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EXIT SYSTEMS[1]

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We associate with a strong Markov process (X_t) and a Borel set B an "exit system." This system provides the structure of the excursions from B of the process (X_t) and gives a new approach to the recent results of Getoor and Sharpe on last exit decompositions and last exit distributions.

1. Introduction. Given a strong Markov process and a regular point x_0, Itô in [6] has defined the process of the excursions from x_0, using a local time at x_0, and has shown that this process is a Poisson point process characterized by a certain entrance law for the semi-group killed at the hitting time of x_0. But for the excursions from a general Borel set B we no longer have the independence properties and do not always have a local time on B. The structure of these excursions has been deduced in [11] from a recent work of Getoor and Sharpe [5], and in [13] from the existence of a Lévy system of the incursion process. Here we give a proof of these results without using the techniques of [5] and [11] (for example raw balayage, well measurable projection of nonadapted additive functionals) and without using the theory of Lévy system which was somewhat difficult to apply to the incursion process. The essential tool will be the "exit system" defined in Section 4. For the results of Getoor and Sharpe [5] about last exit conditional distributions, we give a strong Markov version. We also generalize a result of Meyer, Smythe, Walsh [15]: if L^a is the beginning of the first excursion with length $\geqq a$ ($a \in]0, \infty]$), the process $(X_{L^a+t})_{0<t<a}$ is a nonhomogeneous Markov process; the transition function is related to the semigroup killed at the hitting time of B.

2. General notations and definitions.

(2.1) We shall work with the canonical right continuous realization $(\Omega, \mathscr{F}, \mathscr{F}_t, X_t, \theta_t, P^\mu)$ of a semi-group (P_t) on E that satisfies the "hypothèses droites" of Meyer (the excessive functions are nearly Borel and right continuous on the paths of the process (X_t)). The state space E is compact metric. $\mathscr{B}$, $\mathscr{B}^*$ denote its Borel and universal σ-fields.

We assume that δ is an absorbing state of E and that $\zeta = \inf\{t: X_t = \delta\} = \infty$ P^x-a.s. for each $x \neq \delta$. Actually the point δ is only needed for the definition of the excursions (Section 6).

Let $\mathscr{F}^0 = \sigma\{X_s, s \geqq 0\}$ and let $\mathscr{F}^*$ be the universal completion of $\mathscr{F}^0$: a

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set belongs to $\mathscr{F}^*$ if and only if it belongs to the P-completion of $\mathscr{F}^0$, for all probability measures P on $(\Omega, \mathscr{F}^0)$. One has $\mathscr{F}^0 \subset \mathscr{F}^* \subset \mathscr{F}$.

Let $\mathscr{F}_t^0 = \sigma\{X_s, 0 \leqq s \leqq t\}$. The usual completion procedure with respect to the measure P^μ provides a family $(\mathscr{F}_t^\mu)$ that satisfies the "usual" conditions of [2]. Unless otherwise stated the notions of stopping time, well measurable or predictable processes are taken with respect to all families $(\mathscr{F}_t^\mu)$. For questions of the general theory of processes the reader is referred to Dellacherie [2].

(2.2) We now give a *closed random set* M, *homogeneous in* $]0, \infty[$ *and well measurable*. By random set we mean a mapping from Ω to the set of all subsets of $\mathbb{R}_+$. The random set M is said to be closed if $M(\omega)$ is closed for each ω, and to be homogeneous in $]0, \infty[$ if

$$(M - t) \cap]0, \infty[= (M \circ \theta_t) \cap]0, \infty[, \qquad t \geqq 0 .$$

The canonical example of such a set is $M = \overline{\{t : X_t \in B\}}$, where B is a nearly Borel set.

We associate with M the following random variables

$$R = \inf\{s > 0 : s \in M\} \quad (\inf \phi = +\infty)$$
$$R_t = R \circ \theta_t$$
$$D_t = t + R_t = \inf\{s > t : s \in M\} .$$

Recall that (D_t) is a right continuous, increasing process (such that $D_0 = R$) and that D_t is a stopping time for all t. The family $(\hat{\mathscr{F}}_t) = (\mathscr{F}_{D_t})$ is then right continuous. As a result of [8], Chapter 2, for each $(\hat{\mathscr{F}}_t)$-stopping time S, D_S is a stopping time and $\hat{\mathscr{F}}_S \subset \mathscr{F}_{D_S}$.

In view of [10], one may assume that R is $\mathscr{F}^*$-measurable.

3. The set G.

(3.1) DEFINITION. Let F be the set of the points x such that $P^x\{R = 0\} = 1$. We define the random sets

$$G = \{t > 0 : R_{t-} = 0, R_t > 0\}$$
$$G^r = \{t \in G : X_t \in F\}$$
$$G^i = \{t \in G : X_t \notin F\} .$$

Note that (R_t) has left limits, since (D_t) is increasing, and that $\{t > 0 : R_{t-} = 0\} = M \backslash \{0\}$. Therefore $G(\omega)$ is the set of the left endpoints in $]0, \infty[$ of the intervals contiguous to $M(\omega)$ (maximal in the complement of $M(\omega)$).

The set F is nearly Borel and $\{t : X_t \in F\}$ is well measurable. G, G^r, G^i are $(\hat{\mathscr{F}}_t)$-well measurable by definition, and $(\mathscr{F}_t)$-progressively measurable by ([2] Chapter VI, Theorem 2).

The following result is due to Meyer [11]. We shall give a new proof using the family $(\hat{\mathscr{F}}_t)$. Recall that, for a function $T : \Omega \to [0, \infty]$, $[T]$ denotes the random set defined by $[T](\omega) = \{T(\omega)\}$ if $T(\omega) < \infty$, ϕ otherwise. $[T]$ is called the graph of T.

(3.2) PROPOSITION. (i) *G^i is well measurable.*

(ii) $G^r \cap [T] = \phi$ a.s. *for each stopping time T, and so G^r is progressively measurable but not well-measurable, unless it is empty.*

PROOF. (i) It is sufficient to show that $G^i = M^i$ a.s. where M^i is the well-measurable set $\{t > 0: R_{t-} = 0, X_t \notin F\}$. Let S be an $(\hat{\mathscr{F}}_t)$-stopping time such that $[S] \subset M^i$. By the Markov property at D_S, $X_S \in F$ a.s. on $\{D_S = S < \infty\}$, therefore $R_S > 0$ a.s. and $[S] \subset G$ a.s. The section theorem applied to the $(\hat{\mathscr{F}}_t)$-well-measurable sets M^i and G implies $M^i \subset G$, or $G^i = M^i$ a.s.

(ii) Let T be a stopping time. By the strong Markov property, one has $R_T = 0$ a.s. on $\{X_T \in F, T < \infty\}$ and $G^r \cap [T] = \emptyset$ a.s. □

(3.3) COROLLARY. *For all positive well-measurable processes Z and all positive $\mathscr{F}^*$-measurable functions f*

$$E^\bullet[\textstyle\sum_{s \in G^i} Z_s f \circ \theta_s] = E^\bullet[\sum_{s \in G^i} Z_s E^{X_s}(f)] \, . \tag{3.4}$$

PROOF. G^i is well-measurable and has countable sections. By a theorem of Dellacherie, $G^i = \bigcup [T_n]$, where (T_n) is a sequence of stopping times. One may assume that the graphs $[T_n]$ are disjoint. For each $n Z_{T_n}$ is $\mathscr{F}_{T_n}$-measurable, and (3.4) follows from the Markov property at times T_n. □

(3.5) REMARK. Equation (3.4) would not be true with G^r instead of G^i: for $f = I_{\{R>0\}}$ (recall that R is $\mathscr{F}^*$-measurable), the left side in general is strictly positive, but the right side vanishes.

4. The exit system. For a measure μ, the integral $\int f d\mu$ will be denoted by $\mu(f)$.

(4.1) THEOREM. *There exist a continuous additive functional K, with 1-potential $\leqq 1$, carried by F, and a kernel $\hat{P}$ from $(E, \mathscr{B}^*)$ to $(\Omega, \mathscr{F}^*)$ satisfying $\hat{P}^x\{R = 0\} = 0$ and $\hat{P}^x(1 - e^{-R}) \leqq 1$ for all x such that the following equality*

$$E^\bullet[\textstyle\sum_{s \in G^r} Z_s f \circ \theta_s] = E^\bullet[\int_0^\infty Z_s \hat{P}^{X_s}(f) \, dK_s] \tag{4.2}$$

holds for all positive well-measurable processes Z and all positive $\mathscr{F}^*$-measurable functions f.

(4.3) REMARKS. (i) The condition $\hat{P}^x(1 - e^{-R}) \leqq 1$ implies $\hat{P}^x\{R > t\} < \infty$ for all $t > 0$, and, since $\hat{P}^x$ is carried by $\{R > 0\}$, it is *σ-finite*. The kernel N defined by $N^\bullet(f) = \hat{P}^\bullet((1 - e^{-R})f)$ is submarkovian ($N^\bullet(1) \leqq 1$).

(ii) It is sufficient to require (4.2) for positive *predictable* Z. In fact a positive, bounded, well-measurable Z is different from its predictable projection only on a countable union of graphs of stopping times (see [2] Chapter V, Theorem 19); such a set is not charged by any continuous additive functional and has a.s. no common point with G^r, by Proposition (3.2).

PROOF. (i) For each positive, bounded, $\mathscr{F}^0$-measurable f let us define

$$A_t^f = \textstyle\sum_{t \in G^r, s \leqq t} e^{-s}(1 - e^{-R_s}) f \circ \theta_s \, .$$

A^f is an increasing, non-adapted process, bounded by $\|f\|_\infty$ (note that $\sum_{s\in G} e^{-s}(1 - e^{-R_s}) = \sum_{s\in G} \int_s^{D_s} e^{-u}\, du \leqq 1$). Furthermore $A^f_{T_n} \uparrow A_T{}^f$ a.s. for each sequence $T_n \uparrow T$ of stopping times, since A^f jumps only on G^r and $G^r \cap [T] = \emptyset$ a.s. by Proposition (3.2). By the homogeneity of the set G^r, the function $v_f = E^\bullet(A_\infty{}^f)$ therefore is a regular bounded 1-potential. Consequently there exists a (unique) additive functional K^f such that $v_f = E^\bullet(\int_0^\infty e^{-s}\, dK_s{}^f)$. According to a theorem of Meyer, one has for all positive predictable Z

$$(4.4)\qquad E^\bullet[\textstyle\sum_{s\in G^r} e^{-s} Z_s(1 - e^{-R_s}) f \circ \theta_s] = E^\bullet[\int_0^\infty e^{-s} Z_s\, dK_s{}^f]\,.$$

Equation (4.4) extends to all positive well measurable Z by a previous remark.

(ii) $K = K^1$ is a continuous additive functional with 1-potential $\leqq 1$, and carried by F, since for $Z_s = I_{F^c} \circ X_s$ the left side of (4.4) vanishes.

(iii) In order to define $\hat{P}$, we first show that $K^f \ll K$, that is, for any positive $\mathscr{B}$-measurable g,

$$E^\bullet[\textstyle\int_0^\infty e^{-s} g \circ X_s\, dK_s] = 0 \Rightarrow E^\bullet[\int_0^\infty e^{-s} g \circ X_s\, dK_s{}^f] = 0\,.$$

But this follows from (4.4), with $Z_s = g \circ X_s$ and $f = 1$, then $f = f$. As a consequence of a theorem of Motoo, extended without hypothesis (L) (see [1] or [11]), there exists a positive, finite, universally measurable density Nf of K^f with respect to K. By a "classical" argument using the fact that Ω is universally measurable in a compact metric space (see Appendix), we get a submarkov kernel N from $(E, \mathscr{B}^*)$ to $(\Omega, \mathscr{F}^*)$ such that

$$(4.5)\qquad E^\bullet[\textstyle\sum_{s\in G^r} e^{-s} Z_s((1 - e^{-R}) f) \circ \theta_s] = E^\bullet[\int_0^\infty e^{-s} Z_s N^{X_s}(f)\, dK_s]$$

for all positive well measurable Z and $\mathscr{F}^0$-measurable f. If we let $Z \equiv 1$ and $f = I_{\{R=0\}}$ in (4.5), we obtain $N^x\{R = 0\} = 0$ for K-a.e. x.[2] One may choose the kernel N such that $N^x\{R = 0\} = 0$ for all x.

Define $\hat{P}^x(f) = N^x(f/1 - e^{-R})$. Then $\hat{P}$ is a kernel from $(E, \mathscr{B}^*)$ to $(\Omega, \mathscr{F}^*)$ satisfying $\hat{P}^x\{R = 0\} = 0$, $\hat{P}^x(1 - e^{-R}) \leqq 1$ for all x. Replacing Z_s by $e^s Z_s$ and f by $f/1 - e^{-R}$ in (4.5) we get (4.2) and the proof is complete. □

(4.6) REMARKS. (i) The couple $(K, \hat{P})$ constructed in the proof of Theorem (4.1) satisfies more precisely

$$(4.7)\qquad E^\bullet[\textstyle\int_0^\infty e^{-s}\, dK_s) = E^\bullet[\sum_{s\in G^r} e^{-s}(1 - e^{-R_s})]\,,$$

$$(4.8)\qquad \hat{P}^x(1 - e^{-R}) = 1 \qquad \text{for } K\text{-a.e. } x\,.$$

(Write (4.4) and (4.5) with $Z \equiv 1, f \equiv 1$.)

(ii) We give here a *useful extension of* (4.2): *for all positive $\mathscr{F}^0 \times \mathscr{B}_{R_+} \times \mathscr{F}^0$-measurable or universally measurable functions F one has*

$$(4.9)\qquad E^\bullet[\textstyle\sum_{s\in G^r(\omega)} Z_s(\omega) F(k_s\omega, s, \theta_s\omega)] = E^\bullet[\int_0^\infty Z_s(\omega)(\int F(k_s\omega, s, \omega')\hat{P}^{X_s(\omega)}(d\omega'))\, dK_s(\omega)]\,,$$

[2] Except on a set of K-potential 0.

where k_s is the killing operator at s (see Section 6). It is enough to prove (4.9) for $F(\omega, s, \omega') = e^{-s}H(\omega, s, \omega')(1 - e^{-R(\omega')})$. In the case $H(\omega, s, \omega') = h(\omega)g(s)f(\omega')$, for positive f, g, h, (4.9) follows from (4.2). The general case is obtained by a monotone class argument. Notice that one cannot apply a monotone class argument directly to (4.9), since both sides are not bounded functionals with respect to F.

(4.10) DEFINITION. Let us define the random measures

$$dJ_t = \sum_{s \in G^i} \varepsilon_s(dt)$$
$$dB_t = dJ_t + dK_t$$

and let us change the definition of $\hat{P}$ by setting

$$\hat{P}^x = P^x \qquad \text{if} \quad x \notin F .$$

The couple $(dB_t, \hat{P})$ will be called the exit system of the set M.

Note that the measure dB_t is homogeneous, that $\hat{P}^x\{R = 0\} = 0$ and $\hat{P}(1 - e^{-R}) \leqq 1$ for all x, that (4.2) remains true with this new kernel $\hat{P}$ and that by adding (3.4) and (4.2) one gets

$$E^\cdot[\textstyle\sum_{s \in G} Z_s f \circ \theta_s] = E^\cdot[\int_0^\infty Z_s \hat{P}^{X_s}(f)\, dB_s] . \tag{4.11}$$

In the sequel the kernel $\hat{P}$ will be as in definition (4.10).

5. Markov properties of the measures $\hat{P}^x$. We now derive the Markov properties $\hat{P}^x$ as in [13].

(5.1) THEOREM. *For K-a.e. x, the measure $\hat{P}^x$ is carried by $\{X_0 = x\}$ and the process $(X_t)_{t>0}$ is strong Markov with respect to (P_t) on $(\Omega, \mathscr{F}^0, \hat{P}^x)$, which means: for each $(\mathscr{F}_{t+}^0)$-stopping time T, everywhere strictly positive, all positive $\mathscr{F}_{T+}^0$-measurable a and $\mathscr{F}^0$-measurable b, one has*

$$\hat{P}^x(a \cdot b \circ \theta_T) = \hat{P}^x(a \cdot P^{X_T}(b)) . \tag{5.2}$$

PROOF. (i) In (4.9) let $Z \equiv 1$, and let $F(\omega, s, \omega') = 1$ if $X_0(\omega') \neq X_{\zeta-}(\omega)$ (in particular if $X_{\zeta-}(\omega)$ does not exist), 0 otherwise. Then the left side of (4.9) vanishes, since (X_t) does not jump on G^r; in the right side the integral with respect to $\hat{P}^{X_s(\omega)}$ equals $\hat{P}^{X_s(\omega)}\{X_0 \neq X_{\zeta-}(k_s\omega)\}$ and may be replaced by $\hat{P}^{X_s(\omega)}\{X_0 \neq X_s(\omega)\}$, since $dK_s(\omega)$ does not charge the discontinuity set of ω. Therefore $\hat{P}^x\{X_0 \neq x\} = 0$ for K-a.e. x.

(ii) For $u > 0$, for positive Z, a, b, respectively well measurable, $\mathscr{F}_{u+}^0$- and $\mathscr{F}^0$-mearsurable one has the following equality

$$E^\cdot[\textstyle\sum_{s \in G^r} Z_s(a \cdot b \circ \theta_u) \circ \theta_s] = E^\cdot[\sum_{s \in G^r} Z_s(a \cdot E^{X_u}(b)) \circ \theta_s] . \tag{5.3}$$

In fact let $\varepsilon \in]0, u[$ and let $]G_n^\varepsilon, D_n^\varepsilon[$ be the nth contiguous interval whose length is $> \varepsilon$: $G_n^\varepsilon + \varepsilon$ is a stopping time, and so is $G_n^\varepsilon + u$. Let $Y_s = Z_s a \circ \theta_s I_{\{X_s \in F\}}$: $Y_{G_n^\varepsilon}$ is $\mathscr{F}_{G_n^\varepsilon + u}$-measurable and the Markov property at times $G_n^\varepsilon + u$ yields

$$E^\cdot[\textstyle\sum_n Y_{G_n^\varepsilon} b \circ \theta_{G_n^\varepsilon + u}] = E^\cdot[\sum_n Y_{G_n^\varepsilon} E^{X_{G_n^\varepsilon + u}}(b)] .$$

Equation (5.3) follows by letting $\varepsilon \to 0$.

Apply (4.2) with $f = a \cdot b \circ \theta_u$, then $f = a \cdot E^{X_u}(b)$; from (5.3) we get (5.2) with $T = u$ for K-a.e. x. A priori the exceptional set depends on u, a, b. The reader is referred to Theorem 3 of [13] for the end of proof. □

(5.4) REMARK. For all positive Borel h on E and $s > 0$, set $\hat{P}_s(x, h) = \hat{P}^x(h \circ X_s)$. By (5.2) one has, for K-a.e. x, $\hat{P}_s(x, P_t h) = \hat{P}_{s+t}(x, h)$ for $s > 0$, $t \geqq 0$. But $\hat{P}_s(x, 1)$ is not necessarily finite. As a consequence of $\hat{P}^x\{X_0 \neq x\} = 0$, one can show that $\hat{P}^x(\Omega) = +\infty$ for K-a.e. x (see [13]).

6. Structure of the excursions. The killing operators k_t are defined as usual:

$$\begin{aligned} k_t\,\omega(s) &= X_s(\omega) && \text{if} \quad s < t \\ &= \delta && \text{if} \quad s \geqq t\,. \end{aligned}$$

We denote by k_R the mapping $\omega \to k_{R(\omega)}(\omega)$ from Ω to Ω and we set

$$i_t = k_R \circ \theta_t\,, \qquad t \geqq 0\,.$$

This defines a process (i_t) on Ω that takes its values in Ω: for all ω in Ω and $t \geqq 0$, $i_t(\omega)$ is a right continuous function from $\mathbb{R}_+$ to E. The reader is referred to [12] for the properties of the process (i_t), called *incursion process*.

(6.1) DEFINITION. The collection $\{i_s(\omega),\ s \in G(\omega)\}$ is called the collection of the excursions (with respect to M) of the path ω.

Notice that this definition of the excursions does not require a local time of M as in Itô [6] (see Section 9).

(6.2) DEFINITION. For all positive $\mathscr{F}^*$-measurable f we set

$$\begin{aligned} Q^x(f) &= P^x(f \circ k_R)\,, \\ \hat{Q}^x(f) &= \hat{P}^x(f \circ k_R)\,, \end{aligned}$$

and for all positive $\mathscr{B}^*$-measurable h we set

$$\begin{aligned} Q_t(x, g) &= P^x(g \circ X_t\, I_{\{R>t\}})\,, \\ \hat{Q}_t(x, g) &= \hat{P}^x(g \circ X_t\, I_{\{R>t\}})\,. \end{aligned}$$

Notice that for $x \notin F$, $\hat{Q}^x = Q^x$, $\hat{Q}_t(x, \cdot) = Q_t(x, \cdot)$ since $\hat{P}^x = P^x$.

Recall that (Q_t) is a submarkovian semi-group (called the semi-group killed at time R) and that for the measure Q^x the process $(X_t)_{t\geqq 0}$ is a strong Markov process with respect to the semi-group (Q_t) and with the initial measure ε_x if $x \notin F$, ε_δ if $x \in F$.

(6.3) THEOREM. (i) *For all positive well measurable Z and $\mathscr{F}^*$-measurable f one has*

$$E^\bullet[\textstyle\sum_{s\in G} Z_s f \circ i_s] = E^\bullet[\int_0^\infty Z_s \hat{Q}^{X_s}(f)\, dB_s]\,. \tag{6.4}$$

(ii) *For K-a.e. x, $(\hat{Q}_t(x, \cdot))_{t>0}$ is an entrance law for the semi-group (Q_t) such that $\int_0^\infty e^{-t}\hat{Q}_t(x, 1)\, dt = 1$ and $\hat{Q}_t(x, 1) \uparrow \infty$ as $t \downarrow 0$.*

(iii) *For K-a.e. x, $\hat{Q}^x$ is a non-bounded measure carried by $\{\zeta > 0\}$ and $\{X_0 = x\}$ such that $\{X_t\}_{t>0}$ is strong Markov with respect to (Q_t) and with the entrance law $(\hat{Q}_t(x, \cdot))_{t>0}$.*

PROOF. By writing (4.11) with $f \circ k_R$ instead of f we get (6.4). By definition $\hat{Q}_t(x, 1) = \hat{P}^x\{R > t\}$, so that $\int_0^\infty e^{-t}\hat{Q}_t(x, 1)\, dt = \hat{P}^x(1 - e^{-R})$ and $\hat{Q}_t(x, 1) \uparrow P^x\{R > 0\}$ as $t \downarrow 0$. ii) follows from Theorem (5.1) and (4.8). iii) follows from Theorem (5.1). □

7. Last exit decomposition and conditional distributions.

(7.1) DEFINITION. For $t \geqq 0$ set

$$G_t = \sup\{s \leqq t : s \in M\} \qquad (\sup \emptyset = 0)$$
$$\check{\mathscr{F}}_t = \mathscr{F}_{G_t}\,.$$

Recall that, according to Meyer [11], a random variable is said to be $\mathscr{F}_{G_t}$-measurable if it can be written Z_{G_t}, where Z is well measurable. This defines the σ-field $\mathscr{F}_{G_t}$. One has the following properties: $\check{\mathscr{F}}_t \subset \mathscr{F}_t$; (G_t) is right continuous and adapted to $(\check{\mathscr{F}}_t)$; $(\check{\mathscr{F}}_t)$ is increasing and right continuous.

(7.2) DEFINITION. We set

$$G^0 = \{t \geqq 0 : R_{t-} = 0,\, R_t > 0\} \qquad (R_{0-} = 0)\,,$$
$$dB_t^0 = I_{\{R>0\}}\,\varepsilon_0(dt) + dB_t\,.$$

Notice that $G^0 = G$ if $R = 0$, $G \cup \{0\}$ if $R > 0$. From (4.11) and (4.9) respectively one gets

(7.3) $$E^\cdot\big[\textstyle\sum_{s\in G^0} Z_s f \circ \theta_s\big] = E^\cdot\big[\int_{[0,\infty[} Z_s \hat{P}^{X_s}(f)\, dB_s^0\big]\,,$$

(7.4) $$E^\cdot\big[\textstyle\sum_{s\in G^0} Z_s F(k_s, s, \theta_s)\big] = E^\cdot\big[\int_{[0,\infty[} Z_s(\omega)\hat{P}^{X_s(\omega)}(F(k_s\omega, s, \cdot))\, dB_s^0(\omega)\big]\,.$$

(7.5) PROPOSITION. *For all positive well measurable Z and for all universally measurable g on $\mathbb{R}_+ xE$, h on E one has*

(7.6) $$E^\cdot\big[Z_{G_t} g(t - G_t, X_{G_t}) h \circ X_t I_{\{G_t<t\}}\big] = E^\cdot\big[\int_{[0,t[} Z_s g(t - s, X_s)\hat{Q}_{t-s}(X_s, h)\, dB_s^0\big]\,.$$

Notice that the process $(g(t - s, X_s))_{s>0}$ is well measurable if g is Borel on $\mathbb{R}_+ xE$, but not if g is only universally measurable.

PROOF. The left side of (7.6) equals

$$E^\cdot\big[\textstyle\sum_{s\in G^0} Z_s g(t - s, X_s) h \circ X_t I_{\{0<t-s<R_s\}}\big]\,.$$

Writing (7.4) with

$$F(\omega, s, \omega') = g(t - s, X_0(\omega'))h \circ X_{t-s}(\omega')I_{\{0<t-s<R\}}(\omega')\,,$$

we get (7.6). □

(7.7) COROLLARY. (*Last exit decomposition*). *For all $t > 0$ and $x \in E$*

(7.8) $$P_t(x, h) = E^x[h \circ X_t I_M(t)] + E^x\big[\int_{[0,t[} \hat{Q}_{t-s}(X_s, h)\, dB_s^0\big]\,.$$

Notice that the second term of the right side can also be written $Q_t(x, h) + E^x[\int_{]0,t[} \hat{Q}_{t-s}(X_s, h)\, dB_s]$.

PROOF. Taking $Z \equiv 1$, $g \equiv 1$ in (7.6), we obtain (7.8) by noting that $G_t = t$ is equivalent to $t \in M$ for $t > 0$. □

From Proposition (7.5) we shall now derive an important result of Getoor and Sharpe [5]. We shall give a strong Markov version of this result. We first establish the following lemma.

(7.9) LEMMA. *The mapping* $(a, x) \to \hat{Q}_a(x, h)$ *is* (jointly) *universally measurable on* $]0, \infty[\times E$, *for all positive* $\mathscr{B}^*$*-measurable* h.

PROOF. Let $N^x(f) = \hat{P}^x((1 - e^{-R})f)$ and recall that $N^x(1) \leqq 1$. One has $\hat{Q}_a(x, h) = N^x(H(a, \cdot))$, where $H(a, \cdot) = h \circ X_a I_{\{R>a\}}/1 - e^{-R}$. But $(a, x) \to N^x(H(a, \cdot))$ is universally measurable for a function $H(a, \omega) = g(a)f(\omega)$ (g Borel positive, $f\ \mathscr{F}^0$-measurable positive); this extends to all positive universally measurable H by monotone class and completion arguments. □

(7.10) THEOREM. *Let* T *be an* $(\check{\mathscr{F}}_t)$*-stopping time. One has*

(7.11) $$\hat{Q}_{T-G_T}(X_{G_T}, 1) > 0 \quad \text{a.s. } on \quad \{G_T < T < \infty\},$$

(7.12) $$E^\mu[h \circ X_T \mid \check{\mathscr{F}}_T] = q_{T-G_T}(X_{G_T}, h) \quad \text{a.s. } on \quad \{T < \infty\},$$

where

$$q_a(x, h) = \frac{\hat{Q}_a(x, h)}{\hat{Q}_a(x, 1)} \quad \textit{if} \quad a > 0 \qquad (\tfrac{0}{0} = 0),$$
$$= h(x) \quad \textit{if} \quad a = 0.$$

By $\hat{Q}_{T-G_T}(X_{G_T}, 1)$ we mean the mapping $\omega \to \hat{Q}_{T(\omega)-G_T(\omega)}(X_{G_T}(\omega), 1)$.

PROOF. We shall use the notation $A_t = t \quad G_t$.

(i) Take $g(a, x) = 1$ if $\hat{Q}_a(x, 1) = 0$, 0 otherwise; then for $Z \equiv 1$, $h = 1$ the left side of (7.6) vanishes, showing that $\hat{Q}_{A_t}(X_{G_t}, 1) > 0$ a.s. on $\{G_t < t\}$. For a general stopping time T consider the sets $B_r = \{G_T < T < r < D_T\}$, for r rational. Since $a \to \hat{Q}_a(x, 1)$ is decreasing, one has on B_r

$$\hat{Q}_{A_T}(X_{G_T}, 1) = \hat{Q}_{A_T}(X_{G_r}, 1) \geqq \hat{Q}_{A_r}(X_{G_r}, 1).$$

But $B_r \subset \{G_r < r\}$, and therefore $\hat{Q}_{A_T}(X_{G_T}, 1) > 0$ a.s. on B_r. Equation (7.11) follows by noting that $\{G_T < T < \infty\} = \bigcup_r B_r$.

(ii) On the set $\{G_T = T < \infty\}$, (7.12) is obvious, since $\check{\mathscr{F}}_T \subset \mathscr{F}_T$. We now prove (7.12) *on the set* $B = \{G_T < T < \infty\}$, first recalling the proof of Getoor–Sharpe [5] in the particular case $T = t$. Let Z be a positive well-measurable process; we have to establish that

(7.13) $$E^\mu[Z_{G_t} h \circ X_t I_{\{G_t < t\}}] = E^\mu[Z_{G_t} q_{t-G_t}(X_{G_t}, h) I_{\{G_t < t\}}].$$

By (7.6), written with $g(a, x) = q_a(x, h)$ and $h = 1$, the right side of (7.13) is equal to

$$E^\mu[\int_{[0,t[} Z_s q_{t-s}(Z_t, h)\hat{Q}_{t-s}(X_s, 1)\, dB_s^0] = E^\mu[\int_{[0,t[} Z_s\hat{Q}_{t-s}(X_s, h)\, dB_s^0] \qquad (\tfrac{0}{0} = 0)$$

which equals the left side of (7.13), by (7.6) written with $g = 1$.

So far we have proved (7.12) for $T = t$; (7.12) easily extends to countably valued stopping times T. In order to prove (7.12) on B in the general case, we define for all n

$$T_n = +\infty \quad \text{on} \quad \{T = \infty\}, \qquad \frac{k}{2^n} \quad \text{on} \quad \left\{\frac{k-1}{2^n} \leqq T < \frac{k}{2^n}\right\}.$$

T_n satisfies (7.12); since $\check{\mathscr{F}}_T \subset \check{\mathscr{F}}_{T_n}$, one has for all C in $\check{\mathscr{F}}_T$, $C \subset \{T < \infty\}$

$$\int_{B\cap C} h \circ X_{T_n}\, dP^\mu = \int_{B\cap C} q_{A_{T_n}}(X_{G_{T_n}}, h)\, dP^\mu. \tag{7.14}$$

Suppose h continuous and bounded. The left side of (15) tends to $\int_{B\cap C} h \circ X_T\, dP^\mu$. On the other hand, $X_{G_{T_n}} = X_{G_T}$ for large n on the set B, and $\hat{Q}_{A_{T_n}}(X_{G_{T_n}}, h) \to \hat{Q}_{A_T}(X_{G_T}, h)$ a.s. on B, since $\hat{Q}_\bullet(x, h)$ is right continuous on $]0, \infty[$; this convergence also holds for $h = 1$, and so $q_{A_{T_n}}(X_{G_{T_n}}, h) \to q_{A_T}(X_{G_T}, h)$ a.s. on B by (7.11). By the bounded convergence theorem, the right side of (7.14) tends to $\int_{B\cap C} q_{A_T}(X_{G_T}, h)\, dP^\mu$ and the proof is complete. □

(7.15) REMARKS. (i) From Theorem (7.10) we can derive the Markov property of the process $(\check{X}_t) = (t - G_t, X_{G_t})$, as we did in Chapter IX of [8]. From the strong Markov version of (7.12), we can deduce the *strong Markov* property of $(\check{X}_t)$ and from the quotient form of $q_a(x, \cdot)$ for $a > 0$, we can deduce the *semi-group* property of the transition function (which was not satisfied in [8]).

(ii) Set $(\hat{X}_t) = (R_t, X_{D_t})$. As a consequence of (7.12) one has for all $(\check{\mathscr{F}}_t)$-stopping times T

$$E^\mu[g(\hat{X}_T) \mid \check{\mathscr{F}}_T] = Q(\check{X}_T, g) \qquad \text{a.s. on} \quad \{T < \infty\}, \qquad \text{where} \tag{7.16}$$

$Q(a, x), g) = q_a(x, E^\bullet(g(R, X_R)])$. Such a formula remains true for a regenerative system $(\Omega, \mathscr{F}, \check{\mathscr{F}}_t, X_t, \theta_t, P^\mu; M)$ such that the Markov property is satisfied for all stopping times $\check{T}$ such that $[T] \subset M$ (M is closed): see [9]. And this is enough to ensure the strong Markov property of $(\check{X}_t)$.

8. The post L^a-process. Let a be a fixed number in $]0, \infty]$ and set

$$L^a = \inf\{s\colon R_s \geqq a\}.$$

L^a is the left endpoint of the first interval contiguous to M of length $\geqq a$. As a well known result $T^a = L^a + a$ is a stopping time; therefore $(X_{T^a+t})_{t\geqq 0}$ is a Markov process, for each P^μ, with respect to the semi-group (P_t). The following result gives properties of the process $(X_{L^a+t})_{0<t<a}$.

(8.1) THEOREM. *The process $(X_{L^a+t})_{0<t<a}$ is, for each P^μ, a non-homogeneous*

Markov process with respect to the transition kernels $(Q_u{}^v)_{0<u\leqq v<a}$ *defined by*

$$Q_u{}^v(x, h) = \frac{1}{\phi_u(x)} Q_{v-u}(x, h\phi_v) \qquad (\tfrac{0}{0} = 0)$$

where we set $\phi_u = Q^\bullet\{\zeta \geqq a - u\} = P^\bullet\{R \geqq a - u\}$.[3]

(8.2) REMARK. For $a = +\infty$, $\phi_u = P^\bullet\{R = +\infty\}$ for all $u > 0$. If we set $L = L^\infty$, $\phi = P^\bullet\{R = \infty\}$, $(X_{L+t})_{t>0}$ is then a homogeneous Markov process with respect to the semi-group $(Q_t{}^\phi)$: $Q_t{}^\phi(x, h) = (1/\phi(x))Q_t(x, h\phi)$. This result was first established by Meyer, Smythe and Walsh in [15].

PROOF. For simplicity we shall not use the family $(\mathscr{F}_{L^a+t})_{0<t<a}$, but the natural family of the process $(X_{L^a+t})_{0<t<a}$. Fixing $0 < u \leqq v < a$, we have to prove that for all positive $\mathscr{F}^0_{u+}$-measurable g and $\mathscr{B}$-measurable h

$$(8.3) \qquad E^\bullet[(g \cdot h \circ X_v) \circ i_{L^a} I_{\{L^a<\infty\}}] = E^\bullet[(g \cdot Q_u{}^v(X_u, h)) \circ i_{L^a} I_{\{L^a<\infty\}}] \, .$$

The variable $R^a = D_{L^a}$ is a stopping time, since L^a is an $(\hat{\mathscr{F}}_t)$-stopping time; therefore $Z = I_{[0,R^a[}$ is well measurable, and by noting that $f \circ i_{L^a} I_{\{L^a<\infty\}} = \sum_{s\in G^0} Z_s(fI_{\{\zeta\geqq a\}}) \circ i_s$ we have for all positive $\mathscr{F}^0$-measurable f

$$E^\bullet[f \circ i_{L^a} I_{\{L^a<\infty\}}] = E^\bullet[\textstyle\int_{[0,R^a[} \hat{Q}^{X_s}(fI_{\{\zeta\geqq a\}})\, dB_s{}^0] \, .$$

This follows from (7.3), for $Z = I_{[0,R^a[}$ and $(fI_{\{\zeta\geqq a\}}) \circ k_R$ instead of f. To prove (8.3) it is hence sufficient to prove that for all $x \notin F$ and K-a.e. x one has

$$(8.4) \qquad \hat{Q}^x(g \cdot Q_u{}^v(X_u, h)I_{\{\zeta\geqq a\}}) = \hat{Q}^x(g \cdot h \circ X_v I_{\{\zeta\geqq a\}}) \, .$$

But for all $x \notin F$ and K-a.e. x, $(X_t)_{t>0}$ is a Markov process on $(\Omega, \hat{Q}^x)$ with respect to (Q_t) and the left side of (8.4) equals

$$\begin{aligned} &\hat{Q}^x(g \cdot Q^{X_u}\{\zeta \geqq a - u\}Q_u{}^v(X_u, h)) \\ &\quad = \hat{Q}^x(g \cdot Q_{v-u}(X_u, h\phi_v)) \quad (\phi(y) = 0 \Rightarrow Q_{v-u}(y, h\phi_v) = 0) \\ &\quad = \hat{Q}^x(g \cdot (h\phi_v) \circ X_v) \, , \end{aligned}$$

which equals the right side of (8.4). The proof is complete. □

(8.5) REMARK. The process (i_t) is adapted to $(\hat{\mathscr{F}}_t)$ and *right continuous* when its state space Ω is provided with the topology of convergence in measure ($\mathscr{F}^0$ is then the Borel σ-field) (see [8] or [12]). For every A in $\mathscr{F}^0$ the variable $H_A = \inf\{t > 0: i_t \in A\}$ therefore is an $(\hat{\mathscr{F}}_t)$-stopping time. For $A = \{\zeta \geqq a\}$, $H_A = L^a$. Theorem (8.1) could probably be extended to more general A.

9. The time changed excursions. This section represents a look at Section 6 from Itô's point of view [6]. We assume that M has a.s. no isolated point and that the equilibrium 1-potential $E^\bullet[e^{-R}]$ is *regular*. Then there exists a continuous additive functional L, where 1-potential is $E^\bullet[e^{-R}]$, and which is a *local time* of the set M (the support of L is indistinguishable from M).

[3] We assume here that $P^\delta\{R = 0\} = 1$.

We may carry over to the present situation the definition of the excursions given by Itô [6].

(9.1) DEFINITION. Let us define

$$\sigma_t = \inf\{s : L_s \geqq t\}, \qquad \tau_t = \inf\{s : L_s > t\}.$$

The excursion process (associated with the local time (L_t)) is the process $(\varepsilon_t) = (i_{\sigma_t})_{t>0}$.

Notice that the process (ε_t) takes its values in Ω. It is easily seen that $\varepsilon_{t+} = [\delta]$ $\forall t > 0$, when Ω is provided with the topology of convergence in measure ($[\delta]$ denotes the constant function $t \to \delta$); therefore (ε_t) is not right continuous.

(9.2) PROPOSITION. *There exists a universally measurable family (H^x) of σ-finite measures on $(\Omega, \mathscr{F}^0)$, that are Markov for the semi-group (Q_t), such that*

$$(9.3) \qquad E^\bullet[\textstyle\sum_{0<u\leqq L_\infty} Z_{\sigma_u} f \circ \varepsilon_u] = E^\bullet[\int_0^{L_\infty} Z_{\tau_u} H^{H_{\tau_u}}(f)\, du],$$

for all positive predictable Z and f on Ω such that $f([\delta]) = 0$.

PROOF. (i) For all positive bounded $\mathscr{B}$-measurable h define

$$u_h = E^\bullet[\textstyle\sum_{s\in G^i} e^{-s}(1 - e^{-R_s})h \circ X_s] = E^\bullet[\sum_{s\in G^i} e^{-s}P^{X_s}(1 - e^{-R})h \circ X_s].$$

The equality

$$(9.4) \qquad E^\bullet[e^{-R}] = E^\bullet[\textstyle\int_R^\infty e^{-s}\, ds] = u_1 + v_1 + E^\bullet[\int_0^\infty e^{-s} I_M(s)\, ds],$$

where v_1 is the regular 1-potential defined in the proof of Theorem (4.1), shows that u_1 is also a regular 1-potential. The stopping times (T_n) such that $G^i = \bigcup [T_n]$ therefore are totally inaccessible and u_h is a regular 1-potential for all h. Let H^h be the continuous additive functional whose 1-potential is u_h. For all positive $(\mathscr{F}_t{}^\mu)$-predictable Z one has

$$E^\mu[\textstyle\sum_{s\in G^i} e^{-s}Z_s(1 - e^{-R_s})h \circ X_s] = E^\mu[\int_0^\infty e^{-s}Z_s\, dH_s{}^h].$$

Taking Z to be the $(\mathscr{F}_t{}^\mu)$-predictable projection of $(g \circ X_t)$ (g is Borel positive), one shows that $H^h \ll H^1$. Let nh be the corresponding density. With the densities nh one constructs a kernel n on $(E, \mathscr{B}^*)$. By setting $q(x, h) = n(x, h/P^\bullet(1 - e^{-R}))$ and $H = H^1$ one obtains

$$(9.5) \qquad E^\bullet[\textstyle\sum_{s\in G^i} Z_s h \circ X_s] = E^\bullet[\int_0^\infty Z_s q(X_s, h)\, dH_s]$$

for all positive predictable Z (here one may not extend the equality to well measurable Z).

(ii) Let f be a positive $\mathscr{F}^0$-measurable function. For $h = Q^\bullet(f)$ the left side of (9.5) equals $E^\bullet[\sum_{s\in G^i} Z_s f \circ i_s]$ and (9.5) yields

$$(9.6) \qquad E^\bullet[\textstyle\sum_{s\in G^i} Z_s f \circ i_s] = E^\bullet[\int_0^\infty Z_s q(X_s, Q^\bullet(f))\, dH_s].$$

On the other hand (9.4) implies the following decomposition of the local time (L_t) (recall that v_1 is the 1-potential of K)

$$L_t = H_t + K_t + m(M \cap [0, t]).$$

Therefore H and K have densities h and k with respect to L. Define $H^x(f) = h(x)q(x, Q^{\bullet}(f)) + k(x)\hat{Q}^x(f)$ if $\hat{Q}^x$ is a Markov measure for (Q_t), 0 otherwise. From (4.2) and (9.6) we obtain

$$(9.7) \qquad E^{\bullet}[\textstyle\sum_{s\in G} Z_s f \circ i_s] = E^{\bullet}[\int_0^\infty Z_s H^{X_s}(f)\, dL_s]\,,$$

and (9.3) follows by noting that $G = \{\sigma_u : u > 0,\ \varepsilon_u \neq [\delta]\}$. □

(9.8) REMARKS. (i) The principle of the proof of (9.5) was used by Benveniste and Jacod [1] for proving the existence of a Lévy system under the "hypothèses droites." The function $P^{\bullet}(1 - e^{-R})$ plays the same role as the function h of [1]. Here the simplicity is due to the fact that the points of G^i are isolated from the right, which is not true for all properties of the process (X_t). A similar argument may also be found in Dynkin [3].

(ii) σ_t is a stopping time, so that $Z = I_{]0,\sigma_t]}$ is predictable. For this Z and $f = I_A$, (9.3) yields

$$(9.9) \qquad E^{\bullet}[\textstyle\sum_{0<u\leq L_\infty\wedge t} I_A \circ \varepsilon_u] = E^{\bullet}[\int_0^{L_\infty\wedge t} H^{X_{\tau_u}}(A)\, du]\,.$$

If $M = \overline{\{t : X_t = x_0\}}$ (x_0 is a regular point), the right side of (9.9) equals $H^{x_0}(A)E^{\bullet}[L_\infty \wedge t]$ and the excursion process $(\varepsilon_t)_{t>0}$ is then a Poisson point process with characteristic measure H^{x_0}, absorbed at the exponential time L_∞ (see Meyer [14]).

APPENDIX

10. The "classical" argument. This appendix refers to the proof of Theorem (4.1).

Recall from [12] that, when provided with the topology of convergence in measure, Ω is the complement of an analytic set in a compact metric space $\bar{\Omega}$ and its Borel field is $\mathscr{F}^0$. Actually we shall only use the universal measurability of Ω in $\bar{\Omega}$. Let $\mathscr{H}$ a total sequence in $\mathscr{C}(\bar{\Omega})$ which is a linear space over the rationals, inf-stable, and which contains the function 1. For each positive h in $\mathscr{H}$ we define $\bar{H}h = Nh_\Omega$, where h_Ω is the restriction of h to Ω and Nh_Ω has been defined in the proof of Theorem (4.1) as a density of K^{h_Ω} with respect to K (note that h_Ω is $\mathscr{F}^0$-measurable).

Let H be the set of all x such that $\bar{N}1(x) = 1$, $\bar{N}(h_1 + h_2)(x) = \bar{N}h_1(x) + \bar{N}h_2(x)$ for all positive h_1, h_2 in $\mathscr{H}$. For each x in H the mapping $h \to Nh(x)$ extends to a positive Q-linear functional on $\mathscr{H}$ and by continuity to a positive linear functional on $\mathscr{C}(\bar{\Omega})$, which in turn determines a probability $\bar{N}^x$ on the Borel field of $\bar{\Omega}$ or its universal completion $\bar{\mathscr{F}}^*$. H is universally measurable; if we set $\bar{N}^x \equiv 0$ for $x \notin H$, we get a kernel $\bar{N}$ from $(E, \mathscr{B}^*)$ to $(\bar{\Omega}, \bar{\mathscr{F}}^*)$ such that $\bar{N}^x(1) \leq 1$ for all x. Furthermore H^c is a K-null set and $\bar{N}^{\bullet}(h)$ is the density of K^{h_Ω} with respect to K. By limit, monotone class and completion arguments, we get for all positive bounded $\bar{\mathscr{F}}^*$-measurable $\bar{f}$

$$(10.1) \qquad E^{\bullet}[\textstyle\sum_{s\in G^r} e^{-s}Z_s(1 - e^{-R_s})\bar{f} \circ \theta_s] = E^{\bullet}[\int_0^\infty e^{-s}Z_s \bar{N}^{X_s}(\bar{f})\, dK_s]\,.$$

Taking $Z \equiv 1$, $\bar{f} = I_{\bar{\Omega}\backslash\Omega}$ in (10.1) we see that $\bar{N}^x(\bar{\Omega}\backslash\Omega) = 0$ for K-a.e. x. If f is $\mathscr{F}^*$-measurable, the function $\bar{f} = f$ on Ω, 0 on $\bar{\Omega}\backslash\Omega$ is $\bar{\mathscr{F}}^*$-measurable, and setting $N^\bullet(f) = \bar{N}^\bullet(\bar{f})$, we define a submarkov kernel N from $(E, \mathscr{B}^*)$ to (Ω, F^*) that satisfies (4.5). □

We would not be finished without quoting Kingman who asserts in [7] that "it is a notorious vice of applied probabilists to present their results hidden behind one or more Laplace transforms." We tried to keep it in mind.

ADDENDUM. The argument used at the end of the proof of Theorem (7.10) to extend (7.12) to stopping times was earlier used by J. Jacod in a paper that has appeared recently: Semi-groupes et mesures invariantes pour les processus semi-markoviens à espace d'état quelconque. *Ann. Inst. Henri Poincaré* **9** n° 1 (1973) pages 77–112.

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