Lambda \in \{A^\dagger, N\}$ or $\Lambda \in \{A, T\}$ [cf. (5.1)]. By a "solution of (A-M)" we mean a pair $(X, \mathcal{D})$ consisting of an adapted Fock operator process X and a Fock-adapted space $\mathcal{D}$ contained in $\mathbb{P}\,\mathrm{Dom}\, X$, such that, for each $f \in \mathcal{D}$:

(i) $D_s f \in \mathrm{Dom}\, X_s$ for a.a. s;
(ii) $X.D.f$ is locally square integrable;
(iii) $Q_s f \in \mathrm{Dom}\, H_s$ for a.a. s;
(iv) $H.Q.f$ is locally $\mathcal{R}$-integrable

and the identity (A-M) holds for each $t > 0$. In short, the identity (A-M) should be well defined and valid. Note that condition (i) is redundant since, as $\mathcal{D}$ is adapted, for a.a. s, $D_s f \in \mathcal{D} \subset \mathrm{Dom}\, X_s$. Moreover, by adaptedness (of both X and $\mathcal{D}$) Proposition 3.4 implies that (A-M) is equivalent to

$$ \text{(A-M}'\text{)} \qquad X_t P_t f = \int_0^t X_s D_s f \, d\chi_s + \int_0^t H_s Q_s f \, dr_s; $$

compare Remark (iii) after Theorem 3.1.

THEOREM 7.2. *Let H be an adapted Fock operator process and let Λ be a QS integrator.*

(a) *If $X = {}^r\Lambda.(H)$ and $\mathcal{D}$ is a Fock-adapted space contained in $\mathbb{P}\,\mathrm{Dom}\, X$, then $(X, \mathcal{D})$ solves* (A-M).

(b) *If $(X, \mathcal{D})$ is a solution of* (A-M) *such that $\mathcal{D} \subset \bigcap_{t>0} V^Q(\mathbf{1}_{[0,t[}H)$, then*

$$ \mathcal{D} \subset \mathbb{P}\,\mathrm{Dom}\, {}^r\Lambda(H) \quad \text{and} \quad X_t f = {}^r\Lambda_t(H) f \qquad \forall f \in \mathcal{D}, t > 0. $$

PROOF. (a) This follows immediately by applying the recursion formula (Theorem 5.8) to the process $\mathbf{1}_{[0,t[}H$.

(b) Let $f \in \mathcal{D}$ and let $t > 0$. By the adaptedness of $\mathcal{D}$ and condition (iv), it suffices to show that $f \in \mathrm{Dom}\, \Lambda_t(H)$ and $\Lambda_t(H) f = X_t f$.

CASE $\Lambda = A^\dagger$ or N. Then $dr_s = d\chi_s$ and, by (A-M), $X_t f(\varnothing) = 0$. By conditions (i)–(iv) and (A-M), the following is well defined and valid for a.a. τ:

$$ X_t f(\tau) = \big(X_{t_n \wedge t} D_{t_n} f\big)(t_1, \ldots, t_{n-1}) + \mathbf{1}_{[0,t[}(t_n)\big(H_{t_n} Q_{t_n} f\big)(t_1, \ldots, t_{n-1}), $$

where $\tau = \{t_1 < \cdots < t_n\}$. In turn the following is well defined and valid a.e.:

$$\begin{aligned}\big(X_{t_n \wedge t} D_{t_n} f\big)(t_1, \dots, t_{n-1}) &= \big(X_{t_{n-1} \wedge t} D_{t_{n-1}} D_{t_n} f\big)(t_1, \dots, t_{n-2}) \\ &\quad + \mathbf{1}_{[0,t[}(t_{n-1})\big(H_{t_{n-1}} Q_{t_{n-1}} D_{t_n} f\big)(t_1, \dots, t_{n-2}).\end{aligned}$$

After repeating this n times, the following a.e. identity results:

$$X_t f(\tau) = \sum_{k=1}^{n} \mathbf{1}_{[0,t[}(t_k)\big(H_{t_k} Q_{t_k} D_{t_{k+1}} \cdots D_{t_n} f\big)(t_1, \dots, t_{k-1}).$$

Comparison with (5.2a) shows that $\mathbf{1}_{[0,t[}(\cdot) H^{Q}_{\cdot} f$ is Skorohod integrable with Skorohod integral $X_t f$. Thus, $f \in \operatorname{Dom} \Lambda_t(H)$ and $\Lambda_t(H) f = X_t f$.

CASE $\Lambda = A$ or T. Then $dr_s = ds$ and, by condition (iv) and (A-M), $H.Q.f(\varnothing)$ is locally integrable and

$$\int_0^t (H_s Q_s f)(\varnothing)\, ds = X_t f(\varnothing). \tag{7.2}$$

By conditions (i)–(iv), (A-M) and adaptedness, the following is well defined and valid for a.a. $\tau = \{t_1 < \cdots < t_n\}$:

$$\begin{aligned}X_t f(\tau) &= \big(X_{t_n \wedge t} D_{t_n} f\big)(t_1, \dots, t_{n-1}) + \int_0^t (H_s Q_s f)(t_1, \dots, t_n)\, ds \\ &= \big(X_{t_n \wedge t} D_{t_n} f\big)(t_1, \dots, t_{n-1}) + \int_{t_n \wedge t}^t (H_s Q_s f)(t_1, \dots, t_n)\, ds.\end{aligned}$$

Again, this may be iterated, so that the following is well defined and valid a.e.:

$$\begin{aligned}&\big(X_{t_n \wedge t} D_{t_n} f\big)(t_1, \dots, t_{n-1}) \\ &\quad = \big(X_{t_{n-1} \wedge t} D_{t_{n-1}} D_{t_n} f\big)(t_1, \dots, t_{n-2}) + \int_{t_{n-1} \wedge t}^{t_n \wedge t} \big(H_s Q_s D_{t_n} f\big)(t_1, \dots, t_{n-1})\, ds.\end{aligned}$$

After n steps, using (7.2) applied to $D_{t_1} \cdots D_{t_n} f$, the following a.e. identity results:

$$X_t f(\tau) = \sum_{k=0}^{n} \int_{t_k \wedge t}^{t_{k+1} \wedge t} \big(H_s Q_s D_{t_{k+1}} \cdots D_{t_n} f\big)(t_1, \dots, t_k)\, ds,$$

where $t_0 = 0$ and $t_{n+1} = \infty$. Once again, comparison with (5.2b) shows that $\mathbf{1}_{[0,t[}(\cdot) H^{Q} f$ is time integrable with time integral $X_t f$. Thus, $f \in \operatorname{Dom} \Lambda_t(H)$ and $\Lambda_t(H) f = X_t f$. $\square$

REMARK. By the adaptedness of $\mathcal{D}$ and Proposition 5.1(cii), sufficient conditions for $\mathcal{D} \subset \bigcap_{t>0} V^{Q}(\mathbf{1}_{[0,t[} H)$ to hold are $\mathcal{D} \subset \mathbb{P} \operatorname{Dom} H$ and that $(H_t|_{\mathcal{D}})_{t \geq 0}$ has an adjoint process which is measurable and has dense process domain. The first condition may be arranged by restriction, and the second is a very mild regularity condition on the process H.

Applying part (b) of the theorem to the zero process leads to the following uniqueness result.

COROLLARY 7.3. *Let X be an adapted Fock operator process and let $\mathcal{D}$ be a Fock-adapted space contained in $\mathbb{P}\,\mathrm{Dom}\,X$. Suppose that, for each $t > 0$ and each $f \in \mathcal{D} \cap \mathcal{F}_t$, $\int_0^t X_s D_s f\, d\chi_s$ is well defined and equal to $X_t f$. Then $X_. f = 0$ for all f in $\mathcal{D}$.*

8. Quantum Itô product formula. In this section we show that the composition of QS integrals is given by integration by parts with a correction term when Wick ordering of the integrators is violated. We give the result in two forms, one in which the correction is present and one in which it need not be. Whereas the quantum Itô product formula obtained by Hudson and Parthasarathy is an identity in Fock space inner products (their second fundamental formula), in Theorem 8.1 we have achieved a product formula which is an identity between the Fock space operators. The second form is a basic consequence, for operator products of QS integrals, of our extension of the second fundamental formula—no longer tied to exponential vectors.

For both theorems let F and G be adapted Fock operator processes, let Λ and Λ' be QS integrators and let Λ^{ϵ} equal $A^{\dagger}, N, A$ or T, if the ordered pair (Λ, Λ') is, respectively, $(N, A^{\dagger})$, (N, N), (A, N) or $(A, A^{\dagger})$, and let Λ^{ϵ} equal 0 otherwise. Recall the restricted QS integals ${}^r\Lambda(H)$ defined in (7.1).

THEOREM 8.1. *Let $Z = (XY - W)$, where X, Y and W are, respectively, the processes ${}^r\Lambda_.(F)$, ${}^r\Lambda'_.(G)$ and ${}^r\Lambda_.(FY) + {}^r\Lambda'_.(XG) + {}^r\Lambda^{\epsilon}_.(FG)$, and let $\mathcal{D}$ be a Fock-adapted space contained in $\mathbb{P}\,\mathrm{Dom}\,Z$. If (Λ, Λ') is one of $(N, A^{\dagger})$, (N, N), (A, N) or $(A, A^{\dagger})$, then $Z_. f = 0$ for all f in $\mathcal{D}$.*

PROOF. By Corollary 7.3 it suffices to show that, for each $t > 0$ and each $f \in \mathcal{D} \cap \mathcal{F}_t$, the identity

$$Z_t f = \int_0^t Z_s D_s f\, d\chi_s \tag{8.1}$$

is well defined and valid. Therefore, let f be such a vector, let dr_s denote time or Itô integration according as Λ is A or N and let $Q' = Q^{\Lambda'}$. Then, by Theorem 5.8, the following is well defined and valid:

$$\begin{aligned} W_t f &- \int_0^t W_s D_s f\, d\chi_s \\ &= \int_0^t X_s G_s Q'_s f\, d\chi_s + \int_0^t F_s Y_s D_s f\, dr_s + \int_0^t F_s G_s Q'_s f\, dr_s. \end{aligned} \tag{8.2}$$

Since $f \in \mathcal{F}_t \cap \operatorname{Dom} X_t Y_t$, applying Theorem 5.8 with $Y_t f$ in place of f, we see that the identity

$$X_t Y_t f = \int_0^t X_s D_s Y_t f \, d\chi_s + \int_0^t F_s D_s Y_t f \, dr_s \tag{8.3}$$

is well defined and valid. Since $\mathcal{R}^{\Lambda'} = \mathcal{S}$, applying Theorem 5.8 [and then Theorem 2.5(d)] to f gives

$$D_s Y_t f = Y_s D_s f + G_s Q'_s f \qquad \text{for a.a. } s < t.$$

Substituting this into (8.3) and using the following facts established above:

$\mathbf{1}_{[0,t[}(\cdot) X_{\cdot} G_{\cdot} Q'_{\cdot} f$ is well defined and Itô integrable,

$\mathbf{1}_{[0,t[}(\cdot) F_{\cdot} G_{\cdot} Q'_{\cdot} f$ is well defined and $\mathcal{R}$-integrable,

$X_t Y_t f$ is then expressed as a sum of four integrals. Subtracting the resulting identity from (8.2) and rearranging using the linearity of Itô and $\mathcal{R}$ integration shows that (8.1) is indeed well defined and valid. □

THEOREM 8.2. *Let X, Y and W be, respectively, the QS integrals $\Lambda(F)$, ${}^R\Lambda'(G)$ and ${}^R\Lambda(FY) + {}^R\Lambda'(XG) + {}^R\Lambda^\epsilon(FG)$. If F has an adapted adjoint process $F^\dagger$, for which $X^\dagger := {}^R\Lambda^\dagger(F^\dagger)$ is densely defined, then for all $g \in \operatorname{Dom} Y \cap \operatorname{Dom} W$,*

$$Yg \in \operatorname{Dom}(X^\dagger)^* \quad \textit{and} \quad (X^\dagger)^* Yg = Wg.$$

REMARK. We are using the same notation here (X and Y) for both process and operator.

PROOF OF THEOREM 8.2. If $g \in \operatorname{Dom} Y \cap \operatorname{Dom} W$, then, by Theorem 6.7, Proposition 6.5 and Corollary 6.6,

$$\langle X^\dagger f, Yg\rangle = \langle f, Wg\rangle \qquad \forall f \in \operatorname{Dom}(X^\dagger).$$

Since $X^\dagger$ is densely defined, this implies that $Yg \in \operatorname{Dom}(X^\dagger)^*$ and

$$(X^\dagger)^* Yg = \big(\Lambda(FY) + \Lambda'(XG) + \Lambda^\epsilon(FG)\big)g.$$

The result follows. □

As a consequence of this theorem, we have the quantum Itô product formula

$$\Lambda(F)^R\Lambda'(G)g = \big({}^R\Lambda(FY) + {}^R\Lambda'(XG) + {}^R\Lambda^\epsilon(FG)\big)g,$$

where $X_t = \Lambda_t(F)$, $Y_t = {}^R\Lambda'_t(G)$ and Λ^ϵ is the Itô-correcting QS integrator, whenever g lies in the domain of both left- and right-hand side operators.

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