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ON THE HIGH TEMPERATURE PHASE OF THE SHERRINGTON–KIRKPATRICK MODEL

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We prove the validity of the "replica-symmetric" solution of the Sherrington–Kirkpatrick model in a region that (probably) coincides with the region predicted by the physicists.

1. Introduction. In the Sherrington–Kirkpatrick (SK) model, the energy of a configuration $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_N)$, $\sigma_i = \pm 1$, is given by

$$H_N(\boldsymbol{\sigma}) = -\left(\frac{1}{\sqrt{N}} \sum_{i<j} g_{ij}\sigma_i\sigma_j + h \sum_{i \le N} \sigma_i\right), \tag{1.1}$$

where $(g_{ij})_{i<j}$ are i.i.d. $N(0,1)$. The object of interest is Gibbs' measure, a probability measure on the space $\{-1,1\}^N$ of all configurations, given by

$$G_N(\{\boldsymbol{\sigma}\}) = \frac{\exp(-\beta H_N(\boldsymbol{\sigma}))}{Z_N}, \tag{1.2}$$

where

$$Z_N = Z_N(\beta, h) = \sum_{\boldsymbol{\sigma}} \exp\bigl(-\beta H_N(\boldsymbol{\sigma})\bigr) \tag{1.3}$$

is the normalization factor. The very beautiful structure predicted at low temperature (= large β) by the physicists has fascinated many, but essentially nothing is rigorously known about it. While certainly simpler, the rigorous study of the high temperature phase is apparently also very difficult. It has been the object of intense recent interest [4–6]. In [6], an extremely detailed picture is given, but only in a subregion of the region of parameters where it is believed to hold.

In the present paper, we combine the ideas of [6, 7] with a dazzlingly beautiful recent argument of F. Guerra [1] to extend the validity of the results of [6] to a region of parameters that probably coincides with the region predicted by physicists. The word "probably" here does not mean that the validity of our arguments is dubious, but rather that the description of the region involves properties of certain complicated elementary functions. These properties are likely to be true in the entire region described by the physicists, but it seems difficult to prove this analytically. Probably one should first check these properties numerically (but numerical studies are not my cup of tea).

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Following Guerra [1], we will consider a more general version of the model. In this version, we consider three parameters $s, t, h \geq 0$. The value of h is fixed once and for all. We set

$$U_N(\boldsymbol{\sigma}) = \frac{\sqrt{s}}{\sqrt{N}} \sum_{i<j} g_{ij}\sigma_i\sigma_j + \sqrt{t} \sum_{i \leq N} \xi_i \sigma_i + h \sum_{i \leq N} \sigma_i. \tag{1.4}$$

Here $(\xi_i)_{i \leq N}$ is a new sequence of $N(0, 1)$ variables. If we set $s = \beta^2$, $t = 0$, and h is replaced by βh, we have $U_N(\boldsymbol{\sigma}) = -\beta H_N(\boldsymbol{\sigma})$, where $H_N(\boldsymbol{\sigma})$ is as in (1.1).

We set

$$p_N(s, t) = \frac{1}{N} E \log Z_N, \tag{1.5}$$

where

$$Z_N = \sum_{\boldsymbol{\sigma}} \exp U_N(\boldsymbol{\sigma}), \tag{1.6}$$

and where E denotes expectation in the variables g_{ij}, ξ_i. A main objective is the computation of $\lim_{N\to\infty} p_N(s, t)$ (a quantity closely related to the "free energy" of the physicists). Let us consider the following expression (the "replica-symmetric formula")

$$RS(s, t) = \inf_q \left(\frac{s}{4}(1 - q)^2 + E \log\bigl(2\operatorname{ch}(g\sqrt{sq + t} + h)\bigr) \right), \tag{1.7}$$

where the infimum is over $q \geq 0$, and where g is $N(0, 1)$. The infimum is obtained at a point q such that

$$q = E \operatorname{th}^2\bigl(g\sqrt{sq + t} + h\bigr). \tag{1.8}$$

[This condition simply expresses that the infimum is obtained at a point where the derivative in q of the right hand side of (1.7) is zero. Integration by parts is used to see this.]

When $t = 0$, it is already proved in [5] that if $s \leq s_0$ (where s_0 is a number) we have

$$RS(s, t) = \lim_{N\to\infty} p_N(s, t). \tag{1.9}$$

No changes are required to prove that this is also true for all $t \neq 0$. The physicists conjecture that (1.8) holds whenever

$$s E \frac{1}{\operatorname{ch}^4(g\sqrt{sq + t} + h)} < 1, \tag{1.10}$$

where q is given by (1.7). [Equality in (1.10) defines the so-called A–T line.] A minor difficulty (that announces more serious difficulties of the same nature) is that it is not obvious that (1.8) has a unique solution.

This was clarified recently by F. Guerra [1] and R. Latala [2], who independently proved that the function

$$x \to \frac{1}{x} E \operatorname{th}^2(g\sqrt{x} + h)$$

is strictly decreasing. Thus, if $s, t > 0$, the function

$$q \to \frac{1}{q} E \operatorname{th}^2(g\sqrt{sq+t} + h)$$

is also strictly decreasing, and (1.8) has indeed a unique solution.

The recent progress made by F. Guerra [1] is the following:

THEOREM 1.1 (Guerra's bound). *For all values of s, t, N, we have*

$$p_N(s,t) \le RS(s,t). \tag{1.11}$$

The magically beautiful proof of this statement will be given at the beginning of Section 2. (The only sad point is that at present it is totally unclear whether the argument can be extended to any of the other important models.) Learning (1.11) came as a shock to the author because he had noticed while writing [6] that he could extend (1.8) to all the region (1.10) provided he knew (1.11), not only for the original system, but also for the system made of two coupled versions of it (see also [7]). This is defined as follows. Given a new parameter λ (that represents the intensity of the coupling), for two configurations $\boldsymbol{\sigma}, \boldsymbol{\tau}$, we set

$$V_N(\boldsymbol{\sigma}, \boldsymbol{\tau}) = U_N(\boldsymbol{\sigma}) + U_N(\boldsymbol{\tau}) + \lambda \sum_{i \le N} \sigma_i \tau_i \tag{1.12}$$

and we set

$$r_N(s,t,\lambda) = \frac{1}{N} E \log\left(\sum_{\boldsymbol{\sigma},\boldsymbol{\tau}} \exp V_N(\boldsymbol{\sigma},\boldsymbol{\tau})\right). \tag{1.13}$$

What, in this situation, is the result corresponding to (1.10)? For $x \in \mathbf{R}$, $y > 0$, let us define

$$A(x,y) = E \log\bigl(4(\operatorname{ch}x \operatorname{ch}^2 Y + \operatorname{sh}x \operatorname{sh}^2 Y)\bigr), \tag{1.14}$$

where $Y = g\sqrt{y} + h$ and g is $N(0,1)$. Given $q \ge 0$, $\rho \in \mathbf{R}$, let us define

$$W(s,t,\lambda,q,\rho) = \frac{s}{2}\bigl((1-q)^2 + q^2 - \rho^2\bigr) + A\bigl(s(\rho - q) + \lambda, s, q + t\bigr). \tag{1.15}$$

(Of course such formulas must look like sheer magic to the reader. There is actually no magic, but rather an interesting structure. The author knows how to derive these formulas without appealing to the "replica method" of the physicists. This is however a different topic than the one we pursue here, so we refer the reader to [8] for a detailed account.)

Let us denote by $C(s,t,\lambda)$ the set of critical points of the function $q,\rho \to W(s,t,\lambda,q,\rho)$, that is the values of (q,ρ) such that

$$\frac{\partial W}{\partial q} = \frac{\partial W}{\partial \rho} = 0. \tag{1.16}$$

Elementary (and tedious) computations show that this amounts to

$$\rho = \Phi\big(q, s(\rho - q) + \lambda\big), \qquad q = \Psi\big(q, s(\rho - q) + \lambda\big), \tag{1.17}$$

where

$$\Phi(q,x) = E\left(\frac{\mathrm{sh}x\,\mathrm{ch}^2 Y + \mathrm{ch}x\,\mathrm{sh}^2 Y}{\mathrm{ch}x\,\mathrm{ch}^2 Y + \mathrm{sh}x\,\mathrm{ch}^2 Y}\right), \tag{1.18}$$

$$\Psi(q,x) = E\left(\left(\frac{e^x\,\mathrm{sh}Y\,\mathrm{ch}Y}{\mathrm{ch}x\,\mathrm{ch}^2 Y + \mathrm{sh}x\,\mathrm{sh}^2 Y}\right)^2\right), \tag{1.19}$$

for $Y = g\sqrt{sq+t} + h$. The task of analyzing these explicit equations is in principle elementary. Yet it is not clear how to carry it out. Let us observe that when $\lambda = 0$, if q satisfies (1.8), then $(q,q) \in C(s,t,0)$.

We define

$$RSC(s,t,\lambda) = \inf_{(q,\rho)\in C(s,t,\lambda)} W(s,t,\lambda,q,\rho). \tag{1.20}$$

We have not proved that $C(s,t,\lambda)$ is always non empty [this is very likely, due to the "saddle shape" of the function $q,\rho \to W(s,t,\lambda,q,\rho)$], so we define $RSC(s,t,\lambda) = \infty$ should $C(s,t,\lambda)$ be empty.

Our arguments require some regularity properties of the function *RSC*. We now define a region of parameters where these properties will hold. We say that $(s_0,t_0) \in S$ (remember that h is fixed once for all) if the following occurs:

(1.21) Given $s \le s_0$, $t \ge t_0$, the solution to (1.8) satisfies (1.10).

(1.22) Given $s \le s_0$, $t \ge t_0$ and any $\lambda \in \mathbf{R}$, there is a unique point $q(s,t,\lambda)$, $\rho(s,t,\lambda)$ in $C(s,t,\lambda)$ at which the infimum is obtained in (1.20).

Moreover the map $\lambda \to (q(s,t,\lambda), \rho(s,t,\lambda))$ is differentiable; and the map $\lambda \to \frac{\partial}{\partial\lambda}\rho(s,t,\lambda)$ is continuous over the domain $s \le s_0$, $t \ge t_0$, $\lambda \in \mathbf{R}$.

CONJECTURE 1.2. The region S coincides with the region (1.10).

The rationale for this conjecture is as follows.

1. To say that $(s_0, t_0) \in S$ we require a certain regularity for $s \le s_0$, $t \ge t_0$. This is very mild, because decreasing s and increasing t seem only to improve matters.

2. The requirement (1.22) amounts to little more than the second differentiability of $R(s, t, .)$. One really does not see why any value of λ, except zero, would play a special role and be a singularity. But, as will be explained at the end of this introduction, under (1.10), there can be no singularity at $\lambda = 0$.

The following should be compared to Theorem 1.1.

PROPOSITION 1.3. *If* $(s, t) \in S$, *then*, *for each* $\lambda \in \mathbf{R}$, *we have*

$$\limsup_{N\to\infty} r_N(s, t, \lambda) \le RSC(s, t, \lambda). \tag{1.23}$$

Possibly (1.23) is true for all values of s, t, but there is little incentive to try to prove this. The regularity of $RSC(s, t, .)$ implicit in (1.22) is used as a technical help to establish (1.23). It is possible that by working harder (or being more clever) one could prove (1.23) without using (1.22). As we believe that Conjecture 1.2 holds, there is little motivation to try this.

THEOREM 1.4. *If* $(s, t) \in S$, *then*

$$\lim_{N\to\infty} p_N(s, t) = RS(s, t). \tag{1.24}$$

Moreover, *the system can be described with the same accuracy as in* [6].

The regularity of $RSC(s, t, \cdot)$ at $\lambda \ne 0$ is used only as a technical help in the proof of Proposition 1.3 (and possibly could be dispensed with it). In contrast, the proof of Theorem 1.4 makes crucial use of the fact that

$$RSC(s, t, \lambda) \le 2RS(s, t) + \lambda q + \lambda^2 M \tag{1.25}$$

for small λ, where $M = M(s, t)$ remains bounded when (s, t) stays away from the A–T line. This condition can be proved under (1.10) alone [*without* requiring (1.22)] as follows. Setting $u = \rho - q$, we rewrite (1.17) as

$$u = \Xi(q, su + \lambda), \qquad q = \Psi(q, su + \lambda) \tag{1.26}$$

where

$$\Xi(q, x) = \mathrm{sh}x\, \mathrm{ch}x\, E\, \frac{1}{(\mathrm{ch}x\, \mathrm{ch}^2 Y + \mathrm{sh}x\, \mathrm{sh}^2 Y)^2}. \tag{1.27}$$

As already pointed out, if $q = q_0$ satisfies (1.8), the pair (q_0, q_0) satisfies (1.27) when $\lambda = 0$. Moreover,

$$W(s, t, 0, q_0, q_0) = 2RS(s, t). \tag{1.28}$$

The differential at $(q_0, 0)$ of the map $(q, u) \to (\Xi(q, su), \Psi(q, su))$ is given by the matrix

$$\begin{pmatrix} A & 0 \\ D & C \end{pmatrix}$$

where

$$A = sE\frac{1}{\mathrm{ch}^4 Y}, \qquad C = sE\left(\frac{1 - 2\mathrm{sh}^2 Y}{\mathrm{ch}^4 Y}\right),$$

for $Y = g\sqrt{sq_0 + t} + h$, so that, under (1.10), $A, C < 1$. By the implicit function theorem, for λ small enough, the equations (1.26) have a solution that is differentiable in λ, and this readily implies (1.25). Having proved (1.25) for small λ, there is no loss of generality to assume that it holds for $|\lambda| \le 1$.

2. A priori bounds. In this section, we prove Proposition 1.3. Modulo (very serious) technical difficulties, the proof is the same as that of Theorem 1.1, so we first prove Theorem 1.1.

PROOF OF THEOREM 1.1. Considering $s, t, N, q \ge 0$, we will prove that

$$p_N(s, t) \le \frac{s}{4}(1 - q)^2 + E\log\Bigl(2\mathrm{ch}\bigl(g\sqrt{sq + t} + h\bigr)\Bigr). \tag{2.1}$$

For $u \ge 0$, we set $t(u) = (s - u)q + t$. This value is chosen so that

$$uq + t(u) = sq + t. \tag{2.2}$$

To prove (2.1), we will prove that, for all $u \ge 0$, we have

$$f(u) := p_N(u, t(u)) \le \xi(u) := \frac{u}{4}(1 - q)^2 + E\log\Bigl(2\mathrm{ch}\bigl(g\sqrt{uq + t(u)} + h\bigr)\Bigr).$$

It suffices to prove that

$$f(0) \le \xi(0), \tag{2.3}$$

$$f'(u) \le \xi'(u) = \tfrac{1}{4}(1 - q)^2. \tag{2.4}$$

[The reader certainly observes how (2.2) ensures that ξ has a simple derivative.]

When $s = 0$, Gibbs' measure is a product measure, and it is immediate that

$$p_N(0, t) = E\log\Bigl(2\mathrm{ch}\bigl(g\sqrt{t} + h\bigr)\Bigr),$$

so that there is in fact equality in (2.3). To compute $f'(u)$, we use the basic relations

$$\frac{\partial}{\partial s} p_N(s, t) = \frac{1}{4}\bigl(1 - E\langle R(\boldsymbol{\sigma}, \boldsymbol{\sigma}')^2\rangle\bigr), \tag{2.5}$$

$$\frac{\partial}{\partial t} p_N(s, t) = \frac{1}{2}\bigl(1 - E\langle R(\boldsymbol{\sigma}, \boldsymbol{\sigma}')\rangle\bigr). \tag{2.6}$$

Here, $\boldsymbol{\sigma}, \boldsymbol{\sigma}'$ are two configurations:

$$R(\boldsymbol{\sigma}, \boldsymbol{\sigma}') = \frac{1}{N} \sum_{i \le N} \sigma_i \sigma_i', \tag{2.7}$$

and the bracket represents a double integral with respect to Gibbs' measure, that is,

$$\langle R(\boldsymbol{\sigma}, \boldsymbol{\sigma}') \rangle = \iint R(\boldsymbol{\sigma}, \boldsymbol{\sigma}') \, dG_N(\boldsymbol{\sigma}) \, dG_N(\boldsymbol{\sigma}'), \tag{2.8}$$

for

$$G_N(\{\boldsymbol{\sigma}\}) = \frac{\exp U_N(\boldsymbol{\sigma})}{Z_N}, \tag{2.9}$$

$U_N(\boldsymbol{\sigma})$ and Z_N being given by (1.4) and (1.6) respectively. The relations (2.5) and (2.6) are well known: the proof of similar relations will be given in a more complicated situation in Lemma 2.1 below.

Combining (2.5) and (2.6), we have

$$\begin{aligned} f'(u) &= \frac{1}{4}\big(1 - E\langle R(\boldsymbol{\sigma}, \boldsymbol{\sigma}')^2 \rangle\big) - \frac{q}{2}\big(1 - E\langle R(\boldsymbol{\sigma}, \boldsymbol{\sigma}') \rangle\big) \\ &= \frac{1}{4}(1-q)^2 - \frac{1}{4} E\big\langle (R(\boldsymbol{\sigma}, \boldsymbol{\sigma}') - q)^2 \big\rangle \le \frac{1}{4}(1-q)^2 \end{aligned} \tag{2.10}$$

and this proves (2.4). □

In view of the crucial role played by the relations (2.5), (2.6), we turn to the investigation of similar relations for the function of interest, namely $r_N(s, t, \lambda)$. Gibbs' measure is now a probability on the space of pairs of configurations $(\boldsymbol{\sigma}, \boldsymbol{\tau})$, and is given by

$$G_N\big(\{(\boldsymbol{\sigma}, \boldsymbol{\tau})\}\big) = \frac{1}{Z_N} \exp V_N(\boldsymbol{\sigma}, \boldsymbol{\tau}),$$

where V_N is given by (1.12) and Z_N is the normalization factor. We will again denote by $\langle \cdot \rangle$ integration with respect to Gibbs' measure or its products; we will denote by $(\boldsymbol{\sigma}', \boldsymbol{\tau}')$ a new copy of the system with the same disorder (replica). Thus, we have

$$\langle R^2(\boldsymbol{\sigma}, \boldsymbol{\sigma}') \rangle = \iint \left(\frac{1}{N} \sum_{i \le N} \sigma_i \sigma_i' \right)^2 dG_N(\boldsymbol{\sigma}, \boldsymbol{\tau}) \, dG_N(\boldsymbol{\sigma}', \boldsymbol{\tau}'),$$

and similar expressions.

LEMMA 2.1. *We have*

$$\frac{\partial r_N}{\partial s} = \frac{1}{2}\bigl(1 + E\langle R(\boldsymbol{\sigma}, \boldsymbol{\tau})^2\rangle - 2E\langle R(\boldsymbol{\sigma}, \boldsymbol{\sigma}')^2\rangle\bigr), \tag{2.11}$$

$$\frac{\partial r_N}{\partial t} = 1 + E\langle R(\boldsymbol{\sigma}, \boldsymbol{\tau})\rangle - 2E\langle R(\boldsymbol{\sigma}, \boldsymbol{\sigma}')\rangle, \tag{2.12}$$

$$\frac{\partial r_N}{\partial \lambda} = E\langle R(\boldsymbol{\sigma}, \boldsymbol{\tau})\rangle. \tag{2.13}$$

PROOF. *We have*

$$\frac{\partial r_N}{\partial s} = \frac{1}{2\sqrt{s}N^{1/2}}\left(\sum_{i<j} E g_{ij}\langle \sigma_i\sigma_j + \tau_i\tau_j\rangle\right).$$

We then use the integration by parts formula $E(gf(g)) = E(f'(g))$ to obtain

$$\frac{\partial r_N}{\partial s} = \frac{1}{2N^2}\left(\sum_{i<j} E\bigl(\langle(\sigma_i\sigma_j + \tau_i\tau_j)^2\rangle - \langle(\sigma_i\sigma_j + \tau_i\tau_j)(\sigma_i'\sigma_j' + \tau_i'\tau_j')\rangle\bigr)\right).$$

Next, we note that $\sigma_i^2 = \tau_i^2 = 1$, and that

$$\sum_{i<j} a_i a_j = \frac{1}{2}\Bigl(\sum a_i\Bigr)^2 - \sum a_i^2.$$

We thus get

$$\begin{aligned}\frac{\partial r_N}{\partial s} = \frac{1}{4}\Bigl(&2 + 2E\langle R^2(\boldsymbol{\sigma}, \boldsymbol{\tau})\rangle - E\langle R^2(\boldsymbol{\sigma}, \boldsymbol{\sigma}')\rangle \\ &- E\langle R^2(\boldsymbol{\sigma}, \boldsymbol{\tau}')\rangle - E\langle R^2(\boldsymbol{\tau}, \boldsymbol{\sigma}')\rangle - E\langle R^2(\boldsymbol{\tau}, \boldsymbol{\tau}')\rangle\Bigr)\end{aligned}$$

and the last 4 terms are equal by symmetry.

This yields (2.11). The proof of (2.12) is similar, and (2.13) is obvious. □

At this stage we observe the unpleasant fact that in (2.11) the term $E\langle R^2(\boldsymbol{\sigma}, \boldsymbol{\tau})\rangle$ has the *wrong* sign to make the argument of Theorem 1.1 work. The basic idea to go around this difficulty is to observe that we can control such quantities if we control r_N as a function of λ. [This is why we require control over all values of λ in (1.22).]

LEMMA 2.2. *Consider s, t, λ. Assume that there exists a function $\varphi(\lambda)$ and numbers B, λ_0 such that*

$$\forall \lambda, \qquad r_N(s, t, \lambda) \le \varphi(\lambda), \tag{2.14}$$

$$r_N(s, t, \lambda_0) = \varphi(\lambda_0), \tag{2.15}$$

(2.16) *φ is twice differentiable and $|\varphi''(\lambda)| \le B$ if $|\lambda - \lambda_0| \le N^{-1/4}$.*

Then we have

$$E\langle (R(\boldsymbol{\sigma},\boldsymbol{\tau}) - \varphi'(\lambda_0))^2\rangle \le \frac{L}{\sqrt{N}}(1 + B^2 + s + t), \tag{2.17}$$

where L is a number, and where the Gibbs' measure is for the parameters s, t, λ_0.

PROOF. We fix s, t. We first observe that the random function of λ given by

$$C(\lambda) = \frac{1}{N}\log\left(\sum_{\boldsymbol{\sigma},\boldsymbol{\tau}} \exp V_N(\boldsymbol{\sigma},\boldsymbol{\tau})\right)$$

is convex, and that $r_N = EB$ is also convex as a function of λ. Thus, from (2.14), (2.15), we have

$$\frac{\partial r_N}{\partial \lambda}(s,t,\lambda_0) = \varphi'(\lambda_0), \qquad \frac{\partial^2 r_N}{\partial \lambda^2}(s,t,\lambda_0) \le B. \tag{2.18}$$

Now [starting from (2.13)] one has

$$\frac{\partial^2 r_N}{\partial \lambda^2}(s,t,\lambda_0) = NE(\langle R(\boldsymbol{\sigma},\boldsymbol{\tau})^2\rangle - \langle R(\boldsymbol{\sigma},\boldsymbol{\tau})\rangle^2)$$

so that by (2.18),

$$E\langle (R(\boldsymbol{\sigma},\boldsymbol{\tau}) - \langle R(\boldsymbol{\sigma},\boldsymbol{\tau})\rangle)^2\rangle \le \frac{B}{N}. \tag{2.19}$$

Next,

$$C'(\lambda_0) = \langle R(\boldsymbol{\sigma},\boldsymbol{\tau})\rangle,$$

and by convexity of B, setting $a = N^{-1/4}$, we have

$$\frac{1}{a}(C(\lambda_0) - C(\lambda_0 - a)) \le C'(\lambda_0) \le \frac{1}{a}(C(\lambda_0 + a) - C(\lambda_0)). \tag{2.20}$$

We write

$$\begin{aligned}
&\frac{1}{a}(C(\lambda_0 + a) - C(\lambda_0) - a\varphi'(\lambda_0)) \\
&\quad \le \frac{1}{a}|C(\lambda_0 + a) - EC(\lambda_0 + a)| + \frac{1}{a}|C(\lambda_0) - EC(\lambda_0)| \\
&\qquad + \frac{1}{a}(EC(\lambda_0 + a) - EC(\lambda_0) - a\varphi'(\lambda_0)).
\end{aligned}$$

Now, by (2.14) to (2.16),

$$\begin{aligned}
EC(\lambda_0 + a) - EC(\lambda_0) - a\varphi'(\lambda_0) &= r_N(s,t,\lambda_0 + a) - r_N(s,t,\lambda_0) - a\varphi'(\lambda_0) \\
&\le \varphi(\lambda_0 + a) - \varphi(\lambda_0) - a\varphi'(\lambda_0) \\
&\le \frac{a^2}{2} B
\end{aligned}$$

and proceeding in a similar manner for the left-hand side of (2.20) we have

$$
\begin{aligned}
\big|\langle R(\boldsymbol{\sigma},\boldsymbol{\tau})\rangle - \varphi'(\lambda_0)\big| &= \big|C'(\lambda_0) - EC'(\lambda_0)\big| \\
&\le \frac{a}{2}C + \frac{1}{a}\big|C(\lambda_0+a) - EC(\lambda_0+a)\big| \\
&\quad + \frac{1}{a}\,\big|C(\lambda_0-a) - EC(\lambda_0-a)\big| \\
&\quad + \frac{1}{a}\,\big|C(\lambda_0) - EC(\lambda_0)\big|.
\end{aligned} \tag{2.21}
$$

Now, it follows from "concentration of measure" arguments (see, e.g., [7], Proposition 2.2) that

$$\forall\lambda, \qquad E\,\big|C(\lambda) - EC(\lambda)\big|^2 \le \frac{L}{N}(s+t),$$

where L is a number. Allowing as customary the value of L to change at each occurrence, we get from (2.21) that

$$E\,\big(\langle R(\boldsymbol{\sigma},\boldsymbol{\tau})\rangle - \varphi'(\lambda_0)\big)^2 \le L\biggl(a^2B^2 + \frac{1}{a^2N}(s+t)\biggr).$$

Recalling that $a = N^{-1/4}$ we get the result by combining with (2.19). □

To help the reader get the main idea of Proposition 1.3, we give now the central argument.

LEMMA 2.3. *Consider $s,t>0$. We assume the following*:

$$
\begin{aligned}
&\textit{Given any } \lambda\in\mathbf{R},\ \textit{there is a unique point } q(s,t,\lambda),\ \rho(s,t,\lambda) \\
&\textit{in } C(s,t,\lambda) \textit{ at which the infimum is obtained in } (1.20). \\
&\textit{The map } \lambda\to(q(s,t,\lambda),\rho(s,t,\lambda)) \textit{ is differentiable}; \\
&\textit{the map } \lambda\to\frac{d}{d\lambda}\rho(s,t,\lambda) \textit{ is continuous}.
\end{aligned} \tag{2.22}
$$

Consider $\lambda_0\in\mathbf{R}$, $0<\varepsilon<1$. We set

$$B = \sup_{|\lambda-\lambda_0|\le 1}\left|\frac{\partial\rho}{\partial\lambda}(s,t,\lambda)\right|, \tag{2.23}$$

$$N_0 = \frac{4L^2(1+4B^2+s+t)^2}{\varepsilon^2}, \tag{2.24}$$

where L is the constant occurring in (2.17).

Assume that for some $N\ge N_0$, we have

$$r_N(s,t,\lambda_0) = \varepsilon(1+3s) + (1+\varepsilon)\,RSC(s,t,\lambda_0), \tag{2.25}$$

(2.26) $$\forall\lambda\in\mathbf{R},\qquad r_N(s,t,\lambda)\le\varepsilon(1+3s)+(1+\varepsilon)\,RSC(s,t,\lambda).$$

Then there exist $s'<s$, $t'\ge t$, $\lambda'\in\mathbf{R}$ *with*

(2.27) $$r_N(s',t',\lambda')>\varepsilon(1+3s')+(1+\varepsilon)\,RSC(s',t',\lambda').$$

COMMENT. If $s\le s_0$, $t\ge t_0$, where $(s_0,t_0)\in S$, then (2.22) is a consequence of (1.21).

PROOF. The reader should keep in mind that s,t are fixed throughout the proof, and that ε plays a technical role, so that the main computations are better performed assuming $\varepsilon=0$ at first reading. In the course of this proof, we will use Lemma 2.3 for

$$\varphi(\lambda)=\varepsilon(1+3s)+(1+\varepsilon)RSC(s,t,\lambda).$$

By (1.22), we have

$$\psi(\lambda):=RSC(s,t,\lambda)=W\bigl(s,t,\lambda,q(\lambda),\rho(\lambda)\bigr),$$

where we write $q(\lambda)=q(s,t,\lambda)$, $\rho(\lambda)=\rho(s,t,\lambda)$, so that, since $(q(\lambda),\rho(\lambda))\in C(s,t,\lambda)$; we have

$$\begin{aligned}\psi'(\lambda)&=\frac{\partial W}{\partial\lambda}\bigl(s,t,\lambda,q(\lambda),\rho(\lambda)\bigr)\\&\qquad+q'(\lambda)\,\frac{\partial W}{\partial q}\bigl(s,t,\lambda,q(\lambda),\rho(\lambda)\bigr)+\rho'(\lambda)\frac{\partial W}{\partial\rho}\bigl(s,t,\lambda,q(\lambda),\rho(\lambda)\bigr)\\&=\frac{\partial W}{\partial\lambda}\bigl(s,t,\lambda,q(\lambda),\rho(\lambda)\bigr).\end{aligned}$$

Using (1.15), we get

$$\psi'(\lambda)=\frac{\partial A}{\partial x}\bigl(s\bigl(\rho(\lambda)-q(\lambda)\bigr)+\lambda,sq(\lambda)+t\bigr).$$

Since

$$\frac{\partial A}{\partial x}(x,sq+t)=\Phi(q,x),$$

[the relation behind (1.18)] we see that by (1.17) we have

$$\psi'(\lambda)=\rho(\lambda)$$

so that by (1.22), φ is twice differentiable, $\varphi''(\lambda)=(1+\varepsilon)\,\rho'(\lambda)$ and, by (2.23), $|\varphi''(\lambda)|\le 2B$ if $|\lambda-\lambda_0|\le 1$.

We get by (2.17) that

$$E\big\langle \big(R(\boldsymbol{\sigma},\boldsymbol{\tau}) - (1+\varepsilon)\,\rho_0\big)^2\big\rangle \le \frac{L}{\sqrt{N}}(1+B^2+s+t), \tag{2.28}$$

where $\rho_0 = \rho(\lambda_0)$.

This is the required ingredient to make a suitable version of the proof of Guerra's Theorem 1.1 work. We define $q_0 = q(\lambda_0)$. For $u > 0$, we define

$$t(u) = q_0(s-u) + t,$$
$$\lambda(u) = (s-u)(\rho_0 - q_0) + \lambda_0,$$

that are chosen so that the quantities

$$uq_0 + t(u), \qquad u(\rho_0 - q_0) + \lambda(u) \tag{2.29}$$

do not depend upon u. We consider the functions

$$f(u) = r_N\big(u, t(u), \lambda(u)\big),$$
$$\xi(u) = (1+\varepsilon) RSC\big(u, t(u), \lambda(u)\big).$$

An important observation is that

$$(q_0, \rho_0) \in C\big(u, t(u), \lambda(u)\big).$$

This is true for $u = s$, and the function Φ, Ψ of (1.17) depend upon s, t, λ, q only through $sq + t$ and $s(\rho - q) + \lambda$. For $s = u$, $t = t(u)$, $\lambda = \lambda(u)$, these quantities are independent of u. Thus

$$RCS\big(u, t(u), \lambda(u)\big) \le W\big(u, t(u), \lambda(u), q_0, \rho_0\big) \tag{2.30}$$

and all we have to show is that we can find $u < s$ such that

$$f(u) > \xi(u) + 3\varepsilon(u - s). \tag{2.31}$$

We observe that

$$\xi'(u) = \frac{(1+\varepsilon)}{2}\big((1-q_0)^2 + q_0^2 - \rho_0^2\big),$$

again using that the quantities (2.29) do not depend upon u. Now, Lemma 2.1 shows that

$$\begin{aligned} f'(s) &= \tfrac{1}{2}\big(1 + E\,\langle R(\boldsymbol{\sigma},\boldsymbol{\tau})^2\rangle - 2E\,\langle R\,(\boldsymbol{\sigma},\boldsymbol{\sigma}')^2\rangle\big) \\ &\qquad - q_0\big(1 + E\,\langle R\,(\boldsymbol{\sigma},\boldsymbol{\tau})^2\rangle - 2E\,\langle R\,(\boldsymbol{\sigma},\boldsymbol{\sigma}')\rangle\big) - (\rho_0 - q_0)\,E\,\langle R(\boldsymbol{\sigma},\boldsymbol{\tau})\rangle \\ &= \tfrac{1}{2}\big((1-q_0)^2 + q_0^2 - \rho_0^2\big) - E\big\langle\big(R(\boldsymbol{\sigma},\boldsymbol{\sigma}') - q_0\big)^2\big\rangle + \tfrac{1}{2}\,E\big\langle\big(R(\boldsymbol{\sigma},\boldsymbol{\tau}) - \rho_0\big)^2\big\rangle. \end{aligned}$$

Thus,

$$f'(s) = \frac{1+\varepsilon}{2}\big((1-q_0)^2 + q_0^2 - \rho_0^2\big) + \frac{1}{2}E\big\langle\big(R(\boldsymbol{\sigma},\boldsymbol{\tau}) - (1+\varepsilon)\rho_0\big)^2\big\rangle + \Delta,$$

where

$$\Delta = -\frac{\varepsilon}{2}\big((1-q_0)^2 + q_0^2 - \rho_0^2\big) - E\big\langle\big(R(\boldsymbol{\sigma},\boldsymbol{\sigma}') - q_0\big)^2\big\rangle$$
$$- (\varepsilon^2 + 2\varepsilon)\rho_0^2 + \varepsilon\rho_0 E\langle R(\boldsymbol{\sigma},\boldsymbol{\tau})\rangle \le \frac{3\varepsilon}{2}$$

using that $|\rho_0| \le 1$, by (1.17) (note that $|\Phi| \le 1$) and that $|R(\boldsymbol{\sigma},\boldsymbol{\tau})| \le 1$.

Using (2.28), we get

$$f'(s) \le \xi'(s) + \frac{L}{\sqrt{N}}\big(1 + 4B^2 + (s+t)\big) + \frac{3\varepsilon}{2}. \tag{2.32}$$

Thus, if $N \ge N_0$, we have

$$f'(s) \le \xi'(s) + 2\varepsilon$$

and the existence of u satisfying (2.27) follows. □

LEMMA 2.4. *Given s_0 and $\varepsilon > 0$, there exists a number $D(s_0,\varepsilon)$ such that*: $s \le s_0$, $t \ge 0$, $\lambda \in \mathbf{R}$,

$$r_N(s,t,\lambda) \ge \varepsilon(1+3s) + (1+\varepsilon)RSC(s,t,\lambda) \implies t,\ |\lambda| \le D(s_0,\varepsilon). \tag{2.33}$$

This lemma is technical, so it will be proved after we complete the proof of Proposition 1.3.

PROOF OF PROPOSITION 1.3. Given (s_0,t_0) in S, we consider the number $D(s_0,\varepsilon)$ given by (2.33). We consider

$$B_1 = 4\sup\left\{\left|\frac{\partial\rho}{\partial\lambda}(s,t,\lambda)\right|;\ s \le s_0,\ t_0 \le t \le D(s_0,\varepsilon),\ |\lambda| \le D(s_0,\varepsilon)+1\right\}. \tag{2.34}$$

It follows from (1.21) that $B_1 < \infty$. We consider

$$N_1 = \frac{4L^2(1 + 4B_1^2 + s_0 + D(s_0,\varepsilon))}{\varepsilon^2}, \tag{2.35}$$

where L is again the constant in (2.17).

Since ε is arbitrary, to prove Proposition 1.3, it suffices to show that

$$N \ge N_1, \lambda \in \mathbf{R} \implies r_N(s_0,t_0,\lambda) < \varepsilon(1+3s_0) + (1+\varepsilon)RSC(s_0,t_0,\lambda). \tag{2.36}$$

To prove (2.36) we assume for contradiction that for some $N \geq N_1$, some $\lambda \in \mathbf{R}$, we have

$$r_N(s_0, t_0, \lambda) \geq \varepsilon(1 + 3s_0) + (1 + \varepsilon)RSC(s_0, t_0, \lambda). \tag{2.37}$$

We then set

$$\begin{aligned} s = \inf\{&s' \leq s_0,\ \exists t' \geq t_0,\ \exists \lambda' \in \mathbf{R}, \\ &r_N(s', t', \lambda') \geq \varepsilon(1 + 3s') + (1 + \varepsilon)RSC(s', t', \lambda')\}. \end{aligned} \tag{2.38}$$

Thus, given $n > 0$, we can find $s(n) \leq s' + 2^{-n}$, $t(n) \geq t_0$, $\lambda(n) \in \mathbf{R}$ with

$$r_N\big(s(n), t(n), \lambda(n)\big) \geq \varepsilon\big(1 + 3s(n)\big) + (1 + \varepsilon)RSC\big(s(n), t(n), \lambda(n)\big). \tag{2.39}$$

Lemma 2.4 shows that $t(n) \leq D(s_0, \varepsilon)$, $|\lambda(n)| \leq D(s_0, \varepsilon)$, so that we can find a subsequence $t(n) \to t \geq t_0$, $\lambda(n) \to \lambda_0$. Thus, (2.39) and continuity show that

$$r_N(s, t, \lambda_0) \geq \varepsilon(1 + 3s) + (1 + \varepsilon)RSC(s, t, \lambda_0). \tag{2.40}$$

Next, we observe that $s > 0$. This is because [see (1.14), (1.15)] we have

$$RSC(0, t, \lambda) = A(t, \lambda_0) = r_N(0, t, \lambda_0).$$

The definition of s shows that if $0 \leq s' < s$, then

$$\forall \lambda \in \mathbf{R}, \qquad r_N(s', t, \lambda) \leq \varepsilon(1 + 3s') + (1 + \varepsilon)RSC(s', t, \lambda)$$

so that, by continuity

$$\forall \lambda \in \mathbf{R}, \qquad r_N(s, t, \lambda) \leq \varepsilon(1 + 3s) + (1 + \varepsilon)RSC(s, t, \lambda). \tag{2.41}$$

Since t, $|\lambda_0| \leq D(s_0, \varepsilon)$, $s \leq s_0$, we see that $N_1 \geq N_0$, where N_1 is given by (2.35) and N_0 by (2.24).

Since $N \geq N_1$, we have $N \geq N_0$. But then (2.27) contradicts the definition of s in (2.38). Thus (2.37) cannot hold. This proves (2.36) and Proposition 1.3.

We should note that if $(s_0, t_0) \in S$, the argument proves the uniformity of the limit when $|\lambda| \leq 1$ and (s, t) belong to a compact subset of S. □

PROOF OF LEMMA 2.4. We will prove the following three facts:

$$r_N(s, t, \lambda) \leq 2s + RSC(0, t, \lambda), \tag{2.42}$$

$$RSC(s, t, \lambda) \geq -3s + RSC(0, t, \lambda), \tag{2.43}$$

$$RSC(0, t, \lambda) \geq \frac{1}{L}(|\lambda| + t) - 1. \tag{2.44}$$

It follows from (2.42) and (2.43) that

$$r_N(s, t, \lambda) \leq 5s + RSC(s, t, \lambda)$$

and from (2.44) that $5s \le \varepsilon RSC(s,t,\lambda)$ if either t or $|\lambda|$ is large enough, so that (2.33) will follow.

To prove (2.42), we use Jensen's inequality to integrate in the r.v. g_{ij} inside the log rather than outside, and we obtain

$$r_N(s,t,\lambda) \le 2s + r_N(0,t,\lambda).$$

To prove (2.43), we recall that

$$r_N(0,t,\lambda) = RSC(0,t,\lambda) = E\log\bigl(4(\mathrm{ch}\lambda\,\mathrm{ch}^2 Y + \mathrm{sh}\lambda\,\mathrm{sh}^2 Y)\bigr), \tag{2.45}$$

for $Y = q\sqrt{t} + h$. We observe that

$$\mathrm{ch}\lambda\,\mathrm{ch}^2 Y + \mathrm{sh}\lambda\,\mathrm{sh}^2 Y = \tfrac{1}{2}(e^{\lambda}\,\mathrm{ch}2Y + e^{-\lambda}) \tag{2.46}$$

so that

$$RSC(0,t,\lambda) \ge \log(\mathrm{ch}\lambda)$$

and

$$RSC(0,t,\lambda) \ge \lambda - \log 2 + E\log \mathrm{ch}2Y \ge -|\lambda| - \log 2 + \frac{1}{L}\log\,\mathrm{ch}(\sqrt{t}+h)$$

and these imply (2.44). To prove (2.43), we note that

$$RS(s,t,\lambda) = W(s,t,\lambda,q,\rho),$$

for a certain choice $(q,\rho) \in C(s,t,\lambda)$. Since in particular $|\rho|, |q| \le 1$, we have

$$RS(s,t,\lambda) \ge -\frac{s}{2} + \log 4 + E\log\,(\mathrm{ch}x\,\mathrm{ch}^2 Y + \mathrm{sh}x\,\mathrm{sh}^2 Y), \tag{2.47}$$

where $x = s(\rho - q) + \lambda$, $Y = g\sqrt{sq+t} + h$. It follows from (2.46) that the right-hand side of (2.41) is a convex function of Y. In distribution, we have $Y = g_1\sqrt{sq} + g_2\sqrt{t} + h$, where g_1, g_2 are i.i.d. $N(0,1)$.

Using Jensen's inequality while integrating in g_1, we see that (2.47) remains true if we replace Y by $g\sqrt{t} + h$. This implies (2.43) since $|\rho - q| \le 2$. □

3. High temperature region.

Our first result demonstrates the strength of the information contained in Proposition 1.3.

LEMMA 3.1. *There exists a number L with the following property. Consider $(s,t) \in S$ and set*

$$\Delta_N = \Delta_N(s,t) = RS(s,t) - p_N(s,t). \tag{3.1}$$

Consider M such that (1.25) *holds for* $|\lambda| \le 1$. *Set*

$$\Delta_N^1 = \Delta_N^1(s,t) = \sup_{|\lambda|\le 1}\bigl(r_N(s,t,\lambda) - RSC(s,t,\lambda)\bigr). \tag{3.2}$$

Then, given $v > 0$, $v \le 2M$, we have

$$\begin{aligned} &P\biggl(G_N^{\otimes 2}\bigl(\{(\boldsymbol{\sigma},\boldsymbol{\tau}):|R(\boldsymbol{\sigma},\boldsymbol{\tau})-q|\ge v\}\bigr)\ge \exp N\biggl(2\Delta_N+\Delta_N^1-\frac{v^2}{8M^2}\biggr)\biggr) \\ &\qquad \le \exp\biggl(-\frac{N}{LM}\frac{v^4}{(s+t)^2}\biggr). \end{aligned} \tag{3.3}$$

COMMENT. We know from Section 2 that $\limsup_{N\to\infty}\Delta_N^1(s,t)\le 0$ since $(s,t)\in S$. If we also know that $\limsup_{N\to\infty}\Delta_N=0$ (which, at least "philosophically" means that $E\langle(R(\boldsymbol{\sigma},\boldsymbol{\tau})-q)\rangle^2\to 0$), then we get a strong exponential inequality on the overlap. Lemma 3.1 transforms a control of a second moment into an exponential inequality.

PROOF. The basic observation is that, recalling (1.4) and (1.12), we have

$$\log\langle\exp\lambda N R(\boldsymbol{\sigma},\boldsymbol{\tau})\rangle=\log\sum_{\boldsymbol{\sigma},\boldsymbol{\tau}}\exp V_N(\boldsymbol{\sigma},\boldsymbol{\tau})-2\log\sum_{\boldsymbol{\sigma}}\exp U_N(\boldsymbol{\sigma},\boldsymbol{\tau}). \tag{3.4}$$

Concentration of measure shows that, given $w > 0$

$$P\biggl(\biggl|\log\sum_{\boldsymbol{\sigma},\boldsymbol{\tau}}\exp V_N(\boldsymbol{\sigma},\boldsymbol{\tau})-E\log\sum_{\boldsymbol{\sigma},\boldsymbol{\tau}}\exp V_N(\boldsymbol{\sigma},\boldsymbol{\tau})\biggr|\ge\frac{Nw}{2}\biggr)\le\exp\biggl(-\frac{Nw^2}{L(s+t)}\biggr)$$

and similarly for U_N. Thus, with probability at least

$$1-\exp\biggl(-\frac{Nw^2}{L(s+t)}\biggr), \tag{3.5}$$

we have

$$\begin{aligned} &\log\langle\exp\lambda N R(\boldsymbol{\sigma},\boldsymbol{\tau})\rangle \\ &\quad\le N\bigl(w+r_N(s,t,\lambda)-2p_N(s,t)\bigr) \\ &\quad\le N\bigl(w+\Delta_N^1(s,t)+2\Delta_N(s,t)+RSC(s,t,\lambda)-2RS(s,t)\bigr). \end{aligned} \tag{3.6}$$

Using (1.25) we get that for $|\lambda|\le 1$,

$$\bigl\langle\exp\,\lambda N\bigl(R(\boldsymbol{\sigma},\boldsymbol{\tau})-q\bigr)\bigr\rangle\le\exp\,N(w+\lambda^2M+\Delta_N^1+2\Delta_N) \tag{3.7}$$

and thus, using the Chebyshev inequality,

$$G_N^{\otimes 2}\bigl(\{(\boldsymbol{\sigma},\boldsymbol{\tau}):|R(\boldsymbol{\sigma},\boldsymbol{\tau})-q|\ge v\}\bigr)\le\exp N(w+\lambda^2M+\lambda v_-+\Delta_N^1+2\Delta_N).$$

This holds with probability at least (3.5).

We now take $\lambda=v/2M$, $w=v^2/8M$ to see that

$$G_N^{\otimes 2}\bigl(\{(\boldsymbol{\sigma},\boldsymbol{\tau}):|R(\boldsymbol{\sigma},\boldsymbol{\tau})-q|\ge v\}\bigr)\le\exp\,N\biggl(2\Delta_N+\Delta_N^1-\frac{v^2}{8M}\biggr)$$

with probability at least (3.5). This is the result. □

COROLLARY 3.2. *Under the conditions of Lemma* 3.1, *we have*

$$E\langle (R(\boldsymbol{\sigma},\boldsymbol{\tau})-q)^2\rangle \le 16M^2(\Delta_N+\Delta_N^1)+\delta_N \tag{3.8}$$

where $\delta_N \to 0$ uniformly as s,t,M remain bounded.

PROOF. Since $|R(\boldsymbol{\sigma},\boldsymbol{\tau})-q|\le 2$, (3.3) implies that

$$E\langle (R(\boldsymbol{\sigma},\boldsymbol{\tau})-q)^2\rangle \le v^2+4\exp N\left(2\Delta_N+\Delta_N^1-\frac{v^2}{8M^2}\right) + 4\exp\left(-\frac{Nv^4}{LM(s+t)^2}\right).$$

We simply take v such that $v^2=16M^2((\Delta_N+\Delta_N^1)^+ + N^{-1/8})$, observing that $v\le 2M$ for large N. □

PROOF OF THEOREM 1.3. We consider $(s,t)\in S$, and q such that (1.8) holds. We consider

$$\Delta_N(u)=RS\big(u,t+q(s-u)\big)-p_N\big(u,t+q(s-u)\big).$$

We have shown in the course of the proof of Theorem 1.1 that

$$\frac{d}{du}\Delta_N(u)=E\langle (R(\boldsymbol{\sigma},\boldsymbol{\tau})-q)^2\rangle, \tag{3.9}$$

where the brackets are for the values $s(u)=u$, $t(u)=t+q(s-u)$ of the parameters.

By (1.21), there is a number M such that

$$\forall u\le s,\ \forall\lambda,\ |\lambda|\le 1,\qquad RSC(u,t(u),\lambda)-2RS(u,t(u))\le \lambda q+M\lambda^2.$$

Next, we have shown in Proposition 1.3 that if

$$\Delta_N^1=\sup_{\substack{u\le 1\\ |\lambda|\le 1}}\big(r_N(u,t(u),\lambda)-RSC(u,t(u),\lambda)\big),$$

then $\limsup_{N\to\infty}\Delta_N^1\le 0$. If we combine (3.9) with Corollary 3.2, we see that for some sequence $\varepsilon_N\to 0$, we have

$$\frac{d}{du}\Delta_N(u)\le 16M^2\big(\Delta_N(u)+\varepsilon_N\big)$$

so that

$$\frac{d}{du}\big(\Delta_N(u)+\varepsilon_N\big)\le 16M^2\big(\Delta_N(u)+\varepsilon_N\big)$$

and since $\Delta_N(0) = 0$, we have

$$\Delta_N(u) \le \varepsilon_N \exp 16M^2 u,$$

so that $\lim_{N\to\infty} \Delta_N(u) = 0$ for $u \le s$.

It remains to establish the last claim of Theorem 1.3, on how to get the "complete picture" of [6]. This is simply because, once we know that $\lim_{N\to\infty} \Delta_N = 0$, (3.2) shows that given $v > 0$, for N large enough,

$$E\, G_N^{\otimes 2}\big(\{(\boldsymbol{\sigma}, \boldsymbol{\tau}) : |R(\boldsymbol{\sigma}, \boldsymbol{\tau}) - q| \ge v\}\big) \le \exp\left(-\frac{N}{K}\right), \tag{3.10}$$

where K does not depend upon N. The main difficulty of the cavity method, namely the control of the error terms, vanishes, because under (3.4) we have relations such as

$$E\big\langle\big(R(\boldsymbol{\sigma}, \boldsymbol{\tau}) - q\big)^4\big\rangle \le v\, E\big\langle\big(R(\boldsymbol{\sigma}, \boldsymbol{\tau}) - q\big)^2\big\rangle + \exp\left(-\frac{N}{K}\right) \tag{3.11}$$

(within exponentially small error terms). □

REFERENCES

[1] GUERRA, F. (2000). Sum rules for the free energy of the spin glass model. Conference given at Les Houches, January 2000. To appear.

[2] LATALA, R. (2000). Private communication. August 2000.

[3] MÉZARD, M., PARISI, G. and VIRASORO, M. (1987). *Spin Glass Theory and Beyond.* World Scientific, Singapore.

[4] SCHCHERBINA, M. (1997). On the replica-symmetric solution for the SK model. *Helvetica Phys. Acta.* **70** 848–853.

[5] TALAGRAND, M. (1998). The Sherrington–Kirkpatrick model: a challenge to mathematicians. *Probab. Theory Related Fields* **110** 109–176.

[6] TALAGRAND, M. (2000). Exponential inequalities and replica-symmetry breaking for the Sherrington–Kirkpatrick model. *Ann. Probab.* **28** 1018–1062.

[7] TALAGRAND, M. (2002). *A First Course on Spin Glasses. École d'été de St. Flour 2000. Lecture Notes in Math.* Springer, Berlin. To appear.

[8] TALAGRAND, M. (2001). *Spin Glasses: a Challenge for Mathematicians*. Book in preparation.

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