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STABLE MEASURES AND SEMINORMS

BY ALEJANDRO DE ACOSTA
Universidad Nacional de La Plata, Argentina

The tail behavior of a stable measure with respect to a seminorm is determined. Bounds are obtained for the measure of small spheres. The geometric structure of the support of stable measures with index $p > 1$ is described.

1. Introduction. In this work we study the behavior of stable probability measures on small spheres and complements of large spheres determined by a seminorm on a vector space. We also obtain certain results on the topological support of stable measures.

Section 2 contains some background material on stable measures. In addition, the existence of mean values of stable measures of index $p > 1$ is proved. This generalizes the corresponding fact for Gaussian measures (see Rajput [9]).

In Section 3 we generalize to the case of a stable measure on a vector space and a seminorm the classical result of P. Lévy on the tail behavior of a stable distribution on the real line (see [5], page 182). For the particular case of the norm on a separable Hilbert space the result has been obtained by Kuelbs and Mandrekar [8]; their method of proof depends on a study of the domain of attraction of a stable measure, and ultimately on a representation of its characteristic functional. Our proof rests instead directly on basic principles; in any case, the representation is not known for more general vector spaces. Although technically quite different, our approach is in the same spirit as that used by Fernique [4] in proving his remarkable result on tail bounds for a Gaussian measure on a vector space with respect to a seminorm.

In Section 4 we obtain bounds for the measure of small spheres determined by a seminorm. We prove in particular that all negative powers of the norm are integrable with respect to any truly infinite dimensional stable measure on a separable Banach space.

In Section 5 we prove that the topological support of a symmetric stable measure of index $p > 1$ is a closed subspace. This extends the corresponding result for the Gaussian case (see Kuelbs [7] and Rajput [9]). We also obtain a description of the support of nonsymmetric stable measures ($p > 1$) and establish a relationship between the support and the characteristic functional of a stable measure ($p > 1$).

2. Stable measures. A general framework for the study of stable measures

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on vector spaces has been very recently established by Dudley and Kanter [3]. We state some definitions which are slight variants of those formulated in [3], and refer to [3] for proofs of the facts listed below.

Throughout the paper, unless explicitly stated otherwise, our basic framework will be a measurable space $(E, \mathscr{B})$, with E a real vector space and $\mathscr{B}$ the σ-algebra induced on E by a real vector space F in duality with E. It can be proved that $(E, \mathscr{B})$ has the following properties:

(1) The map $(x, y) \to x + y$ from $(E \times E, \mathscr{B} \otimes \mathscr{B})$ into $(E, \mathscr{B})$ is measurable.

(2) The map $(\lambda, x) \to \lambda x$ from $(R \times E, \mathscr{R} \otimes \mathscr{B})$ into $(E, \mathscr{B})$ is measurable (here $(R, \mathscr{R})$ is the real line with the Borel σ-algebra).

It follows that $\mathscr{B}$ is invariant under translations and homothecies.

Let X be an E-valued random vector (r.v.). The distribution of X will be denoted $\mathscr{L}(X)$.

DEFINITION 2.1. A probability measure (p.m.) μ on $(E, \mathscr{B})$ is *stable* if for every $\alpha > 0$ and for every $\beta > 0$, there exist $\gamma > 0$ and $z \in E$ such that:

(I) if X and Y are independent r.v.'s with $\mathscr{L}(X) = \mathscr{L}(Y) = \mu$, then

$$\mathscr{L}(\alpha X + \beta Y) = \mathscr{L}(\gamma X + z) .$$

The measure μ is called *strictly stable* if for every $\alpha > 0$, $\beta > 0$, the choice $z = 0$ is possible in (I).

Recall that, given a p.m. ν on $\mathscr{B}$, the p.m. $\tilde{\nu}$ is defined by $\tilde{\nu}(B) = \nu(-B)$ for $B \in \mathscr{B}$. A p.m. is *symmetric* if $\nu = \tilde{\nu}$. It can be proved that a symmetric stable p.m. is strictly stable.

We will assume that μ is nondegenerate, that is, μ is not a point mass. For every stable p.m. μ on $(E, \mathscr{B})$ there exists a number $p \in (0, 2]$ such that if α, β, γ are as in Definition 2.1, then $\gamma = (\alpha^p + \beta^p)^{1/p}$ ([3], Theorems 2 and 4). The number p is called the *index* of μ.

The following proposition is concerned with the existence of mean values of stable measures of index $p > 1$. It complements the results of Section 2 of [3]. Recall that, given a p.m. μ on $(E, \mathscr{B})$ such that $\int |\langle x, y\rangle| \mu(dx) < \infty$ for all $y \in F$, a point $z \in E$ is a *mean value* of μ if

$$\langle z, y\rangle = \int \langle x, y\rangle \mu(dx) \qquad \text{for all} \quad y \in F .$$

It is obvious that is F separates points of E and a mean value exists, then it is unique: we will denote it $m(\mu)$. Dudley and Kanter [3] call (E, F) a *semifull* pair if every sequentially $\sigma(F, E)$-continuous linear form on F is defined by an element of E. Examples of semifull pairs are (a) (F', F), where F is a metrizable topological vector space (see Dudley [2]); and (b) (E, E'), where E is a complete separable locally convex Hausdorff topological vector space (see Schaefer [10], page 150).

THEOREM 2.1. *Let (E, F) be a dual system, with E separated by F. Let $\mathscr{B}$ be the σ-algebra induced on E by F.*

(1) *Let μ be a stable measure on $(E, \mathscr{B})$ of index $p > 1$. Then $m(\mu)$ exists. If μ is strictly stable, then $m(\mu) = 0$.*

(2) *Suppose (E, F) is a semifull pair and let μ be a* p.m. *on $(E, \mathscr{B})$ such that $\mu \circ y^{-1}$ is stable of index $p > 1$ for all $y \in F$. Then $m(\mu)$ exists.*

PROOF. For each $y \in F$, $\mu \circ y^{-1}$ is a stable measure of index p in R. It is well known (see [5], page 182) that $\int |t| \mu \circ y^{-1}(dt) < \infty$ (we could also deduce this fact from Theorem 3.2). Let

$$\rho(y) = \int \langle x, y \rangle \mu(dx) \qquad \text{for} \quad y \in F .$$

Given $\alpha > 0$, $\beta > 0$, let $\gamma = \gamma(\alpha, \beta)$, $z = z(\alpha, \beta)$ be as in Definition 2.1. If X and Y are independent r.v.'s with $\mathscr{L}(X) = \mathscr{L}(Y) = \mu$, then for all $y \in F$

$$\begin{aligned} \mathscr{L}(\langle \alpha X + \beta Y, y \rangle) &= \mathscr{L}(\langle \gamma X + z, y \rangle) , \quad \text{and therefore} \\ E(\langle \alpha X + \beta Y, y \rangle) &= E(\langle \gamma X + z, y \rangle) \\ (\alpha + \beta)\rho(y) &= \gamma\rho(y) + \langle z, y \rangle \\ \rho(y) &= \langle (\alpha + \beta - \gamma)^{-1} z(\alpha, \beta), y \rangle . \end{aligned}$$

It follows that $(\alpha + \beta - \gamma)^{-1} z(\alpha, \beta)$ does not depend on (α, β) and is the mean value of μ. If μ is strictly stable, then $z(\alpha, \beta) = 0$, hence $m(\mu) = 0$.

Statement (2) is a consequence of Theorem 5 of [3], where it is proved that the assumptions imply that μ is stable of index p. □

REMARK. It is proved in [3], Theorem 4, that if μ is stable of index $p \neq 1$ then there exists a unique element $z_0 \in E$ such that $\mu \circ \theta_{z_0}^{-1}$ is strictly stable, where $\theta_{z_0}(x) = x + z_0$. Let us observe that if $p > 1$, then $m(\mu) = -z_0$. In fact,

$$0 = E(\langle X + z_0, y \rangle) = \langle m(\mu), y \rangle + \langle z_0, y \rangle \qquad \text{for all} \quad y \in F .$$

The next proposition, stated for easier reference, contains some elementary and essentially known properties of the characteristic functional

$$\hat{\mu}(y) = \int \exp i\langle x, y \rangle \mu(dx) , \qquad y \in F$$

of a symmetric stable measure.

PROPOSITION 2.1. *Let μ be a symmetric stable* p.m. *on $(E, \mathscr{B})$. Then $\hat{\mu} = \exp(-\psi)$, where $\psi : F \to R$ has the properties*:

(a) $\psi(0) = 0$, $\psi \geqq 0$, *and ψ is negative definite*
(b) $\psi(ty) = |t|^p \psi(y)$ *for all $t \in R$, $y \in F$*
(c) *ψ is sequentially $\sigma(F, E)$-continuous*
(d) $\{y : \psi(y) = 0\}$ *is a sequentially $\sigma(F, E)$-closed subspace of F.*

PROOF. Let $y \in F$. Then $\mu \circ y^{-1}$ is symmetric stable in R. Therefore there exists $\psi(y) \geqq 0$ such that

$$(\mu \circ y^{-1})\hat{}\,(t) = \exp(-\psi(y)|t|^p) ,$$

where p is the index of μ. Now

$$\hat{\mu}(ty) = (\mu \circ y^{-1})^{\wedge}(t) = \exp(-\psi(y)|t|^p) \quad \text{and}$$
$$\hat{\mu}(ty) = (\mu \circ (ty)^{-1})^{\wedge}(1) = \exp(-\psi(ty)),$$

Therefore $\psi(ty) = |t|^p \psi(y)$.

Since characteristic functionals are positive definite, it follows that $\exp(-s\psi)$ is positive definite for all $s \geqq 0$. Therefore ψ is negative definite (see e.g. [6], Chapter 3).

(c) follows from the $\sigma(F, E)$ sequential continuity of characteristic functionals.

To prove (d): in the condition of negative definiteness

$$\sum_{i,j=0}^{m} (\psi(x_i) + \overline{\psi(x_j)} - \psi(x_i - x_j))c_i \bar{c}_j \geqq 0,$$

for all nonnegative integers m, all $x_0, \cdots, x_m \in F$, all $c_0, \cdots, c_m \in C$, take $m = 1$, $c_0 = c_1 = 1$ and put $-x_1$ instead of x_1. One obtains

$$\begin{aligned} 0 &\leqq 2\psi(x_0) + 2(\psi(x_0) + \psi(x_1) - \psi(x_0 + x_1)) + 2\psi(x_1) \\ &= 4(\psi(x_0) + \psi(x_1)) - 2\psi(x_0 + x_1). \end{aligned}$$

It is clear from here that $\{y : \psi(y) = 0\}$ is closed under addition. Closure under scalar multiplication is obvious. □

3. The tails of a stable measure. In the following lemmas, μ is a strictly stable p.m. on $(E, \mathscr{B})$ of index p, X is a r.v. with $\mathscr{L}(X) = \mu$, and q is a measurable seminorm on E. Let $\gamma = 2^{1/p}$.

LEMMA 3.1. (a) *Let $s > 0$, $\varepsilon > 0$. Then*

$$P[q(X) > s] \geqq 2P[q(X) > \gamma s(1 + \varepsilon)]P[q(X) \leqq \gamma s\varepsilon].$$

(b) *Let $s > 0$, $0 < a < 1$, $b = 1 - a$. Then*

$$P[q(X) > s] \leqq 2P[q(X) > a\gamma s] + (P[q(X) > b\gamma s])^2.$$

PROOF. (a) Let X, Y be independent r.v.'s with $\mathscr{L}(X) = \mathscr{L}(Y) = \mu$. Then

$$\begin{aligned} P[q(X + Y) > \gamma s] &\geqq P[q(X) > \gamma s(1 + \varepsilon), q(Y) \leqq \gamma s\varepsilon] \\ &\quad + P[q(Y) > \gamma s(1 + \varepsilon), q(X) \leqq \gamma s\varepsilon] \\ &= 2P[q(X) > \gamma s(1 + \varepsilon)]P[q(X) \leqq \gamma s\varepsilon]. \end{aligned}$$

By the hypothesis of strict stability,

$$\begin{aligned} P[q(X + Y) > \gamma s] &= P[q(\gamma^{-1}(X + Y)) > s] \\ &= P[q(X) > s]. \end{aligned}$$

(b) Since

$$[q(X + Y) > t] \subset [q(X) > bt, q(Y) > bt] \cup [q(X) > at] \cup [q(Y) > at],$$

one has

$$P[q(X + Y) > t] \leqq 2P[q(X) > at] + (P[q(X) > bt])^2.$$

Now put $t = \gamma s$ and apply the strict stability of μ. □

LEMMA 3.2. *Let* $s > 0$, $\varepsilon_n > 0$ *for* $n = 1, 2, \cdots$. *Let*

$$\delta_1 = P[q(X) \leqq \gamma s \varepsilon_1],$$
$$\delta_n = P[q(X) \leqq \gamma^n s \prod_{i=1}^{n-1} (1 + \varepsilon_i)\varepsilon_n] \quad \text{for} \quad n \geqq 2.$$

Then for all $n \geqq 1$

$$P[q(X) > s] \geqq 2^n P[q(X) > \gamma^n s \prod_{i=1}^{n} (1 + \varepsilon_i)] \prod_{i=1}^{n} \delta_i.$$

PROOF. Using Lemma 3.1 (a) we have, inductively:

$$\begin{aligned} P[q(X) > s] &\geqq 2P[q(X) > \gamma s(1 + \varepsilon_1)]P[q(X) \leqq \gamma s \varepsilon_1] \\ &\geqq 2(2P[q(X)\gamma^2 s(1 + \varepsilon_1)(1 + \varepsilon_2)]P[q(X) \leqq \gamma^2 s(1 + \varepsilon_1)\varepsilon_2])\delta_1 \\ &= 2^2 P[q(X) > \gamma^2 s(1 + \varepsilon_1)(1 + \varepsilon_2)]\delta_1\delta_2 \\ &\vdots \\ &\geqq 2^n P[q(X) > \gamma^n s \prod_{i=1}^{n} (1 + \varepsilon_i)] \prod_{i=1}^{n} \delta_i. \end{aligned}$$

□

LEMMA 3.3. *For all* $a > 0$, $\beta > 1$,

$$\sum_{n=1}^{\infty} P[q(X) > a\beta^n] < \infty.$$

PROOF. It is clear that it is enough to prove the statement for the case $a = 1$. Choose k so that $\beta^k > \gamma$. Applying Lemma 3.1. (a) with $s = \beta^n$, $\varepsilon = \gamma^{-1}(\beta^k - \gamma)$, we obtain, for all $n \geqq 1$:

$$(1) \qquad P[q(X) > \beta^n] \geqq 2P[q(X) > \beta^{n+k}]P[q(X) \leqq \beta^{n+k} - \gamma\beta^n]$$

Choose N so that $\alpha = P[q(X) \leqq \beta^N(\beta^k - \gamma)] > \frac{1}{2}$. Given $n \geqq N$, write $n = N + dk + r$, with $0 \leqq r \leqq k - 1$. Then using (1), one has

$$\begin{aligned} P[q(X) > \beta^n] &= P[q(X) > \beta^{N+dk+r}] \\ &\leqq (2\alpha)^{-1}P[q(X) > \beta^{N+(d-1)k+r}] \\ &\vdots \\ &\leqq (2\alpha)^{-d}P[q(X) > \beta^{N+r}] \\ &\leqq (2\alpha)^{-d}P[q(X) > \beta^N]. \end{aligned}$$

Let $C = P[q(X) > \beta^N]$. Then

$$\begin{aligned} \sum_{n \geqq N} P[q(X) > \beta^n] &= \sum_{r=0}^{k-1} \sum_{d=0}^{\infty} P[q(X) > \beta^{N+dk+r}] \\ &\leqq \sum_{r=0}^{k-1} \sum_{d=0}^{\infty} C(2\alpha)^{-d} = kC(2\alpha)/(2\alpha - 1) < \infty. \end{aligned}$$

□

THEOREM 3.1. *Let* μ *be a stable* p.m. *on* $(E, \mathscr{B})$ *of index* p. *Let* q *be a measurable seminorm on* E. *Then*

(1) *There exists a constant* $C > 0$, *such that*

$$\mu\{x : q(x) > t\} \leqq Ct^{-p} \quad \text{for all} \quad t > 0.$$

(2) *Let* $p < 2$. *Suppose that there exists a measurable linear form* f *on* E *such that* (a) $|f| \leqq Kq$ *for some constant* K *and* (b) $\mu \circ f^{-1}$ *is nondegenerate. Then there exists a constant* $D > 0$, *such that*

$$\mu\{x : q(x) > t\} \geqq Dt^{-p} \quad \text{for all sufficiently large} \quad t.$$

PROOF. (1) First we prove the statement for strictly stable measures. Choose α so that $\gamma^{-1} < \alpha < 1$, and define $\varepsilon_i = \alpha^i$ for $i = 1, 2, \cdots$. Then $\rho = \prod_{i=1}^\infty (1 + \varepsilon_i) < \infty$. Choose s so that $P[q(X) \leqq s] > 0$. By Lemma 3.2,

$$(1) \qquad \begin{aligned} P[q(X) > s] &\geqq 2^n P[q(X) > \gamma^n s \textstyle\prod_{i=1}^n (1 + \varepsilon_i)] \prod_{i=1}^n \delta_i \\ &\geqq 2^n P[q(X) > \gamma^n s\rho] \textstyle\prod_{i=1}^n \delta_i' \end{aligned}$$

where $\delta_i' = P[q(X) \leqq s(\gamma\alpha)^i]$. By Lemma 3.3,

$$\textstyle\sum_{i=1}^\infty (1 - \delta_i') = \sum_{i=1}^\infty P[q(X) > s(\gamma\alpha)^i] < \infty ;$$

since also $\delta_i' \geqq P[q(X) \leqq s] > 0$ for all i, we have

$$C' = \textstyle\prod_{i=1}^\infty \delta_i' > 0 .$$

From (1) we obtain now: for all $n \geqq 1$

$$P[q(X) > s\rho\gamma^n] \leqq P[q(X) > s](2^n \textstyle\prod_{i=1}^n \delta_i')^{-1} \leqq C''2^{-n}$$

with $C'' = P[q(X) > s]/C'$. A simple interpolation gives the result now. Given $t > 0$, choose n so that $s\rho\gamma^n \leqq t < s\rho\gamma^{n+1}$ (we omit the trivial case of "small" values of t, that is, $t < s\rho\gamma$) and recall $\gamma = 2^{1/p}$. Then

$$P[q(X) > t] \leqq P[q(X) > s\rho\gamma^n] \leqq C''2^{-n} < Ct^{-p}$$

with $C = C''2(s\rho)^p$, because $t < s\rho 2^{(n+1)/p}$ implies $2^{-n} < 2(s\rho)^p t^{-p}$.

To obtain the result for a general stable p.m., we proceed as follows. Let X, Y be independent r.v.'s with $\mathscr{L}(X) = \mathscr{L}(Y) = \mu$. Then $\mathscr{L}(X - Y) = \mu * \bar{\mu}$ is a symmetric stable p.m. of index p, hence a strictly stable p.m. of index p. Then for $t > s$

$$P[q(X) > t]P[q(Y) \leqq s] = P[q(X) > t, q(Y) \leqq s] \leqq P[q(X - Y) > t - s] .$$

There exists a constant C such that for all $t > s$,

$$P[q(X - Y) > t - s] < C(t - s)^{-p} .$$

Therefore

$$P[q(X) > t] \leqq (P[q(X) \leqq s])^{-1}C(t - s)^{-p} .$$

It is clear now that by choosing an appropriate constant C_1 we obtain the inequality:

$$P[q(X) > t] < C_1 t^{-p} \qquad \text{for all} \quad t > 0 .$$

(2) Once more, we prove the statement for strictly stable measures first. We start by obtaining an inequality which corresponds to Lemma 3.2. Let $a_0 = 1$, $0 < a_j < 1$, $b_j = 1 - a_j$ for $j = 1, 2, \cdots$, and put $\varphi(t) = P[q(X) > t]$. Applying the inequality of Lemma 3.1 (b) successively to

$$P[q(X) > \gamma s a_1], \cdots, P[q(X) > \gamma^{n-1} s a_1 \cdots a_{n-1}] ,$$

we get

$$\text{(I)} \quad P[q(X) > s] \leqq 2^n P[q(X) > \gamma^n s a_1 \cdots a_n] + \textstyle\sum_{j=1}^n 2^{j-1}(\varphi(\gamma^j s a_0 \cdots a_{j-1} b_j))^2 .$$

Let $\alpha < 1$, and define $a_i = 1 - \alpha^i$, $i = 1, 2, \cdots$. Then $\rho = \prod_{i=1}^\infty a_i > 0$.

From (I) we obtain: for all $s > 0$, all $n \geqq 1$

(II) $$P[q(X) > s] \leqq 2^n P[q(X) > \gamma^n s\rho] + \sum_{j=1}^{n} 2^{j-1}(\varphi((\gamma\alpha)^j s\rho))^2 .$$

Let $h \in R$, $h \geqq 1$, and consider the statement:

(*) There exists a constant M such that $\varphi(t) \leqq Mt^{-hp}$ for all $t > 0$.

If (*) holds and one chooses α so that $\gamma^{(1/2h)-1} < \alpha < 1$, then for all $s > 0$, all $n \geqq 1$

$$\begin{aligned}\sum_{j=1}^{n} 2^{j-1}(\varphi((\gamma\alpha)^j s\rho))^2 &\leqq \tfrac{1}{2}M^2(s\rho)^{-2hp} \sum_{j=1}^{n} (2(\gamma\alpha)^{-2hp})^j \\ &\leqq M_1 s^{-2hp}, \qquad \text{where} \quad M_1 \quad \text{is a constant.}\end{aligned}$$

From (II) we may obtain now: for all $s > 0$, $n \geqq 1$

(III) $$P[q(X) > \gamma^n s\rho] \geqq 2^{-n}(P[q(X) > s] - M_1 s^{-2hp}) .$$

We shall prove now that if $p < 2$, then (*) cannot hold for $h > 1$. Assume, to the contrary, that (*) holds for some $h > 1$, and suppose that for some $s_0 > 0$, $A = P[q(X) > s_0] - M_1 s_0^{-2hp} > 0$. Then for all $n \geqq 1$,

$$\begin{aligned}M(\gamma^n s_0 \rho)^{-hp} &\geqq \varphi(\gamma^n s_0 \rho) \geqq A2^{-n} , \\ M(s_0\rho)^{-hp} 2^{-hn} &\geqq A2^{-n} , \quad \text{impossible.}\end{aligned}$$

Therefore $P[q(X) > s] \leqq M_1 s^{-2hp}$ for all $s > 0$. We may conclude: if (*) holds for some $h > 1$, then it holds for all $h' > 1$ and consequently $q(X)$ has moments of all orders. In particular, $E(q(X))^2 < \infty$. This fact and the nondegeneracy of $\mu \circ f^{-1}$ yield

$$0 < \operatorname{Var} f(X) < \infty .$$

On the other hand, if X, Y are independent r.v.'s with $\mathscr{L}(X) = \mathscr{L}(Y) = \mu$, we have

$$\begin{aligned}2^{2/p} \operatorname{Var} f(X) &= \operatorname{Var} f(2^{1/p}X) = \operatorname{Var} f(X + Y) \\ &= \operatorname{Var}(f(X) + f(Y)) = \operatorname{Var} f(X) + \operatorname{Var} f(Y) \\ &= 2 \operatorname{Var} f(X) .\end{aligned}$$

We have thus reached a contradiction.

By part (1) of this theorem, (*) is in fact true for $h = 1$. Therefore (III) is true for $h = 1$; also, for some s_0 we must have $A = P[q(X) > s_0] - M_1 s_0^{-2p} > 0$. The result follows by interpolation from the inequality

$$P[q(X) > \gamma^n s_0 \rho] \geqq A2^{-n} .$$

In order to pass to a general stable p.m., consider, as in part (1), independent r.v.'s X, Y with $\mathscr{L}(X) = \mathscr{L}(Y) = \mu$, and use the inequality

$$P[q(X) > t] \geqq \tfrac{1}{2}P[q(X - Y) > 2t] ,$$

which follows from

$$[q(X - Y) > 2t] \subset [q(X) > t] \cup [q(Y) > t] . \qquad \square$$

REMARKS. (1) Part (2) of Theorem 3.1 can be proved trivially by applying to $\mu \circ f^{-1}$ P. Lévy's result on the tail behavior of one-dimensional stable measures. However, we believe that the proof we have given is of interest, since it is elementary and direct.

(2) The theorem remains true if one replaces the hypothesis "μ is a stable p.m." by the slightly weaker one "μ is a p.m. such that $\mu \circ y^{-1}$ is stable for all $y \in F$." In fact, if μ is a p.m. satisfying the latter condition, then $\mu * \bar{\mu}$ is symmetric stable (see [3], Section 2).

(3) If $p = 2$, then μ is Gaussian; in this case the tail behavior is very different and requires a separate treatment. This has been done by Fernique [4]. Incidentally, Remark (1) above could be applied trivially to obtain an inequality of the Fernique type in the opposite direction.

From Theorem 3.1 we obtain at once

THEOREM 3.2. *Let μ be a stable* p.m. *on $(E, \mathscr{B})$ of index p. Let q be a measurable seminorm on E. Then*

(1) *For every $r < p$, $\int q^r \, d\mu < \infty$.*

(2) *Let $p < 2$, and assume that q satisfies the condition of Theorem* 3.2(2). *Then*

$$\int q^r \, d\mu = \infty \qquad \text{for every} \quad r \geqq p .$$

4. The behavior of a stable measure on small spheres. Let μ be a stable p.m. By Proposition 2.1 we have $(\mu * \bar{\mu})^\wedge = \exp(-\psi)$, and therefore $|\hat{\mu}| = ((\mu * \bar{\mu})^\wedge)^{\frac{1}{2}} = \exp(-\frac{1}{2}\psi)$.

THEOREM 4.1. *Let μ be a stable* p.m. *on $(E, \mathscr{B})$, and let $N = \{y : \psi(y) = 0\}$. Let q be a measurable seminorm on E, and let G be the subspace of all q-continuous elements of F.*

Suppose $0 \leqq r \leqq \operatorname{codim}_G (G \cap N)$. Then there exists a constant $M > 0$, such that

$$\mu\{x : q(x - z) \leqq \tau\} \leqq M\tau^r \qquad \text{for all} \quad z \in E, \quad \text{for all} \quad \tau > 0 .$$

PROOF. Let n be a positive integer such that $r \leqq n \leqq \operatorname{codim}_G (G \cap N)$. Choose a linearly independent set $\{f_1, \cdots, f_n\} \subset G$ such that

$$\operatorname{Span}\{f_1, \cdots, f_n\} \cap N = \{0\} .$$

Let $\pi = (f_1, \cdots, f_n) : E \to R^n$, and put $\Phi(t_1, \cdots, t_n) = \frac{1}{2}\psi(\sum_{i=1}^n t_i f_i)$ for $(t_1, \cdots, t_n) \in R^n$. We have

$$\begin{aligned} |(\mu \circ \pi^{-1})^\wedge(t_1, \cdots, t_n)| &= |\hat{\mu}(\textstyle\sum_{i=1}^n t_i f_i)| \\ &= \exp(-\Phi(t)) \qquad t = (t_1, \cdots, t_n) . \end{aligned}$$

Now Φ is a continuous function such that $\Phi(t) > 0$ for $t \neq 0$ and $\Phi(\lambda t) = |\lambda|^p \Phi(t)$. Therefore

$$\int \exp(-\Phi(t)) \, dt < \infty ,$$

and by the Fourier inversion theorem it follows that $\mu \circ \pi^{-1}$ has a bounded continuous density ρ.

Since each f_i is q-continuous, there exist constants c_i such that $|f_i| \leqq c_i q$, and consequently

$$\sum_{i=1}^{n} |f_i| \leqq cq\,, \qquad \text{with} \quad c = \sum_{i=1}^{n} c_i\,.$$

Let $||s|| = \sum_{i=1}^{n} |s_i|$ for $s = (s_1, \cdots, s_n) \in R^n$, let ν be Lebesgue measure on R^n, and let $K = \sup \rho$. We have

$$\begin{aligned}
\mu\{x : q(x - z) \leqq \tau\} &\leqq \mu\{x \in E : \textstyle\sum_{i=1}^{n} |\langle x - z, f_i\rangle| \leqq c\tau\} \\
&= \mu \circ \pi^{-1}\{s \in R^n : ||s - s_0|| \leqq c\tau\}\,, \\
&\qquad\qquad s_0 = (\langle z, f_1\rangle, \cdots, \langle z, f_n\rangle) \\
&= \textstyle\int_{||s-s_0||\leqq c\tau} \rho(s)\nu(ds) \leqq K\nu\{s : ||s - s_0|| \leqq c\tau\} \\
&\leqq M\tau^n \qquad \text{for some constant} \quad M\,. \qquad \square
\end{aligned}$$

As a corollary we obtain:

THEOREM 4.2. *Let (E, q) be a separable Banach space, and let $\mathscr{B}$ be the Borel σ-algebra of E.*

Let μ be a stable measure on $(E, \mathscr{B})$ such that $\mu(S) = 0$ for every finite-dimensional subspace S of E. Then for every $r > 0$, there exists a constant $C > 0$, such that

$$\int (q(x - z))^{-r}\mu(dx) \leqq C \qquad \textit{for all} \quad z \in E\,.$$

PROOF. Recall that $\mathscr{B}$ coincides with the σ-algebra induced on E by $F = E'$. In the present situation $G = F$, so $\operatorname{codim}_G (G \cap N) = \operatorname{codim} N$. Suppose $\operatorname{codim} N = n$, n a positive integer. Then $\dim N^{\perp} = n$. But $\mu * \bar{\mu}(N^{\perp}) = 1$ by the argument in Theorem 5.2, so we have reached a contradiction. Therefore $\operatorname{codim} N$ is infinite and the result follows from Theorem 4.1 by a standard argument. $\square$

REMARK. Theorems 4.1 and 4.2 are true under the weaker hypothesis "μ is a p.m. such that $\mu \circ y^{-1}$ is stable for all $y \in F$" (see Remark (2) following Theorem 3.1).

5. The topological support of a stable measure. Let H be a Hausdorff topological space, $\mathscr{F}$ the Borel σ-algebra of H, that is, the σ-algebra generated by the open subsets of H. The *support* of a p.m. μ on $(H, \mathscr{F})$ is the set

$$S(\mu) = \{x \in H : \mu(U) > 0 \text{ for every open set } U \text{ containing } x\}\,.$$

It is well known and easily proved that $S(\mu)$ is closed and that if D is closed and $\mu(D) = 1$, then $S(\mu) \subset D$. The notion of support is of interest when $\mu(S(\mu)) = 1$; this is true if the topology of H has a countable base, or if μ is a regular Borel measure. However, no assumption ensuring that $S(\mu)$ has full measure is needed in Theorem 5.1.

The following proposition describes the geometric structure of the support of a stable p.m. of index $p > 1$ (the simple method used also yields some information on the case $p \leqq 1$; however, it does not seem satisfactory). Let us call a (nonempty) subset T of E a *truncated cone* if (a) $T + T \subset T$ and (b) $\lambda T \subset T$

for all $\lambda \geqq 1$. It is easily proved that if T is a truncated cone, then $T - T$ is a subspace. It follows that if $T = -T$, then T is a subspace.

THEOREM 5.1. *Let E be a Hausdorff topological vector space. Let $\mathscr{B}$ be the Borel σ-algebra of E, and assume that $\mathscr{B}$ is the σ-algebra induced on E by a vector space F in duality with E.*

Let μ be a stable p.m. *on $(E, \mathscr{B})$ of index $p > 1$. Assume $S(\mu) \neq \phi$. Then*

(a) *$S(\mu) = m(\mu) + T$, where T is a closed truncated cone*
(b) *If μ is symmetric, then $S(\mu)$ is a closed subspace.*

PROOF. By the remark following Theorem 2.1, it is enough to prove (a) for strictly stable measures. Let μ be such a measure, and let X, Y be independent r.v.'s with $\mathscr{L}(X) = \mathscr{L}(Y) = \mu$.

Let V be an open, balanced neighborhood of 0, and let U be an open, balanced neighborhood of 0 such that $U + U \subset V$.

Let $\alpha > 0$, $\beta > 0$ be such that $\alpha^p + \beta^p = 1$. Then for all $x \in E$, $y \in E$,

$$\begin{aligned} P[X \in x + U]P[Y \in y + U] &= P[X \in x + U, Y \in y + U] \\ &\leqq P[\alpha X + \beta Y \in (\alpha x + \beta y) + (\alpha U + \beta U) \\ &\leqq P[\alpha X + \beta Y \in (\alpha x + \beta y) + V] . \end{aligned}$$

Since $\mathscr{L}(\alpha X + \beta Y) = \mu$, we have

$$\mu(x + U)\mu(y + U) \leqq \mu((\alpha x + \beta y) + V) . \tag{1}$$

Take $x = y$ in (1); then

$$[\mu(x + U)]^2 \leqq \mu((\alpha + \beta)x + V) . \tag{2}$$

Since $\{\alpha + \beta \mid \alpha^p + \beta^p = 1\} = (1, 2^{1/q}]$, with $(1/p) + (1/q) = 1$, we may conclude from (2): if $x \in S(\mu)$, then $\lambda x \in S(\mu)$ for all λ with $1 \leqq \lambda \leqq 2^{1/q}$. But this in turn implies $\lambda x \in S(\mu)$ for all $\lambda \geqq 1$.

Take now $\alpha = \beta = 2^{-1/p}$ in (1); then $x \in S(\mu)$, $y \in S(\mu)$ implies $2^{-1/p}(x + y) \in S(\mu)$. By the conclusion of the previous paragraph, this implies $x + y \in S(\mu)$.

If μ is symmetric, then $S(\mu) = -S(\mu)$. By the remarks preceding the theorem, it follows that $S(\mu)$ is a closed subspace. □

Given a stable p.m. μ on $(E, \mathscr{B})$ let ψ be such that $(\mu * \tilde{\mu})^\wedge = \exp(-\psi)$ (Proposition 2.1).

THEOREM 5.2. *Let (E, F) be a dual system of vector spaces. Assume that $\mathscr{B}$ coincides with the σ-algebra generated by the weak topology $\sigma(E, F)$.*

Let μ be a $\sigma(E, F)$-regular stable p.m. *on $(E, \mathscr{B})$ of index $p > 1$. Let $N = \{y : \psi(y) = 0\}$. Then*

(a) $\overline{S(\mu) - S(\mu)} = N^\perp$
(b) *If μ is symmetric, then $S(\mu) = N^\perp$.*

PROOF. It can be proved easily that, given a $\sigma(E, F)$-regular p.m. ν on $(E, \mathscr{B})$

$$[S(\nu)]^{\perp} = \{y \in F : \hat{\nu}(ty) = 1 \quad \text{for all} \quad t \in R\}$$

(see [1], page 275). If ν is a symmetric stable p.m. on $(E, \mathscr{B})$, then it follows that $[S(\nu)]^{\perp} = N$, and consequently $[S(\nu)]^{\perp\perp} = N^{\perp}$. Since $S(\nu)$ is a closed subspace, we have $[S(\nu)]^{\perp\perp} = S(\nu)$. This proves (b).

For a stable measure μ, we have $S(\mu * \tilde{\mu}) = N^{\perp}$. But $S(\mu * \tilde{\mu}) = \overline{S(\mu) - S(\mu)}$ by an elementary property of supports. □

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ESPAÑA 1938
FLORIDA, FCNGBM.
PROVINCIA DE BUENOS AIRES
ARGENTINA