

RATE OF CONVERGENCE OF A PARTICLE METHOD FOR THE SOLUTION OF A 1D VISCOUS SCALAR CONSERVATION LAW IN A BOUNDED INTERVAL

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In this paper, we give a probabilistic interpretation of a viscous scalar conservation law in a bounded interval thanks to a nonlinear martingale problem. The underlying nonlinear stochastic process is reflected at the boundary to take into account the Dirichlet conditions. After proving uniqueness for the martingale problem, we show existence thanks to a propagation of chaos result. Indeed we exhibit a system of N interacting particles, the empirical measure of which converges to the unique solution of the martingale problem as $N \rightarrow +\infty$. As a consequence, the solution of the viscous conservation law can be approximated thanks to a numerical algorithm based on the simulation of the particle system. When this system is discretized in time thanks to the Euler–Lépingle scheme [D. Lépingle, *Math. Comput. Simulation* **38** (1995) 119–126], we show that the rate of convergence of the error is in $\mathcal{O}(\Delta t + 1/\sqrt{N})$, where Δt denotes the time step. Finally, we give numerical results which confirm this theoretical rate.

1. Introduction. We are interested in the following viscous scalar conservation law with nonhomogeneous Dirichlet boundary conditions on the interval $[0, 1]$:

$$(1) \quad \begin{aligned} \frac{\partial}{\partial t} v(t, x) &= \frac{\sigma^2}{2} \frac{\partial^2}{\partial x^2} v(t, x) - \frac{\partial}{\partial x} A(v(t, x)) & \forall (t, x) \in (0, +\infty) \times (0, 1), \\ v(0, x) &= v_0(x) & \forall x \in [0, 1], \\ v(t, 0) &= 0 \quad \text{and} \quad v(t, 1) = 1, & \forall t > 0. \end{aligned}$$

We suppose that $A: \mathbb{R} \rightarrow \mathbb{R}$ is a C^1 function and that the initial data v_0 is the cumulative distribution function of a probability measure U_0 on $[0, 1]$, which is written, $\forall x \in [0, 1]$, $v_0(x) = U_0([0, x]) = H * U_0(x)$, where $H(y) = \mathbb{1}_{\{y \geq 0\}}$ denotes the Heaviside function.

After giving a probabilistic interpretation of the solution of this equation thanks to a nonlinear martingale problem, we want to derive and study a particle approximation of this solution. Our main motivation is that the spatial domain in (1) is bounded. To our knowledge, the only paper about a probabilistic particle interpretation for the solution of a partial differential equation posed in a bounded

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spatial domain is [1], which is dedicated to the two-dimensional incompressible Navier–Stokes equation. In [1], the authors do not prove the convergence of the proposed particle method. By considering the much simpler equation (1), we are able not only to prove the convergence but also to bound the associated rate.

When the viscous scalar conservation law is posed in the spatial domain \mathbb{R} instead of $[0, 1]$, one can show that its unique weak solution is equal to $H * P_t(x)$, where $(P_t)_{t \geq 0}$ denote the time-marginals of the probability measure P on $C([0, +\infty), \mathbb{R})$ characterized the following martingale problem nonlinear in the sense of McKean [4, 8]:

- (i) $P_0 = U_0$,
- (ii) $\forall \varphi \in C_b^2(\mathbb{R})$,

$$\varphi(X_t) - \varphi(X_0) - \int_0^t \left[\frac{\sigma^2}{2} \varphi''(X_s) + A'(H * P_s(X_s)) \varphi'(X_s) \right] ds$$

is a P -martingale, where X denotes the canonical process on $C([0, \infty), \mathbb{R})$.

Here, we follow a similar approach. To take into account the Dirichlet boundary conditions, we work with a diffusion process with reflection. That is why we introduce (X, K) , the canonical process on the sample path space $\mathcal{C} = C([0, +\infty), [0, 1]) \times C([0, +\infty), \mathbb{R})$ (endowed with the topology of uniform convergence on compact sets). For P in $\mathcal{P}(\mathcal{C})$ the set of probability measures on \mathcal{C} , $(\bar{P}_t)_{t \geq 0}$ is the set of time-marginals of the probability measure \bar{P} on $C([0, +\infty), [0, 1])$ defined by $\bar{P} = P \circ X^{-1}$. We associate the following nonlinear problem with (1).

DEFINITION 1.1. A probability measure $P \in \mathcal{P}(\mathcal{C})$ solves the martingale problem (MP) starting at $U_0 \otimes \delta_0 \in \mathcal{P}([0, 1] \times \mathbb{R})$, if the following hold:

- (i) $P \circ (X_0, K_0)^{-1} = U_0 \otimes \delta_0$;
- (ii) $\forall \varphi \in C_b^2(\mathbb{R})$,

$$\begin{aligned} &\varphi(X_t - K_t) - \varphi(X_0 - K_0) \\ &\quad - \int_0^t \left(\frac{\sigma^2}{2} \varphi''(X_s - K_s) + A'(H * \bar{P}_s(X_s)) \varphi'(X_s - K_s) \right) ds \end{aligned}$$

is a P -martingale;

- (iii) P -a.s., $\forall t \geq 0$, $\int_0^t d|K|_s < +\infty$, $|K|_t = \int_0^t \mathbb{1}_{\{0,1\}}(X_s) d|K|_s$ and $K_t = \int_0^t (1 - 2X_s) d|K|_s$.

The finite variation process K_t which increases when $X_t = 0$ and decreases when $X_t = 1$ accounts for reflection and prevents X_t from leaving the interval $[0, 1]$.

In Section 2, we prove that if P solves problem (MP), then $(t, x) \rightarrow H * \bar{P}_t(x)$ is the unique weak solution of (1). We deduce uniqueness for the martingale problem.

Existence is obtained thanks to a propagation of chaos result for a system of weakly interacting diffusion processes.

In Section 3, we discretize this system in time thanks to the version of the Euler scheme introduced by Lépingle [10]. This way, we derive a numerical method to approximate the solution of (1). We prove a theoretical rate of convergence in $\mathcal{O}(\Delta t + 1/\sqrt{N})$, where Δt and N denote respectively the time step and the number of particles. This rate is the same as the one obtained by Bossy [3] when the spatial domain is \mathbb{R} . As an important step in the proof, we show that, when applied to a one-dimensional diffusion reflected at the boundary of $[0, 1]$ with nondegenerate constant diffusion coefficient, the weak error of the Euler–Lépingle scheme is in $\mathcal{O}(\Delta t)$. To our knowledge, this is the first result concerning the weak error of this scheme.

The last section is devoted to numerical experiments which confirm the theoretical rate of convergence of our particle method. The treatment of the reflection by the Euler–Lépingle scheme does not alter the convergence whereas we exhibit a sublinear numerical dependence on the time step Δt when the particle system is discretized thanks to the cruder Euler projection scheme.

To conclude the Introduction, we should mention that, using signed weights as in [8] and [3], we could extend our approach to deal with the following more general boundary conditions in (1): $\forall t > 0, v(t, 0) = a$ and $v(t, 1) = b$, and $\forall x \in [0, 1], v_0(x) = U_0([0, x])$, where U_0 is a bounded signed measure on $[0, 1]$ satisfying the compatibility condition $U_0([0, 1]) = b - a$. However, we restrict ourselves to a simple case without weights to avoid further complication of the already technical developments.

2. Probabilistic interpretation of the viscous scalar conservation law equation. For $T > 0$ let $\mathcal{Q}_T = (0, T) \times (0, 1)$ and let $W_2^{0,1}(\mathcal{Q}_T), W_2^{1,1}(\mathcal{Q}_T)$ denote the Hilbert spaces with respective scalar products (cf. [9])

$$(u, v)_{W_2^{0,1}(\mathcal{Q}_T)} = \int_{\mathcal{Q}_T} (uv + \partial_x u \partial_x v) dx dt,$$

$$(u, v)_{W_2^{1,1}(\mathcal{Q}_T)} = \int_{\mathcal{Q}_T} (uv + \partial_x u \partial_x v + \partial_t u \partial_t v) dx dt.$$

We introduce the Banach space $V_2^{0,1}(\mathcal{Q}_T) = \{u \in W_2^{0,1}(\mathcal{Q}_T) \cap C((0, T), L^2(0, 1))$ such that $\|u\|_{V_2^{0,1}(\mathcal{Q}_T)} = \sup_{0 \leq t \leq T} \|u(t, x)\|_{L^2(0,1)} + \|\partial_x u\|_{L^2(\mathcal{Q}_T)} < +\infty\}$. The corresponding subspaces consisting in elements which vanish on $[0, T] \times \{0, 1\}$ are respectively denoted by $\overset{\circ}{W}_2^{0,1}(\mathcal{Q}_T), \overset{\circ}{W}_2^{1,1}(\mathcal{Q}_T), \overset{\circ}{V}_2^{0,1}(\mathcal{Q}_T)$.

We first prove uniqueness of weak solutions of problem (1) defined in the following way:

DEFINITION 2.1. A weak solution of (1) is a function $v: [0, +\infty) \times [0, 1] \rightarrow \mathbb{R}$ satisfying the boundary conditions and such that, for any $T > 0$,

$v \in V_2^{0,1}(\mathcal{Q}_T) \cap L^\infty(\mathcal{Q}_T)$ and, for all ϕ in $\overset{\circ}{W}_2^{1,1}(\mathcal{Q}_T)$ and all $t \in [0, T]$,

$$(2) \quad \begin{aligned} \int_0^1 v(t, x) \phi(t, x) dx &= \int_0^1 v_0(x) \phi(0, x) dx + \int_0^t \int_0^1 \frac{\partial}{\partial s} \phi(s, x) v(s, x) dx ds \\ &+ \int_0^t \int_0^1 \frac{\partial}{\partial x} \phi(s, x) A(v(s, x)) dx ds \\ &- \int_0^t \int_0^1 \frac{\sigma^2}{2} \frac{\partial}{\partial x} \phi(s, x) \frac{\partial}{\partial x} v(s, x) dx ds. \end{aligned}$$

Then we check that, when P solves the martingale problem (MP), $V(t, x) = H * \bar{P}_t(x)$ is a weak solution of (1). Uniqueness for the martingale problem is derived from uniqueness for this equation. The probabilistic interpretation is completed by a propagation of chaos result which ensures existence for problem (MP).

2.1. Uniqueness result for equation (1).

LEMMA 2.2. *Equation (1) has no more than one weak solution in the sense of Definition 2.1.*

PROOF. Let v^1 and v^2 be two weak solutions of (1) and let $T > 0$. We set $w = v^1 - v^2$. Then w is in $\overset{\circ}{V}_2^{0,1}(\mathcal{Q}_T)$ and $w(0, x) = 0$ for all $x \in [0, 1]$. Moreover, for all $\phi \in \overset{\circ}{W}_2^{1,1}(\mathcal{Q}_T)$,

$$(3) \quad \begin{aligned} \int_0^1 w(t, x) \phi(t, x) dx &= \int_0^t \int_0^1 \frac{\partial}{\partial s} \phi(s, x) w(s, x) dx ds \\ &+ \int_0^t \int_0^1 \frac{\partial}{\partial x} \phi(s, x) \{A(v^1(s, x)) - A(v^2(s, x))\} dx ds \\ &- \int_0^t \int_0^1 \frac{\sigma^2}{2} \frac{\partial}{\partial x} \phi(s, x) \frac{\partial}{\partial x} w(s, x) dx ds. \end{aligned}$$

Thus, w is a generalized solution in the sense of Ladyzenskaja, Solonnikov and Ural'ceva (cf. [9]) of a linear equation with uniformly bounded coefficients. We can apply results of Chapter 3 of [9]. In particular the identity ([9], 2.13) of Section 2 used to establish the energy inequality holds and becomes in our case

$$(4) \quad \begin{aligned} \frac{1}{2} \int_0^1 w^2(x, t) dx + \int_0^t \int_0^1 \frac{\sigma^2}{2} \left(\frac{\partial w}{\partial x} \right)^2 (s, x) dx ds \\ = \int_0^t \int_0^1 \frac{\partial w}{\partial x} (s, x) \{A(v^1(s, x)) - A(v^2(s, x))\} dx ds. \end{aligned}$$

Formally, this identity is obtained by taking $\phi = w$ in (3) and integrating by parts the first term of the right-hand side. As w is not sufficiently smooth to do so,

the proof of (4) relies on two steps. The first consists in working with Steklov averagings in time of functions ϕ and showing that is it possible to integrate by parts the first term of the right-hand side of (3). The second consists in proving that it is possible to replace ϕ by w in the obtained identity.

Following techniques from Chapter 3 of [9], we deduce from (4) that

$$\begin{aligned} & \min\left(\frac{1}{2}, \frac{\sigma^2}{2}\right) \left[\|w(t)\|_{L^2([0,1])}^2 + \int_{\mathcal{Q}_t} \left(\frac{\partial w}{\partial x}\right)^2(s, x) dx ds \right] \\ & \leq \int_{\mathcal{Q}_t} \frac{\partial w}{\partial x}(s, x) \{A(v^1(s, x)) - A(v^2(s, x))\} dx ds. \end{aligned}$$

Now we observe that, for $M = \|v^1\|_{L^\infty(\mathcal{Q}_T)} \vee \|v^2\|_{L^\infty(\mathcal{Q}_T)}$,

$$\begin{aligned} & \int_{\mathcal{Q}_t} \frac{\partial w}{\partial x}(s, x) \{A(v^1(s, x)) - A(v^2(s, x))\} dx ds \\ & \leq \sup_{|x| \leq M} |A'(x)| \int_{\mathcal{Q}_t} |w|(s, x) \left| \frac{\partial w}{\partial x} \right|(s, x) dx ds \\ & \leq \sup_{|x| \leq M} |A'(x)| \sup_{s \leq t} \|w(s)\|_{L^2[0,1]} \int_0^t \left\| \frac{\partial w}{\partial x}(s) \right\|_{L^2[0,1]} ds \\ & \leq \sup_{|x| \leq M} |A'(x)| \frac{\sqrt{t}}{2} \left[\sup_{s \leq t} \|w(s)\|_{L^2(0,1)}^2 + \left\| \frac{\partial w}{\partial x} \right\|_{L^2(\mathcal{Q}_t)}^2 \right] \end{aligned}$$

by using the Cauchy–Schwarz inequality and the upper bound $2ab \leq a^2 + b^2$. Thus

$$\min(1, \sigma^2) \left[\sup_{s \leq t} \|w(s)\|_{L^2(0,1)}^2 + \int_{\mathcal{Q}_t} \left(\frac{\partial w}{\partial x}\right)^2(s, x) dx ds \right] \leq C \sqrt{t} \|w\|_{V_2^{0,1}(\mathcal{Q}_t)}^2$$

and hence

$$\min(1, \sigma^2) \|w\|_{V_2^{0,1}(\mathcal{Q}_t)}^2 \leq C \sqrt{t} \|w\|_{V_2^{0,1}(\mathcal{Q}_t)}^2.$$

Choose t_1 such that

$$t_1 < \frac{\min(1, \sigma^4)}{C^2} \wedge T.$$

Then $\|w\|_{V_2^{0,1}(\mathcal{Q}_{t_1})} = 0$. Now, for $t \geq t_1$, (4) gives

$$\begin{aligned} & \frac{1}{2} \int_0^1 w^2(x, t) dx + \int_{t_1}^t \int_0^1 \frac{\sigma^2}{2} \left(\frac{\partial w}{\partial x}\right)^2(s, x) dx ds \\ & = \int_{t_1}^t \int_0^1 \frac{\partial w}{\partial x}(s, x) \{A(v^1(s, x)) - A(v^2(s, x))\} dx ds \end{aligned}$$

and the previous computation shows that, for any $t_2 \leq T$, such that

$$t_2 - t_1 < \frac{\min(1, \sigma^4)}{C^2},$$

$\|w\|_{V_2^{0,1}(\mathcal{Q}_{t_1, t_2})} = 0$. Finally, we can iterate this procedure to obtain that $\|w\|_{V_2^{0,1}(\mathcal{Q}_T)} = 0$. Since T is arbitrary $v^1 = v^2$. \square

2.2. Uniqueness for the martingale problem (MP) and link with (1).

PROPOSITION 2.3. *If P solves the martingale problem (MP) starting at $U_0 \otimes \delta_0$, then $V(t, x) = H * \bar{P}_t(x)$ is a weak solution of (1). Moreover, uniqueness holds for the martingale problem (MP).*

PROOF. Clearly the function $V(t, x) = H * \bar{P}_t(x)$ is bounded by 1. Let $T > 0$. We check that the function V belongs to $V_2^{0,1}(\mathcal{Q}_T)$ and satisfies the nonhomogeneous Dirichlet boundary conditions in (1) thanks to the following lemma, the proof of which is postponed.

LEMMA 2.4. *If P solves the martingale problem (MP), then, for any $t > 0$, \bar{P}_t has a density \bar{p}_t which belongs to $L^2([0, 1])$ and it holds that*

$$\|\bar{p}_t\|_{L^2([0,1])} \leq C(1 + t^{-1/4}) \exp(Ct).$$

We still have to check that V satisfies the identity (2). Let ϕ be a C^∞ function on $[0, T] \times [0, 1]$ with $\phi(t, 0) = \phi(t, 1) = 0$ for all $t \in [0, T]$. We set $\psi(t, x) = \int_0^x \phi(t, y) dy$. According to Definition 1.1(ii), under the probability measure P , $\frac{1}{\sigma}(X_t - K_t - \int_0^t A'(V(s, X_s)) ds)$ is a local martingale with quadratic variation t , that is, a Brownian motion. By Definition 1.1(iii) and the choice of the boundary conditions for ϕ , P -a.s.,

$$\int_0^t \frac{\partial \psi}{\partial x}(s, X_s) dK_s = \int_0^t \phi(s, X_s) \mathbb{1}_{\{0,1\}}(X_s) dK_s = 0.$$

Thus, by Itô's formula

$$\begin{aligned} \mathbb{E}\psi(t, X_t) &= \mathbb{E}\psi(0, X_0) + \mathbb{E} \int_0^t \left[\frac{\partial \psi}{\partial s} + \frac{\sigma^2}{2} \frac{\partial^2 \psi}{\partial x^2} \right](s, X_s) ds \\ &\quad + \mathbb{E} \int_0^t \frac{\partial \psi}{\partial x}(s, X_s) A'(V(s, X_s)) ds. \end{aligned}$$

As $\bar{p}_s = \frac{\partial V}{\partial x}(s, \cdot)$, we deduce that

$$\begin{aligned} \int_0^1 \psi(t, x) \frac{\partial V}{\partial x}(t, x) dx &= \int_0^1 \psi(0, x) \frac{\partial V_0}{\partial x}(x) dx \\ &+ \int_0^t \int_0^1 \frac{\partial \psi}{\partial s}(s, x) \frac{\partial V}{\partial x}(s, x) dx ds \\ &+ \int_0^t \int_0^1 \frac{\sigma^2}{2} \frac{\partial \phi}{\partial x}(s, x) \frac{\partial V}{\partial x}(s, x) dx ds \\ &+ \int_0^t \int_0^1 \phi(s, x) A'(V(s, x)) \frac{\partial V}{\partial x}(s, x) dx ds. \end{aligned}$$

Applying the Stieltjes integration by parts formula in the spatial integrals in the first and last lines of the previous equality, we get identity (2) for ϕ a C^∞ function vanishing for $x = 0$ and $x = 1$. As V is in $V_2^{0,1}(\mathcal{Q}_T)$ we can extend the identity easily by density for any function ϕ in $\overset{\circ}{W}_2^{1,1}(\mathcal{Q}_T)$.

Hence $V(t, x) = H * \bar{P}_t(x)$ is a weak solution in the sense of Definition 2.1.

Uniqueness for the martingale problem (MP) is derived from the uniqueness result for the problem (1): if P and Q solve (MP), then, for any $(t, x) \in [0, +\infty) \times \mathbb{R}$, $H * \bar{P}_t(x) = H * \bar{Q}_t(x)$. Hence P and Q solve a linear martingale problem with bounded drift term $A'(H * \bar{P}_t(x))$ and by the Girsanov theorem $P = Q$. \square

PROOF OF LEMMA 2.4. We just have to adapt to the case of reflected diffusion processes the proof of Proposition 1.1 of [12]. According to Definition 1.1(ii), under the probability measure P , $\frac{1}{\sigma}(X_t - K_t - \int_0^t A'(V(s, X_s)) ds)$ is a Brownian motion. As $\sup_{[0,1]} |A'(x)| < +\infty$, by the Girsanov theorem, under the probability measure $Q \in \mathcal{P}(\mathcal{C})$ such that $\frac{dQ}{dP} |_{\mathcal{F}_t} = \frac{1}{Z_t}$, where

$$Z_t = \exp\left(\int_0^t \frac{1}{\sigma^2} A'(H * \bar{P}_s(X_s)) d(X_s - K_s) - \frac{1}{2\sigma^2} A'^2(H * \bar{P}_s(X_s)) ds\right),$$

$\beta_t = \frac{1}{\sigma}(X_t - K_t)$ is a Brownian motion starting at $\frac{1}{\sigma}X_0$ and $(X_t)_{t \geq 0}$ is the doubly reflected process associated with $(\sigma\beta_t)_{t \geq 0}$.

For ψ bounded and measurable, since $\mathbb{E}^P(\psi(X_t)) = \mathbb{E}^Q(\psi(X_t)Z_t)$, by the Cauchy–Schwarz inequality,

$$\mathbb{E}^P(\psi(X_t)) \leq \left(\int_0^1 \psi^2(x) u_t(x) dx\right)^{1/2} \exp\left(\frac{t}{2\sigma^2} \sup_{[0,1]} |A'^2(x)|\right),$$

where $u_t(x) = \int_0^1 p_{\sigma^2 t}(z, x) U_0(dz)$ and

$$p_t(z, x) = \frac{1}{\sqrt{2\pi t}} \sum_{n \in \mathbb{Z}} (e^{-(x-z-2n)^2/2t} + e^{-(x+z+2n)^2/2t})$$

denotes the transition density of the doubly reflected Brownian motion in $[0, 1]$. For any $(z, x) \in \mathbb{R}^2$ we easily check that $p_t(z, x) \leq \frac{2}{\sqrt{2\pi t}} + 1$. Thus,

$$\mathbb{E}^P(\psi(X_t)) \leq C(1 + t^{-1/4}) \exp(Ct) \|\psi\|_{L^2([0,1])},$$

which gives the lemma. \square

2.3. The propagation of chaos result. The system of weakly interacting diffusion processes with normal reflecting boundary conditions is given by the N -dimensional stochastic differential equation

$$\begin{aligned} X_t^{i,N} &= X_0^{i,N} + \sigma W_t^i + \int_0^t A'(H * \bar{\mu}_s^N(X_s^{i,N})) ds + K_t^{i,N}, \\ (5) \quad |K^{i,N}|_t &= \int_0^t \mathbb{1}_{[0,1]}(X_s^{i,N}) d|K^{i,N}|_s, \\ K_t^{i,N} &= \int_0^t (1 - 2X_s^{i,N}) d|K^{i,N}|_s, \quad i \leq N, \end{aligned}$$

where $\bar{\mu}_s^N = \frac{1}{N} \sum_{j=1}^N \delta_{X_s^{j,N}}$ and (W^1, \dots, W^N) is an N -dimensional Brownian motion independent of the initial variables $(X_0^{1,N}, \dots, X_0^{N,N})$, which are i.i.d. with law U_0 . As $\sup_{[0,1]} |A'(x)|$ is bounded, by the Girsanov theorem, this equation admits a unique weak solution. Existence for problem (MP) is ensured by the following propagation of chaos result:

THEOREM 2.5. *The particle systems $((X^{1,N}, K^{1,N}), \dots, (X^{N,N}, K^{N,N}))$ are P -chaotic, where P denotes the unique solution of the martingale problem (MP) starting at $U_0 \otimes \delta_0$; that is, for fixed $j \in \mathbb{N}^*$ the law of $((X^{1,N}, K^{1,N}), \dots, (X^{j,N}, K^{j,N}))$ converges weakly to $P^{\otimes j}$ as $N \rightarrow +\infty$.*

PROOF. Except in the treatment of the discontinuity of the Heaviside function, we follow the proof given by Sznitman [13], Theorem 1.4. When possible, we take advantage of the particular form of our diffusion domain (the interval $[0, 1]$) to simplify the arguments.

By Proposition 2.3, uniqueness holds for problem (MP). As the particles $(X^{i,N}, K^{i,N})_{i \leq N}$ are exchangeable, the propagation of chaos result is equivalent to the weak convergence of the law π^N of the empirical measure $\mu^N = \frac{1}{N} \sum_{i=1}^N \delta_{(X^{i,N}, K^{i,N})}$ to a probability measure concentrated on solutions of problem (MP) when $N \rightarrow +\infty$ (see [14] and the references cited therein).

Again by exchangeability the tightness of the sequence $(\pi^N)_N$ is equivalent to the tightness of the laws of the couples $(X^{1,N}, K^{1,N})$. As $\sup_{[0,1]} |A'(x)| < +\infty$, the laws of the processes $Y^{1,N} = X^{1,N} - K^{1,N}$ are tight. Since the map sending $y \in C([0, +\infty), \mathbb{R})$ to the solution $(x, k) \in \mathcal{C}$ of the Skorokhod problem is continuous

(see [11]), we deduce that the laws of the couples $(X^{1,N}, K^{1,N})$ are tight. Hence $(\pi^N)_N$ is a tight sequence.

Let π^∞ be the limit of a converging subsequence that we still index by N for simplicity and let Q denote the canonical variable on $\mathcal{P}(\mathcal{C})$. We are going to prove that, π^∞ -a.s., Q solves problem (MP). Clearly, π^∞ -a.s., $Q \circ (X_0, K_0)^{-1} = U_0(dx) \otimes \delta_0$; that is, π^∞ -a.s., condition (i) in Definition 1.1 is satisfied. To deal with condition (ii), we set $\varphi \in C_b^2(\mathbb{R})$, $p \geq 1$, $t \geq s \geq s_1 \geq \dots \geq s_p \geq 0$, $g \in C_b(\mathbb{R}^{2p})$ and define a mapping F on the set $\mathcal{P}(\mathcal{C})$ of probability measures on \mathcal{C} by

$$F(Q) = \left\langle Q, g(X_{s_1}, K_{s_1}, \dots, X_{s_p}, K_{s_p}) \right. \\ \times \left(\varphi(X_t - K_t) - \varphi(X_s - K_s) \right. \\ \left. \left. - \int_s^t \frac{\sigma^2}{2} \varphi''(X_r - K_r) + A'(H * \bar{Q}_r(X_r)) \varphi'(X_r - K_r) dr \right) \right\rangle.$$

The mapping F_k defined like F with the Heaviside function H replaced by the Lipschitz continuous approximation $H_k(x) = k(x + \frac{1}{k})\mathbb{1}_{\{-1/k < x < 0\}} + \mathbb{1}_{\{x \geq 0\}}$ is continuous and bounded. Hence the weak convergence of π^N to π^∞ implies

$$(6) \quad \mathbb{E}^{\pi^\infty} |F(Q)| \leq \limsup_k \mathbb{E}^{\pi^\infty} |F - F_k(Q)| \\ + \limsup_k \limsup_N \mathbb{E}^{\pi^N} |F - F_k(Q)| + \limsup_N \mathbb{E}^{\pi^N} |F(Q)|.$$

As the mappings F_k converge pointwise to F and are bounded uniformly in k , the first term of the right-hand side is equal to 0. Applying Itô's formula, we check that the third term is also nil. By the Lipschitz continuity of A' on $[0, 1]$,

$$\mathbb{E}^{\pi^N} |F - F_k(Q)| \leq C \mathbb{E} \left(\left| \bar{\mu}^N, \int_s^t (H_k - H) * \bar{\mu}_r^N(X_r) dr \right| \right).$$

Using the exchangeability of the particles $X^{i,N}$, $i \leq N$, we deduce that

$$(7) \quad \limsup_N \mathbb{E}^{\pi^N} |F - F_k(Q)| \leq C \limsup_N \mathbb{E} \left(\int_s^t (H_k - H)(X_r^{1,N} - X_r^{2,N}) dr \right).$$

Using the Girsanov theorem as in the proof of Lemma 2.4, we obtain that, $\forall N \geq 2$, the couple $(X_r^{1,N}, X_r^{2,N})$ has a density that belongs to $L^2([0, 1] \times [0, 1])$ with a norm smaller than $C(1 + r^{-1/2}) \exp(Ct)$. Hence

$$\forall r \in [0, t], \quad \mathbb{E}((H_k - H)(X_r^{1,N} - X_r^{2,N})) \leq C(t)(1 + r^{-1/2})k^{-1/4}.$$

By (7), we deduce that $\limsup_k \limsup_N \mathbb{E}^{\pi^N} |F - F_k(Q)| = 0$. Hence each term of the right-hand side of (6) is nil and $\mathbb{E}^{\pi^\infty} |F(Q)| = 0$. As a consequence, π^∞ -a.s., Q satisfies condition (ii) in Definition 1.1.

Let us check condition (iii). Since the following subset of \mathcal{C} ,

$$F^{T,M} = \left\{ (x, k) : |k|_T \leq M, |k|_T = \int_0^T \mathbb{1}_{\{0,1\}}(x_s) d|k|_s, \right. \\ \left. \forall t \leq T, k_t = \int_0^t (1 - 2x_s) d|k|_s \right\},$$

is closed as stated in Lemma 2.6, $\{Q \in \mathcal{P}(\mathcal{C}) : Q(F^{T,M}) \geq 1 - \varepsilon\}$ is also closed. By the weak convergence of π^N to π^∞ , we deduce

$$(8) \quad \begin{aligned} \pi^\infty(\{Q : Q(F^{T,M}) \geq 1 - \varepsilon\}) &\geq \limsup_N \pi^N(\{Q : Q(F^{T,M}) \geq 1 - \varepsilon\}) \\ &= 1 - \liminf_N \pi^N(\{Q : Q(\{|k|_T > M\}) > \varepsilon\}) \\ &\geq 1 - \liminf_N \mathbb{E}^{\pi^N} \left(\frac{\langle Q, |k|_T \rangle}{M\varepsilon} \right) \\ &= 1 - \frac{\liminf_N \mathbb{E}(|K^{1,N}|_T)}{M\varepsilon}. \end{aligned}$$

As $|K^{1,N}|_T = \int_0^T (1 - 2X_s^{1,N}) dK_s^{1,N}$, by Itô's formula, we obtain

$$\begin{aligned} |K^{1,N}|_T &= \left(X_0^{1,N} - \frac{1}{2} \right)^2 - \left(X_T^{1,N} - \frac{1}{2} \right)^2 \\ &\quad + \int_0^T (2X_s^{1,N} - 1) A'(H * \bar{\mu}_s^N(X_s^{1,N})) ds \\ &\quad + \int_0^T \sigma(2X_s^{1,N} - 1) dW_s^1 + \sigma^2 T. \end{aligned}$$

Hence $\sup_N \mathbb{E}(|K^{1,N}|_T) < +\infty$.

With (8), we deduce that $\pi^\infty(\{Q : Q(\bigcup_{M>0} F^{M,T}) \geq 1 - \varepsilon\}) = 1$. As ε is arbitrary, we conclude that

$$\pi^\infty \left(\left\{ Q : Q \left(\bigcap_{T>0} \bigcup_{M>0} F^{M,T} \right) = 1 \right\} \right) = 1,$$

that is, π^∞ -a.s., Q satisfies Definition 1.1(iii), which completes the proof. \square

LEMMA 2.6. *The subset $F^{T,M}$ of \mathcal{C} which consists in the couples (x, k) such that $|k|_T \leq M$, $|k|_T = \int_0^T \mathbb{1}_{\{0,1\}}(x_s) d|k|_s$ and $\forall t \leq T, k_t = \int_0^t (1 - 2x_s) d|k|_s$ is closed.*

PROOF. Let $(x^n, k^n) \in F^{T,M}$ converge to (x, k) in \mathcal{C} . As $\forall n \geq 0, |k^n|_T \leq M$, by extraction of a subsequence, we can suppose that the measures $d|k|^n$ (resp. dk^n) on the compact set $[0, T]$ converge weakly to da , a positive measure with

mass smaller than M (resp. db a signed measure). As k^n converges uniformly to k on $[0, T]$, $t \in [0, T] \rightarrow k_t$ is the cumulative distribution function of the measure db .

If $f : [0, T] \rightarrow \mathbb{R}$ is continuous, as x^n converges uniformly to x on $[0, T]$,

$$\begin{aligned} \int_0^T f(s) db_s &= \lim_n \int_0^T f(s) dk_s^n \\ &= \lim_n \int_0^T f(s)(1 - 2x_s^n) d|k^n|_s = \int_0^T f(s)(1 - 2x_s) da_s. \end{aligned}$$

Hence $(1 - 2x_s)$ is a density of db w.r.t. da and

$$(9) \quad \forall t \in [0, T], \quad k_t = \int_0^t (1 - 2x_s) da_s.$$

As $(x^n, k^n) \in F^{T,M}$, $\int_0^T x_s^n(1 - x_s^n) d|k^n|_s = 0$.

Letting $n \rightarrow +\infty$, we get $\int_0^T x_s(1 - x_s) da_s = 0$, that is, da_s a.e. $x_s \in \{0, 1\}$ and $|1 - 2x_s| = 1$. With (9), we deduce that da is the total variation of dk and conclude that $(x, k) \in F^{T,M}$. \square

COROLLARY 2.7. *It is possible to approximate the weak solution $V(t, x) = H * \bar{P}_t(x)$ of (1) thanks to the empirical cumulative distribution function $H * \bar{\mu}_t^N(x)$ of the particle system. More precisely*

$$\forall (t, x) \in [0, +\infty) \times [0, 1], \quad \lim_{N \rightarrow +\infty} \mathbb{E}|V(t, x) - H * \bar{\mu}_t^N(x)| = 0.$$

PROOF. For $t > 0$ and $x \in [0, 1]$, according to Lemma 2.4, the function $Q \in \mathcal{P}(\mathcal{C}) \rightarrow |H * \bar{P}_t(x) - H * \bar{Q}_t(x)| \in \mathbb{R}$ is continuous at P . The weak convergence of the sequence $(\pi^N)_N$ to $\pi^\infty = \delta_P$ implies

$$\lim_{N \rightarrow +\infty} \mathbb{E}|H * \bar{P}_t(x) - H * \bar{\mu}_t^N(x)| = \mathbb{E}^{\pi^\infty} |H * \bar{P}_t(x) - H * \bar{Q}_t(x)| = 0.$$

In the case $t = 0$, we conclude with the strong law of large numbers. \square

3. Particle method. In this section we describe a numerical particle method to approximate the solution V of (1) on $[0, T] \times [0, 1]$ (where T is a positive constant) and analyze its rate of convergence. According to Corollary 2.7, it is possible to approximate $V(t, x)$ by the empirical cumulative distribution function $H * \bar{\mu}_t^N(x) = \frac{1}{N} \sum_{j=1}^N H(x - X_t^{j,N})$ of the particle system (5). To transform this convergence result into a numerical approximation procedure, we need to discretize in time the N -dimensional stochastic differential equation (5). To do so we use the version of the Euler scheme introduced by Lépingle [10] which mimics the reflection at the boundary. We choose $\Delta t > 0$ and $L \in \mathbb{N}$ such that $T = L \Delta t$ and denote by Y_t^i the position of the i th particle ($1 \leq i \leq N$) at the

discretization time $t_l = l \Delta t$ ($0 \leq l \leq L$). The Euler–Lépingle scheme consists in setting $0 < \alpha_0 < \alpha_1 < 1$ and in generating exact reflection on the lower boundary on $[t_l, t_{l+1}]$ when $Y_{t_l}^i \leq \alpha_0$ and exact reflection on the upper boundary on $[t_l, t_{l+1}]$ when $Y_{t_l}^i \geq \alpha_1$. When at the end of the time step the computed position is smaller than 0 or greater than 1, then $Y_{t_{l+1}}^i$ is respectively chosen equal to 0 or equal to 1. We will actually let α_0 and α_1 depend on Δt to reduce the computational effort, but to simplify notation we do not emphasize this dependence unless necessary. Taking advantage of the one-dimensional space domain, we invert the initial cumulative distribution function $V_0(x) = H * U_0(x)$ to construct the set of initial positions of the numerical particles:

$$(10) \quad y_0^i = \inf \left\{ z : H * U_0(z) \geq \frac{i}{N} \right\} \quad \text{for } 1 \leq i \leq N.$$

At time t_l , the function $V(t_l, x)$ is approximated thanks to the empirical cumulative distribution function

$$\bar{V}(t_l, x) = \frac{1}{N} \sum_{i=1}^N H(x - Y_{t_l}^i)$$

and the positions of the i th particle are given inductively by

$$(11) \quad \begin{aligned} Y_{t_0}^i &= y_0^i \quad \text{and} \quad \forall 0 \leq l \leq L - 1, \forall t \in [t_l, t_{l+1}], \\ Y_t^i &= 0 \vee (Y_{t_l}^i + \sigma(W_t^i - W_{t_l}^i) + (t - t_l)A'(\bar{V}(t_l, Y_{t_l}^i)) + C_t^i) \wedge 1, \\ C_t^i &= \mathbb{1}_{\{Y_{t_l}^i \leq \alpha_0\}} \sup_{s \in [t_l, t]} (Y_{t_l}^i + \sigma(W_s^i - W_{t_l}^i) + (s - t_l)A'(\bar{V}(t_l, Y_{t_l}^i)))^- \\ &\quad - \mathbb{1}_{\{Y_{t_l}^i \geq \alpha_1\}} \sup_{s \in [t_l, t]} (Y_{t_l}^i - 1 + \sigma(W_s^i - W_{t_l}^i) + (s - t_l)A'(\bar{V}(t_l, Y_{t_l}^i)))^+. \end{aligned}$$

Since it is possible to simulate jointly the Brownian increment $(W_{t_{l+1}}^i - W_{t_l}^i)$ and the corresponding $\sup_{s \in [t_l, t_{l+1}]} (W_s^i - W_{t_l}^i + (s - t_l)\alpha)$, this discretization scheme is feasible.

To obtain the optimal rate of convergence $\mathcal{O}(1/\sqrt{N} + \Delta t)$ we are going to make rather strong assumptions on the initial condition $v_0(x) = H * U_0(x)$ ensuring that the weak solution of (1) is in fact a classical solution. These hypotheses are possibly too restrictive but they avoid further complications of the already technical proof. For the solution of (1) to be classical, that is, $C^{1,2}$ (C^1 in the time variable t and C^2 in the space variable x), it is necessary that v_0 is C^2 . Moreover, since the Dirichlet boundary conditions are constant in time, for $x = 0$ or $x = 1$, $\frac{\sigma^2}{2} \partial_{xx} v(t, x) - A'(v(t, x)) \partial_x v(t, x) = \partial_t v(t, x) = 0$. At time $t = 0$, we obtain the necessary compatibility conditions $\sigma^2 v_0''(0) = 2A'(0)v_0'(0)$ and

$\sigma^2 v_0''(1) = 2A'(1)v_0'(1)$. Our hypothesis

$$v_0 \in C^{2+\beta}([0, 1])$$

(H) $(v_0''$ Hölder continuous with exponent β), where $\beta \in]0, 1[$,

$$\sigma^2 v_0''(0) = 2A'(0)v_0'(0) \quad \text{and} \quad \sigma^2 v_0''(1) = 2A'(1)v_0'(1), \quad A \text{ is a } C^3 \text{ function,}$$

is slightly stronger than these necessary conditions. Combining Theorem 6.1 of [9, pages 452–453], which gives existence of a classical solution on $[0, T] \times [0, 1]$ for (1), and the proof of Lemma 2.2, which gives uniqueness of weak solutions on $[0, T] \times [0, 1]$, we obtain the following lemma.

LEMMA 3.1. *Under hypothesis (H), the solution $V(t, x) = H * \bar{P}_t(x)$ of (1) belongs to $C^{1,2}([0, T] \times [0, 1])$ and $\partial_x V(t, x)$ is Hölder continuous with exponent $(1 + \beta)/2$ in the time variable t on $[0, T] \times [0, 1]$.*

To reduce the effort needed to compute the correction terms C_t^i in (11), it is interesting to let α_0 and α_1 depend on the time step Δt and converge respectively to 0 and 1 as $\Delta t \rightarrow 0$. Supposing that these convergences are not too quick, we obtain the following estimate for the convergence rate of the particle method.

THEOREM 3.2. *Under hypothesis (H), if we assume that $0 < \alpha_0(\Delta t) \leq \alpha_1(\Delta t) < 1$ satisfy $\alpha_0(\Delta t) \wedge (1 - \alpha_1(\Delta t)) \geq a \Delta t^\gamma$ for $0 \leq \gamma < 1/2$ and $a > 0$, then there exists a strictly positive constant C depending on A, U_0, T, σ, a and γ such that*

$$\forall 0 \leq l \leq L, \quad \sup_{x \in [0, 1]} \mathbb{E}|V(t_l, x) - \bar{V}(t_l, x)| \leq C \left(\frac{1}{\sqrt{N}} + \Delta t \right).$$

The proof follows the main ideas of Bossy [3], who deals with the convergence rate of a particle approximation for the solution of the scalar conservation law with spatial domain \mathbb{R} similar to (1) even if some new difficulties arise in the present framework because of the reflection. Let W_t denote a standard Brownian motion. To analyze the convergence rate, for $y \in [0, 1]$, we introduce the stochastic differential equation with normal reflection, constant diffusion coefficient and drift coefficient $A'(V(s, x))$:

$$(12) \quad \begin{aligned} X_t^y &= y + \sigma W_t + \int_0^t A'(V(s, X_s^y)) ds + K_t^y, \\ |K^y|_t &= \int_0^t \mathbb{1}_{\{0,1\}}(X_s^y) d|K^y|_s, \quad K_t^y = \int_0^t (1 - 2X_s^y) d|K^y|_s. \end{aligned}$$

Under hypothesis (H), according to Lemma 3.1, the function $b(s, x) \stackrel{\text{def}}{=} A'(V(s, x))$ is Lipschitz continuous in the space variable x uniformly for $s \in [0, T]$ and is bounded. As a consequence, for any $y \in [0, 1]$ the above stochastic differential equation has a unique solution (see, e.g., [11], Remark 3.3, page 525).

We are interested in the upper bound of

$$\begin{aligned}
 \text{Error}(t_l) &= \sup_{x \in [0,1]} \mathbb{E} |V(t_l, x) - \bar{V}(t_l, x)| \\
 (13) \quad &\leq \sup_{x \in [0,1]} \left| V(t_l, x) - \frac{1}{N} \sum_{i=1}^N \mathbb{E} \left(H(x - X_{t_l}^{y_0^i}) \right) \right| \\
 &\quad + \sup_{x \in [0,1]} \mathbb{E} \left| \frac{1}{N} \sum_{i=1}^N \mathbb{E} \left(H(x - X_{t_l}^{y_0^i}) \right) - \bar{V}(t_l, x) \right|.
 \end{aligned}$$

The first term of the right-hand side is an initialization error, which we upper-bound in the next paragraph. Since the dynamics of the particle system (11) on $[t_l, t_{l+1}]$ depends on the approximate solution $\bar{V}(t_l, \cdot)$ of (1) at time t_l , the analysis of the second term of the right-hand side is more complicated than the analysis of the weak error of the discretization by the Euler–Lépingle scheme of the stochastic differential equation (12), where the drift coefficient $b(s, x) = A'(V(s, x))$ is supposed to be known. We are going to deal with the latter problem as a model problem: the results obtained are useful in solving the former problem. They are also interesting by themselves because, although limited to the case of a constant diffusion coefficient, they form the first study of the weak rate of the Euler–Lépingle scheme to our knowledge.

3.1. Initialization error.

LEMMA 3.3. *Under hypothesis (H), the solution (X_t^y) of (12) can be chosen continuous in $(t, y) \in [0, T] \times [0, 1]$ and nondecreasing in y for fixed $t \in [0, T]$. Moreover, $\forall (t, x) \in [0, T] \times [0, 1]$, $V(t, x) = \mathbb{E}(\int_0^1 H(x - X_t^y) U_0(dy))$.*

PROOF. As $(X_t^x - X_t^y)_{t \in [0, T]}$ is a continuous process with bounded variation, $(X_t^x - X_t^y)^+$ is equal to

$$(x - y)^+ + \int_0^t \mathbb{1}_{\{X_s^x > X_s^y\}} (b(s, X_s^x) - b(s, X_s^y)) ds + \int_0^t \mathbb{1}_{\{X_s^x > X_s^y\}} (dK_s^x - dK_s^y).$$

The third term of the right-hand side is nonpositive. By the Lipschitz continuity of $x \rightarrow b(s, x)$ and Gronwall’s lemma, we deduce that, for some real constant C_T ,

$$(14) \quad \text{a.s.,} \quad \sup_{t \in [0, T]} (X_t^x - X_t^y)^+ \leq C_T (x - y)^+.$$

Using the symmetric inequality for $(X_t^y - X_t^x)^+$, we obtain that a.s. $\sup_{t \in [0, T]} |X_t^x - X_t^y| \leq C_T |x - y|$. According to the Kolmogorov continuity theorem, the $C([0, T], [0, 1])$ -valued process $(t \rightarrow X_t^y)$ indexed by $y \in [0, 1]$ admits a continuous version that we still denote by X_t^y to simplify notation. By (14), a.s., $\forall q \leq q' \in [0, 1] \cap \mathbb{Q}$, $\forall t \in [0, T]$, $X_t^q \leq X_t^{q'}$. With the a.s. continuity of

$y \in [0, 1] \rightarrow X_t^y$ in the variable y , we conclude that this function is a.s. nondecreasing.

Now let X_0 be an initial variable with law U_0 independent of the Brownian motion W . We easily check that the law of $(X_t^{X_0}, K_t^{X_0})_{t \in [0, T]}$ on $C([0, T], [0, 1] \times \mathbb{R})$ solves the following martingale problem:

- (i) $Q_0 = U_0 \otimes \delta_0$;
- (ii) $\varphi(X_t - K_t) - \varphi(X_0 - K_0) - \int_0^t \frac{\sigma^2}{2} \varphi''(X_s - K_s) + A'(V(s, X_s)) \varphi'(X_s - K_s) ds$ is a Q martingale for any function $\varphi \in C_b^2(\mathbb{R})$;
- (iii) Q a.s., $\forall t \geq 0, \int_0^t d|K|_s < +\infty, |K|_t = \int_0^t \mathbb{1}_{[0, 1]}(X_s) d|K|_s$ and $K_t = \int_0^t (1 - 2X_s) d|K|_s$.

By the Girsanov theorem, uniqueness holds for this martingale problem. Since the image of the solution P of problem (MP) starting at $U_0 \otimes \delta_0$ by the canonical restriction from $C([0, +\infty), [0, 1] \times \mathbb{R})$ to $C([0, T], [0, 1] \times \mathbb{R})$ solves this martingale problem, we deduce that $\forall (t, x) \in [0, T] \times [0, 1], V(t, x) = H * \bar{P}_t(x) = \mathbb{E}(H(x - X_t^{X_0}))$. By independence of X_0 and W , we conclude that $V(t, x) = \mathbb{E}(\int_0^1 H(x - X_t^y) U_0(dy))$. \square

We easily deduce that the initialization error is smaller than $1/N$.

LEMMA 3.4. For all $t \geq 0$,

$$\sup_{x \in [0, 1]} \left| V(t, x) - \frac{1}{N} \sum_{i=1}^N \mathbb{E}(H(x - X_t^{y_0^i})) \right| \leq \frac{1}{N}.$$

PROOF. Let $\bar{U}_0 = \frac{1}{N} \sum_{i=1}^N \delta_{y_0^i}$. Clearly,

$$\begin{aligned} \text{a.s., } \int_{[0, 1]} H(x - X_t^y) U_0(dy) - \frac{1}{N} \sum_{i=1}^N H(x - X_t^{y_0^i}) \\ = (U_0 - \bar{U}_0)(\{y \in [0, 1]: X_t^y \leq x\}). \end{aligned}$$

Since, by Lemma 3.3, $y \rightarrow X_t^y$ is a.s. continuous and increasing, if nonempty, the set $\{y : X_t^y \leq x\}$ is equal to $[0, \phi_t(x)]$, where $\phi_t(x) = \inf\{y : X_t^y > x\}$. By definition of the initial positions y_0^i [see (10)], $\forall y \in [0, 1], 0 \leq (U_0 - \bar{U}_0)([0, y]) \leq 1/N$. Hence

$$0 \leq \int_{[0, 1]} H(x - X_t^y) U_0(dy) - \frac{1}{N} \sum_{i=1}^N H(x - X_t^{y_0^i}) \leq \frac{1}{N}.$$

We conclude by taking expectations. \square

3.2. *Weak error of the Euler–Lépingle scheme.* We recall that $T = L \Delta t$ ($\Delta t > 0$, $L \in \mathbb{N}$) and $t_l = l \Delta t$ for $0 \leq l \leq L$. The Euler–Lépingle discretization of the stochastic differential equation

$$(15) \quad \begin{aligned} X_t^y &= y + \sigma W_t + \int_0^t b(s, X_s^y) ds + K_t^y, \\ |K^y|_t &= \int_0^t \mathbb{1}_{\{0,1\}}(X_s^y) d|K^y|_s, \quad K_t^y = \int_0^t (1 - 2X_s^y) d|K^y|_s \end{aligned}$$

is given by

$$\begin{aligned} \tilde{X}_{t_0}^y &= y, \\ \tilde{X}_t^y &= 0 \vee (\tilde{X}_{t_l}^y + \sigma(W_t - W_{t_l}) + b(t_l, \tilde{X}_{t_l}^y)(t - t_l) + C_t) \wedge 1, \\ C_t &= \mathbb{1}_{\{\tilde{X}_{t_l}^y \leq \alpha_0\}} \sup_{s \in [t_l, t]} (\tilde{X}_{t_l}^y + \sigma(W_s - W_{t_l}) + b(t_l, \tilde{X}_{t_l}^y)(s - t_l))^- \\ &\quad - \mathbb{1}_{\{\tilde{X}_{t_l}^y \geq \alpha_1\}} \sup_{s \in [t_l, t]} (\tilde{X}_{t_l}^y - 1 + \sigma(W_s - W_{t_l}) + b(t_l, \tilde{X}_{t_l}^y)(s - t_l))^+ \\ &\quad \forall t \in [t_l, t_{l+1}]. \end{aligned}$$

In the next proposition, assuming a regularity condition on the drift coefficient $b(s, x)$ which is satisfied by $A'(V(s, x))$ under hypothesis (H) (see Lemma 3.1), we upper-bound the weak convergence rate of this scheme:

PROPOSITION 3.5. *Assume that b is $C^{1,2}$ on $[0, T] \times [0, 1]$, that for some $\alpha > 0$, $\partial_x b(t, x)$ is Hölder continuous with exponent α in t and that $0 < \alpha_0(\Delta t) \leq \alpha_1(\Delta t) < 1$ satisfy $\alpha_0(\Delta t) \wedge (1 - \alpha_1(\Delta t)) \geq a \Delta t^\gamma$ for $\gamma \in [0, 1/2)$ and $a > 0$. Then there is a constant C depending on σ, T, b, a, γ but not on y and Δt such that, when $f : [0, 1] \rightarrow \mathbb{R}$ is a function with bounded variation and m denotes its distribution derivative,*

$$\forall l \leq L, \quad |\mathbb{E}(f(X_{t_l}^y) - f(\tilde{X}_{t_l}^y))| \leq C \Delta t \int_0^1 |m|(dx).$$

The error proceeds from two sources. The first one is the usual Euler discretization of the drift coefficient. The second contribution is the inexact treatment of the reflection on the lower boundary (resp. the upper boundary, resp. both boundaries) between t_l and t_{l+1} when $\tilde{X}_{t_l}^y \geq \alpha_1$ [resp. $\tilde{X}_{t_l}^y \leq \alpha_0$, resp. $\tilde{X}_{t_l}^y \in (\alpha_0, \alpha_1)$], which will turn out to be negligible. To get rid of it, we introduce the Euler–Peano discretization of (15). The Euler–Peano scheme is a theoretical discretization scheme which consists in freezing the drift coefficient on each interval $[t_l, t_{l+1}]$ whereas the normal reflection remains exact:

$$(16) \quad \begin{aligned} \hat{X}_t^y &= y + \sigma W_t + \int_0^t b(\tau_s, \hat{X}_{\tau_s}^y) ds + \hat{K}_t^y, \\ |\hat{K}^y|_t &= \int_0^t \mathbb{1}_{\{0,1\}}(\hat{X}_s^y) d|\hat{K}^y|_s, \quad \hat{K}_t^y = \int_0^t (1 - 2\hat{X}_s^y) d|\hat{K}^y|_s, \end{aligned}$$

where $\tau_s = \Delta t \lceil \frac{s}{\Delta t} \rceil$ and $[x]$ denotes the integral part of x .

LEMMA 3.6. Assume that $b(\cdot, \cdot)$ is bounded and that $0 < \alpha_0(\Delta t) \leq \alpha_1(\Delta t) < 1$ satisfy $\alpha_0(\Delta t) \wedge (1 - \alpha_1(\Delta t)) \geq a \Delta t^\gamma$ for $\gamma \in [0, 1/2)$ and $a > 0$. Then, for some positive constants c and C independent of Δt and y ,

$$(17) \quad \forall l \leq L, \quad \mathbb{P}(\exists k \leq l, \hat{X}_{t_k}^y \neq \tilde{X}_{t_k}^y) \leq C \Delta t^{-\gamma-1/2} e^{-c \Delta t^{2\gamma-1}}.$$

PROOF. The proof follows the ideas of [10] even if this upper bound is not stated. To simplify notation, we do not emphasize the dependence of α_0 and α_1 on Δt ,

$$(18) \quad \mathbb{P}(\exists k \leq l, \hat{X}_{t_k}^y \neq \tilde{X}_{t_k}^y) \leq \sum_{k=0}^{l-1} \mathbb{P}(\hat{X}_{t_k}^y = \tilde{X}_{t_k}^y, \hat{X}_{t_{k+1}}^y \neq \tilde{X}_{t_{k+1}}^y).$$

When $\alpha_0 < \hat{X}_{t_k}^y = \tilde{X}_{t_k}^y < \alpha_1$, we remark that $\forall t \in [t_k, t_{k+1}]$, $\hat{X}_t^y = \tilde{X}_t^y$ unless both processes reach 0 or 1 before t_{k+1} . As a consequence,

$$\begin{aligned} & \mathbb{P}(\alpha_0 < \hat{X}_{t_k}^y = \tilde{X}_{t_k}^y < \alpha_1, \hat{X}_{t_{k+1}}^y \neq \tilde{X}_{t_{k+1}}^y) \\ & \leq \mathbb{P}\left(\sup_{t \in [t_k, t_{k+1}]} |W_t - W_{t_k}| > (a \Delta t^\gamma - \sup |b(\cdot, \cdot)| \Delta t) / \sigma\right). \end{aligned}$$

Since $\gamma < 1$, for Δt small enough, $a \Delta t^\gamma - \sup |b(\cdot, \cdot)| \Delta t \geq a \Delta t^\gamma / 2$. Then

$$\begin{aligned} & \mathbb{P}(\alpha_0 < \hat{X}_{t_k}^y = \tilde{X}_{t_k}^y < \alpha_1, \hat{X}_{t_{k+1}}^y \neq \tilde{X}_{t_{k+1}}^y) \\ & \leq \mathbb{P}\left(\sup_{s \in [t_k, t_{k+1}]} |W_s - W_{t_k}| \geq \frac{a \Delta t^\gamma}{2\sigma}\right) \\ & \leq 2\sqrt{\frac{2}{\pi \Delta t}} \int_{a \Delta t^\gamma / (2\sigma)}^{+\infty} e^{-z^2/2 \Delta t} dz \leq \frac{4\sigma}{a} \sqrt{\frac{2}{\pi}} \Delta t^{1/2-\gamma} e^{-a^2 \Delta t^{2\gamma-1} / 8\sigma^2}. \end{aligned}$$

Since $\alpha_0 \leq \alpha_1$, $\alpha_1 \wedge (1 - \alpha_0) \geq a \Delta t^\gamma$ and remarking that $\forall t \in [t_k, t_{k+1}]$, $C_t \leq \sigma \sup_{s \in [t_k, t_{k+1}]} |W_s - W_{t_k}| + \sup |b(\cdot, \cdot)| \Delta t$, we easily obtain similar bounds for $\mathbb{P}(\hat{X}_{t_k}^y = \tilde{X}_{t_k}^y \geq \alpha_1, \hat{X}_{t_{k+1}}^y \neq \tilde{X}_{t_{k+1}}^y)$ and $\mathbb{P}(\hat{X}_{t_k}^y = \tilde{X}_{t_k}^y \leq \alpha_0, \hat{X}_{t_{k+1}}^y \neq \tilde{X}_{t_{k+1}}^y)$ and we conclude with (18) since $l \leq L = T/\Delta t$. \square

Let $l \geq 1$, let $f : [0, 1] \rightarrow \mathbb{R}$ be a function with bounded variation and let m denote its distribution derivative. According to the previous lemma,

$$\begin{aligned} & |\mathbb{E}(f(X_t^y) - f(\tilde{X}_t^y))| \\ & \leq |\mathbb{E}(f(X_t^y) - f(\hat{X}_t^y))| \\ & \quad + \sup_{z, z' \in [0, 1]} |f(z) - f(z')| C \Delta t^{-\gamma-1/2} e^{-c \Delta t^{2\gamma-1}} \\ & \leq |\mathbb{E}(f(X_t^y) - f(\hat{X}_t^y))| + C \Delta t^{-\gamma-1/2} e^{-c \Delta t^{2\gamma-1}} \int_0^1 |m|(dy). \end{aligned}$$

The set D_f of discontinuity points of f is countable. For $y \leq z \in [0, 1] \setminus D_f$, $f(z) = f(y) + \int_y^z m(dx) = f(y) + \int_0^1 H(x - y) - H(x - z)m(dx)$. As the function $b(\cdot, \cdot)$ is bounded, by the Girsanov theorem both variables $X_{t_l}^y$ and $\hat{X}_{t_l}^y$ have densities w.r.t. Lebesgue measure. Hence $\mathbb{P}(X_{t_l}^y \in D_f) + \mathbb{P}(\hat{X}_{t_l}^y \in D_f) = 0$ and

$$|\mathbb{E}(f(X_{t_l}^y) - f(\hat{X}_{t_l}^y))| = \left| \int_0^1 \mathbb{E}(H(x - X_{t_l}^y) - H(x - \hat{X}_{t_l}^y))m(dx) \right|.$$

Therefore the proof of Proposition 3.5 will complete as soon as we obtain the following weak convergence rate for the Euler–Peano scheme:

PROPOSITION 3.7. *Under the assumptions of Proposition 3.5, there is a constant C depending on σ, T, b but not on x, y and Δt such that*

$$\forall l \leq L, \forall x, y \in [0, 1], \quad |\mathbb{E}(H(x - X_{t_l}^y) - H(x - \hat{X}_{t_l}^y))| \leq C \Delta t.$$

In the case $l = 0$, the conclusion is clear. As for all other values of l the proof is the same; we only deal with the case $l = L$, that is, $t_l = T$. By the Girsanov theorem both variables X_T^y and \hat{X}_T^y admit densities with respect to Lebesgue measure. Hence

$$\mathbb{E}(H(0 - X_T^y) - H(0 - \hat{X}_T^y)) = \mathbb{P}(X_T^y = 0) - \mathbb{P}(\hat{X}_T^y = 0) = 0 - 0 = 0,$$

$$\mathbb{E}(H(1 - X_T^y) - H(1 - \hat{X}_T^y)) = \mathbb{P}(X_T^y \leq 1) - \mathbb{P}(\hat{X}_T^y \leq 1) = 1 - 1 = 0$$

and the conclusion holds for $x \in \{0, 1\}$.

From now on, we assume that $x \in (0, 1)$. We follow the idea first introduced by Talay and Tubaro: if the function v solves the parabolic problem

$$(19) \quad \begin{aligned} \partial_t v + \frac{\sigma^2}{2} \partial_z^2 v + b(t, z) \partial_z v &= 0, & (t, z) \in [0, T) \times [0, 1], \\ v(T, z) &= H(x - z) & \forall z \in [0, 1], \\ \partial_z v(t, 0) = \partial_z v(t, 1) &= 0 & \forall t \in [0, T], \end{aligned}$$

then formally computing $H(x - \hat{X}_T^y) - H(x - X_T^y) = v(T, \hat{X}_T^y) - v(T, X_T^y)$ by Itô's formula and taking expectations we obtain

$$\mathbb{E}(H(x - \hat{X}_T^y) - H(x - X_T^y)) = \mathbb{E} \left(\int_0^T (b(s, \hat{X}_s^y) - b(\tau_s, \hat{X}_{\tau_s}^y)) (-\partial_z v(s, \hat{X}_s^y)) ds \right).$$

The function v appears only through the opposite of its spatial derivative, which justifies our interest in the parabolic problem satisfied by $w = -\partial_z v$,

$$(20) \quad \begin{aligned} \partial_t w + \frac{\sigma^2}{2} \partial_z^2 w + b(t, z) \partial_z w + \partial_z b(t, z) w &= 0, & (t, z) \in [0, T) \times [0, 1], \\ w(T, \cdot) &= \delta_x(\cdot), & w(t, 0) = w(t, 1) = 0 & \forall t \in [0, T]. \end{aligned}$$

According to [9], Section IV.16, and [7], Section 3.7, which are dedicated to Green’s functions, the following holds:

LEMMA 3.8. *Under the assumptions of Proposition 3.5, there is a continuous function $(x, t, z) \in (0, 1) \times [0, T] \times [0, 1] \rightarrow w(x, t, z) \in \mathbb{R}$ such that the following hold:*

(i) *For fixed $x \in (0, 1)$, $(t, z) \rightarrow w(x, t, z) \in C^{1,2}([0, T] \times [0, 1])$ solves*

$$(21) \quad \begin{aligned} \partial_t w + \frac{\sigma^2}{2} \partial_z^2 w + b(t, z) \partial_z w + \partial_z b(t, z) w &= 0, & (t, z) \in [0, T] \times [0, 1], \\ w(x, t, 0) = w(x, t, 1) &= 0 & \forall t \in [0, T] \end{aligned}$$

and takes the terminal value $w(x, T, \cdot) = \delta_x(\cdot)$ in the distribution sense.

(ii) *For any integers r and s such that $2r + s \leq 2$,*

$$\forall t \in [0, T), \forall x, z,$$

$$|\partial_t^r \partial_z^s w(x, t, z)| \leq C(T - t)^{-(1+2r+s)/2} \exp\left(-c \frac{(z - x)^2}{T - t}\right).$$

(iii) *For any function φ continuous on $[0, 1]$, the function*

$$w^\varphi(t, z) = \begin{cases} \int_0^1 w(x', t, z) \varphi(x') dx', & \text{if } t < T, \\ \varphi(z), & \text{if } t = T, \end{cases}$$

is continuous on $[0, T] \times [0, 1]$ and satisfies (21).

Thanks to these results, we rigorously express $\mathbb{E}(H(x - \hat{X}_T^y) - H(x - X_T^y))$ in terms of w :

LEMMA 3.9.

$$\mathbb{E}(H(x - \hat{X}_T^y) - H(x - X_T^y)) = \mathbb{E}\left(\int_0^T (b(s, \hat{X}_s^y) - b(\tau_s, \hat{X}_{\tau_s}^y)) w(x, s, \hat{X}_s^y) ds\right).$$

PROOF. For $\varepsilon > 0$, we set $\varphi^\varepsilon(x') = e^{-(x'-x)^2/(2\varepsilon)}/\sqrt{2\pi\varepsilon}$ and $v^\varepsilon(t, z) = \int_z^1 w^{\varphi^\varepsilon}(t, z') dz'$. By Lemma 3.8, the function v^ε is continuous on $[0, T] \times [0, 1]$ and satisfies

$$\begin{aligned} \forall (t, z) \in [0, T] \times [0, 1], \quad \partial_t v^\varepsilon + \frac{\sigma^2}{2} \partial_z^2 v^\varepsilon + b(t, z) \partial_z v^\varepsilon + \frac{\sigma^2}{2} \partial_z w^{\varphi^\varepsilon}(t, 1) &= 0, \\ \forall t \in [0, T), \quad \partial_z v^\varepsilon(t, 0) = \partial_z v^\varepsilon(t, 1) &= 0. \end{aligned}$$

Hence for $t < T$, by Itô's formula,

$$\begin{aligned} v^\varepsilon(t, \hat{X}_t^y) - v^\varepsilon(t, X_t^y) &= \int_0^t \sigma(\partial_z v^\varepsilon(s, \hat{X}_s^y) - \partial_z v^\varepsilon(s, X_s^y)) dW_s \\ &\quad + \int_0^t (b(s, \hat{X}_s^y) - b(\tau_s, \hat{X}_{\tau_s}^y))(-\partial_z v^\varepsilon(s, \hat{X}_s^y)) ds. \end{aligned}$$

This equation still holds for $t = T$ by continuity of both sides, since for $s < T$, by the upper bound of w given in Lemma 3.8 and the convolution property of Gaussian kernels,

$$(22) \quad |\partial_z v^\varepsilon(s, z)| \leq \int_0^1 \varphi^\varepsilon(x') |w(x', s, z)| dx' \leq C/\sqrt{\varepsilon + (T - s)}.$$

Taking expectations, we deduce that

$$(23) \quad \begin{aligned} &\mathbb{E}(v^\varepsilon(T, \hat{X}_T^y) - v^\varepsilon(T, X_T^y)) \\ &= \mathbb{E}\left(\int_0^T (b(s, \hat{X}_s^y) - b(\tau_s, \hat{X}_{\tau_s}^y))(-\partial_z v^\varepsilon(s, \hat{X}_s^y)) ds\right). \end{aligned}$$

The function $z \rightarrow v^\varepsilon(T, z) = \int_z^1 \varphi^\varepsilon(z') dz'$ is bounded by 1 uniformly in ε and converges pointwise to $\mathbb{1}_{\{z=x\}}/2 + \mathbb{1}_{\{z < x\}}$ as $\varepsilon \rightarrow 0$. By the Lebesgue theorem, the left-hand side of (23) converges to

$$\begin{aligned} &\mathbb{E}(H(x - \hat{X}_T^y) - H(x - X_T^y)) + \frac{1}{2}(\mathbb{P}(X_T^y = x) - \mathbb{P}(\hat{X}_T^y = x)) \\ &= \mathbb{E}(H(x - \hat{X}_T^y) - H(x - X_T^y)) \end{aligned}$$

since by the Girsanov theorem both variables X_T^y and \hat{X}_T^y admit densities w.r.t. Lebesgue measure.

By continuity of the function w , $\forall (s, z) \in [0, T) \times [0, 1]$, $-\partial_z v^\varepsilon(s, z) = \int_0^1 w(x', s, z) \varphi^\varepsilon(x') dx'$ converges to $w(x, s, z)$ as $\varepsilon \rightarrow 0$. Using (22), we obtain by the Lebesgue theorem that the right-hand side of (23) converges to $\mathbb{E}(\int_0^T (b(s, \hat{X}_s^y) - b(\tau_s, \hat{X}_{\tau_s}^y)) w(x, s, \hat{X}_s^y) ds)$. Hence Lemma 3.9 holds. \square

In the sequel $x \in (0, 1)$ is fixed and we write $w(t, z)$ instead of $w(x, t, z)$.

Applying Itô's formula to the function $g_l(t, z) = (b(t, z) - b(t_l, \hat{X}_{t_l}^y))w(t, z)$ we get that, for $l \leq L - 1$ and $s \in [t_l, t_{l+1})$,

$$(24) \quad \begin{aligned} &(b(s, \hat{X}_s^y) - b(t_l, \hat{X}_{t_l}^y))w(s, \hat{X}_s^y) \\ &= \int_{t_l}^s \left(\partial_t + \frac{\sigma^2}{2} \partial_z^2 + b(t_l, \hat{X}_{t_l}^y) \partial_z \right) g_l(\theta, \hat{X}_\theta^y) d\theta \\ &\quad + \int_{t_l}^s \partial_z g_l(\theta, \hat{X}_\theta^y) d\hat{K}_\theta^y + \sigma \int_{t_l}^s \partial_z g_l(\theta, \hat{X}_\theta^y) dW_\theta \\ &\stackrel{\text{def}}{=} T_s^1 + T_s^2 + T_s^3. \end{aligned}$$

We divide the integral on $[0, T]$ in the right-hand side of the equality given by Lemma 3.9 into integrals on $[t_l, t_{l+1}]$, $0 \leq l \leq L - 1$, and treat separately the first and last terms,

$$\begin{aligned}
 & |\mathbb{E}(H(x - \hat{X}_T^y) - H(x - X_T^y))| \\
 & \leq \left| \mathbb{E} \left(\int_0^{t_1} (b(s, \hat{X}_s^y) - b(0, y)) w(s, X_s^y) ds \right) \right| \\
 (25) \quad & + \left| \mathbb{E} \left(\sum_{l=1}^{L-2} \int_{t_l}^{t_{l+1}} T_s^1 + T_s^2 + T_s^3 ds \right) \right| \\
 & + \left| \mathbb{E} \left(\int_{t_{L-1}}^T (b(s, \hat{X}_s^y) - b(t_{L-1}, \hat{X}_{t_{L-1}}^y)) w(s, X_s^y) ds \right) \right|.
 \end{aligned}$$

To upper-bound the last term we need the following lemma, the proof of which is postponed.

LEMMA 3.10. For all $0 \leq t \leq s \leq T$,

$$\mathbb{E} \left(\sup_{\theta \in [t, s]} (\hat{X}_\theta^y - \hat{X}_s^y)^2 + (|\hat{K}^y|_s - |\hat{K}^y|_t)^2 \right) \leq C(s - t).$$

Combining this result, the regularity assumptions on b and the upper bound $|w(s, \cdot)| \leq C(T - s)^{-1/2}$ given in Lemma 3.8, we get

$$\begin{aligned}
 & \left| \mathbb{E} \left(\int_{t_{L-1}}^T (b(s, \hat{X}_s^y) - b(t_{L-1}, \hat{X}_{t_{L-1}}^y)) w(s, \hat{X}_s^y) ds \right) \right| \\
 & \leq \mathbb{E} \sup_{[t_{L-1}, T]} |b(s, \hat{X}_s^y) - b(t_{L-1}, \hat{X}_{t_{L-1}}^y)| \int_{t_{L-1}}^T C(T - s)^{-1/2} ds \leq C\Delta t.
 \end{aligned}$$

The same bound is valid for $|\mathbb{E}(\int_0^{t_1} (b(s, \hat{X}_s^y) - b(0, y)) w(s, \hat{X}_s^y) ds)|$.

LEMMA 3.11.

$$(26) \quad \forall s \in [0, T], \quad \mathbb{E}|T_s^1| \leq C \int_{\tau_s}^s ((T - \theta)^{-1/2} + (T - \theta)^{-2/3} \theta^{-1/3}) d\theta;$$

$$\begin{aligned}
 (27) \quad & \forall \eta \in (0, 1/2), \forall 1 \leq l \leq L - 1, \forall s \in [t_l, t_{l+1}[, \\
 & \mathbb{E}|T_s^2| \leq C t_l^{-(1-2\eta)/4(1-\eta)} \Delta t^{(5-4\eta)/4} (T - s)^{-1};
 \end{aligned}$$

$$(28) \quad \mathbb{E} \left(\int_{t_1}^{t_{L-1}} T_s^3 ds \right) = 0.$$

Once we check that Lemma 3.11 holds, we deduce from (25) that

$$\begin{aligned} & |\mathbb{E}(H(x - \hat{X}_T^y) - H(x - X_T^y))| \\ & \leq C\Delta T + C \int_{t_1}^{t_{L-1}} \int_{\tau_s}^s ((T - \theta)^{-1/2} + (T - \theta)^{-2/3}\theta^{-1/3}) d\theta ds \\ & \quad + C\Delta t^{(5-4\eta)/4} \sum_{l=1}^{L-2} t_l^{-(1-2\eta)/4(1-\eta)} (T - t_{l+1})^{-1} \Delta t \\ & \leq C(\Delta t + \Delta t^{(5-4\eta)/4} |\ln \Delta t|) \leq C\Delta t \quad \text{by choosing } \eta < 1/4; \end{aligned}$$

that is, Proposition 3.7 holds.

PROOF OF LEMMA 3.10. Let $\phi : x \in \mathbb{R} \rightarrow 1 - |1 - x + 2[x/2]| \in [0, 1]$ and $0 \leq t \leq T$. By the Girsanov theorem, since b is bounded, the stochastic differential equation

$$Y_\theta = \hat{X}_t^y + \sigma(W_\theta - W_t) + \int_t^\theta (-1)^{\lfloor Y_r \rfloor} b(\tau_r, \phi(Y_{\tau_r})) dr$$

has a unique weak solution. Moreover the processes $(\phi(Y_\theta))_{\theta \geq t}$ and $(\hat{X}_\theta^y)_{\theta \geq t}$ have the same law. Since ϕ is Lipschitz continuous with constant 1, we deduce that

$$\mathbb{E} \left(\sup_{\theta \in [t, s]} (\hat{X}_\theta^y - \hat{X}_t^y)^2 \right) \leq \mathbb{E} \left(\sup_{\theta \in [t, s]} (Y_\theta - \hat{X}_t^y)^2 \right) \leq C(s - t).$$

Now applying Itô's formula to compute $(\hat{X}_s^y - \frac{1}{2})^2 - (\hat{X}_t^y - \frac{1}{2})^2$, we get

$$\begin{aligned} |\hat{K}^y|_s - |\hat{K}^y|_t &= \int_t^s (2\hat{X}_r^y - 1)(\sigma dW_r + b(\tau_r, \hat{X}_{\tau_r}^y) dr) \\ &\quad + (s - t) + (\hat{X}_t^y - \hat{X}_s^y)(\hat{X}_t^y + \hat{X}_s^y - 1). \end{aligned}$$

Using the previous upper bound, we conclude that $\mathbb{E}((|\hat{K}^y|_s - |\hat{K}^y|_t)^2) \leq C(s - t)$. □

PROOF OF LEMMA 3.11. Using (20) we check that $\forall 0 \leq l \leq L - 1, \forall s \in [t_l, t_{l+1})$,

$$\begin{aligned} T_s^1 &= \int_{t_l}^s w(\theta, \hat{X}_\theta^y) \left(\partial_t + \frac{\sigma^2}{2} \partial_z^2 + (2b(t_l, \hat{X}_{t_l}^y) - b(\theta, \hat{X}_\theta^y)) \partial_z \right) b(\theta, \hat{X}_\theta^y) d\theta \\ &\quad + \int_{t_l}^s (\sigma^2 \partial_z b(\theta, \hat{X}_\theta^y) - (b(\theta, \hat{X}_\theta^y) - b(t_l, \hat{X}_{t_l}^y))^2) \partial_z w(\theta, \hat{X}_\theta^y) d\theta. \end{aligned}$$

By the regularity assumptions on the function $b(\cdot, \cdot)$ and the upper bound $|w(\theta, \cdot)| \leq C(T - \theta)^{-1/2}$ given in Lemma 3.8, we deduce that $\mathbb{E}|T_s^1| \leq$

$C \int_t^s (T - \theta)^{-1/2} + \mathbb{E}|\partial_z w(\theta, \hat{X}_\theta^y)| d\theta$. Since the rough upper bound $C/(T - \theta)$ of $\partial_z w(\theta, \cdot)$ is not integrable for $\theta \in [0, T]$ we make use of the Girsanov theorem: for a well-chosen exponential martingale \hat{Z}_t , if β_t^y denotes the doubly reflected process associated with the Brownian motion $y + \sigma W_t$, we have

$$\mathbb{E}|\partial_z w(\theta, \hat{X}_\theta^y)| = \mathbb{E}(|\partial_z w(\theta, \beta_\theta^y)| \hat{Z}_T) \leq (\mathbb{E}(\hat{Z}_T^3))^{1/3} (\mathbb{E}|\partial_z w(\theta, \beta_\theta^y)|^{3/2})^{2/3}.$$

Since $\forall \theta \in]0, T], \forall y \in [0, 1]$, the density of β_θ^y is smaller than $C\theta^{-1/2}$, by Lemma 3.8,

$$\begin{aligned} \mathbb{E}|\partial_z w(\theta, \beta_\theta^y)|^{3/2} &\leq C \int_0^1 (T - \theta)^{-3/2} \exp\left(-c \frac{(z - x)^2}{T - \theta}\right) \theta^{-1/2} dz \\ &\leq C(T - \theta)^{-1} \theta^{-1/2} \end{aligned}$$

and we deduce that $\mathbb{E}|\partial_z w(\theta, \hat{X}_\theta^y)| \leq C(T - \theta)^{-2/3} \theta^{-1/3}$. Hence (26) holds.

We turn to the proof of (27). Let $0 \leq l \leq L - 1$ and $s \in [t_l, t_{l+1})$.

As $d|\hat{K}^y|_\theta = \mathbb{1}_{\{0,1\}}(\hat{X}_\theta^y) d|\hat{K}^y|_\theta$ and $w(t, 0) = w(t, 1) = 0$,

$$T_s^2 = \int_{t_l}^s (b(\theta, \hat{X}_\theta^y) - b(t_l, \hat{X}_{t_l}^y)) \partial_z w(\theta, \hat{X}_\theta^y) \mathbb{1}_{\{0,1\}}(\hat{X}_\theta^y) d\hat{K}_\theta^y.$$

We deduce that

$$\begin{aligned} \mathbb{E}|T_s^2| &\leq \sup_{[t_l, s]} (|\partial_z w(\theta, 0)| \vee |\partial_z w(\theta, 1)|) \times (\mathbb{E}(|\hat{K}^y|_{t_{l+1}} - |\hat{K}^y|_{t_l})^2)^{1/2} \\ (29) \quad &\times \left(\mathbb{E} \left(\sup_{[t_l, t_{l+1}]} ((b(\theta, \hat{X}_\theta^y) - b(t_l, \hat{X}_{t_l}^y))^2 \mathbb{1}_{\{0,1\}}(\hat{X}_\theta^y)) \right) \right)^{1/2}. \end{aligned}$$

Let us upper-bound the three terms of the right-hand side. By Lemma 3.8, $\sup_{[t_l, s]} |\partial_z w(\theta, 0)| \vee |\partial_z w(\theta, 1)| \leq C(T - s)^{-1}$. For $\eta \in (0, 1/2)$,

$$\begin{aligned} &\mathbb{E} \left(\sup_{[t_l, t_{l+1}]} ((b(\theta, \hat{X}_\theta^y) - b(t_l, \hat{X}_{t_l}^y))^2 \mathbb{1}_{\{\hat{X}_\theta^y=0\}}) \right) \\ &\leq 4 \sup |b(\cdot, \cdot)|^2 \mathbb{P} \left(\hat{X}_{t_l}^y \geq \Delta t^{(1-\eta)/2}, \inf_{[t_l, t_{l+1}]} \hat{X}_\theta^y = 0 \right) \\ &\quad + \mathbb{E} \left(\mathbb{1}_{\{\hat{X}_{t_l}^y \leq \Delta t^{(1-\eta)/2}\}} \sup_{[t_l, t_{l+1}]} (b(\theta, 0) - b(t_l, \hat{X}_{t_l}^y))^2 \right). \end{aligned}$$

Following the same approach as in the proof of (17), we upper-bound the first term of the right-hand side by $C\Delta t^{\eta/2} e^{-c/\Delta t^{2\eta}}$. The second term is smaller than $C\Delta t^{1-\eta} \mathbb{P}(\hat{X}_{t_l}^y \leq \Delta t^{(1-\eta)/2})$ and by using the Girsanov theorem as in the derivation of (26), when $l \geq 1$, we get

$$\begin{aligned} \mathbb{P}(\hat{X}_{t_l}^y \leq \Delta t^{(1-\eta)/2}) &\leq \mathbb{E}(\hat{Z}_T^{(1-\eta)/\eta})^{\eta/1-\eta} \mathbb{P}(\beta_{t_l}^y \leq \Delta t^{(1-\eta)/2})^{(1-2\eta)/(1-\eta)} \\ &\leq C t_l^{-(1-2\eta)/2(1-\eta)} \Delta t^{(1-2\eta)/2}. \end{aligned}$$

Treating in a symmetric way $\mathbb{E}(\sup_{[t_l, t_{l+1}]} ((b(\theta, \hat{X}_\theta^y) - b(t_l, \hat{X}_{t_l}^y))^2) \mathbb{1}_{\{\hat{X}_\theta^y=1\}})$, we deduce that, when $l \geq 1$,

$$\mathbb{E} \left(\sup_{[t_l, t_{l+1}]} (\mathbb{1}_{\{0,1\}}(\hat{X}_\theta^y) (b(\theta, \hat{X}_\theta^y) - b(t_l, \hat{X}_{t_l}^y))^2) \right) \leq C t_l^{-(1-2\eta)/2(1-\eta)} \Delta t^{(3-4\eta)/2}.$$

Since, according to Lemma 3.10, the second term of the right-hand side of (29) is smaller than $C \Delta t^{1/2}$ we conclude that (27) holds.

Finally, let us check (28). By the integration-by-parts formula, for $l \leq L - 2$, $\int_{t_l}^{t_{l+1}} T_s^3 ds$ is equal to

$$\sigma \int_{t_l}^{t_{l+1}} (t_{l+1} - \theta) ((b(\theta, \hat{X}_\theta^y) - b(t_l, \hat{X}_{t_l}^y)) \partial_z w(\theta, \hat{X}_\theta^y) + \partial_z b(\theta, \hat{X}_\theta^y) w(\theta, \hat{X}_\theta^y)) dW_\theta.$$

Since, according to Lemma 3.8, $|w(\theta, \cdot)| \leq C(T - \theta)^{-1/2}$, $|\partial_z w(\theta, \cdot)| \leq C(T - \theta)^{-1}$ and the functions b and $\partial_z b$ are bounded, we deduce that $\forall 0 \leq l \leq L - 2$, $\mathbb{E}(\int_{t_l}^{t_{l+1}} T_s^3 ds) = 0$. Hence (28) holds. \square

REMARK 3.12. Our proof only works in the case that the diffusion coefficient is constant because otherwise the analysis of the error would involve higher order derivatives of the Green’s function w .

3.3. *Proof of Theorem 3.2.* We come back to the analysis of the stochastic particle method and the estimation of $\sup_{x \in [0,1]} \mathbb{E}|V(t_l, x) - \bar{V}(t_l, x)| = \text{Error}(t_l)$, for $0 \leq l \leq L$. Now, we set

$$b(t, x) = A'(V(t, x)).$$

By Lemma 3.1, this drift function satisfies the regularity assumptions made in the study of the weak error of the Euler–Lépingle scheme. By (13) and Lemma 3.4,

$$\text{Error}(t_l) \leq \frac{1}{N} + \sup_{x \in [0,1]} \mathbb{E} \left| \frac{1}{N} \sum_{i=1}^N \mathbb{E} \left(H(x - X_{t_l}^{y_i^0}) \right) - \bar{V}(t_l, x) \right|.$$

To deal with the inexact treatment of reflection by the Euler–Lépingle scheme, we introduce the system of processes $(Z^i, i = 1, \dots, N)$ evolving according to the Euler–Peano scheme on $[t_l, t_{l+1})$ and reinitialized at the positions $(Y_{t_{l+1}}^i, 1 \leq i \leq N)$ at time t_{l+1} (for $0 \leq l \leq L - 1$):

$$\begin{aligned} & \forall 0 \leq l \leq L - 1, \forall t \in [t_l, t_{l+1}), \\ (30) \quad & Z_t^i = Y_{t_l}^i + \sigma(W_t^i - W_{t_l}^i) + (t - t_l)A'(\bar{V}(t_l, Y_{t_l}^i)) + \hat{K}_t^i - \hat{K}_{t_l}^i, \\ & |\hat{K}^i|_t = \int_0^t \mathbb{1}_{\{0,1\}}(Z_s^i) d|\hat{K}^i|_s \quad \text{and} \quad \hat{K}_t^i = \int_0^t (1 - 2Z_s^i) d|\hat{K}^i|_s. \end{aligned}$$

Since we assume that $\gamma < 1/2$, according to Lemma 3.6, $\forall 1 \leq i \leq N$, $\mathbb{P}(\exists k \leq l : Z_{t_k^-}^i \neq Y_{t_k}^i) \leq C \Delta t$. Hence

$$\begin{aligned}
 \text{Error}(t_l) &\leq \frac{1}{N} + C \Delta t \\
 (31) \quad &+ \sup_{x \in [0,1]} \mathbb{E} \left| \frac{1}{N} \sum_{i=1}^N \mathbb{1}_{\{\forall 1 \leq k \leq l, Z_{t_k^-}^i = Y_{t_k}^i\}} \right. \\
 &\quad \left. \times \left(\mathbb{E}(H(x - X_{t_l}^{y_0^i})) - H(x - Z_{t_l^-}^i) \right) \right|.
 \end{aligned}$$

We introduce the solution $w(x, t, z)$ of the parabolic problem

$$\begin{aligned}
 \partial_t w + \frac{\sigma^2}{2} \partial_z^2 w + b(t, z) \partial_z w + \partial_z b(t, z) w &= 0, \quad (t, z) \in [0, t_l] \times [0, 1], \\
 w(t_l, \cdot) &= \delta_x(\cdot), \quad w(t, 0) = w(t, 1) = 0 \quad \forall t \in [0, t_l].
 \end{aligned}$$

Lemma 3.8 remains valid with t_l replacing T .

LEMMA 3.13.

$$\begin{aligned}
 \mathbb{E} \left| \frac{1}{N} \sum_{i=1}^N \mathbb{1}_{\{\forall 1 \leq k \leq l, Z_{t_k^-}^i = Y_{t_k}^i\}} \right. &\left(\mathbb{E}(H(x - X_{t_l}^{y_0^i})) - H(x - Z_{t_l^-}^i) \right. \\
 \left. - \sum_{k=0}^{l-1} \int_{t_k}^{t_{k+1}} w(x, s, Z_s^i) (A'(\bar{V}(t_k, Y_{t_k}^i)) - b(s, Z_s^i)) ds \right) &\left. \right| \leq C \left(\frac{1}{\sqrt{N}} + \Delta t \right),
 \end{aligned}$$

where the constant $C \geq 0$ does not depend on $x \in [0, 1]$.

PROOF. For $\varepsilon > 0$, we set $\varphi^\varepsilon(x') = e^{-(x'-x)^2/(2\varepsilon)}/\sqrt{2\pi\varepsilon}$ and

$$v^\varepsilon(t, z) = \int_z^1 w^{\varphi^\varepsilon}(t, z') dz' = \int_z^1 \int_0^1 w(x', t, z') \varphi^\varepsilon(x') dx' dz'.$$

Applying Itô's formula as in the proof of Lemma 3.9, we obtain that if $\forall 1 \leq k \leq l$, $Z_{t_k^-}^i = Y_{t_k}^i$,

$$\begin{aligned}
 (32) \quad &\mathbb{E} v^\varepsilon(t_l, X_{t_l}^{y_0^i}) - v^\varepsilon(t_l, Z_{t_l}^i) \\
 &- \sum_{k=0}^{l-1} \int_{t_k}^{t_{k+1}} (b(s, Z_s^i) - A'(\bar{V}(t_k, Y_{t_k}^i))) \partial_z v^\varepsilon(s, Z_s^i) ds \\
 &= -\sigma \int_0^{t_l} \partial_z v^\varepsilon(s, Z_s^i) dW_s^i = -\sigma \int_0^{t_l} \partial_z v^\varepsilon(s, \hat{Z}_s^i) dW_s^i,
 \end{aligned}$$

where \hat{Z}^i is the continuous process satisfying

$$\begin{aligned} \hat{Z}_0^i &= y_0^i, \\ \forall t \in [t_k, t_{k+1}], \quad \hat{Z}_t^i &= \hat{Z}_{t_k}^i + \sigma(W_t^i - W_{t_k}^i) + (t - t_k)A'(\bar{V}(t_k, Y_{t_k}^i)) + \hat{K}_t^i, \\ |\hat{K}^i|_t &= \int_0^t \mathbb{1}_{\{0,1\}}(\hat{Z}_s^i) d|\hat{K}^i|_s \quad \text{and} \quad \hat{K}_t^i = \int_0^t (1 - 2\hat{Z}_s^i) d|\hat{K}^i|_s. \end{aligned}$$

Since, according to Lemma 3.8, $(\partial_z v^\varepsilon(s, z))^2 \leq \frac{C}{t-s} \exp(-c\frac{(z-x)^2}{t-s})$, the inequality

$$\mathbb{E}(\partial_z v^\varepsilon(s, \hat{Z}_s^i))^2 \leq C(t-s)^{-2/3} s^{-1/3},$$

where the constant C does not depend on ε and x , is obtained like the upper bound of $\mathbb{E}|\partial_z w(\theta, \hat{X}_\theta^y)|$ in the proof of Lemma 3.11. Hence $\mathbb{E}(\int_0^{t_l} (\partial_z v^\varepsilon(s, \hat{Z}_s^i))^2 ds) \leq C$ and

$$\begin{aligned} & \mathbb{E} \left| \frac{1}{N} \sum_{i=1}^N \mathbb{1}_{\{\forall 1 \leq k \leq l, Z_{t_k}^i = Y_{t_k}^i\}} \int_0^{t_l} \partial_z v^\varepsilon(s, \hat{Z}_s^i) dW_s^i \right| \\ & \leq \mathbb{E} \left| \frac{1}{N} \sum_{i=1}^N \int_0^{t_l} \partial_z v^\varepsilon(s, \hat{Z}_s^i) dW_s^i \right| \\ & \quad + \frac{1}{N} \sum_{i=1}^N \mathbb{E} \left| \mathbb{1}_{\{\exists 1 \leq k \leq l, Z_{t_k}^i \neq Y_{t_k}^i\}} \int_0^{t_l} \partial_z v^\varepsilon(s, \hat{Z}_s^i) dW_s^i \right| \\ & \leq \left(\frac{1}{N^2} \sum_{i=1}^N \mathbb{E} \int_0^{t_l} (\partial_z v^\varepsilon(s, \hat{Z}_s^i))^2 ds \right)^{1/2} \\ & \quad + \frac{1}{N} \sum_{i=1}^N \left(\mathbb{P}(\exists 1 \leq k \leq l, Z_{t_k}^i \neq Y_{t_k}^i) \right)^{1/2} \left(\mathbb{E} \int_0^{t_l} (\partial_z v^\varepsilon(s, \hat{Z}_s^i))^2 ds \right)^{1/2} \\ & \leq C \left(\frac{1}{\sqrt{N}} + \Delta t \right) \quad \text{according to Lemma 3.6.} \end{aligned}$$

By (32), we deduce that

$$\begin{aligned} & \mathbb{E} \left| \frac{1}{N} \sum_{i=1}^N \mathbb{1}_{\{\forall 1 \leq k \leq l, Z_{t_k}^i = Y_{t_k}^i\}} \left(\mathbb{E}(v^\varepsilon(t_l, X_{t_l}^{y_0^i})) - v^\varepsilon(t_l, Z_{t_l}^i) \right) \right. \\ & \quad \left. - \sum_{k=0}^{l-1} \int_{t_k}^{t_{k+1}} (A'(\bar{V}(t_k, Y_{t_k}^i)) - b(s, Z_s^i)) (-\partial_z v^\varepsilon(s, Z_s^i)) ds \right| \leq C \left(\frac{1}{\sqrt{N}} + \Delta t \right) \end{aligned}$$

and we conclude by taking the limit $\varepsilon \rightarrow 0$ as in the proof of Lemma 3.9. \square

By (31) and the previous lemma, we obtain that

$$\begin{aligned} \text{Error}(t_l) &\leq C\left(\frac{1}{\sqrt{N}} + \Delta t\right) \\ &\quad + \sup_{x \in [0,1]} \mathbb{E} \left| \frac{1}{N} \sum_{i=1}^N \mathbb{1}_{\{\forall 1 \leq k \leq l, Z_{t_k}^i = Y_{t_k}^i\}} \right. \\ &\quad \quad \quad \left. \times \sum_{k=0}^{l-1} \int_{t_k}^{t_{k+1}} w(x, s, Z_s^i) (A'(\bar{V}(t_k, Y_{t_k}^i)) - b(s, Z_s^i)) ds \right|. \end{aligned}$$

Since, according to Lemma 3.8, $w(x, s, z) \leq C/\sqrt{t_l - s}$, using Lemma 3.6 once more, we get

$$\begin{aligned} \text{Error}(t_l) &\leq C\left(\frac{1}{\sqrt{N}} + \Delta t\right) \\ &\quad + \sup_{x \in [0,1]} \mathbb{E} \left| \frac{1}{N} \sum_{i=1}^N \sum_{k=0}^{l-1} \int_{t_k}^{t_{k+1}} w(x, s, Z_s^i) (A'(\bar{V}(t_k, Y_{t_k}^i)) - b(s, Z_s^i)) ds \right|. \end{aligned}$$

We now consider the last term in the upper bound of $\text{Error}(t_l)$. We split it into two parts to introduce the difference between the drift function $b(t_k, \cdot) = A'(V(t_k, \cdot))$ and its approximation $A'(\bar{V}(t_k, \cdot))$ at the same point $Y_{t_k}^i$:

As $\forall 0 \leq k \leq L - 1, Z_{t_k}^i = Y_{t_k}^i$, we get

$$\begin{aligned} &\mathbb{E} \left| \frac{1}{N} \sum_{i=1}^N \sum_{k=0}^{l-1} \int_{t_k}^{t_{k+1}} w(x, s, Z_s^i) (A'(\bar{V}(t_k, Y_{t_k}^i)) - b(s, Z_s^i)) ds \right| \\ (33) \quad &\leq \mathbb{E} \left| \frac{1}{N} \sum_{i=1}^N \sum_{k=0}^{l-1} \int_{t_k}^{t_{k+1}} w(x, s, Z_s^i) (b(s, Z_s^i) - b(t_k, Z_{t_k}^i)) ds \right| \\ &\quad + \mathbb{E} \left| \frac{1}{N} \sum_{i=1}^N \sum_{k=0}^{l-1} \int_{t_k}^{t_{k+1}} w(x, s, Z_s^i) (b(t_k, Y_{t_k}^i) - A'(\bar{V}(t_k, Y_{t_k}^i))) ds \right|. \end{aligned}$$

The first term in the right-hand side of (33) is a time discretization error. To obtain an error bound of order $\mathcal{O}(\Delta t)$, we need an expectation inside the absolute value. If for $k \in \{0, \dots, L\}$ we write $\mathcal{F}_{t_k} \stackrel{\text{def}}{=} \sigma(W_s^i; 0 \leq s \leq t_k, i = 1, \dots, N)$, then for all $s \in [t_k, t_{k+1})$ the variables $(R_{t_k, s}^i \stackrel{\text{def}}{=} w(x, s, Z_s^i)[b(s, Z_s^i) - b(t_k, Z_{t_k}^i)])$,

$i = 1, \dots, N$), are \mathcal{F}_{t_k} -independent. Hence

$$\begin{aligned} \mathbb{E} \left| \frac{1}{N} \sum_{i=1}^N R_{t_k, s}^i - \mathbb{E}^{\mathcal{F}_{t_k}}(R_{t_k, s}^i) \right| &\leq \frac{1}{\sqrt{N}} \sqrt{\frac{1}{N} \sum_{i=1}^N \mathbb{E}(R_{t_k, s}^i)^2} \\ &\leq \frac{C}{\sqrt{N}} \sqrt{\frac{1}{N} \sum_{i=1}^N \mathbb{E}(w(x, s, Z_s^i))^2}. \end{aligned}$$

Using once more that $w(x, s, z) \leq C/\sqrt{t_l - s}$, we easily obtain that

$$\begin{aligned} &\mathbb{E} \left| \frac{1}{N} \sum_{i=1}^N \sum_{k=0}^{l-1} \int_{t_k}^{t_{k+1}} w(x, s, Z_s^i) (b(s, Z_s^i) - b(t_k, Z_{t_k}^i)) ds \right| \\ &\leq \mathbb{E} \left| \frac{1}{N} \sum_{i=1}^N \sum_{k=0}^{l-1} \int_{t_k}^{t_{k+1}} \mathbb{E}^{\mathcal{F}_{t_k}} \{w(x, s, Z_s^i) [b(s, Z_s^i) - b(t_k, Z_{t_k}^i)]\} ds \right| + \frac{C}{\sqrt{N}}. \end{aligned}$$

To obtain an upper bound of order $\mathcal{O}(\Delta t)$ for the first term in the right-hand side of the previous inequality, we just have to remark that we are now in the same context as in the proof of Proposition 3.7: equality (24) and Lemma 3.11 are valid replacing \hat{X} by Z^i and T by t_l . Following the proof of Proposition 3.7 we conclude that

$$\begin{aligned} (34) \quad &\sup_{x \in [0, 1]} \mathbb{E} \left| \frac{1}{N} \sum_{i=1}^N \sum_{k=0}^{l-1} \int_{t_k}^{t_{k+1}} w(x, s, Z_s^i) (b(s, Z_s^i) - b(t_k, Z_{t_k}^i)) ds \right| \\ &\leq C \Delta t + \frac{C}{\sqrt{N}}. \end{aligned}$$

For the second term in (33), by the upper bound $C/\sqrt{t_l - s}$ for $w(x, s, z)$ and since by definition $b(t, \cdot) = A'(V(t, \cdot))$, we get

$$\begin{aligned} (35) \quad &\sup_{x \in [0, 1]} \mathbb{E} \left| \frac{1}{N} \sum_{i=1}^N \sum_{k=0}^{l-1} \int_{t_k}^{t_{k+1}} w(x, s, Z_s^i) (b(t_k, Y_{t_k}^i) - A'(\bar{V}(t_k, Y_{t_k}^i))) ds \right| \\ &\leq C \sup_{v \in [0, 1]} |A'(v)| \sum_{k=0}^{l-1} \left(\int_{t_k}^{t_{k+1}} \frac{1}{\sqrt{t_l - s}} ds \right) \\ &\quad \times \frac{1}{N} \sum_{i=1}^N \mathbb{E} |V(t_k, Y_{t_k}^i) - \bar{V}(t_k, Y_{t_k}^i)|. \end{aligned}$$

LEMMA 3.14. For all $0 \leq l \leq L$,

$$\frac{1}{N} \sum_{i=1}^N \mathbb{E} |V(t_l, Y_{t_l}^i) - \bar{V}(t_l, Y_{t_l}^i)| \leq C \left(\frac{1}{\sqrt{N}} + \Delta t \right).$$

Using Lemma 3.14 in (35), with (34) we come back to (33) and deduce that

$$\text{Error}(t_l) \leq C \left(\frac{1}{\sqrt{N}} + \Delta t \right),$$

which ends the proof of Theorem 3.2.

We now concentrate on the upper bound of

$$\frac{1}{N} \sum_{i=1}^N \mathbb{E} |V(t_l, Y_{t_l}^i) - \bar{V}(t_l, Y_{t_l}^i)|$$

given in Lemma 3.14. Because of the complex form of $\bar{V}(t_l, Y_{t_l}^i)$, we need to introduce another auxiliary family of discretized time processes: let $(\bar{X}_t^i, t \in [0, T], i = 1, \dots, N)$ denote the solution of the following Euler–Peano equations:

$$(36) \quad \begin{aligned} \forall t \in [t_l, t_{l+1}], \quad & \bar{X}_0^i = y_0^i, \\ & \bar{X}_t^i = \bar{X}_{t_l}^i + \sigma(W_t^i - W_{t_l}^i) + (t - t_l)b(t_l, \bar{X}_{t_l}^i) + \bar{K}_t^i, \\ & |\bar{K}^i|_t = \int_0^t \mathbb{1}_{(0,1)}(\bar{X}_s^i) d|\bar{K}^i|_s \quad \text{and} \\ & \bar{K}_t^i = \int_0^t (1 - 2\bar{X}_s^i) d|\bar{K}^i|_s. \end{aligned}$$

We will compare $\bar{V}(t_l, Y_{t_l}^i) = \frac{1}{N} \sum_{j=1}^N H(Y_{t_l}^i - Y_{t_l}^j)$ with the same expression written with the system of independent particles $\frac{1}{N} \sum_{j=1}^N H(\bar{X}_{t_l}^i - \bar{X}_{t_l}^j)$.

First, we note that

$$\begin{aligned} \mathbb{E} |V(t_l, Y_{t_l}^i) - V(t_l, \bar{X}_{t_l}^i)| &= \mathbb{E} |V(t_l, Z_{t_l}^i) - V(t_l, \bar{X}_{t_l}^i)| \\ &\leq \mathbb{E} |V(t_l, Z_{t_l}^i) - V(t_l, \bar{X}_{t_l}^i)| + C \Delta t^2 \\ &\leq C |Z_{t_l}^i - \bar{X}_{t_l}^i| + C \Delta t^2. \end{aligned}$$

The first inequality is obtained thanks to Lemma 3.6, which compares the Peano and Lépingle schemes. The second one uses the Lipschitz property of V stated in Lemma 3.1. Now, using arguments similar to those given at the beginning of the proof of Lemma 3.3, one can easily check that

$$\begin{aligned} |Z_{t_l}^i - \bar{X}_{t_l}^i| &\leq |Y_{t_{l-1}}^i - \bar{X}_{t_{l-1}}^i| + \Delta t C |\bar{V}(t_{l-1}, Y_{t_{l-1}}^i) - V(t_{l-1}, \bar{X}_{t_{l-1}}^i)| \\ &\leq |Z_{t_{l-1}}^i - \bar{X}_{t_{l-1}}^i| + C \Delta t^2 + \Delta t C |\bar{V}(t_{l-1}, Y_{t_{l-1}}^i) - V(t_{l-1}, \bar{X}_{t_{l-1}}^i)|. \end{aligned}$$

By induction, we deduce that

$$(37) \quad \begin{aligned} & \frac{1}{N} \sum_{i=1}^N \mathbb{E} |V(t_l, Y_{t_l}^i) - V(t_l, \bar{X}_{t_l}^i)| \\ & \leq C \Delta t \sum_{m=0}^{l-1} \frac{1}{N} \sum_{i=1}^N \mathbb{E} |\bar{V}(t_m, Y_{t_m}^i) - V(t_m, \bar{X}_{t_m}^i)| + C \Delta t. \end{aligned}$$

For all $k \in \{0, \dots, L\}$ we set

$$\bar{E}(t_k) \stackrel{\text{def}}{=} \frac{1}{N} \sum_{i=1}^N \mathbb{E} |\bar{V}(t_k, Y_{t_k}^i) - V(t_k, \bar{X}_{t_k}^i)|,$$

so that, by (37),

$$(38) \quad \frac{1}{N} \sum_{i=1}^N \mathbb{E} |V(t_l, Y_{t_l}^i) - \bar{V}(t_l, Y_{t_l}^i)| \leq \bar{E}(t_l) + C \Delta t \sum_{m=0}^{l-1} \bar{E}(t_m) + C \Delta t.$$

We have transformed the estimation of $\frac{1}{N} \sum_{i=1}^N \mathbb{E} |V(t_l, Y_{t_l}^i) - \bar{V}(t_l, Y_{t_l}^i)|$ into the estimation of each $\bar{E}(t_m)$ for $0 \leq m \leq L$.

For $m = 0$, $\bar{E}(0) = \frac{1}{N} \sum_{i=1}^N |\bar{V}(0, y_0^i) - V(0, y_0^i)| \leq \frac{1}{N}$, by Lemma 3.4. For $m \geq 1$, we insert the term $\frac{1}{N} \sum_{j=1}^N H(\bar{X}_{t_l}^i - \bar{X}_{t_l}^j)$ in the expression of $\bar{E}(t_m)$ to split it into two parts:

$$(39) \quad \begin{aligned} \bar{E}(t_m) & \leq \frac{1}{N} \sum_{i=1}^N \mathbb{E} \left| \bar{V}(t_m, Y_{t_m}^i) - \frac{1}{N} \sum_{j=1}^N H(\bar{X}_{t_m}^i - \bar{X}_{t_m}^j) \right| \\ & \quad + \frac{1}{N} \sum_{i=1}^N \mathbb{E} \left| \frac{1}{N} \sum_{j=1}^N H(\bar{X}_{t_m}^i - \bar{X}_{t_m}^j) - V(t_m, \bar{X}_{t_m}^i) \right| \\ & \leq \frac{1}{N^2} \sum_{i,j=1}^N \mathbb{E} |H(Y_{t_m}^i - Y_{t_m}^j) - H(\bar{X}_{t_m}^i - \bar{X}_{t_m}^j)| \\ & \quad + \frac{1}{N} \sum_{i=1}^N \mathbb{E} \left| \frac{1}{N} \sum_{j=1}^N H(\bar{X}_{t_m}^i - \bar{X}_{t_m}^j) - V(t_m, \bar{X}_{t_m}^i) \right|. \end{aligned}$$

To deal with the first term in the right-hand side, we introduce the errors $\bar{E}(t_k)$ for $k \leq m - 1$. The second term is very similar to error terms we have already treated. The upper bounds of these terms are respectively given in the following lemmas, the proofs of which are postponed.

LEMMA 3.15.

$$\frac{1}{N^2} \sum_{i,j=1}^N \mathbb{E} |H(Y_{t_m}^i - Y_{t_m}^j) - H(\bar{X}_{t_m}^i - \bar{X}_{t_m}^j)| \leq C \Delta t + C \Delta t \sum_{k=0}^{m-1} \max_{l \leq k} \bar{E}(t_l) / \sqrt{t_m - t_k}.$$

LEMMA 3.16.

$$\frac{1}{N} \sum_{i=1}^N \mathbb{E} \left| \frac{1}{N} \sum_{j=1}^N H(\bar{X}_{t_m}^i - \bar{X}_{t_m}^j) - V(t_m, \bar{X}_{t_m}^i) \right| \leq \frac{C}{\sqrt{N}} + C \Delta t.$$

Coming back to (39), we have obtained that

$$\max_{l \leq m} \bar{E}(t_l) \leq C \Delta t + \frac{C}{\sqrt{N}} + C \Delta t \sum_{k=0}^{m-1} \max_{l \leq k} \bar{E}(t_l) / \sqrt{t_m - t_k}.$$

Using a discrete time version of Gronwall’s lemma, we obtain that

$$\max_{m \leq L} \bar{E}(t_m) \leq C \Delta t + \frac{C}{\sqrt{N}}.$$

By (38), we conclude that Lemma 3.14 holds.

PROOF OF LEMMA 3.15. The main difficulty is to deal with the non-Lipschitz Heaviside function H . To overcome this difficulty, the idea consists in smoothing H thanks to the probability transition density of the Euler–Peano scheme. First, we note that

$$\begin{aligned} & H(Y_{t_m}^i - Y_{t_m}^j) - H(\bar{X}_{t_m}^i - \bar{X}_{t_m}^j) \\ &= \sum_{k=0}^{m-1} \left(H(\bar{X}_{t_m}^{i,t_{m-k}, Y_{t_{m-k}}^i} - \bar{X}_{t_m}^{j,t_{m-k}, Y_{t_{m-k}}^j}) \right. \\ & \quad \left. - H(\bar{X}_{t_m}^{i,t_{m-k-1}, Y_{t_{m-k-1}}^i} - \bar{X}_{t_m}^{j,t_{m-k-1}, Y_{t_{m-k-1}}^j}) \right), \end{aligned}$$

where, for $0 \leq k \leq L$, $y \in [0, 1]$ and $1 \leq i \leq N$, $(\bar{X}_t^{i,t_k,y})_{t \in [t_k, T]}$ denotes the Euler–Peano process starting from $\bar{X}_{t_k}^{i,t_k,y} = y$ at time t_k and with posterior evolution given by (36). By Lemma 3.6, replacing $Y_{t_{m-k}}^i$ by $Z_{t_{m-k}}^i$ in the expression above has a cost of order $\mathcal{O}(\Delta t^2)$. Hence, using the inequality

$$\mathbb{E}|H(A) - H(B)| = \mathbb{P}(A \geq 0, B < 0) + \mathbb{P}(A < 0, B \geq 0) \leq \mathbb{P}(|B| \leq |B - A|),$$

$$\begin{aligned} & \mathbb{E} \left| H(\bar{X}_{t_m}^{i,t_{m-k}, Y_{t_{m-k}}^i} - \bar{X}_{t_m}^{j,t_{m-k}, Y_{t_{m-k}}^j}) \right. \\ & \quad \left. - H(\bar{X}_{t_m}^{i,t_{m-k-1}, Y_{t_{m-k-1}}^i} - \bar{X}_{t_m}^{j,t_{m-k-1}, Y_{t_{m-k-1}}^j}) \right| \\ & \leq C \Delta t^2 + \mathbb{E} \left| H(\bar{X}_{t_m}^{i,t_{m-k}, Z_{t_{m-k}}^i} - \bar{X}_{t_m}^{j,t_{m-k}, Z_{t_{m-k}}^j}) \right. \\ & \quad \left. - H(\bar{X}_{t_m}^{i,t_{m-k-1}, Y_{t_{m-k-1}}^i} - \bar{X}_{t_m}^{j,t_{m-k-1}, Y_{t_{m-k-1}}^j}) \right| \end{aligned}$$

$$\begin{aligned}
 &\leq \mathbb{P} \left[\left| \bar{X}_{t_m}^{i,t_{m-k-1}, Y_{t_{m-k-1}}^i} - \bar{X}_{t_m}^{j,t_{m-k-1}, Y_{t_{m-k-1}}^j} \right| \right. \\
 &\quad \leq \left| \bar{X}_{t_m}^{i,t_{m-k-1}, Y_{t_{m-k-1}}^i} - \bar{X}_{t_m}^{j,t_{m-k-1}, Y_{t_{m-k-1}}^j} \right. \\
 &\quad \quad \left. \left. - \bar{X}_{t_m}^{i,t_{m-k}, Z_{t_{m-k}}^i} + \bar{X}_{t_m}^{j,t_{m-k}, Z_{t_{m-k}}^j} \right| \right] + C \Delta t^2 \\
 &\leq \mathbb{P} \left[\left| \bar{X}_{t_m}^{i,t_{m-k-1}, Y_{t_{m-k-1}}^i} - \bar{X}_{t_m}^{j,t_{m-k-1}, Y_{t_{m-k-1}}^j} \right| \leq C \Delta t (|V - \bar{V}|(t_{m-k-1}, Y_{t_{m-k-1}}^i) \right. \\
 &\quad \left. + |V - \bar{V}|(t_{m-k-1}, Y_{t_{m-k-1}}^j)) \right] + C \Delta t^2
 \end{aligned}$$

as for $i = 1, \dots, N$,

$$\begin{aligned}
 \left| \bar{X}_{t_m}^{i,t_{m-k-1}, Y_{t_{m-k-1}}^i} - \bar{X}_{t_m}^{i,t_{m-k}, Z_{t_{m-k}}^i} \right| &\leq C \left| \bar{X}_{t_{m-k}}^{i,t_{m-k-1}, Y_{t_{m-k-1}}^i} - Z_{t_{m-k}}^i \right| \\
 &\leq C \Delta t |V - \bar{V}|(t_{m-k-1}, Y_{t_{m-k-1}}^i).
 \end{aligned}$$

The variable $|V - \bar{V}|(t_{m-k-1}, Y_{t_{m-k-1}}^i) + |V - \bar{V}|(t_{m-k-1}, Y_{t_{m-k-1}}^j)$ is $\mathcal{F}_{t_{m-k-1}}$ -measurable. Moreover, for $i \neq j$, conditionally on $\mathcal{F}_{t_{m-k-1}}$, the variables $\bar{X}_{t_m}^{i,t_{m-k-1}, Y_{t_{m-k-1}}^i}$ and $\bar{X}_{t_m}^{j,t_{m-k-1}, Y_{t_{m-k-1}}^j}$ are independent and admit densities with respect to Lebesgue measure with an L^2 norm smaller than $C/(t_{k+1})^{1/4}$ (by the Girsanov theorem, as in the proof of Lemma 2.4). Hence, conditionally on $\mathcal{F}_{t_{m-k-1}}$, the variable $\bar{X}_{t_m}^{i,t_{m-k-1}, Y_{t_{m-k-1}}^i} - \bar{X}_{t_m}^{j,t_{m-k-1}, Y_{t_{m-k-1}}^j}$ admits a density with respect to Lebesgue measure with an L^∞ norm smaller than $C/\sqrt{t_{k+1}}$ and we deduce that

$$\begin{aligned}
 &\mathbb{P} \left[\left| \bar{X}_{t_m}^{i,t_{m-k-1}, Y_{t_{m-k-1}}^i} - \bar{X}_{t_m}^{j,t_{m-k-1}, Y_{t_{m-k-1}}^j} \right| \leq C \Delta t (|V - \bar{V}|(t_{m-k-1}, Y_{t_{m-k-1}}^i) \right. \\
 &\quad \left. + |V - \bar{V}|(t_{m-k-1}, Y_{t_{m-k-1}}^j)) \right] \\
 &\leq C \frac{\Delta t}{\sqrt{t_{k+1}}} \mathbb{E} (|V - \bar{V}|(t_{m-k-1}, Y_{t_{m-k-1}}^i) + |V - \bar{V}|(t_{m-k-1}, Y_{t_{m-k-1}}^j)).
 \end{aligned}$$

We conclude by (28) that

$$\begin{aligned}
 &\frac{1}{N^2} \sum_{i,j=1}^N \mathbb{E} |H(Y_{t_m}^i - Y_{t_m}^j) - H(\bar{X}_{t_m}^i - \bar{X}_{t_m}^j)| \\
 &\leq C \Delta t + C \Delta t \sum_{k=0}^{m-1} \max_{l \leq m-k-1} \bar{E}(t_l) / \sqrt{t_{k+1}}. \quad \square
 \end{aligned}$$

PROOF OF LEMMA 3.16. We decompose the expression of interest in order to introduce error terms that we have already bounded:

$$\begin{aligned}
 & \frac{1}{N} \sum_{i=1}^N \mathbb{E} \left| V(t_m, \bar{X}_{t_m}^i) - \frac{1}{N} \sum_{j=1}^N H(\bar{X}_{t_m}^i - \bar{X}_{t_m}^j) \right| \\
 & \leq \frac{1}{N} \sum_{i=1}^N \mathbb{E} \left(\left| V(t_m, x) - \frac{1}{N} \sum_{j=1}^N \mathbb{E} H(x - X_{t_m}^{y_0^j}) \right|_{x=\bar{X}_{t_m}^i} \right) \\
 & \quad + \frac{1}{N} \sum_{i=1}^N \mathbb{E} \left(\frac{1}{N} \sum_{j=1}^N \left| \mathbb{E} H(x - X_{t_m}^{y_0^j}) - \mathbb{E} H(x - \bar{X}_{t_m}^j) \right|_{x=\bar{X}_{t_m}^i} \right) \\
 & \quad + \frac{1}{N} \sum_{i=1}^N \mathbb{E} \left(\left| \frac{1}{N} \sum_{j=1}^N (\mathbb{E} H(x - \bar{X}_{t_m}^j) |_{x=\bar{X}_{t_m}^i} - H(\bar{X}_{t_m}^i - \bar{X}_{t_m}^j)) \right| \right) \\
 & \leq \sup_{x \in [0,1]} \left| V(t_m, x) - \frac{1}{N} \sum_{j=1}^N \mathbb{E} H(x - X_{t_m}^{y_0^j}) \right| \\
 & \quad + \sup_{x \in [0,1], j \leq N} \left| \mathbb{E} H(x - X_{t_m}^{y_0^j}) - \mathbb{E} H(x - \bar{X}_{t_m}^j) \right| \\
 & \quad + \frac{1}{N} \sum_{i=1}^N \left(\mathbb{E} \left(\frac{1}{N^2} \sum_{j,k=1}^N (\mathbb{E} H(x - \bar{X}_{t_m}^j) |_{x=\bar{X}_{t_m}^i} - H(\bar{X}_{t_m}^i - \bar{X}_{t_m}^j)) \right. \right. \\
 & \qquad \qquad \qquad \left. \left. \times (\mathbb{E} H(x - \bar{X}_{t_m}^k) |_{x=\bar{X}_{t_m}^i} - H(\bar{X}_{t_m}^i - \bar{X}_{t_m}^k)) \right) \right)^{1/2}.
 \end{aligned}$$

The first term in the right-hand side is the initialization error bounded in Lemma 3.4 by $\frac{1}{N}$. The second term is the weak time discretization error for the Euler–Peano scheme bounded in Proposition 3.7 by $C \Delta t$. The last term is a statistical error: it is smaller than $1/\sqrt{N}$ since, by independence of the variables $(\bar{X}_{t_m}^j, 1 \leq j \leq N)$, each term of the summation $\sum_{j,k=1}^N$ with $j \neq k$ is nil. \square

4. Numerical experiments. As a numerical benchmark, we consider the following Dirichlet problem for the viscous Burgers equation which corresponds to the choice $A(x) = x^2/2$:

$$\begin{aligned}
 & \frac{\partial}{\partial t} v(t, x) = \frac{\partial^2 v}{\partial x^2}(t, x) - v(t, x) \frac{\partial v}{\partial x}(t, x), \quad t > 0, x \in [0, 2\pi], \\
 (40) \quad & v(0, x) = \frac{2 \sin(x)}{\cos(x) + e}, \quad x \in [0, 2\pi], \\
 & v(t, 0) = 0, v(t, 2\pi) = 0 \quad \forall t \geq 0.
 \end{aligned}$$

The exact solution is (see [2]) $V(t, x) = 2 \sin(x)/(\cos(x) + e^{(1+t)})$.

The spatial domain $[0, 2\pi]$ is different from the one considered so far but our results remain true for any bounded interval replacing $[0, 1]$. The fact that the distribution derivative $m_0(x) dx$ of the initial data $v(0, x)$ given by $m_0(x) = (2 + 2e \cos(x))/(\cos(x) + e)^2$ is not a probability measure but a bounded signed measure represents a more significant modification. In fact, we could not find any explicit solution when $v(0, \cdot)$ is the cumulative distribution function of a probability measure.

To take this modification into account, we use weighted particles $(Y_t^i, w^i)_{1 \leq i \leq N}$ (see, e.g., [8], which deals with a spatial domain equal to \mathbb{R}). The N initial locations $y_0^i = \inf\{y; H * |m_0|/\|m_0\|_{L^1([0, 2\pi])}(y) = i/N\}$ are chosen in order to approximate the cumulative distribution function of the probability measure $|m_0|(x) dx/\|m_0\|_{L^1([0, 2\pi])}$ and the corresponding weights are $w_i = \|m_0\|_{L^1([0, 2\pi])} \text{sign}(m_0(y_0^i))$. The approximate solution is given by the weighted cumulative distribution function of the particle system $\bar{V}(t, x) = \frac{1}{N} \sum_{i=1}^N w^i \times H(x - Y_t^i)$, where the successive positions are defined inductively by (11) but with $\wedge 1$ (resp. -1) replaced by $\wedge 2\pi$ (resp. -2π) in the second (resp. last) line.

The parameters of the Lépingle scheme are $\alpha_0 = 0.25$ and $\alpha_1 = 2\pi - 0.25$. In Figure 1 we have plotted the numerical solution at time $t = 1$. As the dependence of the error on the number of particles is standard and corresponds to the usual central limit theorem rate (see [4, 5, 8] for numerical results in the case where the spatial domain is \mathbb{R}), we concentrate our numerical study on the dependence on the time step. That is why we take a large number of particles $N = 10^6$. According to Theorem 3.2, $\mathbb{E}\|V(1, \cdot) - \bar{V}(1, \cdot)\|_{L^1([0, 2\pi])} \leq 2\pi \sup_{x \in [0, 2\pi]} \mathbb{E}|V(1, x) - \bar{V}(1, x)| \leq C(\Delta t + N^{-1/2})$. Since it is not possible to compute the last quantity,

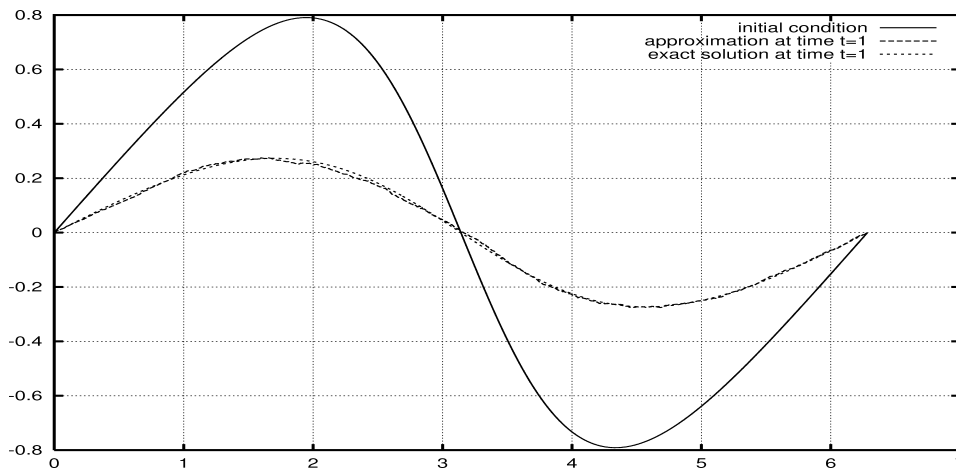


FIG. 1. Exact and numerical solutions of (40) obtained at time $t = 1$, for 10^4 particles and $\Delta t = 10^{-2}$ with the Lépingle scheme.

TABLE 1
 Expectation of L^1 norm of error at $t = 1$ for $N = 10^6$ particles ($\|V(\cdot, 1)\|_{L^1([0, 2\pi])} = 1.09$)

Δt	Lépingle scheme	Confidence interval at 95%	Projection scheme	Confidence interval at 95%
2^{-1}	0.0940	[0.0933, 0.0946]	0.2510	[0.2501, 0.2519]
2^{-2}	0.0585	[0.0579, 0.0591]	0.2320	[0.2309, 0.2329]
2^{-3}	0.0329	[0.0322, 0.0336]	0.1964	[0.1953, 0.1975]
2^{-4}	0.0173	[0.0166, 0.0180]	0.1568	[0.1557, 0.1578]
2^{-5}	0.0083	[0.0076, 0.0090]	0.1241	[0.1227, 0.1254]
2^{-6}	0.0053	[0.0045, 0.0060]	0.0982	[0.0969, 0.0995]
2^{-7}	0.0049	[0.0043, 0.0055]	0.0779	[0.0765, 0.0793]
2^{-8}	0.0050	[0.0042, 0.0058]	0.0635	[0.0627, 0.0643]

we compute the first one by averaging $\|V(1, \cdot) - \bar{V}(1, \cdot)\|_{L^1([0, 2\pi])}$ over 20 runs of the particle method and give the dependence of the result on Δt in Table 1 and Figure 2.

We need to check that our test case (40) produces a significant rate of effective reflections. If this rate is too small, we only observe the effect of the classical Euler scheme (without reflection) with weak convergence also in Δt , and we cannot conclude on the convergence of the Lépingle scheme. The rate of effective reflections is around 10% for this test case: more precisely there are about 10% of the particles in $[0, \alpha_0] \cup [\alpha_1, 2\pi]$ at each time step. For these particles, we compute the correction term C in (11). When we discretize the particle system according to the projected Euler scheme, which treats the reflection simply by projection onto $[0, 1]$, we clearly observe a sublinear convergence in Δt (see Table 1 and Figure 2). The projected Euler scheme does not use the correction term C whatever

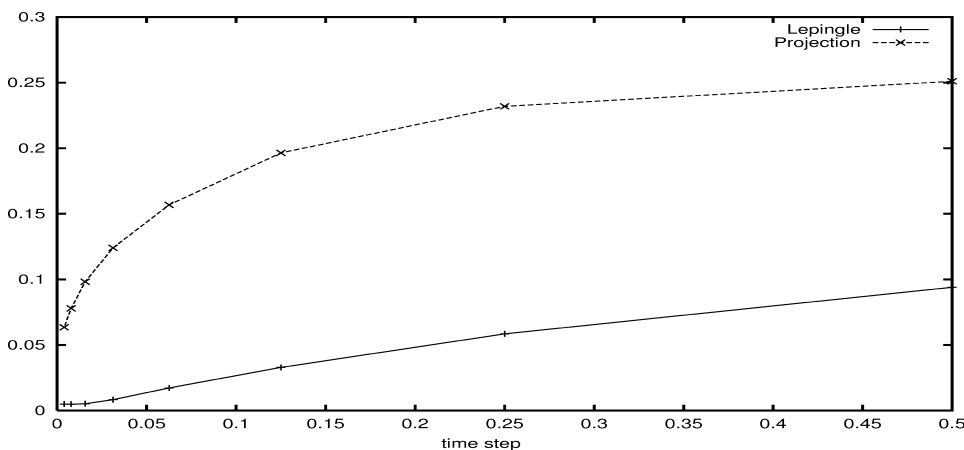


FIG. 2. $\mathbb{E}\|V(\cdot, 1) - \bar{V}(\cdot, 1)\|_{L^1(\mathbb{R})}$ in terms of Δt ($N = 10^6$).

the position of the particle and its weak convergence rate is in $\mathcal{O}(\Delta t^{1/2})$ (see [6]). Therefore we can conclude that the quasilinear decreasing of the error for the Lépingle scheme confirms our theoretical analysis.

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