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ENTROPIC REPULSION AND THE MAXIMUM OF THE TWO-DIMENSIONAL HARMONIC CRYSTAL

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We consider the lattice version of the free field in two dimensions (also called *harmonic crystal*). The main aim of the paper is to discuss quantitatively the entropic repulsion of the random surface in the presence of a hard wall. The basic ingredient of the proof is the analysis of the maximum of the field which requires a multiscale analysis reducing the problem essentially to a problem on a field with a tree structure.

1. Introduction. Let $V_N \stackrel{\text{def}}{=} \{1, \dots, N\}^2$, and ∂V_N be the inner boundary, that is, the points in V_N which have a nearest neighbor outside. We also set $\operatorname{int}(V_N) \stackrel{\text{def}}{=} V_N \backslash \partial V_N$. Let $\Phi_N = \{\phi_x\}_{x \in V_N}$ be the two-dimensional free field with zero boundary conditions: Φ_N is a family of centered Gaussian random variables with covariances given by the discrete Green's function of the (discrete) Laplacian on $\operatorname{int}(V_N)$, that is,

$$G_N(x, y) = \mathbb{E}_x\left(\sum_{i=0}^{\tau_{\delta V_N}} 1_{\eta_i = y}\right), \qquad x, y \in \operatorname{int}(V_N).$$

Here $\{\eta_i\}_{i \geq 0}$ is a standard symmetric nearest neighbor random walk on the two-dimensional lattice $\mathbb{Z}^2$, starting in x under the law $\mathbb{P}_x$, and $\tau_{\partial V_N}$ is the first entrance time in $\partial V_N (\phi_x = 0$ for $x \in \partial V_N)$. We will always write $\mathbb{P}$ and $\mathbb{E}$ for the law of this symmetric random walk, and P_N and E_N for the law of Φ_N, sometimes dropping the index $N \cdot P_N$ is the finite volume Gibbs measure [14] on $\mathbb{R}^{V_N}$ with (formal) Hamiltonian

$$\frac{1}{16} \sum_{x, y:\ |x-y|=1} (\phi_x - \phi_y)^2.$$

We will always assume that N is odd so that there is a point x_N or x_{V_N} in the center of the square V_N. For $y \in \mathbb{Z}^2$, we set

$$\sigma^2(N, y) = \operatorname{var}(\phi_y),$$
$$\sigma^2(N, y) = 0 \quad \text{if } y \notin \operatorname{int}(V_N)$$

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and

$$\sigma^2(N) = \mathrm{var}(\phi_{x_N}).$$

It is well known that the diagonal terms of $G_N(x, x)$ are logarithmically divergent if x is not too close to the boundary. If $\delta \in [0, 1/2)$, let

$$V_N^\delta \stackrel{\mathrm{def}}{=} \{x \in V_N: \mathrm{dist}(x, V_N^{\mathsf{C}}) \geq \delta N\}.$$

LEMMA 1. *Let* $g = \frac{2}{\pi}$, *then*

(a) $$\sup_y \sigma^2(N, y) \leq g \log N + c.$$

(b) *For any* $\delta \in (0, 1/2)$,

$$\sup_{x \in V_N^\delta} |\sigma^2(N, y) - g \log N| \leq c(\delta).$$

Here and later on we use c, c_1, etc. as generic positive constants, not necessarily the same in different contexts. If depending on further parameters like δ, η, we denote them by $c(\delta)$, $c(\delta, \eta)$.

PROOF OF LEMMA 1. Let $d(x, N) = \min\{\mathrm{dist}(x, y) : y \in \partial V_N\}$ and at the same time $D(x, N) = \max\{\mathrm{dist}(x, y): y \in \partial V_N\}$ and denote by $\overline{G}_N(x, x)$ the Green function of the discrete Laplacian on the ball $B_N(x)$ centered at x of radius N. Then

$$\overline{G}_{d(x, N)}(x, x) \leq G_N(x, x) \leq \overline{G}_{D(x, N)}(x, x).$$

Now the result follows from Theorem 1.6.6 of [15], since

$$\overline{G}_N(x, x) = \frac{2}{\pi} \log N + c + O(N^{-1}). \qquad \square$$

Our first result states that the maximum of the free field behaves trivially in the sense that in first approximation it is of the same order as if the random variables were independent:

THEOREM 2.

(a) $$\lim_{N \to \infty} P_N\left(\sup_{x \in V_N} \phi_x \geq 2\sqrt{g} \log N\right) = 0.$$

(b) *For any* $\eta > 0$ *and any* $\delta \in [0, 1/2)$, *there exists* $c = c(\delta, \eta) > 0$ *such that*

$$P_N\left(\sup_{x \in V_N^\delta} \phi_x \leq (2\sqrt{g} - \eta) \log N\right) \leq \exp\left[-c(\log N)^2\right],$$

if N *is large enough* ($N \geq N_0(\delta, \eta)$).

PROOF OF (a). This is a trivial consequence of Lemma 1:

$$P_N\Big(\sup_x \phi_x \geq 2\sqrt{g}\log N\Big) \leq N^2 \sup_{x\in V_N} P_N(\phi_x \geq 2\sqrt{g}\log N)$$
$$\leq N^2 \frac{\sqrt{\sup_x G_N(x,x)}}{\sqrt{2\pi}2\sqrt{g}\log N}\exp\bigg[-\frac{(2\sqrt{g}\log N)^2}{2\sup_x G_N(x,x)}\bigg] = o(1). \qquad \square$$

The proof of (b) is much more involved and will be given in the next section. It might be somewhat surprising that the above trivial estimate in part (a) of Theorem 2 is sharp in first order. This means that the maximum of the highly correlated free field is essentially the same as if the variables were independent. We will discuss this aspect and the relation with a hierarchical model at the end of this section.

Theorem 2 is the basis for proving our results on entropic repulsion. If D is a subset of V_N then we define the event $\Omega_D^+ \stackrel{\text{def}}{=} \{\phi_x \geq 0,\ x \in D\}$. We would like to have information about $P_N(\Omega_D^+)$. The most natural choice would be $D = V_N$. In that case, it was proved in [10] that $P_N(\Omega_{V_N}^+)$ is of order $\exp[-cN]$. The rapid decay of this probability, however, is a pure boundary effect: the zero boundary condition essentially decouples the field near the boundary, so that the behavior, say of the first layer inside V_N, behaves roughly as if the random variables were independent, and therefore, the probability that this layer V_N is positive everywhere is already of order $\exp[-cN]$. This boundary effect hides the interplay between long-range correlations and local fluctuations which is the main topic of this paper. To see this effect, one has to consider sets D which are a bit away from the boundary. In three and higher dimensions one can first consider the thermodynamic limit $P_\infty = \lim_{N\to\infty} P_N$ and then discuss $P_\infty(\Omega_{V_N}^+)$. This was the topic treated in [5]. In two dimensions, P_∞ of course does not exist, but we can investigate $P_N(\Omega_D^+)$ for nice subsets D instead.

To formulate the problem, let $V = [0,1]^2$ and $D \subset V$ be an open subset with smooth boundary which has positive (Euclidean) distance to the complement of V. We then put $D_N = ND \cap \mathbb{Z}^2$.

THEOREM 3.

$$\lim_{N\to\infty} \frac{1}{(\log N)^2} \log P_N(\Omega_{D_N}^+) = -4g\,\mathrm{cap}_V(D),$$

where $\mathrm{cap}_V(D)$ *is the relative capacity of* D *with respect to* V,

$$\mathrm{cap}_V(D) \stackrel{\text{def}}{=} \inf\big\{\tfrac{1}{2}\|\nabla f\|_2^2 \colon f \in H_o^1(V),\ f \geq 1 \text{ on } D\big\}.$$

Here, $H_o^1(V)$ *is the Sobolev space of* (*weakly*) *differentiable function* f *with square integrable gradient and* $f|_{\partial V} = 0$.

The proof of Theorem 3 will be given in Section 3. The result should be compared with the results in [5] and [10] for dimensions $d \geq 3$ where $P_N(\Omega_{D_N}^+)$

decays exponentially in $N^{d-2}\log N$,

$$\lim_{N\to\infty}\frac{1}{N^{d-2}\log N}\log P_N(\Omega^+_{D_N}) = -4G\ \mathrm{cap}_V(D),$$

where $G = \lim_{N\to\infty}\sigma^2(N)$. See [4] for related large deviation principles in $d \geq 3$.

What lies behind the above result is the following effect. The "easiest" way in which the field can achieve its goal of being positive on D_N is to have a shift of the whole field "on small macroscopic averages" on this set D_N to a level which leaves enough room for local spikes which still are present. In order to understand the picture, one has to look at the field on different scales. On the one hand, one has the macroscopic scale where one regards the field on subset of V_N with length scales of order N (but possibly small). On the other hand, one has to look at microscopic scales, that is, the ones of order 1 and on mesoscopic scales with length scales of order N^α, $0 < \alpha < 1$. The delicate point of the two-dimensional case is coming from the fact that the spikes are living on mesoscopic scales, and actually a precise analysis requires the multiscale considerations which do the job for proving Theorem 2 (reflected by a tree approximation). It is evident from this theorem that on length scales of order N^α one observes spikes of height $2\alpha\sqrt{g}\log N$. If α is close to 1 then this is essentially $2\sqrt{g}\log N$. Therefore, it is plausible that the field on small macroscopic scales has to be shifted to this level in order to have enough room for the "large mesoscopic" spikes. The form of the tail probabilities as described in Theorem 2 is actually absolutely instrumental for this simple picture. In three and higher dimensions, the situation in this respect is more delicate as there emerges a nontrivial competition between the macroscopic shift and the tail behavior for the local spikes (see [5, 10]). However, in other respects, the two-dimensional case is much more delicate, mainly because the analysis of the spikes requires a multiscale decomposition, whereas in three and higher dimensions, the spikes can be understood on a purely microscopical level.

Given these observations, it is plausible that $P_N(\Omega^+_{D_N})$ is in first order just the probability that the field is shifted on (small) macroscopic scales to $2\sqrt{g}\log N$. This probability is then not difficult to analyze, and leads to the statement of Theorem 3.

It should be plausible that these considerations also lead to some description of the conditioned field, that is, $P_N(\cdot|\Omega^+_{D_N})$. We can prove the following result.

THEOREM 4. *For any* $\varepsilon > 0$,

$$\lim_{N\to\infty}\sup_{x\in D_N} P_N\big(|\phi_x - 2\sqrt{g}\log N| \geq \varepsilon\log N|\Omega^+_{D_N}\big) = 0.$$

To understand Theorem 2 better, we consider the standard "hierarchical approximation" of the free field: here V_N is replaced by a binary tree T_n of depth n: the elements α of T_n are sequences $\alpha = \alpha_1\alpha_2\cdots\alpha_n$ where $\alpha_i \in \{0, 1\}$. Of course, T_n has 2^n elements. In order to make comparisons with the free field one should therefore think as n being such that $\#T_n = \#V_N$; that is,

$n = [2 \log N / \log 2]$. We consider the following family of normally distributed random variables,

$$X_\alpha = \xi^1_{\alpha_1} + \xi^2_{\alpha_1 \alpha_2} + \cdots + \xi^n_{\alpha_1 \cdots \alpha_n},$$

where the $\xi^k_{\alpha_1 \cdots \alpha_k}$, $k \le n$, $\alpha_1 \cdots \alpha_k \in \{0, 1\}^k$ are standard independent Gaussian random variables with variance 1. (In order to have a better match with the free field, one should take $\mathrm{var}(\xi) = (g \log 2)/2$, but this is, of course, of no importance). Clearly, the X_α then have variance n and

$$E(X_\alpha X_\beta) = n - d_H(\alpha, \beta),$$

where $d_H(\alpha, \beta) = n - \max\{k \le n\colon \alpha_1 \cdots \alpha_k = \beta_1 \cdots \beta_k\}$ is the hierarchical distance (or "ultrametric" in more fashionable expression). In many respects, this hierarchical field resembles the two-dimensional free field: if we match the number of points in V_N and T_n, that is, setting $n = [2 \log N / \log 2]$, then for $x \in V_N$ (not too close to the boundary), the decay of $G_N(x, y) = E_N(\phi_x \phi_y)$ as a function of the Euclidean distance is roughly $g\bigl(n \frac{\log 2}{2} - \log |x - y|\bigr)$ (see Theorem 1.6.6 of [15]). Therefore, essentially, our hierarchical field is obtained by replacing the Euclidean distance by the hierarchical one (and a trivial scaling).

The behavior of the maximum of the above hierarchical field is known up to great precision, including corrections of smaller order (see Remark 5 below). The leading order of the maximum is again the same as for completely independent random variables: it is in fact not difficult to see that

$$\lim_{n \to \infty} \frac{\max_{\alpha \in T_n} X_\alpha}{n} = \sqrt{2 \log 2} \tag{1}$$

in probability. This is well known and there are a number of different proofs (the earliest one seems to be the one by Biggins [3]).

Perhaps the easiest way to *understand* (1) (but perhaps not to prove it) is to replace the binary tree of depth n by one with a fixed (large) number K of branching levels. So we consider variables

$$X^{(K)}_\alpha \stackrel{\text{def}}{=} \xi^1_{\alpha_1} + \xi^2_{\alpha_1 \alpha_2} + \cdots + \xi^K_{\alpha_1 \cdots \alpha_K}, \tag{2}$$

where $\alpha_i \in \{1, \dots, 2^{n/K}\}$, and the $\xi^i_{\alpha_1 \cdots \alpha_i}$ are normally distributed with expectation 0 and variance n/K. Then, as $n \to \infty$ (K fixed), we have

$$\frac{1}{n} \max_{\alpha_1} \xi^1_{\alpha_1} \to \frac{\sqrt{2 \log 2}}{K}$$

in probability, and for any $\alpha_1 \cdots \alpha_i$,

$$\frac{1}{n} \max_{\alpha_{i+1}} \xi^{i+1}_{\alpha_1 \cdots \alpha_{i+1}} \to \frac{\sqrt{2 \log 2}}{K}.$$

From this, one gets

$$\lim_{n \to \infty} \frac{1}{n} \max_\alpha X^{(K)}_\alpha \ge \sqrt{2 \log 2}$$

for any fixed K. The upper bound follows directly from Slepian's lemma (also for the binary tree case). This of course does not prove (1) in the binary tree case, but as K is arbitrary, it makes it plausible. We will base our proof of Theorem 2 on a refinement and extension of the above "finite K" argument.

There is no point in discussing the binary tree case separately, as most of the facts are well known. However, we would like to make some (sidetracking) remarks.

REMARK 5. Much more than just (1) and even the statements of Theorem 2 is known to be true in the binary tree case: for n and u large enough, one has

$$P\biggl(\max_{\alpha\in T_n} X_\alpha \ge \sqrt{2\log 2}\,n - \frac{3}{2\sqrt{2\log 2}}\log n + u\biggr) \le \exp[-cu], \tag{3}$$

$$P\biggl(\max_{\alpha\in T_n} X_\alpha \le \sqrt{2\log 2}\,n - \frac{3}{2\sqrt{2\log 2}}\log n - u\biggr) \le \exp[-cu^2]. \tag{4}$$

There exists no published proof of the $\frac{3}{2\sqrt{2\log 2}}\log n$ correction which is different from the correction in the case of independent variables (where it is $\frac{1}{2\sqrt{2\log 2}}\log n$). The result is, however, close to a result of Bramson [7] on branching Brownian motions, and can be proved by an adaptation of his approach.

REMARK 6. Sidetracking still a bit further, let us observe that the binary tree case is the border line case where the above triviality of the maximum (in leading order) is correct, that is, where the maximum of the field of random variables is in first order at the same level as if they were independent. To give this a precise meaning, consider again the above binary tree, but where the variances of the variables $\xi^k_{\alpha_1\cdots\alpha_k}$, $k\le n$, may vary with k, but still remain independent. For instance, consider a continuous function $f\colon [0,1]\to(0,\infty)$, satisfying $\int f(x)\,dx=1$, and set $\mathrm{var}(\xi^k_{\alpha_1\cdots\alpha_k})=f(k/n)$. Then the variances of the variables X_α is still n (approximately). One may ask under which conditions on f (1) remains true. One can prove that this is the case if and only if f is nondecreasing. The binary tree case discussed before is the case with $f\equiv 1$. For a discussion of various aspects of this and related models, see [9].

REMARK 7. This remark should be skipped by readers not familiar with spin glass jargon. Our proof of Theorem 2 proceeds by introducing a tree structure with the help of a suitable conditioning procedure. Although the free field is not "ultrametric," we show that it is ultrametrically well approximated in the sense that the "nonultrametric" part is negligible, at least for the leading order approximation of the maximum (including the correct constant). The procedure probably does not shed much light on the much more delicate claims concerning ultrametrical approximations in spin glass theory but it might give some insights into the problem of how such ultrametric structures can appear in the $N\to\infty$ limit, at least in a very special case. It should be remarked that the problem here is trivial from the spin glass point of view, since there is no nontrivial "replica symmetry breaking" and the field is asymptotically equivalent to the random energy model. Whether or not the more refined properties predicted by spin glass theory (like the distribution on the notorious "pure states") have (provable) interpretations for this lattice free field is an interesting question which we cannot answer.

2. Proof of Theorem 2(b). We start the section by fixing some notations and providing some elementary properties of the free field. Generalizing slightly the situation introduced before, we define for any finite subset B of $\mathbb{Z}^2$ the free field $\Phi_B = (\phi_x)_{x\in \mathrm{int}\, B}$ as the centered Gaussian field with covariances $\mathrm{cov}_B(\phi_x, \phi_y) = \mathbb{E}_x\big(\sum_{j=0}^{\tau_{\partial B}} 1_{\eta_j = y}\big)$, where ∂B again is the inner boundary of B. We may extend the field to all of $\mathbb{Z}^2$ by 0. We will write P_B for the corresponding measure on $\mathbb{R}^{\mathbb{Z}^2}$.

For any subset C of $\mathbb{Z}^2$ we denote by $\mathscr{F}_C$ the σ-field generated by ϕ_x, $x \in C$. We write $\mathrm{var}_{\mathscr{F}_C}(\cdot)$ and $\mathrm{cov}_{\mathscr{F}_C}(\cdot,\cdot)$ for the conditional variances and covariances. Remark that for $x, y \in B$, $\mathrm{cov}_{\mathscr{F}_{B^\complement}}(\phi_x, \phi_y)$ is nonrandom and just $\mathrm{cov}_B(\phi_x, \phi_y)$. If $x \in B \subset C$, we have by a standard decomposition

$$\begin{aligned}\mathrm{var}_C(\phi_x) &= \mathrm{var}_C(E_C(\phi_x|\mathscr{F}_{\partial B})) + E_C(\mathrm{var}_{\mathscr{F}_{\partial B}}(\phi_x))\\ &= \mathrm{var}_C(E_C(\phi_x|\mathscr{F}_{\partial B})) + \mathrm{var}_B(\phi_x).\end{aligned} \tag{5}$$

If $x \in \mathrm{int}\, B$, let $\alpha_{\partial B}(x, y)$ be the first exit distribution

$$\alpha_{\partial B}(x, y) \stackrel{\mathrm{def}}{=} \mathbb{P}_x(\eta_{\tau_{\partial B}} = y).$$

If $x \in B \subset C$, then

$$E_C(\phi_x|\mathscr{F}_{\partial B}) = \sum_{y\in\partial B} \alpha_{\partial B}(x, y)\phi_y.$$

In case $B \subset V_N$ is an $n \times n$-box (n odd, this we always assume in all such situations) and $x = x_B$ the midpoint of this box, then we write

$$\phi_B \stackrel{\mathrm{def}}{=} E_N(\phi_{x_B}|\mathscr{F}_{\partial B}) = \sum_{y\in\partial B} \alpha_{\partial B}(x_B, y)\phi_y. \tag{6}$$

Specializing (5) to a $n \times n$ square $B \subset V_N$, and $x = x_B$, we get

$$\sigma^2(N, x_B) = \mathrm{var}_N(\phi_B) + \sigma^2(n). \tag{7}$$

If B is in the center of V_N, that is, when $x_B = x_{V_N}$ then we get

$$\mathrm{var}_N(\phi_B) = \sigma^2(N) - \sigma^2(n). \tag{8}$$

Through this paper we will need intermediate scales N^α, $\alpha \in (0, 1)$, and subboxes of our main box V_N of that side length. We then patch V_N^δ, $\delta \equiv (1/2) - \bar\delta \in [0, 1/2)$ chosen once for all in this proof, with these smaller boxes having overlapping boundaries. In order to avoid endless repetitions of trivial adjustments and corrections, we always assume that N^α is an odd integer (so that boxes of side length N^α have a midpoint on the lattice) and that $N^\alpha - 1$ divides $2\bar{\delta}N-1$, which we assume to be integer too. For $i=(i_1,i_2)$, $1 \le i_1,\ i_2 \le \frac{2\bar{\delta}N-1}{N^\alpha-1}$ we consider subboxes,

$$\begin{aligned} B_i^\alpha = {} & [(i_1-1)(N^\alpha-1)+1, i_1(N^\alpha-1)+1] \\ & \times [(i_2-1)(N^\alpha-1)+1, i_2(N^\alpha-1)+1]. \end{aligned}$$

Remark that each of these boxes contains $N^{2\alpha}$ points. Boundaries of neighboring boxes do intersect, and

$$\bigcup_i \partial B_i^\alpha = \Delta_N^\alpha \stackrel{\text{def}}{=} \left\{ k(N^\alpha - 1) + 1\colon\ 0 \le k \le \frac{2\bar{\delta}N-1}{N^\alpha - 1} \right\}^2.$$

We call the boxes B_i^α just α-boxes. The notion depends on N, but we suppress this in the notation. We denote by Π_α the set of α-boxes in V_N^δ, and by $\mathscr{F}_\alpha$ the σ-field generated by ϕ_x, $x \in \Delta_N^\alpha$.

We will also have to consider different "mesoscopic" scales, say N^{α_i}, $1 > \alpha_1 > \cdots > 0$. We will then always assume that the above assumptions are in force on all scales and that $N^{\alpha_{i+1}} - 1$ divides $N^{\alpha_i} - 1$.

In all proofs, inequalities involving N are required to hold only for large enough N, where the notion of "large enough" may depend on all the parameters involved.

Before giving the technical details, we outline the strategy of the proof: we consider mesoscopic scales with parameters $1 > \alpha_1 > \cdots > \alpha_K > 0$. We then want to show that the field reaches $2\sqrt{g}\alpha_i \log N$ "on scale" N^{α_i}. To give this a precise meaning, we consider the variables ϕ_B, $B \in \Pi_{\alpha_i}$. We would like to argue as follows: given that $\max_{B\in\Pi_{\alpha_i}} \phi_B \sim 2\sqrt{g}\alpha_i \log N$, we take the α_i-box, say $\widehat{B}$, where this maximum is achieved and then investigate the maximum of the variables $\phi_C - \phi_{\widehat{B}}, C$ any α_{i+1}-box inside $\widehat{B}$. We then would like to show that conditionally on $\mathscr{F}_{\alpha_i}$, this maximum is approximately $2\sqrt{g}(\alpha_i - \alpha_{i+1}) \log N$.

This is exactly the kind of procedure which worked for the K-level tree (2). In our case, there is, however, the problem that the variables $\phi_C - \phi_{\widehat{B}}, C$ being α_{i+1}-boxes inside $\widehat{B}$, are not independent, conditionally on $\mathscr{F}_{\alpha_i}$. In order to overcome this difficulty, we need not *one* $\widehat{B}$ where $\phi_{\widehat{B}}$ reaches $2\sqrt{g}\alpha_i \log N$, but *many* which reach a level close to that. These "many" then allow getting rid of the problem of this lack of independence inside $\widehat{B}$, essentially because what is happening inside different $\widehat{B}$'s is independent, conditionally on $\mathscr{F}_{\alpha_i}$.

We will need a basis for these considerations, telling that, on the first scale N^{α_1}, there are sufficiently many boxes where ϕ_B is positive. This is the content of the next lemma. Recall that we are working on V_N^δ, for a chosen δ.

LEMMA 8. *Given $\alpha \in (1/2, 1)$, there exist $\kappa(\alpha), a(\delta,\alpha) > 0$ such that*

$$P(\#\{B \in \Pi_\alpha\colon\ \phi_B \ge 0\} \le N^\kappa) \le \exp[-a(\log N)^2].$$

PROOF. We choose $\alpha' = (1+\alpha)/2$ and consider

$$\Lambda \stackrel{\text{def}}{=} \left\{ B \in \Pi_{\alpha'}\colon\ \phi_B \geq -\frac{(1-\alpha')\sqrt{g}\log N}{2} \right\},$$

and A the event

$$A \stackrel{\text{def}}{=} \{|\Lambda| \geq N^{1-\alpha'}\}.$$

Then

$$P(A^{\mathsf{C}}) \leq P\Big(A^{\mathsf{C}}, \max_{B\in\Pi_{\alpha'}} \phi_B \leq (\log N)^2\Big) + P\Big(\max_{B\in\Pi_{\alpha'}} \phi_B > (\log N)^2\Big).$$

By (6) and by applying Lemma 1 we obtain that

$$P\Big(\max_{B\in\Pi_{\alpha'}} \phi_B > (\log N)^2\Big) \leq P\Big(\max_{x\in V_N} \phi_x > (\log N)^2\Big) \leq \exp[-c(\log N)^3]. \tag{9}$$

On $A^{\mathsf{C}} \cap \{\max_{B\in\Pi_{\alpha'}} \phi_B \leq (\log N)^2\}$ we have (assume $\delta > 0$)

$$\begin{aligned} |\Pi_{\alpha'}|^{-1} \sum_{B\in\Pi_{\alpha'}} \phi_B \ \leq\ & -\frac{(1-\alpha')\sqrt{g}\log N}{2} \\ & + \left(\frac{2\bar{\delta}N-1}{N^{\alpha'}-1}\right)^{-2}\left((\log N)^2 + \frac{1-\alpha'}{2}\sqrt{g}\log N\right)N^{1-\alpha'} \\ \leq\ & -\frac{(1-\alpha')\sqrt{g}\log N}{3}. \end{aligned} \tag{10}$$

From Lemma 13 (with $F \equiv 1_{(\delta,\,1-\delta)^2}$), we get that there exists $c = c(\delta, \alpha)$ such that

$$P\left(A^{\mathsf{C}} \cap \left\{\max_{B\in\Pi_{\alpha'}} \phi_B \leq (\log N)^2\right\}\right) \leq \exp[-c(\log N)^2],$$

and together with (9) this yields

$$P(A^{\mathsf{C}}) \leq \exp[-c(\log N)^2]. \tag{11}$$

If $\delta = 0$ just restrict the sum in (10) to α'-boxes in a set $V_N^{\delta'}$, with $\delta' > 0$ sufficiently small and repeat the very same argument: note that $c(\delta, \alpha)$ can be chosen bounded away from zero and infinity for δ in a neighborhood of zero.

For any α'-box, we consider the α-box whose center coincides with that of the α'-box. We denote by $\Pi_{\alpha',\alpha}$ the set of these special α-boxes. For the proof of the lemma, we concentrate on them:

$$\begin{aligned} &P(\#\{B \in \Pi_\alpha\colon\ \phi_B \geq 0\} \leq N^\kappa) \\ &\quad \leq P(\#\{B \in \Pi_{\alpha',\alpha}\colon\ \phi_B \geq 0\} \leq N^\kappa) \\ &\quad \leq E[P(\#\{B \in \Pi_{\alpha',\alpha}\colon\ \phi_B \geq 0\} \leq N^\kappa | \mathscr{F}_{\alpha'}); A] + P(A^{\mathsf{C}}). \end{aligned}$$

We choose now $\kappa < 1-\alpha'$. On A, there are at least $N^{1-\alpha'}$ α'-boxes B' which satisfy $\phi_{B'} \geq -\frac{(1-\alpha')}{2}\sqrt{g}\log N$. Evidently, we have $\phi_{B'} = E(\phi_B|\mathscr{F}_{\alpha'})$, when B is the α-box which has the same center as B'. Conditionally on $\mathscr{F}_{\alpha'}$, the variables

$\{\phi_B - \phi_{B'}\}_{B' \in \Pi_{\alpha'}}$ are i.i.d. centered Gaussian random variables with conditional variance $\mathrm{var}_{B'}(\phi_B)$ which according to (8) and Lemma 1 satisfies

$$\mathrm{var}_{B'}(\phi_B) = g(\alpha' - \alpha) \log N + O(1).$$

Using this, we have on A,

$$\begin{aligned} &P(\#\{B \in \Pi_{\alpha', \alpha}\colon\ \phi_B \geq 0\} \leq N^\kappa | \mathscr{F}_{\alpha'}) \\ &\quad \leq P\left(\sum_{i=1}^{N^{1-\alpha'}} \mathbb{I}\left[\xi_i \geq \frac{1-\alpha'}{2} \sqrt{g} \log N \right] \leq N^\kappa \right), \end{aligned} \tag{12}$$

ξ_i are i.i.d. centered Gaussian variables with variance $\mathrm{var}_{B'}(\phi_B)$, and $\mathbb{I}[A]$ denotes the indicator function of an event A. We have

$$P\left(\xi_i \geq \frac{1-\alpha'}{2} \sqrt{g} \log N \right) \geq N^{-(1-\alpha')/3}.$$

Therefore, after centering the indicator functions and choosing $\kappa = \frac{1-\alpha'}{2}$, we get

$$P\left(\sum_{i=1}^{N^{1-\alpha'}} \mathbb{I}\left[\xi_i \geq \frac{1-\alpha'}{2} \sqrt{g} \log N \right] \leq N^\kappa \right) \leq P\left(\left| \sum_{i=1}^{N^{1-\alpha'}} (\theta_i - E\theta_i) \right| \geq \frac{N^{2(1-\alpha')/3}}{2} \right),$$

where $\theta_i = \mathbb{I}[\xi_i \geq \frac{1-\alpha'}{2} \sqrt{g} \log N]$. Applying standard estimates for binomial distributions (e.g., Lemma 11), we get that the right-hand side of this is less than or equal to $\exp[-cN^{(1-\alpha')/3}]$, which is much better than required. Together with (11), this proves the lemma. $\square$

PROOF OF THEOREM 2(b). We fix $1/2 < \alpha < 1$ and take $\kappa = \kappa(\alpha)$, $a = a(\delta, \alpha)$ according to Lemma 8. We choose $K \in \mathbb{N}$ and set $\alpha_i = \frac{K-i+1}{K}\alpha$, $1 \leq i \leq K$. We now define collections of subsets of the set of α_i-boxes which we denote by Γ_{α_i}, defined recursively. $\Gamma_{\alpha_1} \stackrel{\text{def}}{=} \Pi_{\alpha_1}$. Assume Γ_{α_i} has been chosen $(1 \leq i \leq K-1)$. For any $B \in \Gamma_{\alpha_i}$, we draw a square of side length $(N^{\alpha_i} - 1)/2$ which has the same center as B. The collection of α_{i+1}-boxes inside the square is denoted by $\Gamma_{B, \alpha_{i+1}}$. We then set

$$\Gamma_{\alpha_{i+1}} \stackrel{\text{def}}{=} \bigcup_{B \in \Gamma_{\alpha_i}} \Gamma_{B, \alpha_{i+1}}.$$

We define a sequence of events $C_1, \dots, C_K$, $C_k \in \mathscr{F}_{\alpha_k}$, $1 \leq k \leq K$ in the following way:

$$C_1 \stackrel{\text{def}}{=} \{\#\{B \in \Gamma_{\alpha_1}\colon\ \phi_B \geq 0\} \geq N^\kappa\}.$$

To define C_k, $k \geq 2$ we consider sequences $\underline{B}^{(k)} = (B_1, \dots, B_k)$ of boxes satisfying $B_1 \supset B_2 \supset \cdots \supset B_k$, and $B_i \in \Gamma_{\alpha_i}$. Then

$$C_k \stackrel{\text{def}}{=} \left\{ \#\left\{ \underline{B}^{(k)}\colon\ \phi_{B_i} \geq (i-1)\alpha\left(\frac{2\sqrt{g}}{K} - \frac{1}{K^2} \right) \log N,\ 1 \leq i \leq k \right\} \geq N^\kappa \right\}.$$

From Lemma 8 we know that

$$P(C_1) \geq 1 - \exp[-a(\delta, \alpha)(\log N)^2] \tag{13}$$

[with $\kappa = \kappa(\alpha)$]. We define $\widetilde{\mathscr{F}}_k = \sigma(\phi_x\colon\ x \in \cup_{B\in\Pi_{\alpha_k}} \partial B)$. Remark that by our construction, $\widetilde{\mathscr{F}}_1 \subset \widetilde{\mathscr{F}}_2 \subset \cdots \subset \widetilde{\mathscr{F}}_K$.

On C_k we have at least N^κ sequences $\underline{B}^{(k)} = (B_1, \ldots, B_k)$, $B_i \in \Gamma_{\alpha_i}$, which are nested, and satisfy $\phi_{B_1} \geq 0$, $\phi_{B_i} \geq (i-1)\alpha(\frac{2\sqrt{g}}{K} - \frac{1}{K^2})$, $2 \leq i \leq k$. We denote these sequences by

$$\underline{B}_j^{(k)} = (B_{j1}, \ldots, B_{jk}), \qquad 1 \leq j \leq N^\kappa.$$

(We select N^κ if there are more.) We use the splitting

$$P(C_{k+1}^{\complement}) \leq E\big(P(C_{k+1}^{\complement}|\widetilde{\mathscr{F}}_k); C_k\big) + P(C_k^{\complement}). \tag{14}$$

We have

$$\begin{aligned} C_k \cap C_{k+1}^{\complement} \subset C_k \cap &\left\{ \sum_{j=1}^{N^\kappa} \sum_{B\in\Gamma_{B_{jk},\alpha_{k+1}}} \mathbb{1}\left[\phi_B - \phi_{B_{jk}} \geq \alpha\left(\frac{2\sqrt{g}}{K} - \frac{1}{K^2}\right)\log N\right] \leq N^\kappa \right\} \\ \subset C_k \cap &\left\{ \sum_{j=1}^{N^\kappa} \frac{1}{\left|\Gamma_{B_{jk},\alpha_{k+1}}\right|} \sum_{B\in\Gamma_{B_{jk},\alpha_{k+1}}} \mathbb{1}\left[\phi_B - \phi_{B_{jk}} \geq \alpha\left(\frac{2\sqrt{g}}{K} - \frac{1}{K^2}\right)\log N\right] \right. \\ &\qquad\left. \leq \frac{4N^\kappa}{(N^{2\alpha/K} - 1)} \right\} \end{aligned}$$

We write

$$\zeta_j \stackrel{\text{def}}{=} \frac{1}{\left|\Gamma_{B_{jk},\,\alpha_{k+1}}\right|} \sum_{B\in\Gamma_{B_{jk},\,\alpha_{k+1}}} \mathbb{1}\left[\phi_B - \phi_{B_{jk}} \geq \alpha\left(\frac{2\sqrt{g}}{K} - \frac{1}{K^2}\right)\log N\right]$$

Remark that conditionally on $\widetilde{\mathscr{F}}_k$, the ζ_j are i.i.d. Then

$$E(\zeta_j|\widetilde{\mathscr{F}}_k) \geq \inf_{B\in\Gamma_{B_{jk},\,\alpha_{k+1}}} P\left(\phi_B - \phi_{B_{jk}} \geq \alpha\left(\frac{2\sqrt{g}}{K} - \frac{1}{K^2}\right)\log N \middle| \widetilde{\mathscr{F}}_k\right).$$

The conditional variance of $\phi_B - \phi_{B_{jk}}$ for $B \in \Gamma_{B_{jk},\,\alpha_{k+1}}$ is

$$\begin{aligned} \operatorname{var}_{\widetilde{\mathscr{F}}_k}(\phi_B - \phi_{B_{jk}}) &= \operatorname{var}_{B_{jk}}(\phi_B) \\ &= g(\alpha_k - \alpha_{k+1})\log N + O(1) \\ &= \frac{g\alpha}{K}\log N + O(1). \end{aligned} \tag{15}$$

Indeed, applying (5) with B and $C \stackrel{\text{def}}{=} B_{jk}$,we have

$$\operatorname{var}_{B_{jk}}(\phi_{x_B}) = \sigma^2(N^{\alpha_k} - 1) + \operatorname{var}_{B_{jk}}(\phi_B),$$

and applying Lemma 1, this implies (15). Therefore if we choose $K > 1/2\sqrt{g}$,

$$E\big(\zeta_j|\widetilde{\mathscr{F}}_k\big) \geq N^{-\frac{2\alpha}{K}+\frac{\alpha}{\sqrt{g}K^2}}.$$

In this case we have

$$\begin{aligned} C_k \cap C_{k+1}^{\mathsf{C}} &\subset C_k \cap \left\{ \sum_{j=1}^{N^\kappa} \big(\zeta_j - E\big(\zeta_j|\widetilde{\mathscr{F}}_k\big)\big) \leq \frac{4N^\kappa}{N^{2\alpha/K}} - N^\kappa N^{-\frac{2\alpha}{K}+\frac{\alpha}{\sqrt{g}K^2}} \right\} \\ &\subset C_k \cap \left\{ \left| \sum_{j=1}^{N^\kappa} \big(\zeta_j - E\big(\zeta_j|\widetilde{\mathscr{F}}_k\big)\big) \right| \geq \frac{1}{2} N^\kappa N^{-\frac{2\alpha}{K}+\frac{\alpha}{\sqrt{g}K^2}} \right\} \end{aligned}$$

Applying Lemma 11 we therefore get on C_k,

$$P\big(C_{k+1}^{\mathsf{C}}|\widetilde{\mathscr{F}}_k\big) \leq 2\exp\Big[-cN^{\kappa-\frac{4\alpha}{K}+\frac{2\alpha}{\sqrt{g}K^2}}\Big]. \tag{16}$$

Let now $\eta > 0$ be given, as in the statement of Theorem 3. Then we can choose $\alpha < 1$ such that $2\sqrt{g} - \eta < 2\sqrt{g}\alpha$. To this α we choose $\kappa(\alpha)$ according to Lemma 8, and then we choose K large enough such that $\kappa(\alpha) - 2\alpha/K > 0$, $(K-1)\alpha(2\sqrt{g}/K - 1/K^2) > 2\sqrt{g} - \eta$ and $K > 1/2\sqrt{g}$ (which we imposed before) are satisfied. Then

$$\begin{aligned} P_N\Big(\sup_x \phi_x \leq (2\sqrt{g}-\eta)\log N\Big) &\leq P_N\big(C_K^{\mathsf{C}}\big) \\ &\leq e^{-a(\delta,\alpha)(\log N)^2} + 2K\exp\big[-cN^{\kappa-2\alpha/K}\big] \\ &\leq \exp\big[-c(\delta,\eta)(\log N)^2\big], \end{aligned}$$

as required. □

3. Proof of Theorem 3.

PROOF OF THE LOWER BOUND. We have

$$\mathrm{cap}_V(D) = \inf\big\{\tfrac{1}{2}\|\nabla f\|^2\colon\ f \in C_0^1(V),\ f \geq 1 \text{ on } D\big\},$$

where $C_0^1(V)$ is the set of once (continuously) differentiable functions, vanishing at the boundary. Let $f \in C_0^1(V)$ be such that $f \geq 1$ on D and $f \geq 0$. Let us denote by P_N^a the Gaussian measure with covariance G_N and mean $E_N^a[\phi_x] = (a\log N)f(x/N)$ where $a > 2\sqrt{g}$. Let $\mathbf{H}_N(P_N^a|P_N)$ denote the relative entropy of P_N^a with respect to P_N. By Lemma 2.4 of [1] we see that

$$\lim_{N\to\infty} \frac{1}{(\log N)^2}\mathbf{H}_N(P_N^a|P_N) = \frac{a^2}{2}\|\nabla f\|^2.$$

On the other hand,

$$P_N^a\big((\Omega_{D_N}^+)^{\complement}\big) \le \sum_{x\in D_N} P_N^a(\phi_x < 0) = \sum_{x\in D_N} P_N(\phi_x < -a\log N)$$
$$\le \sum_{x\in D_N} \exp\left[-\frac{a^2(\log N)^2}{2\sigma^2(N,x)}\right] \le N^2 \exp\left[-\frac{a^2(\log N)^2}{2g\log N + c}\right] \to 0$$

as $N\to\infty$. Using the standard entropy inequality (cf. Lemma 5.4.21 of [13]),

$$-\log\frac{P_N(\Omega_{D_N}^+)}{P_N^a(\Omega_{D_N}^+)} \ge -\frac{\mathbf{H}_N(P_N^a|P_N) + e^{-1}}{P_N^a(\Omega_{D_N}^+)},$$

we see that

$$\liminf_{N\to\infty} \frac{1}{(\log N)^2}\log P_N(\Omega_{D_N}^+) \ge -\inf_{a>2\sqrt{g}}\inf_f \frac{a^2}{2}\|\nabla f\|^2 = -4g\,\mathrm{cap}_V(D).$$

PROOF OF THE UPPER BOUND. The argument is roughly as follows. We consider boxes B of side length N^α where $\alpha\in(0,1)$ is close to 1. Conditioned on $\mathscr{F}_\alpha$, if ϕ_B is not at least close to $2\sqrt{g}\log N$, the probability that the field is staying positive inside the box is estimated by Theorem 2 (replacing the maximum by the minimum). Even if α is very close to 1, there are of course many such boxes, and Theorem 2 then tells us that on $\Omega_{D_N}^+$ there cannot be more than a finite number of boxes where ϕ_B is *not* close to $2\sqrt{g}\log N$ except for situations which have a negligeable probability.

To fix this procedure precisely, we choose an arbitrary $\beta>0$. If $K\in\mathbb{N}$, $\alpha\in(1/2,1)$ we define the event

$$A_{K,\beta,\alpha} \overset{\text{def}}{=} \{\#\{B\in\Pi_\alpha\colon\ B\subset D_N,\ \phi_B\le(2\sqrt{g}-\beta)\log N\}\le K\}.$$

In this section we set $\delta=0$ in defining Π_α; that is, Π_α is the set of all the α-boxes in V_N. We will, however, use Theorem 2 with different values of δ.

The proof of the upper bound of Theorem 3 is split into two parts.

LEMMA 9. *For given β, we can choose $\alpha=\alpha(\beta)\in(0,1)$ (close to 1), and $K=K(\beta)\in\mathbb{N}$ such that*

$$P\big(A_{K,\beta,\alpha}^{\complement}\cap\Omega_{D_N}^+\big)\le 2\exp\big[-(4g\,\mathrm{cap}_V(D)+1)(\log N)^2\big]. \tag{17}$$

LEMMA 10. *For given β and any $\alpha\in(0,1)$, $K\in\mathbb{N}$, we have*

$$\limsup_{N\to\infty}\frac{1}{(\log N)^2}\log P(A_{K,\beta,\alpha}\cap\Omega_{D_N}^+)\le -(2\sqrt{g}-\beta)^2\,\mathrm{cap}_V(D).$$

It is evident that the two lemmas together prove the upper bound in Theorem 3.

PROOF OF LEMMA 9. If $\eta > 0$, $\varepsilon \in (0, 1/2]$ and $\alpha \in (0, 1)$, we consider the event

$$F \stackrel{\text{def}}{=} \bigcup_{B \in \Pi_\alpha} \bigcup_{x \in B^{(\varepsilon)}} \{|\phi_B - E(\phi_x|\mathscr{F}_\alpha)| \geq \eta \log N\},$$

where $B^{(\varepsilon)}$ is the set of point in B which are in a box of side length εN^α with center x_B.Then, by Lemma 12,

$$P(F) \leq N^2 \exp\left[-c\frac{\eta^2(\log N)^2}{\varepsilon}\right] \leq \exp\left[-c'\frac{\eta^2(\log N)^2}{\varepsilon}\right]. \tag{18}$$

We will choose $\varepsilon = \varepsilon(\eta)$ such that

$$c'\eta^2/\varepsilon \geq 4g \ \mathrm{cap}_V(D) + 1. \tag{19}$$

Remark that there is no dependency of ε on α (but we, however, have the usual convention that (18) has to hold only for large N, where this notion of course may depend on all the parameters including α). We then have

$$\begin{aligned} P(A^{\mathsf{C}}_{K,\beta,\alpha} \cap \Omega_D^+) \leq{} & E(P(A^{\mathsf{C}}_{K,\beta,\alpha} \cap \Omega_D^+|\mathscr{F}_\alpha); F^{\mathsf{C}}) \\ & + \exp[-(4g \ \mathrm{cap}_V(D) + 1)(\log N)^2]. \end{aligned}$$

Conditionally on $\mathscr{F}_\alpha$ and for fixed $B \in \Pi_\alpha$, the field $(\phi_x - E(\phi_x|\mathscr{F}_\alpha))_{x \in B}$ is just the free field on the box B (with side length N^α). Therefore, for N sufficiently large, by Theorem 2,

$$\begin{aligned} & P\Big(\sup_{x \in B^{(\varepsilon)}} (\phi_x - E(\phi_x|\mathscr{F}_\alpha)) \leq (2\sqrt{g} - \beta) \log N \big| \mathscr{F}_\alpha\Big) \\ & \quad \leq P\Big(\sup_{x \in B^{(\varepsilon)}} (\phi_x - E(\phi_x|\mathscr{F}_\alpha)) \leq (2\sqrt{g} - \beta/2) \log \varepsilon N^\alpha \big| \mathscr{F}_\alpha\Big) \\ & \quad \leq \exp[-c(\varepsilon, \beta/2)(\log \varepsilon N^\alpha)^2] \leq \exp[-c_1(\varepsilon, \beta)(\log N)^2], \end{aligned}$$

if $\alpha_o(\beta) \leq \alpha < 1$. Therefore on $F^{\mathsf{C}} \cap \{\phi_B \leq (2\sqrt{g} - \beta) \log N\}$ we have that

$$\begin{aligned} P\Big(\inf_{x \in B} \phi_x \geq 0 \big| \mathscr{F}_\alpha\Big) & \leq P\Big(\inf_{x \in B^{(\varepsilon)}} \phi_x \geq 0 \big| \mathscr{F}_\alpha\Big) \\ & \leq P\Big(\inf_{x \in B^{(\varepsilon)}} (\phi_x - E(\phi_x|\mathscr{F}_\alpha)) \geq -(2\sqrt{g} - \beta + \eta) \log N \big| \mathscr{F}_\alpha\Big) \\ & \leq \exp[-c_1(\varepsilon, \beta)(\log N)^2], \end{aligned}$$

if $\eta \leq \beta/2$ and $\alpha \geq \alpha_o(\beta/2)$. Using this, we get ($c_2(\varepsilon, \beta) = c_1(\varepsilon, \beta/2)$)

$$\begin{aligned} P(A^{\mathsf{C}}_{K,\beta,\alpha} \cap \Omega_D^+) \leq{} & \binom{N^{2-2\alpha}}{K} \{\exp[-c_2(\varepsilon, \beta)(\log N)^2]\}^K \\ & + \exp[-(4g \ \mathrm{cap}_V(D) + 1)(\log N)^2] \end{aligned}$$

$$
\begin{aligned}
&\le \exp\big[(2-2\alpha)K\log N - c_2(\varepsilon,\beta)K(\log N)^2\big]\\
&\quad + \exp\big[-\big(4g\ \mathrm{cap}_V(D)+1\big)(\log N)^2\big]\\
&\le \exp\bigg[-\frac{c_2(\varepsilon,\beta)K}{2}(\log N)^2\bigg]\\
&\quad + \exp\big[-\big(4g\ \mathrm{cap}_V(D)+1\big)(\log N)^2\big].
\end{aligned}
$$

If we choose now K large enough such that $c_2(\varepsilon,\beta)K/2 \ge 4g\ \mathrm{cap}_V(D)+1$, we get the desired inequality (17). $\square$

PROOF OF LEMMA 10. Take $f \ge 0$, $f \in C^1(D)$, then on $A_{K,\beta,\alpha}\cap\Omega^+_{D_N}$ we have

$$
\begin{aligned}
&\frac{1}{|\Pi_\alpha|}\sum_{B\in\Pi_\alpha,\,B\subset D_N} f(x_B/N)\phi_B\\
&\qquad\ge (2\sqrt{g}-\beta)\log N\bigg(\frac{1}{|\Pi_\alpha|}\sum_{B\in\Pi_\alpha,\,B\subset D_N} f(x_B/N) - \frac{K\|f\|_\infty}{|\Pi_\alpha|}\bigg).
\end{aligned}
$$

Thus

$$
P(A_{K,\beta,\alpha}\cap\Omega^+_{D_N}) \le \exp\Bigg[-\frac{\big((2\sqrt{g}-\beta)\log N\big(\frac{1}{|\Pi_\alpha|}\sum_B f(x_B/N) - cN^{-2(1-\alpha)}\big)\big)^2}{2\ \mathrm{var}_N\big(\frac{1}{|\Pi_\alpha|}\sum_B f(x_B/N)\phi_B\big)}\Bigg].
$$

In view of Lemma 13, we see that

$$
\limsup_{N\to\infty}\frac{1}{(\log N)^2}\log P\big(A_{K,\beta,\alpha}\cap\Omega^+_{D_N}\big) \le -(2\sqrt{g}-\beta)^2\frac{\big(\int_D f(x)\,dx\big)^2}{2\sigma_V^2(f1_D)}
$$

and we get the result with the alternative definition of the capacity

$$
\mathrm{cap}_V(D) = \sup\bigg\{\frac{\big(\int_D f(x)\,dx\big)^2}{2\sigma_V^2(f1_D)}:\ f\in C^1(D)\bigg\},
$$

compare Lemma 2.2 [4]. $\square$

4. Proof of Theorem 4.

PROOF OF THE UPPER BOUND. We are using the notations of our Theorem 3. The upper bound is quite simple: choose $a = 2\sqrt{g}+\varepsilon/2$ and define P^a_N as above. Then, using the Fortuin–Kasteleyn–Ginibre (FKG) inequality, we have

$$
E_N\big[\phi_x|\Omega^+_{D_N}\big] \le E^a_N\big[\phi_x|\Omega^+_{D_N}\big] \le \frac{E^a_N[|\phi_x|]}{P^a_N(\Omega^+_{D_N})} \le \frac{a\log N + \sigma(N,x)}{P^a_N(\Omega^+_{D_N})}.
$$

Since $\lim_{N\to\infty} P^a_N(\Omega^+_{D_N}) = 1$ and $\sigma(N,x)\le c\sqrt{\log N}$, we see that

$$
\limsup_{N\to\infty}\frac{E_N\big[\phi_x|\Omega^+_{D_N}\big]}{\log N} \le 2\sqrt{g}+\varepsilon/2.
$$

Next, using the Brascamp–Lieb inequality (see [8]) for the conditioned measure $P_N^+ = P_N(\cdot|\Omega_{D_N}^+)$ (see the introduction of [11]), we have, for large N,

$$P_N^+\big(\phi_x \geq (2\sqrt{g}+\varepsilon)\log N\big) \leq P_N^+\bigg(\phi_x - E_N^+[\phi_x] \geq \frac{\varepsilon}{3}\log N\bigg)$$
$$\leq \exp\bigg(-\frac{(\varepsilon^2/9)(\log N)^2}{2\sigma^2(N,x)}\bigg) \leq \exp\big(-c\varepsilon^2\log N\big),$$

and this concludes the proof of the upper bound. □

PROOF OF THE LOWER BOUND. The lower bound is more delicate. For $\delta > 0$, let $D_N^\delta = \{x \in D_N\colon\ \mathrm{dist}(x, D_N^{\complement}) \geq \delta N\}$. Since the boundary of D is smooth, in view of the argument in the proof of Lemma 3.3 in [11], it is sufficient to show that for any $a < 2\sqrt{g}$ and $\delta \in (0,1)$,

$$\lim_{N\to\infty} \sup_{x\in D_N^\delta} P_N^+(\phi_x < a\log N) = 0.$$

For $x \in D_N^\delta$ set $\widetilde{D}_N(x,\rho) = \{y \in D_N\colon |y-x| \leq \rho N\}$. Then, for each $y \in \mathbb{Z}^2$ with $|y| \leq \frac{\delta}{4}N$, we have by FKG,

$$\begin{aligned}P_{V_N}\big(\phi_x \leq a\log N\big|\Omega_{D_N}^+\big) &= P_{V_N+y}\big(\phi_{x+y} \leq a\log N\big|\Omega_{D_N+y}^+\big)\\ &\leq \widetilde{P}_N\Big(\phi_{x+y} \leq a\log N\big|\Omega^+_{\widetilde{D}_N(x,3\delta/4)}\Big)\\ &\leq \widetilde{P}_N\Big(\phi_{x+y} \leq a\log N\big|\Omega^+_{\widetilde{D}_N(x,\delta/2)}\Big)\\ &= \widetilde{P}_N^+\big(\phi_{x+y} \leq a\log N\big),\end{aligned}$$

where we write $\widetilde{P}_N = P(\cdot|\phi_y = 0,\ y \notin \widetilde{D}_N(x, 3\delta/4))$ and $\widetilde{P}_N^+ = \widetilde{P}_N\ (\cdot|\Omega^+_{\widetilde{D}_N(x,\delta/2)})$.

Let $\alpha \in (0,1)$ to be chosen later. We may assume that $x = x_B$ for some box $B \in \Pi_\alpha$ (otherwise just move the grid!). Let $\Lambda = \{x_{B'}\colon |x_B - x_{B'}| \leq \frac{\delta}{4}N\}$, and set $\varepsilon = \frac{2\sqrt{g}-a}{2}$, using the above

$$\begin{aligned}P_N\big(\phi_x \leq a\log N|\Omega_{D_N}^+\big) &\leq \widetilde{E}_N^+\bigg[\frac{1}{|\Lambda|}\sum_{x_{B'}\in\Lambda} 1_{\phi_{x_{B'}} < a\log N}\bigg]\\ &\leq \widetilde{E}_N^+\bigg[\frac{1}{|\Lambda|}\sum_{x_{B'}\in\Lambda} 1_{\phi_{B'} < (2\sqrt{g}-\varepsilon)\log N}\bigg]\\ &\quad + \widetilde{E}_N^+\bigg[\frac{1}{|\Lambda|}\sum_{x_{B'}\in\Lambda} 1_{|\phi_{x_{B'}} - \phi_{B'}| > \varepsilon\log N}\bigg].\end{aligned}$$

Next, define $A'_{K,\varepsilon,\alpha}$ in terms of $\widetilde{D}_N(x,\delta/2)$ as in Lemma 9, then we can choose $\alpha(\varepsilon)$ and $K(\varepsilon)$, such that

$$\begin{aligned}\widetilde{E}_N^+\biggl[\frac{1}{|\Lambda|}\sum_{B'\subset\Lambda}1_{\phi_{B'}<(2\sqrt{g}-\varepsilon)\log N}\biggr] &\le \frac{K}{|\Lambda|}+\frac{\widetilde{P}_N\Bigl(\bigl(A'_{K,\varepsilon,\alpha}\bigr)^{\mathsf{C}}\cap\Omega^+_{\widetilde{D}_N(x,\delta/2)}\Bigr)}{\widetilde{P}_N\Bigl(\Omega^+_{\widetilde{D}_N(x,\delta/2)}\Bigr)}\\ &\le \exp(-c\log N),\end{aligned}$$

where in the last inequality we have used the lower bound in Theorem 6. For the second term, note that $\{\phi_{B'}-\phi_{x_{B'}}, x_{B'}\in\Lambda\}$ are independent under $\widetilde{P}_N(\cdot|\mathscr{F}_\alpha)$ with

$$\begin{aligned}\widetilde{P}_N\bigl(|\phi_{B'}-\phi_{x_{B'}}|>\varepsilon\log N|\mathscr{F}_\alpha\bigr) &\le 2\exp\biggl[-\frac{\varepsilon^2(\log N)^2}{2\sigma^2(N^\alpha,x_{B'})}\biggr]\\ &\le 2\exp\bigl(-c_1\varepsilon^2\log N\bigr).\end{aligned}$$

Set

$$C_{N,\varepsilon,\alpha}=\biggl(\#\bigl\{B'\in\Lambda\colon |\phi_{x_{B'}}-\phi_{B'}|>\varepsilon\log N\bigr\}\ge\frac{|\Lambda|}{\log N}\biggr),$$

then, again using Lemma 11, we see that

$$\widetilde{P}_N\bigl(C_{N,\varepsilon,\alpha}|\mathscr{F}_\alpha\bigr)\le\exp\biggl[-c_1\frac{|\Lambda|}{(\log N)^2}\biggr]\le\exp\bigl(-c_1'N^{(1-\alpha)}\bigr).$$

However, this together with the lower bound in Theorem 3 implies

$$\widetilde{E}_N^+\biggl[\frac{1}{|\Lambda|}\sum_{B'\in\Lambda}1_{|\phi_{x_{B'}}-\phi_{B'}|>\varepsilon\log N}\biggr]\le\frac{1}{\log N}+\frac{\widetilde{P}_N(C_{N,\varepsilon,\alpha})}{\widetilde{P}_N\Bigl(\Omega^+_{D_N^{\text{"}}}\Bigr)}\le\frac{c_2}{\log N},$$

and concludes the proof. $\square$

5. Technical lemmas.

LEMMA 11. *Let $Z_1,\dots,Z_n$ be i.i.d. real valued random variables satisfying $EZ_i=0$, $\sigma^2=EZ_i^2$, $\|Z_i\|_\infty\le 1$. Then for any $t>0$,*

$$P\biggl(\biggl|\sum_{i=1}^n Z_i\biggr|\ge t\biggr)\le 2\exp\biggl[-\frac{t^2}{2n\sigma^2+2t/3}\biggr].$$

This is a standard large deviation estimate, see, for example, [2].

LEMMA 12. *Let $0<n<N$, and B be a $n\times n$-square of side length n with $x_B=x_N$. If $x\in B$ satisfies $|x-x_B|\le\varepsilon n$, where $\varepsilon\le\frac{1}{2}$, then*

$$\operatorname{var}_N\bigl(E(\phi_x|\mathscr{F}_{\partial B})-\phi_B\bigr)\le c\varepsilon.$$

PROOF.

$$
\begin{aligned}
\operatorname{var}_N\big(E[\phi_x|_{\partial B}]-\phi_B\big) &= \operatorname{var}_N\big(E[\phi_x-\phi_{x_B}|_{\partial B}]\big)\\
&= \operatorname{var}_N(\phi_x-\phi_{x_B})-\operatorname{var}_B(\phi_x-\phi_{x_B})\\
&= G_N(x,x)+G_N(x_B,x_B)-2G_N(x,x_B)\\
&\quad -G_B(x,x)-G_B(x_B,x_B)+2G_B(x,x_B)\\
&= \mathbb{E}_x\Bigg[\sum_{i=\tau_{\partial B}}^{\tau_{\partial V_N}}\big(1_x(\eta_i)-1_{x_B}(\eta_i)\big)\Bigg]\\
&\quad +\mathbb{E}_{x_B}\Bigg[\sum_{i=\tau_{\partial B}}^{\tau_{\partial V_N}}\big(1_{x_B}(\eta_i)-1_x(\eta_i)\big)\Bigg].
\end{aligned}
$$

Note also that

$$
\operatorname{var}_N\big(E[\phi_x-\phi_{x_B}|_{\partial B}]\big)\le \operatorname{var}_{N+1}\big(E[\phi_x-\phi_{x_B}|_{\partial B}]\big),
$$

so that, using the strong Markov property,

$$
\begin{aligned}
\operatorname{var}_N\big(E[\phi_x|_{\partial B}]-\phi_B\big) &\le \mathbb{E}_{x_B}\Bigg[\sum_{i=\tau_{\partial B}}^{\infty}\big(1_{x_B}(\eta_i)-1_x(\eta_i)\big)\Bigg]\\
&\quad +\mathbb{E}_x\Bigg[\sum_{i=\tau_{\partial B}}^{\infty}\big(1_{x_B}(\eta_i)-1_x(\eta_i)\big)\Bigg]\\
&= \mathbb{E}_{x_B}\Bigg[\mathbb{E}_{\eta_{\tau_{\partial B}}}\Bigg[\sum_{i=0}^{\infty}\big(1_{x_B}(\eta_i)-1_x(\eta_i)\big)\Bigg]\Bigg]\\
&\quad +\mathbb{E}_x\Bigg[\mathbb{E}_{\eta_{\tau_{\partial B}}}\Bigg[\sum_{n=0}^{\infty}\big(1_{x_B}(\eta_i)-1_x(\eta_i)\big)\Bigg]\Bigg].
\end{aligned}
$$

Let $a(x)=\sum_{n=0}^{\infty}(P_n(0,0)-P_n(0,x))$, where $P_n(x,y)=\mathbb{P}_x(\eta_n=y)$. Fix $y=\eta_{\tau_{\partial B}}\in\partial B$, then in view of Theorem 1.6.2 in [15],

$$
\begin{aligned}
\mathbb{E}_y\Bigg[\sum_{n=0}^{\infty}(1_{x_B}(\eta_i)-1_x(\eta_i))\Bigg] &= \mathbb{E}_0\Bigg[\sum_{n=0}^{\infty}(1_{x_B-y}(\eta_i)-1_{x-y}(\eta_i))\Bigg]\\
&= a(x-y)-a(x_B-y)\\
&= \frac{2}{\pi}\log(|x-y|)-\frac{2}{\pi}\log(|x_B-y|)+O(|x-y|^{-2})\\
&= \frac{2}{\pi}\log\left(\frac{|x-y|}{|x_B-y|}\right)+O(n^{-2}),
\end{aligned}
$$

where $\log(1-\varepsilon)\le\log\left(\frac{|x-y|}{|x_B-y|}\right)\le\log(1+\varepsilon)$. $\square$

In what follows we will consider the class $\mathscr{C}$ of functions defined as follows: $F\colon V \to \mathbb{R}$ belongs to $\mathscr{C}$ if there exist two sequences of nonnegative functions, $\{\underline{F}_n\}$ and $\{\overline{F}_n\}$, in $C_0(V)$ such that $\underline{F}_n \le F \le \overline{F}_n$ for every n and such that $\lim_{n\to\infty} \underline{F}_n(x) = \lim_{n\to\infty} \overline{F}_n(x) = F(x)$ for every x. We observe that if $D \subset V$ has a piecewise smooth boundary which does not intersect the boundary of V, then $F1_D \in \mathscr{C}$ for any continuous function F.

LEMMA 13. *For any $\alpha \in [0, 1)$ and $F \in \mathscr{C}$ we have*

$$\lim_{N\to\infty} \operatorname{var}\biggl(|\Pi_\alpha|^{-1} \sum_{B\in\Pi_\alpha} F(x_B/N)\phi_B\biggr) = \sigma_V^2(F)$$
$$= \int_V \int_V F(x)\mathscr{G}_V(x, y)F(y)\,dx\,dy,$$

which is independent of α, where $\mathscr{G}_V$ is the Green function of the Brownian motion, killed as it exits V. For $\alpha = 0$ the above mean has to be interpreted as the sum of the ϕ_x, $x \in V_N$.

PROOF. We start with the case $\alpha = 0$, and show that

$$\lim_{N\to\infty} \operatorname{var}\biggl(|V_N|^{-1} \sum_{x\in V_N} F(x/N)\phi_x\biggr) = \sigma_V^2(F). \tag{20}$$

But this follows from the invariance principle (cf. Lemma 2.10 of [1]) if $F \in C_0(V)$. The validity of (20) is extended to $F \in \mathscr{C}$ by observing that, if we set $F_n = \underline{F}_n$ or $F_n = \overline{F}_n$, then $\sigma_V^2(F_n) \to \sigma_V^2(F)$ and by using the positivity of the correlations of the free field.

Next, note that

$$\operatorname{var}\biggl(|\Pi_\alpha|^{-1} \sum_{B\in\Pi_\alpha} F(x_B/N)\phi_B\biggr) = \operatorname{var}\biggl(|\Pi_\alpha|^{-1} \sum_{B\in\Pi_\alpha} F(x_B/N)\phi_{x_B}\biggr)$$
$$- \operatorname{var}\biggl(|\Pi_\alpha|^{-1} \sum_{B\in\Pi_\alpha} F(x_B/N)\phi_{x_B}|\mathscr{F}_\alpha\biggr)$$

Using the independence of the $\{\phi_{x_B}\}$ under $P(\cdot|\mathscr{F}_\alpha)$, and $\operatorname{var}(\phi_{x_B}|\mathscr{F}_\alpha) \le g\log(N^\alpha) + c$, we see that the second term is given by

$$\operatorname{var}\biggl(|\Pi_\alpha|^{-1} \sum_{B\in\Pi_\alpha} F(x_B/N)\phi_{x_B}\big|\mathscr{F}_\alpha\biggr) = |\Pi_\alpha|^{-2} \sum_{B\in\Pi_\alpha} F(x_B/N)^2 \operatorname{var}(\phi_{x_B}|\mathscr{F}_\alpha)$$
$$\le |\Pi_\alpha|^{-1}(g\alpha\log N + c) = o(1)$$

as $N \to \infty$. Thus all we need to show is

$$\begin{aligned}
&\lim_{N\to\infty} \operatorname{var}\biggl(|\Pi_\alpha|^{-1} \sum_{B\in\Pi_\alpha} F(x_B/N)\phi_{x_B}\biggr) \\
&\quad = \lim_{N\to\infty} |\Pi_\alpha|^{-2} \sum_{B,\, B'\in\Pi_\alpha} F(x_B/N) G_N(x_B, x_{B'}) F(x_{B'}/N) \\
&\quad \equiv \lim_{N\to\infty} \sigma^2_{V,\,N}(F) = \sigma^2_V(F).
\end{aligned}$$

The idea is to apply again the invariance principle: by the same argument as before, it is sufficient to consider the case $F \in C_0(V)$. Let $\Lambda = N^\alpha \mathbb{Z}^2 + ([N^\alpha/2], [N^\alpha/2])$ be the grid of mash N^α containing the set $\{x_B\colon\ B \in \Pi_\alpha\}$ and introduce the rescaled embedded random walk $\{\hat\eta_n = \eta_{\tau(n)}/N^\alpha,\ n \geq 0\}$ where $\tau(0) = 0$ and

$$\tau(n) = \inf\{k > \tau(n-1)\colon\ \eta_k \in \Lambda\}, \qquad n \geq 1.$$

Next let $\widehat{P}(y, z) = \mathbb{P}(\hat\eta_n = z | \hat\eta_0 = y)$ be the corresponding transition kernels and write $\widehat{\mathbb{E}}_y$ for the law of $\{\hat\eta\}$ starting at $\hat\eta_0 = y$. For $\theta \in \mathbb{R}^d$ with $|\theta| = 1$, set for some fixed z,

$$\widehat{A}(\theta) = \sum_{y\in\Lambda} ((y - z)\cdot\theta)^2 \widehat{P}_1(z, y).$$

We claim that $\widehat{A}(\theta) = A(\theta) = \frac{1}{2}$, the variance of the simple random walk: define

$$\hat a(x) = \sum_{n=0}^{\infty} \bigl(\widehat{P}_n(z, z) - \widehat{P}_n(z, z + x)\bigr),$$

then we know that

$$\widehat{A}(\theta) = \lim_{|y|\to\infty} \frac{\log|y|}{\pi \hat a(y)}$$

(cf. [16], P12.3). On the other hand, if P_n denotes the transition kernels of the simple random walk, then

$$\begin{aligned}
a(N^\alpha x) &= \sum_{k=0}^{\infty} \bigl(P_k(0,0) - P_k(0, N^\alpha x)\bigr) \\
&= \mathbb{E}_0\biggl[\sum_{k=0}^{\infty} \bigl(1_0(\eta_k) - 1_{N^\alpha x}(\eta_k)\bigr)\biggr] \\
&= \mathbb{E}_z\biggl[\sum_{n=0}^{\infty} \bigl(1_z(\hat\eta_n) - 1_{z+x}(\hat\eta_n)\bigr)\biggr] = \hat a(x),
\end{aligned}$$

and therefore,

$$A(\theta) = \lim_{|x|\to\infty} \frac{\log|x|}{\pi a(x)} = \lim_{|y|\to\infty} \frac{\log|y|}{\pi \hat a(y)} = \widehat{A}(\theta).$$

Next let $\widehat{V}_N = \{y = \frac{x}{N^\alpha},\ x \in \Lambda \cap V_N\}$, introduce the exit time $\hat{\tau}_N = \inf\{n \geq 0\colon \hat{\eta}_n \notin \widehat{V}_N\}$ and the Green function

$$\widehat{G}_N(y, z) = \widehat{\mathbb{E}}_y\Bigg[\sum_{n=0}^{\hat{\tau}_N - 1} 1_z(\hat{\eta}_n)\Bigg], \qquad z, y \in \widehat{V}_N.$$

Then applying the invariance principle for the random walk $\{\hat{\eta}\}$, and using the fact that $A(\theta) = \widehat{A}(\theta)$, we have

$$\begin{aligned}
\lim_{N\to\infty} \hat{\sigma}^2_{V,N}(F) &\equiv \lim_{N\to\infty} |\Pi_\alpha|^{-2} \sum_{y, z \in \widehat{V}_N} F(y/N^{1-\alpha})\widehat{G}_N(y, z)F(z/N^{1-\alpha})\\
&= \lim_{N\to\infty} |\Pi_\alpha|^{-2} \sum_{y \in \widehat{V}_N} F(y/N^{1-\alpha})\widehat{\mathbb{E}}_y\Bigg[\sum_{n=0}^{\hat{\tau}_N - 1} F(\hat{\eta}_n/N^{1-\alpha})\Bigg].\\
&= \sigma^2_V(F).
\end{aligned}$$

On the other hand, setting $\tilde{\tau}_N = \inf\{n \geq 0\colon \tau(n) \geq \tau_{\partial_{V_N}}\}$, we have

$$\sigma^2_{V,N}(F) = |\Pi_\alpha|^{-2} \sum_{y \in \widehat{V}_N} F(y/N^{1-\alpha})\widehat{\mathbb{E}}_y\Bigg[\sum_{n=0}^{\tilde{\tau}_N - 1} F(\hat{\eta}_n/N^{1-\alpha})\Bigg].$$

Obviously $\tilde{\tau}_N \leq \hat{\tau}_N$, and we get our result as soon as we show that

$$\begin{aligned}
&\lim_{N\to\infty}\Big[\hat{\sigma}^2_{N,V}(F) - \sigma^2_{N,V}(F)\Big]\\
&\quad = \lim_{N\to\infty} |\Pi_\alpha|^{-2} \sum_{y \in \widehat{V}_N} F(y/N^{1-\alpha})\widehat{E}_y\Bigg[\sum_{n=\tilde{\tau}_N}^{\hat{\tau}_N - 1} F(\hat{\eta}_n/N^{1-\alpha})\Bigg] = 0.
\end{aligned}$$

Since F is bounded with compact support, we can assume that $F(x) = 1_{V^\delta}(x)$, where $V^\delta = [\delta, 1-\delta]^2$. Let $\varepsilon \in (0, \delta)$, and introduce the stopping time $\hat{\tau}^\varepsilon_N = \inf\{n \geq 0\colon \hat{\eta}_n \notin \widehat{V}^\varepsilon_N\}$, for some $\varepsilon < \delta$, where $\widehat{V}^\varepsilon_N = \{y = N^{-\alpha}x\colon x \in \Lambda \cap V^\varepsilon_N\}$, then

$$\begin{aligned}
\widehat{\mathbb{E}}_y\Bigg[\sum_{n=\tilde{\tau}_N}^{\hat{\tau}_N - 1} F(\hat{\eta}_n/N^{1-\alpha})\Bigg] &\leq \widehat{\mathbb{E}}_y\Bigg[\sum_{n=\tilde{\tau}_N}^{\hat{\tau}^\varepsilon_N - 1} F(\hat{\eta}_n/N^{1-\alpha}); \hat{\tau}^\varepsilon_N > \bar{\tau}_N\Bigg]\\
&\quad + \widehat{\mathbb{E}}_y\Bigg[\sum_{n=\hat{\tau}^\varepsilon_N}^{\hat{\tau}_N - 1} F(\hat{\eta}_n/N^{1-\alpha})\Bigg].
\end{aligned}$$

Thus

$$\begin{aligned}
0 &\leq \hat{\sigma}^2_{N,V}(F) - \sigma^2_{N,V}(F) \leq \hat{\sigma}^2_{N,V}(F) - \hat{\sigma}^2_{N,V^\varepsilon}(F)\\
&\quad + |\Pi_\alpha|^{-2} \sum_{y \in \widehat{V}_N} F(y/N^{1-\alpha})\widehat{\mathbb{E}}_y\Bigg[\sum_{n=\tilde{\tau}_N}^{\hat{\tau}^\varepsilon_N - 1} F(\hat{\eta}_n/N^{1-\alpha}); \hat{\tau}^\varepsilon_N > \bar{\tau}_N\Bigg],
\end{aligned}$$

where

$$\lim_{N\to\infty}\left[\hat{\sigma}^2_{N,V}(F)-\hat{\sigma}^2_{N,V^\varepsilon}(F)\right]=\sigma^2_V(F)-\sigma^2_{V^\varepsilon}(F)\longrightarrow 0 \quad \text{as } \varepsilon\to 0,$$

and we get our result once we show that

$$\lim_{N\to\infty}|\Pi_\alpha|^{-1}\sup_{y\in\widehat{V}^\delta_N}\widehat{\mathbb{E}}_y\left[\sum_{n=\tilde{\tau}_N}^{\hat{\tau}^\varepsilon_N-1}1_{\widehat{V}^\delta_N}(\hat{\eta}_n);\hat{\tau}^\varepsilon_N>\tilde{\tau}_N\right]=0.$$

Write $\tau^\varepsilon_N=\inf\{k\ge 0\colon\ \eta_k\in\Lambda\backslash V^\varepsilon_N\}$ for the first time that the random walk $\{\eta\}$ gets trapped on Λ outside of V^ε_N, then

$$\widehat{\mathbb{E}}_y\left[\sum_{n=\tilde{\tau}_N}^{\hat{\tau}^\varepsilon_N-1}1_{\widehat{V}^\delta_N}(\hat{\eta}_n);\hat{\tau}^\varepsilon_N>\tilde{\tau}_N\right]=\mathbb{E}_{N^\alpha y}\left[\sum_{k=\tau_{\partial V_N}}^{\tau^\varepsilon_N-1}1_{V^\delta_N\cap\Lambda}(\eta_k);\tau^\varepsilon_N>\tau_{\partial V_N}\right].$$

Using the strong Markov property, we see that

$$|\Pi_\alpha|^{-1}\mathbb{E}_{yN^\alpha}\left[\sum_{k=\tau_{\partial V_N}}^{\tau^\varepsilon_N-1}1_{V^\delta_N\cap\Lambda}(\eta_k);\tau^\varepsilon_N>\tau_{\partial V_N}\right]\le C\mathbb{P}_{N^\alpha y}(\tau^\varepsilon_N>\tau_{\partial V_N}),$$

where, in view of the invariance principle,

$$C=\sup_{N,\,\varepsilon}\sup_{z\in\partial V_N}|\Pi_\alpha|^{-1}\mathbb{E}_z\left[\sum_{n=0}^{\tau^\varepsilon_N-1}1_{V^\delta_N\cap\Lambda}(\eta_n)\right]<\infty.$$

Finally, we can use the result of Lemma 3.3 of [12], and show that for each fixed $0<\varepsilon<\delta$, the probability for the random walk $\{\eta\}$ starting in V^δ_N to reach the boundary ∂V_N before getting trapped in $\Lambda\cap(V_N\backslash V^\varepsilon_N)$ goes to 0,

$$\lim_{N\to\infty}\sup_{y\in V^\delta_N}\mathbb{P}_{N^\alpha y}(\bar{\tau}_N<\hat{\tau}^\varepsilon_N)=0.$$

This shows the result. $\square$

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