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ON THE GLIVENKO-CANTELLI THEOREM FOR WEIGHTED EMPIRICALS BASED ON INDEPENDENT RANDOM VARIABLES

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For $X_1, \cdots, X_n$ independent real valued random variables and for $\alpha \in [0, 1]$, let $F_j(x) = \alpha P[X_j < x] + (1-\alpha)P[X_j \leqq x]$ and $Y_j(x) = \alpha I_{[X_j<x]} + (1-\alpha)I_{[X_j \leqq x]}$, where I_A is the indicator function of the set A. For numbers $w_1, w_2, \cdots, w_n$, let $D_n = \sup_{x,\alpha} \max_{N \leqq n} |\sum_1^N w_j(Y_j(x) - F_j(x))|$. We will obtain an exponential bound for $P[D_n \geqq a]$ and a rate for *almost sure* convergence of D_n. When $w_j \equiv 1$ the bound and the rate become, respectively, $4a \exp\{-2((a^2/n) - 1)\}$ and $O((n \log n)^{\frac{1}{2}})$.

1. Introduction. Let $X_1, \cdots, X_n$ be independent real valued random variables. If $X_1, \cdots, X_n$ are *identically* distributed, Theorem 2 of Kiefer (1961) leads to

$$\sup_x |\textstyle\sum_1^n (I_{[X_j<x]} - P[X_1 < x])| = O((n \log\log n)^{\frac{1}{2}}) \quad \text{w.p. } 1\,, \tag{1}$$

where (and hereinafter) indicator of a set A is denoted by I_A and convergence is wrt $n \to \infty$. The analogue of (1) in the non-identically distributed case would be

$$\sup_x |\textstyle\sum_1^n (I_{[X_j<x]} - P[X_j < x])| = O((n \log\log n)^{\frac{1}{2}}) \quad \text{w.p. } 1\,, \tag{2}$$

but this is unproved. Neither of the two proofs given in Kiefer (1961) for (1) works for (2). Nor are we able to supply a proof here. However, using a simple proof, we derive a result whose specialization shows that, if for $\alpha \in [0, 1]$

$$F_j(x) = \alpha P[X_j < x] + (1-\alpha)P[X_j \leqq x]$$

and

$$Y_j(x) = \alpha I_{[X_j<x]} + (1-\alpha)I_{[X_j \leqq x]}\,,$$

then

$$\sup_{x,\alpha} \max_{N \leqq n} |\textstyle\sum_1^N (Y_j(x) - F_j(x))| = O((n \log n)^{\frac{1}{2}}) \quad \text{w.p. } 1\,. \tag{3}$$

THEOREM. *Let $w_1, \cdots, w_n$ be any numbers. Set $\|\mathbf{w}_n\| = \sum_1^n |w_j|$ and $\|\mathbf{w}_n\|_2^2 = \sum_1^n w_j^2$. For any sequence $\{a_n, n \geqq 1\}$ for which $a_n \geqq \|\mathbf{w}_n\|_2$ and*

$$\textstyle\sum_1^\infty \left\{a_n \frac{\|\mathbf{w}_n\|}{\|\mathbf{w}_n\|_2^2} \exp\left(-2\left(\frac{a_n}{\|\mathbf{w}_n\|_2}\right)^2\right)\right\} < \infty\,, \tag{4}$$

we have

$$D_n = \sup_{x,\alpha} \max_{N \leqq n} |\textstyle\sum_1^N w_j(Y_j(x) - F_j(x))| = O(a_n) \quad \text{w.p. } 1\,. \tag{5}$$

2. Proof of Theorem. We assume, without loss of generality, that $w_j \geqq 0$

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(since, otherwise, we can always work with w_j^+ and w_j^- separately). In view of the Borel–Cantelli lemma and our assumption (4) we complete the proof of the theorem by proving the following lemma.

LEMMA. *For each $n \geqq 1$ and for any $a \geqq ||\mathbf{w}_n||_2$,*

$$P[D_n \geqq a] < \frac{4a||\mathbf{w}_n||}{||\mathbf{w}_n||_2^2} \exp\left\{-2\left(\left(\frac{a}{||\mathbf{w}_n||_2}\right)^2 - 1\right)\right\}. \tag{6}$$

PROOF OF THE LEMMA. For each $j = 1, \cdots, n$, let $w_j' = w_j/||\mathbf{w}_n||_2$ and $W = \sum_1^n w_j'$. Set $H_n = \sum_1^n w_j' F_j$, $H_n^* = \sum_1^n w_j' Y_j$ and $S = \sup_{x,\alpha} \max_{N \leqq n} |H_N^*(x) - H_N(x)|$. Thus, to complete the proof of the lemma, it suffices to show that, with $M = a/||\mathbf{w}_n||_2$,

$$P[S \geqq M] < 4WM \exp(-2(M^2 - 1)). \tag{7}$$

Let $\Delta = \max_{N \leqq n} (H_N^* - H_N)$ and $S^+ = \sup_{x,\alpha} \Delta(x)$. The remark following (2.17) of Hoeffding (1963) page 17, and Theorem 2 therein, applied to random variables $w_j' Y_j$ with $\alpha = 1$ give

$$P[\Delta(x-) \geqq \eta] \leqq \exp(-2\eta^2) \qquad \forall x \in R \text{ and } \forall \eta > 0. \tag{8}$$

Fix (temporarily) $0 < \gamma \leqq M$ and partition R into k intervals with endpoints $-\infty = x_0 < x_1 < \cdots < x_k = \infty$ such that $H_n(x_{j-1}, x_j) \leqq \gamma$ for $j = 1, \cdots, k$. Since $0 \leqq H_n(\cdot) \leqq W$, we can (and do) take $k < W\gamma^{-1} + 1$. Since $H_N(x_{j-1}, x_j) \leqq H_n(x_{j-1}, x_j) \leqq \gamma$ for $N \leqq n$, using the monotonicity of H_N and H_N^*, we get

$$\begin{aligned} \sup_{x_{j-1} < x < x_j} \Delta(x) &\leqq \max_{N \leqq n}(H_N^*(x_j-) - H_N(x_{j-1}+)) \\ &\leqq \Delta(x_j-) + \gamma. \end{aligned} \tag{9}$$

Note that the rhs of (9) is independent of α.

Now observe that $\Delta(x) \leqq \Delta(x+) \vee \Delta(x-) \leqq \sup_{x \in A} \sup_\alpha \Delta(x)$, where A is any dense subset of R. Therefore, $S^+ \doteq \sup_{x,\alpha} \Delta(x) \leqq \sup_\alpha \max_{1 \leqq j \leqq k} \sup_{x_{j-1} < x < x_j} \Delta(x)$, and from (9), (8) and $\Delta(x_k-) = 0$, we have

$$\begin{aligned} P[S^+ \geqq M] &\leqq P(\textstyle\bigcup_1^{k-1} [\Delta(x_j-) \geqq M - \gamma]) \\ &< W\gamma^{-1} \exp(-2(M - \gamma)^2). \end{aligned} \tag{10}$$

Since the lhs of (10) is independent of γ, substituting γ on the rhs of (10) by $\gamma_0 = M(1 - (1 - M^{-2})^{\frac{1}{2}})$ and noting that $\gamma_0^{-1} \leqq 2M$, we get from (10)

$$P[S^+ \geqq M] < 2WM \exp(-2(M^2 - 1)). \tag{11}$$

Let S^- be defined by interchanging H_N^* and H_N in S^+. Then, since $S^-(\mathbf{X}_n) = S^+(-\mathbf{X}_n)$ where $\mathbf{X}_n = (X_1, \cdots, X_n)$, the arguments used for (11) lead to

$$P[S^- \geqq M] < \text{rhs of (11)}. \tag{12}$$

Since $S = S^+ \vee S^-$, the proof of (7) (and hence of the lemma) is complete by (11) and (12).

3. Remarks. Consideration of certain nonparametric test-statistics is the motivation of the present consideration of weighted empiricals, (e.g., the weights $w_1, \cdots, w_n$ could be regression constants of certain nonparametric test statistics based on $X_1, \cdots, X_n$). The result of the theorem with $\alpha = 0$ and $w_j \equiv 1$ is needed in another paper (Singh (1974)) on nonparametric estimation of derivatives of average of densities.

The choice, γ_0, of γ in the proof of the lemma is made so that $\gamma_0^{-1}\exp(-2M - \gamma_0)^2)$ is quite close to the $\inf_{0<\gamma\leqq M}\gamma^{-1}\exp(-2(M-\gamma)^2)$, and the resulting bound is not a complicated one. This choice is suggested by Professor James F. Hannan.

When in the theorem $a_n = ||\mathbf{w}_n||_2\{1 + \log(n|\mathbf{w}_n|^{\frac{1}{2}})\}^{\frac{1}{2}}$ where $|\mathbf{w}_n| = ||\mathbf{w}_n||/||\mathbf{w}_n||_2$ and $n|\mathbf{w}_n|^{\frac{1}{2}} \geqq 1$, then, since by the c_r inequality (Loève (1963) page 155) $\{1 + \log(n|\mathbf{w}_n|^{\frac{1}{2}})\}^{\frac{1}{2}} \leqq 1 + \{\log(n|\mathbf{w}_n|^{\frac{1}{2}})\}^{\frac{1}{2}}$, (4) reduces to a simple condition

$$(4') \qquad \sum_1^\infty n^{-2}\{\log(n|\mathbf{w}_n|^{\frac{1}{2}})\}^{\frac{1}{2}} < \infty .$$

Thus, as a special case of the theorem we have: If $n|\mathbf{w}_n|^{\frac{1}{2}} \geqq 1$ for all sufficiently large n, and if (4′) holds, then

$$(5') \qquad D_n = O(||\mathbf{w}_n||_2\{1 + \log(n|\mathbf{w}_n|^{\frac{1}{2}})\}^{\frac{1}{2}}) \quad \text{w.p. } 1 .$$

In particular, with $w_j \equiv 1$, (5′) gives (3).

It is proved by Dvoretzky, Kiefer and Wolfowitz (1956) (and later generalized to the multivariate case by Kiefer and Wolfowitz (1958)) that there is a universal constant c such that, for all $r \geqq 0$, $P[\text{lhs } (1) \geqq r] \leqq c\exp(-2r^2/n)$. This bound is stronger than the one obtained for the *larger probability* in (6) (with $w_j \equiv 1$), (omission of the condition that $a \geqq ||\mathbf{w}_n||_2$ in the lemma here results in a slight change in the bound). The question of whether an inequality of the type $P[\text{lhs } (2) \geqq r] \leqq c_1\exp(-c_2 r^2/n)$, where c_1, c_2 are universal constants, holds and that whether lhs of (2) is $O((n\log\log n)^{\frac{1}{2}})$ w.p. 1 are still open. The affirmative answers of these questions, however, may not lead to similar results concerning the lhs of (3) (and hence of (5)), because (3) is a special case of (5) and the lhs of (3) could be much larger than that of (2).

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