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ORLICZ NORMS OF SEQUENCES OF RANDOM VARIABLES

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Let f_i, $i = 1, \dots, n$, be copies of a random variable f and let N be an Orlicz function. We show that for every $x \in \mathbb{R}^n$ the expectation $\mathbf{E}\|(x_i f_i)_{i=1}^n\|_N$ is maximal (up to an absolute constant) if f_i, $i = 1, \dots, n$, are independent. In that case we show that the expectation $\mathbf{E}\|(x_i f_i)_{i=1}^n\|_N$ is equivalent to $\|x\|_M$, for some Orlicz function M depending on N and on distribution of f only. We provide applications of this result.

1. Introduction and main results. Let f_i, $i = 1, \dots, n$, be identically distributed random variables. We investigate here expectations

$$\mathbf{E}\big\|\big(x_i f_i(\omega)\big)_{i=1}^n\big\|_N,$$

where $\|\cdot\|_N$ is an Orlicz norm. We find out that these expressions are maximal (up to an absolute constant) if the random variables are, in addition, required to be independent.

In case the random variables are independent, we get quite precise estimates for the above expectations. In particular, let $f_1, \dots, f_n$ be independent standard Gauss variables and let the norm on $\mathbb{R}^n$ be defined by $\|z\|_{k,*} = \sum_{i=1}^k z_i^*$, where $(z_i^*)_i$ is the nonincreasing rearrangement of the sequence $(|z_i|)_i$. Then we have, for all $x \in \mathbb{R}^n$,

$$c_1\|x\|_M \le \mathbf{E}\big\|\big(x_i f_i(\omega)\big)_{i=1}^n\big\|_{k,*} \le c_2\|x\|_M,$$

where the Orlicz function is $M(t) = \frac{1}{k}e^{-1/(kt)^2}$, $t < 1/(2k)$, $M(1) = 1$. This case is of particular interest to us. In a forthcoming paper [2] these estimates are applied to obtain estimates for various parameters associated to the local theory of convex bodies. Let us note that in the case $k = 1$ the norm $\|\cdot\|_{k,*}$ is just the ℓ_∞-norm.

Some of the methods that are used here have been developed by Kwapień and Schütt (cf. [4, 5, 9] and [10]).

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In this paper we consider random variables with finite first moments only. In the proofs of our results we assume that the random variables have continuous distributions; that is, $P\{\omega \mid f(\omega) = t\} = 0$ for every $t \in \mathbb{R}$. The general case follows by approximation. We define the following parameters of the distribution. Let f be a random variable with a continuous distribution and with $\mathbf{E}|f| < \infty$. Let $t_n = t_n(f) = 0$, $t_0 = t_0(f) = \infty$ and, for $j = 1, \dots, n-1$,

$$t_j = t_j(f) = \sup\left\{t \,\middle|\, P\{\omega \,|\, |f(\omega)| > t\} \ge \frac{j}{n}\right\}. \tag{1}$$

Since f has the continuous distribution we have, for every $j \ge 1$,

$$P\{\omega \,|\, |f(\omega)| \ge t_j\} = \frac{j}{n}.$$

We define the sets

$$\Omega_j = \Omega_j(f) = \{\omega \mid t_j \le |f(\omega)| < t_{j-1}\} \tag{2}$$

for $j = 1, \dots, n$. Clearly, for all $j = 1, \dots, n$,

$$P(\Omega_j) = \frac{1}{n}.$$

Indeed,

$$\Omega_j = \{\omega \mid t_j \le |f(\omega)| < t_{j-1}\} = \{\omega \mid t_j \le |f(\omega)|\} \setminus \{\omega \mid t_{j-1} \le |f(\omega)|\}.$$

Therefore we get

$$P(\Omega_j) = \frac{j}{n} - \frac{j-1}{n} = \frac{1}{n}.$$

We put, for $j = 1, \dots, n$,

$$y_j = y_j(f) = \int_{\Omega_j} |f(\omega)| \, dP(\omega). \tag{3}$$

We have

$$\sum_{j=1}^{n} y_j = \mathbf{E}|f| \quad \text{and} \quad t_j \le n y_j < t_{j-1} \qquad \text{for all } j = 1, \dots, n.$$

We recall briefly the definitions of an Orlicz function and an Orlicz norm (see, e.g., [3] and [6]). A convex function $M\colon \mathbb{R}^+ \to \mathbb{R}^+$ with $M(0) = 0$ and $M(t) > 0$ for $t \ne 0$ is called an Orlicz function. Then the Orlicz norm on $\mathbb{R}^n$ is defined by

$$\|x\|_M = \inf\left\{\rho > 0 : \sum_{i=1}^{n} M(|x_i|/\rho) \le 1\right\}.$$

Clearly, if two Orlicz functions M, N satisfy $M(t) \le aN(bt)$ for every positive t, then $\|x\|_M \le ab\|x\|_N$ for every $x \in \mathbb{R}^n$. Thus equivalent Orlicz functions generate equivalent norms. In other words, to prove the equivalence of $\|x\|_M$ and $\|x\|_N$, it is enough to prove the equivalence of M and N. Moreover, to define an Orlicz norm $\|\cdot\|_M$, it is enough to define an Orlicz function M on $[0, T]$, where $M(T) = 1$.

Any Orlicz function M can be represented as

$$M(t) = \int_0^t p(s)\,ds,$$

where $p(t)$ is a nondecreasing function continuous from the right. If $p(t)$ satisfies

$$p(0) = 0 \quad \text{and} \quad p(\infty) = \lim_{t\to\infty} p(t) = \infty, \tag{4}$$

then we define the dual Orlicz function M^* by

$$M^*(t) = \int_0^t q(s)\,ds,$$

where $q(s) = \sup\{t : p(t) \le s\}$. Such a function M^* is also an Orlicz function and

$$\|x\|_M \le |||x||| \le 2\|x\|_M,$$

where $|||\cdot|||$ is the dual norm to $\|\cdot\|_{M^*}$ (see, e.g., [6]). Note that condition (4), in fact, excludes only the case $M(t)$ is equivalent to t. Note also that q satisfies condition (4) as well and that $q = p^{-1}$ if p is an invertible function.

We shall need the following property of M and M^* (see, e.g., 2.10 of [3]):

$$s < M^{*-1}(s)M^{-1}(s) \le 2s \tag{5}$$

for every positive s.

The aim of this paper is to prove the following theorem.

THEOREM 1. *Let $f_1, \dots, f_n$ be independent, identically distributed random variables with $\mathbf{E}|f_1| < \infty$. Let N be an Orlicz function and let s_k, $k = 1, \dots, n^2$, be the nonincreasing rearrangement of the numbers*

$$\left|y_i\left(N^{*-1}\left(\frac{j}{n}\right) - N^{*-1}\left(\frac{j-1}{n}\right)\right)\right|, \qquad i, j = 1, \dots, n,$$

where y_i, $i = 1, \dots, n$, is given by (3). *Let M be an Orlicz function such that, for all $\ell = 1, \dots, n^2$,*

$$M^*\left(\sum_{k=1}^{\ell} s_k\right) = \frac{\ell}{n^2}.$$

Then, for all $x \in \mathbb{R}^n$,

$$\frac{1}{8}\|x\|_M \le \mathbf{E}\big\|\big(x_i f_i(\omega)\big)_{i=1}^n\big\|_N \le 8\frac{e}{e-1}\|x\|_M.$$

COROLLARY 2. *Let $f_1, \dots, f_n$ be independent, identically distributed random variables with $\mathbf{E}|f_1| < \infty$. Let M be an Orlicz function such that, for all $k = 1, \dots, n$,*

$$M^*\left(\sum_{j=1}^{k} y_j\right) = \frac{k}{n}.$$

Then, for all $x \in \mathbb{R}^n$,

$$c_1\|x\|_M \le \mathbf{E} \max_{1\le i\le n} |x_i f_i(\omega)| \le c_2\|x\|_M,$$

where c_1, c_2 are absolute positive constants.

PROOF. We choose p large enough so that the ℓ_p-norm $\|\cdot\|_p$ approximates the supremum norm $\|\cdot\|_\infty$ well enough ($p = n$ suffices). We consider $N(t) = |t|^p$. This means that for all $t > 0$ we have

$$N'(t) = pt^{p-1} \quad \text{and} \quad N'^{-1}(t) = \left(\frac{1}{p}t\right)^{1/(p-1)}.$$

Therefore

$$N^*(t) = \int_0^t N'^{-1}(s)\,ds = \int_0^t \left(\frac{1}{p}s\right)^{1/(p-1)} ds = p^{-1/(p-1)}\left(1 - \frac{1}{p}\right)t^{1+1/(p-1)}.$$

Thus

$$N^{*-1}(t) = p^{1/p}\left(\frac{p}{p-1}\right)^{(p-1)/p} t^{1-1/p}.$$

With this we get

$$N^{*-1}\left(\frac{j}{n}\right) - N^{*-1}\left(\frac{j-1}{n}\right) = p^{1/p}\left(\frac{p}{p-1}\right)^{(p-1)/p}\left(\left(\frac{j}{n}\right)^{1-1/p} - \left(\frac{j-1}{n}\right)^{1-1/p}\right).$$

By the mean value theorem we get, for $j \ge 2$,

$$\begin{aligned} p^{1/p}\left(1-\frac{1}{p}\right)^{1/p} n^{-1+1/p} j^{-1/p} &\le N^{*-1}\left(\frac{j}{n}\right) - N^{*-1}\left(\frac{j-1}{n}\right) \\ &\le p^{1/p}\left(1-\frac{1}{p}\right)^{1/p} n^{-1+1/p}(j-1)^{-1/p}. \end{aligned}$$

For sufficiently large p we have, for all j with $1 \le j \le n$,

$$\frac{1}{n} \le N^{*-1}\left(\frac{j}{n}\right) - N^{*-1}\left(\frac{j-1}{n}\right) \le \frac{2}{n}.$$

Now we choose $\ell = kn$ and get

$$\sum_{i=1}^{k} y_i \le \sum_{j=1}^{\ell} s_j \le 2\sum_{i=1}^{k} y_i,$$

which implies the corollary. □

COROLLARY 3. *Let $f_1, \dots, f_n$ be independent, identically distributed random variables with $\mathbf{E}|f_i| = 1$. Let $k \in \mathbb{N}$, $1 \le k \le n$, and let the norm $\|\cdot\|_{k,*}$ on $\mathbb{R}^n$ be given by*

$$\|x\|_{k,*} = \sum_{i=1}^{k} x_i^*,$$

where x_i^*, $i = 1, \dots, n$, is the decreasing rearrangement of the numbers $|x_i|$, $i = 1, \dots, n$. Let M be an Orlicz function such that $M^*(1) = 1$ and, for all $m = 1, \dots, n-1$,

$$M^*\left(\sum_{j=1}^{m} y_j\right) = \frac{m}{kn}.$$

Then, for all $x \in \mathbb{R}^n$,

$$c_1\|x\|_M \le \mathbf{E}\big\|\big(x_i f_i(\omega)\big)_{i=1}^n\big\|_{k,*} \le c_2\|x\|_M,$$

where c_1, c_2 are absolute positive constants.

Clearly, Corollary 3 implies Corollary 2. We state them separately here, since the proof of Corollary 3 is more involved. We could argue in the proof of this corollary in the same way as in the proof of Corollary 2. But it is less cumbersome to use the lemmas on which Theorem 1 is based.

PROOF. Let $\varepsilon > 0$ which will be specified later. Consider the vector

$$z = \frac{(1, \dots, 1, \varepsilon, \dots, \varepsilon)}{[n/k] + (n - [n/k])\varepsilon},$$

where the vector contains $[n/k]$ coordinates that are equal to 1. (For technical reasons we require that all the coordinates of z are nonzero; otherwise, the function M^* might not be well defined.) First we show that if ε is small enough then, for every $x \in \mathbb{R}^n$,

$$c_1\|x\|_{k,*} \le n^{-n+1} \sum_{1\le j_1,\dots,j_n\le n} \max_{1\le i\le n} |x_i z_{j_i}| \le c_2\|x\|_{k,*}. \tag{6}$$

To obtain this, we observe first that we can choose ε so small that we can actually consider the vector $\bar{z} = (1, \dots, 1, 0, \dots, 0)/[n/k]$ instead. By Lemma 7 we have

$$c_n \sum_{i=1}^{n} s_i(x, \bar{z}) \le n^{-n+1} \sum_{1 \le j_1, \dots, j_n \le n} \max_{1 \le i \le n} |x_i \bar{z}_{j_i}| \le \sum_{i=1}^{n} s_i(x, \bar{z}),$$

where $s_l(x, \bar{z})$ is the decreasing rearrangement of the numbers $|x_i \bar{z}_j|$, $i, j = 1, \dots, n$. On the other hand,

$$\sum_{i=1}^{k} x_i^* \le \sum_{i=1}^{n} s_i(x, \bar{z}) \le \frac{n/k}{[n/k]} \sum_{i=1}^{k} x_i^* \le 2 \sum_{i=1}^{k} x_i^*.$$

Let N be an Orlicz function that satisfies

$$N^*\left(\sum_{i=1}^{k} z_i\right) = \frac{k}{n}.$$

Lemmas 5 and 9 and inequality (6) imply

$$c_3 \|x\|_N \le \|x\|_{k,*} \le c_4 \|x\|_N$$

for some absolute constants c_3, c_4. Clearly,

$$N^{*-1}\left(\frac{j}{n}\right) - N^{*-1}\left(\frac{j-1}{n}\right) = z_j.$$

Now we apply Theorem 1 to the Orlicz function N and obtain the numbers s_k and the function M as in the statement of Theorem 1. Choosing ε small enough we obtain

$$s_1 = \cdots = s_{[n/k]} = \left(\left[\frac{n}{k}\right] + \left(n - \left[\frac{n}{k}\right]\right)\varepsilon\right)^{-1} y_1,$$

$$s_{[n/k]+1} = \cdots = s_{2[n/k]} = \left(\left[\frac{n}{k}\right] + \left(n - \left[\frac{n}{k}\right]\right)\varepsilon\right)^{-1} y_2,$$

$$\vdots$$

$$s_{(n-1)[n/k]+1} = \cdots = s_{n[n/k]} = \left(\left[\frac{n}{k}\right] + \left(n - \left[\frac{n}{k}\right]\right)\varepsilon\right)^{-1} y_n.$$

The following numbers s_k, $k = n[n/k] + 1, \dots, n^2$, are all smaller than εy_1. Since $\sum_{j=1}^{n} y_j = \mathbf{E}|f_i| = 1$, we get $\sum_{k=1}^{n^2} s_k = 1$, which means $M^*(1) = 1$ and

$$\sum_{i=1}^{j[n/k]} s_i = \frac{[n/k]}{[n/k] + (n - [n/k])\varepsilon} \sum_{i=1}^{j} y_i.$$

This means that, for $j = 1, \dots, n$,

$$M^*\left(\frac{[n/k]}{[n/k] + (n - [n/k])\varepsilon} \sum_{i=1}^{j} y_i\right) = \frac{j[n/k]}{n^2}.$$

Therefore there are absolute constants c and C such that

$$c\frac{j}{kn} \le M^*\left(\sum_{i=1}^{j} y_i\right) \le C\frac{j}{kn}.$$

Theorem 1 implies the result. $\square$

REMARK. In particular in the proof we get that, for every $x \in \mathbb{R}^n$,

$$c_n\|x\|_{k,*} \le n^{-n+1} \sum_{1\le j_1,\dots,j_n\le n} \max_{1\le i\le n} |x_i \bar{z}_{j_i}| \le c_{n,k}\|x\|_{k,*}, \tag{7}$$

where $c_n = 1 - (1 - 1/n)^n$ and $c_{n,k} = \frac{n}{k}/[\frac{n}{k}] \le 2$.

THEOREM 4. *Let $f_1, \dots, f_n$, $g_1, \dots, g_n$ be identically distributed random variables. Suppose that $g_1, \dots, g_n$ are independent. Let M be an Orlicz function. Then we have, for all $x \in \mathbb{R}^n$,*

$$\mathbf{E}\big\|\big(x_i f_i(\omega)\big)_{i=1}^n\big\|_M \le \frac{16e}{e-1}\mathbf{E}\big\|\big(x_i g_i(\omega)\big)_{i=1}^n\big\|_M.$$

REMARK. The subspaces of L_1 with a symmetric basis or symmetric structure can be written as an average of Orlicz spaces; more precisely, the norm in such a space is equivalent to an average of Orlicz norms. Thus our theorems and corollaries extend naturally (for subspaces of L_1 with a symmetric basis, see [1]; for the case of symmetric lattices, see [7]).

2. Proofs of the theorems. To approximate Orlicz norms on $\mathbb{R}^n$, we will use the following norm. Given a vector $z \in \mathbb{R}^m$ with $z_1 \ge z_2 \ge \dots \ge z_m > 0$, denote

$$\|x\|_z = \max_{\sum_{i=1}^n k_i = m} \sum_{i=1}^{n} \left(\sum_{j=1}^{k_i} z_j\right)|x_i|.$$

In this definition we allow some of the k_i to be 0 (setting $\sum_{i=1}^{0} z_j = 0$).

The following lemma was proved by Kwapień and Schütt (Lemma 2.1 of [5]).

LEMMA 5. *Let $n, m \in \mathbb{N}$ with $n \le m$, let $y \in \mathbb{R}^m$ with $y_1 \ge y_2 \ge \dots \ge y_m > 0$ and let M be an Orlicz function that satisfies, for all $k = 1, \dots, m$,*

$$M^*\left(\sum_{i=1}^{k} y_i\right) = \frac{k}{m}.$$

Then we have, for every $x \in \mathbb{R}^n$,

$$\tfrac{1}{2}\|x\|_y \le \|x\|_M \le 2\|x\|_y.$$

REMARK. Note that for every Orlicz function M there exists a sequence $y_1 \ge y_2 \ge \cdots \ge y_m > 0$ such that

$$M^*\left(\sum_{i=1}^{k} y_i\right) = \frac{k}{m}$$

for every $k \le m$.

To prove both our theorems, it is enough to prove the following proposition because of Lemma 5.

PROPOSITION 6. *Let* $f_1, \dots, f_n$ *be identically distributed random variables* (*not necessarily independent*). *Let* N *be an Orlicz function and denote*

$$z_j = N^{*-1}\left(\frac{j}{n}\right) - N^{*-1}\left(\frac{j-1}{n}\right), \qquad j = 1, \dots, n.$$

Let $s = (s_k)_k \in \mathbb{R}^{n^2}$ *be the nonincreasing rearrangement of the numbers* $|y_i z_j|$, $i, j = 1, \dots, n$, *where the numbers* y_i, $i = 1, \dots, n$, *are given by* (3). *Then, for all* $x \in \mathbb{R}^n$,

$$\mathbf{E}\big\|(x_i f_i(\omega))_{i=1}^n\big\|_z \le \frac{2}{c_n}\|x\|_s,$$

where $c_n = 1 - (1 - 1/n)^n > 1 - 1/e$.

Moreover, if the random variables $f_1, \dots, f_n$ *are independent, then, for all* $x \in \mathbb{R}^n$,

$$\tfrac{1}{2}\|x\|_s \le \mathbf{E}\big\|(x_i f_i(\omega))_{i=1}^n\big\|_z.$$

To prove this proposition, we need Lemmas 7–11.

LEMMA 7. *Let* $a_{i,j}$, $i, j = 1, \dots, n$, *be a matrix of real numbers. Let* $s(k)$, $k = 1, \dots, n^2$, *be the decreasing rearrangement of the numbers* $|a_{i,j}|$, $i, j = 1, \dots, n$. *Then*

$$\frac{c_n}{n}\sum_{k=1}^{n} s(k) \le n^{-n} \sum_{j_1,\dots,j_n=1}^{n} \max_{1\le i\le n} |a_{i,j_i}| \le \frac{1}{n}\sum_{k=1}^{n} s(k),$$

where $c_n = 1 - (1 - 1/n)^n$. *Both inequalities are optimal.*

PROOF. Both expressions

$$n^{-n} \sum_{j_1,\dots,j_n=1}^{n} \max_{1\le i\le n} |a_{i,j_i}| \quad \text{and} \quad \sum_{k=1}^{n} s(k)$$

are norms on the space of $n \times n$ matrices. We show first the right-hand inequality. The extreme points of the unit ball of the norm $\sum_{k=1}^{n} s(k)$ are—up to a permutation of the coordinates—of the form

$$(\varepsilon_1 a, \varepsilon_2 b, \varepsilon_3 b, \dots, \varepsilon_{n^2} b),$$

with $a \ge b \ge 0$, $a + (n-1)b = 1$ and $\varepsilon_i = \pm 1$, $i \le n^2$. This means that such a matrix has the property: the absolute values of the coordinates are b except for one coordinate which is a. We get

$$n^{-n} \sum_{j_1,\dots,j_n=1}^{n} \max_{1\le i\le n} |a_{i,j_i}| = \frac{1}{n} a + \frac{n-1}{n} b = \frac{1}{n}.$$

Now we show the left-hand inequality. Clearly, we may assume that at most n coordinates of the matrix are different from 0. Next we observe that we may assume that for each row in the matrix there is at most one entry that is different from 0. In fact, we may assume that this is the first coordinate in the row. Now we average the nonzero entries, leaving us with the case that all nonzero coordinates are equal. In fact, we may assume that these coordinates equal 1.

Thus $\max_{1\le i\le n} |a_{i,j_i}|$ takes the value 0 or 1. In fact, it takes the value 0 exactly

$$(n-1)^n$$

out of n^n times. It follows

$$n^{-n} \sum_{j_1,\dots,j_n=1}^{n} \max_{1\le i\le n} |a_{i,j_i}| = 1 - \left(1 - \frac{1}{n}\right)^n,$$

which proves the lemma. □

LEMMA 8. *Let $a_{i,j,k}$, $i, j, k = 1, \dots, n$, be nonnegative real numbers. Let s_ℓ, $\ell = 1, \dots, n^3$, be the decreasing rearrangement of the numbers $a_{i,j,k}$, $i, j, k = 1, \dots, n$. Then*

$$\frac{1}{2n^2} \sum_{\ell=1}^{n^2} s_\ell \le n^{-2n} \sum_{\substack{1\le j_1,\dots,j_n\le n \\ 1\le k_1,\dots,k_n\le n}} \max_{1\le i\le n} a_{i,j_i,k_i} \le \frac{1}{n^2} \sum_{\ell=1}^{n^2} s_\ell.$$

PROOF. The right-hand inequality is shown as in Lemma 7. For the left-hand inequality we use here a counting argument.

Note that without loss of generality we may assume that the sequence $\{s_k\}$ is strongly decreasing. There are exactly n^{2n-2} out of n^{2n} multi-indices $(j_1, \dots, j_n, k_1, \dots, k_n)$ such that

$$\max_{1 \le i \le n} a_{i,j_i,k_i} = s_1.$$

Now we estimate for $k \ge 2$ how many multi-indices there are such that

$$\max_{1 \le i \le n} a_{i,j_i,k_i} = s_k.$$

Clearly, one of the coordinates a_{i,j_i,k_i} has to equal s_k, but none of these coordinates may equal s_j for $j = 1, \dots, k-1$. The second condition means that for every i (except for the row with the coordinate equal to s_k) there are j_i^k coordinates that have to be avoided and $\sum_{i=1}^n j_i^k = k-1$. Let us assume that the coordinate that equals s_k is an element of the first row. This leaves us with

$$\prod_{i=2}^n (n^2 - j_i^k)$$

multi-indices. Therefore we get

$$\begin{aligned} n^{-2n} \sum_{\substack{1 \le j_1, \dots, j_n \le n \\ 1 \le k_1, \dots, k_n \le n}} \max_{1 \le i \le n} a_{i,j_i,k_i} &\ge \frac{1}{n^2} \sum_{k=1}^{n^2} s_k \prod_{i=2}^n \left(1 - \frac{j_i^k}{n^2}\right) \\ &\ge \frac{1}{n^2} \sum_{k=1}^{n^2} s_k \left(1 - \frac{k-1}{n^2}\right) \ge \frac{1}{n^2} \frac{n^2+1}{2n^2} \sum_{k=1}^{n^2} s_k, \end{aligned}$$

since

$$\begin{aligned} \sum_{k=1}^{n^2} k s_k = \sum_{j=1}^{n^2} \sum_{k=j}^{n^2} s_k &= \sum_{j=1}^{n^2} \left(\sum_{k=1}^{n^2} s_k - \sum_{k=1}^{j-1} s_k \right) \\ &\le \sum_{j=1}^{n^2} \left(\sum_{k=1}^{n^2} s_k - \frac{j-1}{n^2} \sum_{k=1}^{n^2} s_k \right) = \frac{n^2+1}{2} \sum_{k=1}^{n^2} s_k. \end{aligned}$$

That completes the proof. $\square$

LEMMA 9. *Let $n \in \mathbb{N}$ and let $y \in \mathbb{R}^n$ with $y_1 \ge y_2 \ge \cdots \ge y_n > 0$. Then we have, for $x \in \mathbb{R}^n$,*

$$c_n \|x\|_y \le n^{-n+1} \sum_{1 \le j_1, \dots, j_n \le n} \max_{1 \le i \le n} |x_i y_{j_i}| \le \|x\|_y,$$

where $c_n = 1 - (1 - 1/n)^n$.

PROOF. We show the right-hand inequality. By Lemma 7,

$$n^{-n+1} \sum_{1 \le j_1, \dots, j_n \le n} \max_{1 \le i \le n} |x_i y_{j_i}| \le \sum_{k=1}^{n} s_k(x, y),$$

where $\{s_k(x, y)\}_{k \le n^2}$ is the nonincreasing rearrangement of $\{|x_i y_j|\}_{i,j \le n}$. Therefore there are numbers k_i, $i = 1, \dots, n$, with $\sum_{i=1}^{n} k_i = n$ such that

$$n^{-n+1} \sum_{1 \le j_1, \dots, j_n \le n} \max_{1 \le i \le n} |x_i y_{j_i}| \le \sum_{i=1}^{n} |x_i| \sum_{k=1}^{k_i} y_k \le \|x\|_y.$$

Now we show the left-hand inequality. By Lemma 7,

$$c_n \sum_{k=1}^{n} s_k(x, y) \le n^{-n+1} \sum_{j_1, \dots, j_n = 1}^{n} \max_{1 \le i \le n} |x_i y_{j_i}|.$$

Therefore we have, for all numbers k_i, $i = 1, \dots, n$, with $\sum_{i=1}^{n} k_i = n$,

$$c_n \sum_{i=1}^{n} |x_i| \sum_{k=1}^{k_i} y_k \le n^{-n+1} \sum_{j_1, \dots, j_n = 1}^{n} \max_{1 \le i \le n} |x_i y_{j_i}|.$$

The result follows by the definition of $\| \cdot \|_y$. $\square$

LEMMA 10. *Let $f_1, \dots, f_n$ be independent, identically distributed random variables with $\mathbf{E}|f_1| < \infty$. Let y_j, $j = 1, \dots, n$, be defined as in* (3). *Let $\| \cdot \|$ be a one-unconditional norm on $\mathbb{R}^n$. Then we have, for all $x \in \mathbb{R}^n$,*

$$n^{-n+1} \sum_{j_1, \dots, j_n = 1}^{n} \|(x_i y_{j_i})_{i=1}^n\| \le \mathbf{E}\big\|\big(x_i f_i(\omega)\big)_{i=1}^n\big\|.$$

PROOF. Let $t_j(f_i)$ and $\Omega_j^i := \Omega_j(f_i)$, $i, j \le n$, be defined by (1) and (2). Since the functions f_i, $i = 1, \dots, n$, are identically distributed, the numbers $t_i(f_j)$ do not depend on the functions f_j. Below we will write just t_j.

For $j_1, \dots, j_n$ with $1 \le j_1, \dots, j_n \le n$ we put

$$\Omega_{j_1, \dots, j_n} = \bigcap_{i=1}^{n} \Omega_{j_i}^i.$$

Since $f_1, \dots, f_n$ are independent we have

$$P(\Omega_{j_1, \dots, j_n}) = n^{-n}.$$

For $(j_1, \dots, j_n) \ne (i_1, \dots, i_n)$ we have

$$\Omega_{j_1, \dots, j_n} \cap \Omega_{i_1, \dots, i_n} = \varnothing.$$

Using this and the unconditionality of the norm, we obtain

$$\begin{aligned}\mathbf{E}\big\|\big(x_i f_i(\omega)\big)_{i=1}^n\big\| &= \sum_{j_1,\dots,j_n=1}^{n} \int_{\Omega_{j_1,\dots,j_n}} \big\|\big(x_i f_i(\omega)\big)_{i=1}^n\big\|\, dP(\omega)\\ &\geq \sum_{j_1,\dots,j_n=1}^{n} \left\|\left(x_i \int_{\Omega_{j_1,\dots,j_n}} |f_i(\omega)|\, dP(\omega)\right)_{i=1}^n\right\|\\ &= n^{-n+1} \sum_{j_1,\dots,j_n=1}^{n} \|(x_i y_{j_i})_{i=1}^n\|.\end{aligned}$$

For the last equality we have to show

$$\int_{\Omega_{j_1,\dots,j_n}} |f_i(\omega)|\, dP(\omega) = n^{-n+1} y_{j_i}.$$

We check this. The functions

$$|f_i|\chi_{\Omega^i_{j_i}}, \chi_{\Omega^1_{j_1}}, \dots, \chi_{\Omega^{i-1}_{j_{i-1}}}, \chi_{\Omega^{i+1}_{j_{i+1}}}, \dots, \chi_{\Omega^n_{j_n}}$$

are independent. Therefore we get

$$\begin{aligned}\int_{\Omega_{j_1,\dots,j_n}} |f_i(\omega)|\, dP(\omega) &= \int_{\Omega} |f_i(\omega)| \chi_{\Omega^1_{j_1}} \cdots \chi_{\Omega^n_{j_n}}\, dP(\omega)\\ &= n^{-n+1} \int_{\Omega^i_{j_i}} |f_i(\omega)|\, dP(\omega). \qquad \square\end{aligned}$$

LEMMA 11. *Let $f_1, \dots, f_n$ be identically distributed random variables* (*not necessarily independent*) *with $\mathbf{E}|f_1| < \infty$. Let y_j, $j = 1, \dots, n$, be defined as in* (3). *Let $z_1 \geq z_2 \geq \cdots \geq z_n \geq 0$. Let $s_k(x, y, z)$, $k = 1, \dots, n^3$, be the decreasing rearrangement of the numbers $|x_i y_j z_k|$, $i, j, k = 1, \dots, n$. Then we have, for all $x \in \mathbb{R}^n$*,

$$n^{-n} \sum_{1 \leq k_1,\dots,k_n \leq n} \mathbf{E} \max_{1 \leq i \leq n} |x_i z_{k_i} f_i(\omega)| \leq \frac{2}{n} \sum_{k=1}^{n^2} s_k(x, y, z).$$

PROOF. Let μ be the normalized counting measure on $\{\mathbf{k} = (k_1, \dots, k_n) \mid 1 \leq k_1, \dots, k_n \leq n\}$. For $i = 1, \dots, n$ define the functions ζ_i: $\{\mathbf{k} = (k_1, \dots, k_n) \mid 1 \leq k_1, \dots, k_n \leq n\} \to \mathbb{R}$, $i = 1, \dots, n$, by $\zeta_i(\mathbf{k}) = z_{k_i}$ and put

$$\Lambda_i = \left\{(\omega, \mathbf{k}) \,\middle|\, |x_i \zeta_i(\mathbf{k}) f_i(\omega)| = \max_{1 \leq \ell \leq n} |x_\ell \zeta_\ell(\mathbf{k}) f_\ell(\omega)|\right\}.$$

We may assume that the sets Λ_i, $i = 1, \dots, n$, are disjoint. In case they are not disjoint, we make them disjoint. Therefore

$$\sum_{i=1}^{n} P \times \mu(\Lambda_i) = 1.$$

We define numbers λ_i and sets $\tilde{\Lambda}_i$, $i = 1, \dots, n$, by

$$P \times \mu\{(\omega, \mathbf{k}) \,|\, |\zeta_i(\mathbf{k}) f_i(\omega)| \geq \lambda_i\} = P \times \mu(\Lambda_i)$$

and

$$\tilde{\Lambda}_i = \{(\omega, \mathbf{k}) \,|\, |\zeta_i(\mathbf{k}) f_i(\omega)| \geq \lambda_i\}.$$

The existence of these numbers λ_i follows from the continuity of distribution of the functions f_i [cf. the definition of $t_j(f)$]. We have

$$\sum_{i=1}^{n} P \times \mu(\tilde{\Lambda}_i) = 1$$

and

$$\tilde{\Lambda}_i = \bigcup_{\ell=1}^{n} \{\mathbf{k} \mid \zeta_i(\mathbf{k}) = z_\ell\} \times \left\{\omega \,\middle|\, |f_i(\omega)| \geq \frac{\lambda_i}{z_\ell}\right\}.$$

Since $\mu\{\mathbf{k} \mid \mathbf{k}_i = \ell\} = \mu\{\mathbf{k} \mid \zeta_i(\mathbf{k}) = z_\ell\} = \frac{1}{n}$ we get

$$P \times \mu(\tilde{\Lambda}_i) = \frac{1}{n} \sum_{\ell=1}^{n} P\left\{\omega \,\middle|\, |f_i(\omega)| \geq \frac{\lambda_i}{z_\ell}\right\}.$$

As in the previous lemma we denote

$$\Omega_j^i = \Omega_j(f_i) = \{\omega \mid t_j \leq f_i(\omega) < t_{j-1}\}.$$

For (i, ℓ) we choose $j_{i,\ell} = 1$ if $t_1 \leq \lambda_i / z_\ell$ and $j_{i,\ell}$ with

$$t_{j_{i,\ell}} \leq \frac{\lambda_i}{z_\ell} < t_{j_{i,\ell}-1}$$

otherwise. Then we have

$$\left\{\omega \,\middle|\, |f_i(\omega)| \geq \frac{\lambda_i}{z_\ell}\right\} \subseteq \{\omega \,|\, |f_i(\omega)| \geq t_{j_{i,\ell}}\} = \bigcup_{i=1}^{j_{i,\ell}} \Omega_j^i$$

and

$$\left\{\omega \,\middle|\, |f_i(\omega)| \geq \frac{\lambda_i}{z_\ell}\right\} \supseteq \{\omega \,|\, |f_i(\omega)| \geq t_{j_{i,\ell}-1}\} = \bigcup_{i=1}^{j_{i,\ell}-1} \Omega_j^i,$$

setting $\bigcup_{j=1}^{0} \Omega_j^i = \varnothing$. Therefore we have

$$1 = \sum_{i=1}^{n} P \times \mu(\tilde{\Lambda}_i) = \sum_{i=1}^{n} \frac{1}{n} \sum_{\ell=1}^{n} P\left\{\omega \,\Big|\, |f_i(\omega)| \geq \frac{\lambda_i}{z_\ell}\right\} \geq \sum_{i=1}^{n} \frac{1}{n} \sum_{\ell=1}^{n} P\left(\bigcup_{j=1}^{j_{i,\ell}-1} \Omega_j^i\right).$$

Thus we get

$$n^2 \geq \sum_{i,\ell=1}^{n} (j_{i,\ell} - 1),$$

which gives us

$$2n^2 \geq \sum_{i,\ell=1}^{n} j_{i,\ell}.$$

By the definitions of the sets Λ_i and $\tilde{\Lambda}_i$ we obtain

$$\begin{aligned} n^{-n} \sum_{\mathbf{k}} \mathbf{E} \max_{1 \leq i \leq n} |x_i \zeta_i(\mathbf{k}) f_i(\omega)| &= \sum_{i=1}^{n} \int_{\Lambda_i} |x_i \zeta_i(\mathbf{k}) f_i(\omega)| \, dP(\omega) \, d\mu(\mathbf{k}) \\ &\leq \sum_{i=1}^{n} \int_{\tilde{\Lambda}_i} |x_i \zeta_i(\mathbf{k}) f_i(\omega)| \, dP(\omega) \, d\mu(\mathbf{k}). \end{aligned}$$

Since $\tilde{\Lambda}_i \subseteq \bigcup_{\ell=1}^{n} (\{\mathbf{k} \mid \zeta_i(\mathbf{k}) = z_\ell\} \times \bigcup_{j=1}^{j_{i,\ell}} \Omega_j^i)$,

$$\begin{aligned} n^{-n} \sum_{\mathbf{k}} \mathbf{E} \max_{1 \leq i \leq n} |x_i \zeta_i(\mathbf{k}) f_i(\omega)| &\leq \frac{1}{n} \sum_{i=1}^{n} \sum_{\ell=1}^{n} |x_i z_\ell| \int_{\bigcup_{j=1}^{j_{i,\ell}} \Omega_j^i} |f_i(\omega)| \, dP(\omega) \\ &\leq \frac{1}{n} \sum_{i=1}^{n} \sum_{\ell=1}^{n} |x_i z_\ell| \sum_{j=1}^{j_{i,\ell}} y_j. \end{aligned}$$

Since $2n^2 \geq \sum_{i,\ell=1}^{n} j_{i,\ell}$, we get

$$n^{-n} \sum_{\mathbf{k}} \mathbf{E} \max_{1 \leq i \leq n} |x_i \zeta_i(\mathbf{k}) f_i(\omega)| \leq \frac{1}{n} \sum_{i=1}^{2n^2} s_i(x, y, z) \leq \frac{2}{n} \sum_{i=1}^{n^2} s_i(x, y, z). \qquad \square$$

PROOF OF PROPOSITION 6. Let t_ℓ, $\ell = 1, \ldots, n^3$, denote the decreasing rearrangement of the numbers

$$\left| x_i y_j \left(N^{*-1}\left(\frac{k}{n}\right) - N^{*-1}\left(\frac{k-1}{n}\right) \right) \right|, \qquad i, j, k = 1, \ldots, n.$$

Then, by the definitions of the numbers s_l, there are numbers k_i with $\sum_{i=1}^n k_i = n^2$ such that

$$\sum_{\ell=1}^{n^2} t_\ell = \sum_{i=1}^{n} |x_i| \sum_{\ell=1}^{k_i} s_\ell,$$

setting $\sum_{\ell=1}^{0} s_\ell = 0$. Moreover, for all numbers m_i with $\sum_{i=1}^n m_i = n^2$, we have

$$\sum_{\ell=1}^{n^2} t_\ell \geq \sum_{i=1}^{n} |x_i| \sum_{\ell=1}^{m_i} s_\ell,$$

which means

$$\sum_{\ell=1}^{n^2} t_\ell = \|x\|_s.$$

By Lemma 9,

$$\mathbf{E}\big\|\big(x_i f_i(\omega)\big)_{i=1}^n\big\|_z \leq \frac{1}{c_n} n^{-n+1} \sum_{1\leq k_1,\ldots,k_n\leq n} \mathbf{E} \max_{1\leq i\leq n} |x_i z_{k_i} f_i(\omega)|.$$

By Lemma 11,

$$\mathbf{E}\big\|\big(x_i f_i(\omega)\big)_{i=1}^n\big\|_z \leq \frac{2}{c_n} \sum_{\ell=1}^{n^2} t_\ell = \frac{2}{c_n} \|x\|_s.$$

Now we show the “moreover” part of the proposition. By Lemma 10,

$$\mathbf{E}\big\|\big(x_i f_i(\omega)\big)_{i=1}^n\big\|_z \geq n^{-n+1} \sum_{j_1,\ldots,j_n=1}^{n} \|(x_i y_{j_i})_{i=1}^n\|_z.$$

By Lemma 9,

$$\mathbf{E}\big\|\big(x_i f_i(\omega)\big)_{i=1}^n\big\|_z \geq n^{-2n+2} \sum_{\substack{1\leq j_1,\ldots,j_n\leq n\\ 1\leq k_1,\ldots,k_n\leq n}} \max_{1\leq i\leq n} |(x_i y_{j_i} z_{k_i})_{i=1}^n|.$$

By Lemma 8,

$$\mathbf{E}\big\|\big(x_i f_i(\omega)\big)_{i=1}^n\big\|_z \geq \tfrac{1}{2} \sum_{\ell=1}^{n^2} t_\ell = \tfrac{1}{2} \|x\|_s,$$

which proves the proposition. $\square$

REMARK. Using (7) and repeating the proof of Proposition 6, we can obtain estimates for the constants in Corollary 3. Namely, for every $f_1, \dots, f_n$ satisfying the condition of the proposition, we have

$$\mathbf{E}\big\|\big(x_i f_i(\omega)\big)_{i=1}^n\big\|_{k,*} \le \frac{2}{c_n}\|x\|_s,$$

where $s = (s_l)_{l=1}^{n^2}$ is the nonincreasing rearrangement of the numbers $|y_i z_j|$, $1 \le i, j \le n$, $z = (1, \dots, 1, 0, \dots, 0)/[n/k]$. Moreover, if $f_1, \dots, f_n$ are independent, then

$$\|x\|_s \le 2c_{n,k}\mathbf{E}\big\|\big(x_i f_i(\omega)\big)_{i=1}^n\big\|_{k,*}.$$

In particular, we have the variant of Theorem 4 for $\|\cdot\|_{k,*}$:

$$\mathbf{E}\big\|\big(x_i f_i(\omega)\big)_{i=1}^n\big\|_{k,*} \le \frac{4c_{n,k}}{c_n}\mathbf{E}\big\|\big(x_i g_i(\omega)\big)_{i=1}^n\big\|_{k,*}, \tag{8}$$

where $f_1, \dots, f_n$ satisfy the condition of Proposition 6, $g_1, \dots, g_n$ are independent copies of f_1 and $c_{n,k} = n/k/[n,k] < 2$, $c_n = 1 - (1 - 1/n)^n > 1 - 1/e$. Let us note that taking $m = k([n/k] + 1)$ and applying (8) for the sequences $(\bar{x}_i f_i)_{i \le m}$, $(\bar{x}_i g_i)_{i \le m}$, where $\bar{x} = (x_1, x_2, \dots, x_n, 0, \dots, 0)$, we obtain

$$\mathbf{E}\big\|\big(x_i f_i(\omega)\big)_{i=1}^n\big\|_{k,*} \le \frac{4e}{e-1}\mathbf{E}\big\|\big(x_i g_i(\omega)\big)_{i=1}^n\big\|_{k,*}, \tag{9}$$

since $c_{m,k} = 1$.

3. Examples.

In this section we provide a few examples. We need the following two lemmas about the normal distribution.

LEMMA 12. *For all x with $x > 0$,*

$$\frac{\sqrt{2\pi}}{(\pi-1)x + \sqrt{x^2+2\pi}}e^{-x^2/2} \le \sqrt{\frac{2}{\pi}}\int_x^\infty e^{-s^2/2}\,ds \le \sqrt{\frac{2}{\pi}}\frac{1}{x}e^{-x^2/2}.$$

The left-hand inequality can be found in [8]. The right-hand inequality is trivial.

LEMMA 13. *Let f be a Gauss variable with distribution $N(0,1)$. Let the numbers t_j and y_j be defined by* (1) *and* (3). *Then there are absolute positive constants c_1, c_2, c_3 such that*:

(i) *for all $1 \le j \le n/e$ we have*

$$\sqrt{\frac{1}{2}\ln\frac{n}{j}} \le t_j \le \sqrt{2\ln\frac{n}{j}} \quad \textit{and} \quad \frac{c_1 n}{\sqrt{\ln n}} \le \exp\Big(\frac{t_1^2}{2}\Big) \le \frac{c_2 n}{\sqrt{\ln n}};$$

(ii) *for all* $2 \le j \le n/e$ *we have*

$$\frac{1}{n}\sqrt{\frac{1}{2}\ln\frac{n}{j}} \le y_j \le \frac{1}{n}\sqrt{2\ln\frac{n}{j-1}} \quad \textit{and} \quad \frac{\sqrt{\ln n}}{n} \le y_1 \le \frac{c_3\sqrt{\ln n}}{n}.$$

PROOF. The inequalities for t_1 and y_1 follow by direct computation. The inequalities for the y_j's follow from the inequalities for the t_j's, since $t_j/n \le y_j \le t_{j-1}/n$ for every $2 \le j \le n$. Let us prove the inequalities for the t_j's. By definition,

$$P\{\omega \,|\, |f(\omega)| \ge t_j\} = \frac{j}{n}.$$

This means

$$\sqrt{\frac{2}{\pi}}\int_{t_j}^{\infty} e^{-s^2/2}\,ds = \frac{j}{n}.$$

By Lemma 12 we get

$$\frac{\sqrt{2\pi}}{(\pi-1)t_j + \sqrt{t_j^2 + 2\pi}} e^{-t_j^2/2} \le \frac{j}{n} \le \sqrt{\frac{2}{\pi}}\frac{1}{t_j} e^{-t_j^2/2}. \tag{10}$$

First we show $t_j \le \sqrt{2\ln\frac{n}{j}}$. For this we observe that $\frac{1}{s}e^{-s^2/2}$ is decreasing on $(0,\infty)$. Suppose now that for some j we have $t_j > \sqrt{2\ln\frac{n}{j}}$. Therefore, using (10), we get

$$\frac{j}{n} \le \sqrt{\frac{2}{\pi}}\frac{1}{t_j}e^{-t_j^2/2} \le \sqrt{\frac{2}{\pi}}\frac{1}{\sqrt{2\ln\frac{n}{j}}}\frac{j}{n}.$$

Thus we have

$$\sqrt{2\ln\frac{n}{j}} \le \sqrt{\frac{2}{\pi}},$$

which is not true if $ej \le n$.

We show now that $\sqrt{\frac{1}{2}\ln\frac{n}{j}} \le t_j$. The function

$$\frac{\sqrt{2\pi}}{(\pi-1)x + \sqrt{x^2+2\pi}} e^{-x^2/2}$$

is decreasing on $(0,\infty)$. Suppose now

$$t_j < \sqrt{\frac{1}{2}\ln\frac{n}{j}}.$$

Then we have, by (10),

$$\frac{j}{n} \geq \frac{\sqrt{2\pi}}{(\pi-1)t_j + \sqrt{t_j^2 + 2\pi}} e^{-t_j^2/2} \geq \frac{\sqrt{2\pi}}{(\pi-1)\sqrt{\frac{1}{2}\ln\frac{n}{j}} + \sqrt{\frac{1}{2}\ln\frac{n}{j} + 2\pi}} \left(\frac{j}{n}\right)^{1/4},$$

which is false for $j \leq n/e$. That proves the lemma. □

EXAMPLE 14. *Let $f_1, \dots, f_n$ be independent Gauss variables with distribution $N(0,1)$. Let M be the Orlicz function given by*

$$M(t) = \begin{cases} 0, & t = 0, \\ e^{-3/(2t^2)}, & t \in (0,1), \\ e^{-3/2}(3t-2), & t \geq 1. \end{cases}$$

Then we have, for all $x \in \mathbb{R}^n$,

$$c\|x\|_M \leq \mathbf{E} \max_{1\leq i\leq n} |x_i f_i(\omega)| \leq C\|x\|_M,$$

where c and C are absolute positive constants.

PROOF. It is easy to see that there are absolute constants c_1, c_2 such that

$$c_1 k\sqrt{\ln(en/k)} \leq \sum_{j=1}^{k} \sqrt{\ln(n/j)} \leq c_2 k\sqrt{\ln(en/k)}$$

for every $k \leq n$. Since $\sum_{j=1}^n y_j = \mathbf{E}|f_1| = \sqrt{2/\pi}$, Lemma 13 implies that, for every $k \leq n$,

$$c_3 \frac{k\sqrt{\ln(en/k)}}{n} \leq \sum_{j=1}^{k} y_j \leq c_4 \frac{k\sqrt{\ln(en/k)}}{n}, \tag{11}$$

where c_3, c_4 are absolute constants.

By the condition of the example, $M^{-1}(t) = \sqrt{-3/(2\ln t)}$ on $(0, e^{-3/2})$. Thus $M^{-1}(t) \approx \sqrt{3/(2\ln(e/t))}$ on $(0,1)$. By (5) we observe

$$t\sqrt{2\ln(e/t)}/\sqrt{3} \leq M^{*-1}(t) \leq 2t\sqrt{2\ln(e/t)}/\sqrt{3}.$$

Taking $t = k/n$ and using (11), we get, for every $k \leq n$,

$$c_5 \sum_{j=1}^{k} y_j \leq M^{*-1}\left(\frac{k}{n}\right) \leq c_6 \sum_{j=1}^{k} y_j,$$

where c_5, c_6 are absolute constants. Applying Corollary 2 we obtain the result. □

The next example is proved in the same way as the previous one; we just use Corollary 3 instead of Corollary 2 at the end.

EXAMPLE 15. *Let g_i, $i = 1, \dots, n$, be independent Gauss variables with distribution $N(0,1)$, $k \le n$ and $\|x\| = \sum_{i=1}^k x_i^*$. Let*

$$M(t) = \begin{cases} 0, & t = 0, \\ \dfrac{1}{k} e^{-3/(2k^2t^2)}, & t \in \left(0, \dfrac{1}{k}\right), \\ e^{-3/2}\left(3t - \dfrac{2}{k}\right), & t \ge \dfrac{1}{k}. \end{cases}$$

Then for all $\lambda \in \mathbb{R}^n$ we have

$$c_1 \|\lambda\|_M \le \mathbf{E}\big\|\big(\lambda_i g_i(\omega)\big)_{i=1}^n\big\| \le c_2 \|\lambda\|_M,$$

where c_1 and c_2 are positive absolute constants.

The following example deals with the moments of Gauss variables.

EXAMPLE 16. *Let $0 < q \le \ln n$, $a_q = \max\{1, q\}$, g_i, $i = 1, \dots, n$, be independent Gauss variables with distribution $N(0,1)$ and $f_i = |g_i|^q$, $i = 1, \dots, n$. Let*

$$M(t) = \begin{cases} 0, & t = 0, \\ \dfrac{1}{k} \exp\left(-\dfrac{a_q}{(kt)^{2/q}}\right), & t \in (0, t_0), \\ at - b, & t \ge t_0, \end{cases}$$

where

$$t_0 = \frac{1}{k}\left(\frac{2a_q}{q+2}\right)^{q/2}, \qquad a = \frac{q+2}{eqkt_0} e^{-q/2}, \qquad b = \frac{2}{eqk} e^{-q/2}.$$

Then for all $\lambda \in \mathbb{R}^n$ we have

$$c(a_q)^{q/2} \|\lambda\|_M \le \mathbf{E}\big\|\big(\lambda_i f_i(\omega)\big)_{i=1}^n\big\| \le C(Ca_q)^{q/2} \|\lambda\|_M,$$

where $0 < c < 1 < C$ are absolute constants and $\|x\| = \sum_{i=1}^k x_i^*$.

This example is proved in the same way as the previous two examples. We use that

$$k\left(\sqrt{\ln(n/k)}\right)^{q/2} \le \sum_{j=1}^k \left(\sqrt{\ln(n/j)}\right)^{q/2} \le 2k\left(\sqrt{\ln(n/k)}\right)^{q/2}$$

for every $k \le n/e^q$ and that

$$ca_q \le \big(\mathbf{E}|g(\omega)|^q\big)^{2/q} \le Ca_q$$

for some absolute positive constants c, C.

Finally, we apply our theorem to the p-stable random variables. Let us recall that a random variable f is called p-stable, $p \in (0,2]$, if the Fourier transform of f satisfies

$$\mathbf{E}\exp(-itf) = \exp(-c|t|^p)$$

for some positive constant c (in the case $p = 2$ we obtain the Gauss variable).

EXAMPLE 17. *Let $p \in (1,2)$. Let $f_1, \dots, f_n$ be p-stable, independent, random variables with $\mathbf{E}|f_i| = 1$. Let $k \le n$ and $\|x\| = \sum_{i=1}^k x_i^*$. Let*

$$M(t) = \begin{cases} \frac{1}{k}(kt)^p, & t \in \left[0, \frac{1}{k}\right], \\ pt + \frac{p-1}{k}, & t > \frac{1}{k}. \end{cases}$$

Then, for all $x \in \mathbb{R}^n$,

$$c_p\|x\|_M \le \mathbf{E}\big\|\big(\lambda_i f_i(\omega)\big)_{i=1}^n\big\| \le C_p\|x\|_M,$$

where c_p, C_p are positive constants depending on p only.

In particular,

$$c_p|x|_p \le \mathbf{E}\max_{1\le i\le n}|x_i f_i(\omega)| \le C_p|x|_p,$$

where $|\cdot|_p$ denotes the standard ℓ_p-norm.

PROOF. There are positive constants c_1 and c_2 depending on p only such that, for all $t > 1$,

$$c_1 t^{-p} \le P\{\omega \,|\, |f(\omega)| \ge t\} \le c_2 t^{-p}.$$

Thus

$$\left(c_1\frac{n}{j}\right)^{1/p} \le t_j \le \left(c_2\frac{n}{j}\right)^{1/p}.$$

Repeating the proof of Example 14, we obtain the desired result. □

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