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DOMINOS AND THE GAUSSIAN FREE FIELD

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We define a scaling limit of the height function on the domino tiling model (dimer model) on simply connected regions in $\mathbb{Z}^2$ and show that it is the "massless free field," a Gaussian process with independent coefficients when expanded in the eigenbasis of the Laplacian.

1. Introduction. A *domino tiling* of a polyomino P in $\mathbb{Z}^2$ is a tiling of P with 2×1 and 1×2 rectangles. For a polyomino P let $\mu = \mu(P)$ denote the uniform measure on the set of all domino tilings of P.

Let $U \subset \mathbb{R}^2$ be a Jordan domain with smooth boundary. We study uniform random domino tilings of polyominos P_ε in $\varepsilon\mathbb{Z}^2$ which approximate U (and using dominos which are $2\varepsilon \times \varepsilon$ and $\varepsilon \times 2\varepsilon$ rectangles).

A domino tiling of a polyomino P_ε in $\varepsilon\mathbb{Z}^2$ can be thought of as a random map from $\varepsilon\mathbb{Z}^2 \cap P_\varepsilon$ to $\mathbb{Z}$ in the following way. Let $V_\varepsilon = \varepsilon\mathbb{Z}^2 \cap P_\varepsilon$ be the set of lattice points in the polyomino P_ε. Let $h: V_\varepsilon \to \mathbb{Z}$ be a function which has the property that around every lattice square of P_ε the four values of h are four consecutive integers h_0, h_0+1, h_0+2, h_0+3, with the values on any two adjacent boundary vertices of P_ε differing by 1. The set of such functions h (up to additive constants and a global sign change) is in bijection with the set of domino tilings of P_ε: dominos cross exactly those edges whose h-difference is 3. The function h associated to a tiling is called its *height function* [16]. See Figure 1. Note that the height function takes values in $\mathbb{Z}$, not in $\varepsilon\mathbb{Z}$.

Our aim is to prove that in the limit as $\varepsilon \to 0$ the height function on a random tiling of P_ε tends to a random (generalized) function which has a succinct description in terms of the eigenbasis of the Laplacian operator on U.

THEOREM 1.1. *Let U be a Jordan domain with smooth boundary in $\mathbb{R}^2$. For each $\varepsilon > 0$ sufficiently small let P_ε be a Temperleyan polyomino approximating U as described below. Let h_ε be the height of a random domino tiling of P_ε and $\bar{h}_\varepsilon$ be its mean value. Then as ε tends to 0, $h_\varepsilon - \bar{h}_\varepsilon$ tends weakly in distribution to $4/\sqrt{\pi}$ times the "massless free field" F on U, in the sense that for any smooth function ϕ on U, the random variable $\varepsilon^2 \sum_{x \in V_\varepsilon} \phi(x)(h_\varepsilon(x) - \overline{h_\varepsilon(x)})$ tends in distribution to $\frac{4}{\sqrt{\pi}} \int_U \phi F \, dx\, dy$.*

For the definition of Temperleyan polyominos see below. The *massless free field* F on U is a random variable taking values in the space of distributions

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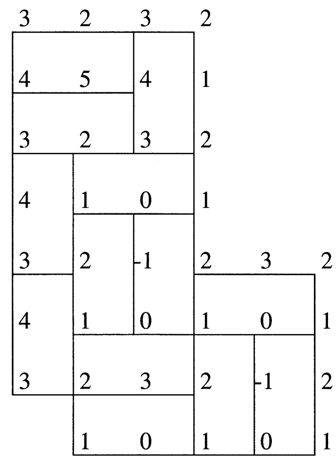

FIG. 1. *Height function of a domino tiling.*

(henceforth we will refer to these objects as "generalized functions" to avoid confusion) which are continuous linear functionals on the space of C^1 functions on U (with a C^1-norm). For background on the massless free field see [13]. It can be defined as follows: let $\{f_i\}_{i\geq 1}$ be an L^2-orthonormal eigenbasis for the Laplacian $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ on U with Dirichlet boundary conditions (that is, $f_i \equiv 0$ on ∂U). Let λ_i be the eigenvalue of f_i. Then

$$F = \sum_{i\geq 1} \frac{c_i f_i}{(-\lambda_i)^{1/2}}, \tag{1}$$

where the c_i are i.i.d. Gaussian random variables of mean 0 and variance 1. Here this expression is interpreted as the generalized function F satisfying, for any C^1 function ϕ,

$$\int_U \phi F \, dx \, dy = \sum_{i\geq 1} \frac{c_i}{(-\lambda_i)^{1/2}} \int_U \phi f_i \, dx \, dy,$$

series which converges almost surely. The expression (1) does not define a function since the series diverges almost everywhere.

REMARKS.

1. The above theorem describes the limiting value of $h_\varepsilon - \bar{h}_\varepsilon$. The limiting average value $\bar{h} = \lim \bar{h}_\varepsilon$ was computed in [5]: it is a harmonic function whose boundary values are given by $\frac{2}{\pi}$ times the angle of turning of the boundary tangent counterclockwise from a fixed basepoint. (Regarding the choice of basepoint, see the definition of "Temperleyan" polyomino below.)

2. Theorem 1.1 has a well-known one-dimensional analog: let X be the sets of random maps h from $0, \frac{1}{n}, \frac{2}{n}, \dots, 1$ to $\mathbb{Z}$ satisfying $h(0) = h(1) = 0$ and $|h(\frac{i+1}{n}) - h(\frac{i}{n})| = 1$. A random element of X, when divided by $\sqrt{n}$, converges to a random function known as the "Brownian bridge" [8]. In the eigenbasis of the one-dimensional Laplacian $\frac{\partial^2}{\partial x^2}$ the coefficients of the Brownian bridge are again independent Gaussians. One difference between the one-dimensional case and Theorem 1.1, however, is that the height function h of Theorem 1.1 is *unnormalized*. It is therefore all the more surprising that the integer-valued function h of Theorem 1.1 converges to a continuous-valued object.

3. Theorem 1.1 was known (nonrigorously) to physicists: see for example papers of Nienhuis [9, 10]. A related model where a similar result is shown rigorously is [11].

4. An open problem is to compute the distribution of the height function on a nonsimply connected domain, even an annulus. In particular for an annulus the distribution of the height difference between the two boundary components (in the limit $\varepsilon \to 0$) is unknown although it was shown in [5] to depend only on the conformal modulus of the annulus.

5. Temperley [14] gave a bijection between the *uniform spanning tree process* on subgraphs of $\mathbb{Z}^2$ and domino tilings. The function h of Theorem 1.1 corresponds under this bijection to the "winding number" of the branches of a spanning tree [5], as first conjectured by I. Benjamini. As it is an open question to show that a scaling limit exists for the uniform spanning tree process [1, 12], one might hope that the reconstruction of the tree from its winding numbers, which is possible for $\varepsilon > 0$, also works in the limit $\varepsilon = 0$. So far this remains an open problem.

6. The result of Theorem 1.1 depends strongly on the choice of boundary conditions for the approximating polyominos P_ε. For even slight generalizations of these boundary conditions our methods will not work; see [5] for a discussion of this issue.

7. When the region U is a rectangle $U = [0, a] \times [0, b]$, the orthonormal eigenvectors of Δ with Dirichlet boundary conditions are $\frac{4}{ab} \sin \frac{\pi jx}{a} \sin \frac{\pi ky}{b}$, where j, k are positive integers. So in this case the massless free field has independent Fourier coefficients.

8. Most of the work to prove Theorem 1.1 was done in [5], where we proved Proposition 2.2, below.

If we consider the massless free field F to be a continuous linear functional on the space of smooth 2-forms on U (rather than on the space of smooth functions on U) then F is *conformally invariant*, in the following sense.

PROPOSITION 1.2. *Let ω be a smooth 2-form on U and let $f: V \to U$ be a conformal bijection. Let F_U, F_V be the massless free fields on U and V, respectively. Let $X = \int_U F_U(z)\omega(z)$ and $Y = \int_V F_V(z) f^*\omega(z)$, where $f^*\omega$ is the pullback of ω to V. Then the random variables X and Y are equal in distribution.*

For the proof see Section 4.

2. Background and preliminaries.

2.1. *Temperleyan polyominos and approximation.* Define the (i, j)-lattice square in $\mathbb{Z}^2$ to be the lattice square whose lower left corner is (i, j). A lattice square is said to be *even* if the coordinates of its lower left corner are even. A *polyomino* is a union of lattice squares which is bounded by a simple closed lattice curve. A polyomino is *even* if all of its *corner squares* are even, where by corner squares we mean those lattice squares adjacent to the corners and containing the interior angle bisector at the corner. In particular note that an edge of an even polyomino P' has odd length if its two extremities are both concave or both convex corners; if the extremities consist of one concave and one convex corner the edge length is even. Let P be a polyomino obtained from an even polyomino P' by removing one lattice square b adjacent to its boundary, where b is of the same parity as the corners of P'. Such a polyomino is called *Temperleyan*, and the removed square b is called its *root*. In Figure 1, the polyomino is Temperleyan with root the lower left (removed) square.

All Temperleyan polyominos have domino tilings ([5], Section 7). The term *Temperleyan* comes from the bijection due to Temperley between the set of spanning trees of a rectangle in $\mathbb{Z}^2$ and the set of domino tilings of a rectangular region with a corner removed [14]. This bijection was generalized in [3] and further in [7].

Let U be a smooth Jordan domain with a marked point $b \in \partial U$. For each $\varepsilon > 0$ let P_ε be a Temperleyan polyomino in $\varepsilon\mathbb{Z}^2$ approximating U as follows.

1. The boundary of P_ε lies within $O(\varepsilon)$ of ∂U, and the counterclockwise boundary path of P_ε points locally into the same half-space as the (directed) tangent to ∂U which it is near.
2. The root b_ε of P_ε is within $O(\varepsilon)$ of b.
3. Somewhere on the boundary of P_ε there must be a segment of length δ on which the boundary of P_ε is straight (exactly vertical or exactly horizontal), where δ tends to zero sufficiently slowly: in such a way that $\delta/\varepsilon \to \infty$.

This last requirement is a technical one necessary to make the proof of [5], Theorem 13, upon which Proposition 2.2 below relies, work.

2.2. *The Green's function.* Let U be a Jordan domain with basepoint $b \in \partial U$. The Green's function with Dirichlet boundary conditions, or simply Green's function, $g(z_1, z_2)$, is defined as follows. For fixed z_1 in the interior of U, $g(z_1, z_2)$ is the unique real-valued function of z_2 satisfying $\Delta g(z_1, z_2) = \delta_{z_1}(z_2)$ (the Dirac delta), and which is zero when $z_2 \in \partial U$ (the Laplacian is with respect to z_2). This function is well defined and when z_2 is near z_1 has the form $g(z_1, z_2) = \frac{1}{2\pi} \log|z_2 - z_1| + O(1)$.

The Green's function has the following simple expression in the basis of eigenfunctions of the Laplacian on U.

LEMMA 2.1.

$$g(z_1, z_2) = \sum_{i\geq 1} \frac{f_i(z_1)f_i(z_2)}{\lambda_i}.$$

PROOF. Since the eigenbasis $\{f_i\}$ of Δ is an orthonormal basis for $L^2(U)$, it suffices to show that for each i, $\langle f_i(z_2), g(z_1, z_2)\rangle = \frac{f_i(z_1)}{\lambda_i}$. But

$$\begin{aligned}\langle f_i x(z_2), g(z_1, z_2)\rangle &= \frac{1}{\lambda_i}\langle \lambda_i f_i(z_2), g(z_1, z_2)\rangle \\ &= \frac{1}{\lambda_i}\langle \Delta f_i(z_2), g(z_1, z_2)\rangle \\ &= \frac{1}{\lambda_i}\langle f_i(z_2), \Delta g(z_1, z_2)\rangle \\ &= \frac{1}{\lambda_i}\langle f_i(z_2), \delta_{z_1}(z_2)\rangle \\ &= \frac{1}{\lambda_i} f_i(z_1).\end{aligned}$$

□

Let $\hat{g}(z_1, z_2)$ be the harmonic conjugate (with respect to the second variable) of $g(z_1, z_2)$. This function is only defined up to an additive constant and is moreover multiply valued, increasing by 1 when z_2 turns counterclockwise around z_1. We define the additive constant so that $\hat{g}(z_1, b)$ is locally independent of z_1 [since b is on the boundary $g(z_1, b)$ is single-valued as z_1 varies]. The function $\tilde{g}(z_1, z_2) := g(z_1, z_2) + i\hat{g}(z_1, z_2)$ is analytic in z_2 except at z_1, and is the *analytic Green's function*. It is also multiply valued.

As examples of these functions, on the upper half-plane $\mathbb{H}$ with $b = \infty$ we have

$$g(z_1, z_2) = \frac{1}{2\pi} \log\left|\frac{z_2 - z_1}{z_2 - \bar{z}_1}\right| \tag{2}$$

and

$$\tilde{g}(z_1, z_2) = \frac{1}{2\pi} \log\left(\frac{z_2 - z_1}{z_2 - \bar{z}_1}\right). \tag{3}$$

For a more general Jordan domain V, let f be a Riemann map from V to the upper half-plane sending b (the base point of V) to ∞. Then the analytic Green's function on V satisfies $\tilde{g}^V(z_1, z_2) = \tilde{g}^{\mathbb{H}}(f(z_1), f(z_2))$.

2.3. *Moment formula.* For a region U with basepoint $b \in \partial U$ and analytic Green's function $\tilde{g}$, define the functions $F_+(z_1, z_2)$ and $F_-(z_1, z_2)$ by

$$-4d\tilde{g}(z_1, z_2) = F_+(z_1, z_2)dz_1 + F_-(z_1, z_2)d\overline{z_1}, \tag{4}$$

where d is exterior differentiation with respect to the first variable. These functions $F_\pm$ are single-valued and zero at $z_2 = b$. The function $F_+(z_1, z_2)$ is analytic in both variables (or rather, meromorphic with a pole at $z_2 = z_1$), and $F_-(z_1, z_2)$ is analytic in z_2 and antianalytic in z_1 (and F_- has no poles).

Let $h_0(x) = h(x) - \overline{h(x)}$. The following proposition appeared in [5] in a more general form. In that paper we were interested in computing height moments of points lying on different boundary components of a nonsimply connected domain. In particular in [5] there are extra hypotheses put on the structure of these boundary components (and are similar to the third condition on approximation discussed in Section 2.1). These hypotheses are unnecessary in the present case where the points at which we are evaluating the height h_0 lie in the interior of the domain. Indeed the proof can be simplified in this case.

PROPOSITION 2.2 [5]. *Under the hypotheses of Theorem* 1.1, *let* $z_1, \dots, z_k$ *be distinct points of* U, *and* $\gamma_1, \dots, \gamma_k$ *disjoint paths running from the boundary of* U *to* $z_1, \dots, z_k$, *respectively. Let* $h(z_1)$ *denote the height of a point of* P_ε *lying within* $O(\varepsilon)$ *of* z_1. *Then*

$$\begin{aligned}(5)\qquad &\lim_{\varepsilon\to 0}\mathbb{E}(h_0(z_1)\cdots h_0(z_k))\\ &= (-i)^k \sum_{\varepsilon_1,\dots,\varepsilon_k\in\{\pm 1\}} \varepsilon_1\cdots\varepsilon_k \int_{\gamma_1}\cdots\int_{\gamma_k} \det_{i,j\in[1,k]}(F_{\varepsilon_i,\varepsilon_j}(z_i,z_j))\,dz_1^{(\varepsilon_1)}\cdots dz_k^{(\varepsilon_k)},\end{aligned}$$

where $z_j^{(1)} = z_j$ *and* $z_j^{(-1)} = \overline{z_j}$, *and*

$$F_{\varepsilon_i,\varepsilon_j}(z_i,z_j) = \begin{cases} 0, & \text{if } i = j,\\ F_+(z_i,z_j), & \text{if } (\varepsilon_i,\varepsilon_j) = (1,1),\\ F_-(z_i,z_j), & \text{if } (\varepsilon_i,\varepsilon_j) = (-1,1),\\ \overline{F_-(z_i,z_j)}, & \text{if } (\varepsilon_i,\varepsilon_j) = (1,-1),\\ \overline{F_+(z_i,z_j)}, & \text{if } (\varepsilon_i,\varepsilon_j) = (-1,-1).\end{cases}$$

Note that $F_{\varepsilon_i,\varepsilon_j}(z_i, z_j)$ is a meromorphic function of $z_i^{(\varepsilon_i)}$ and $z_j^{(\varepsilon_j)}$.

3. Proof of Theorem 1.1. When U is the upper half-plane with basepoint at ∞, the derivative of the analytic Green's function (3) is

$$d\tilde{g}(z_1, z_2) = \frac{dz_1}{2\pi(z_1 - z_2)} - \frac{d\overline{z_1}}{2\pi(\overline{z_1} - z_2)}.$$

Thus from (4) we have $F_+(z_1, z_2) = \frac{2}{\pi(z_2 - z_1)}$ and $F_-(z_1, z_2) = -\frac{2}{\pi(z_2 - \overline{z_1})}$. In (5), the matrix has ij entry

$$F_{\varepsilon_i,\varepsilon_j}(z_i, z_j) = \frac{2\varepsilon_i\varepsilon_j}{\pi(z_j^{(\varepsilon_j)} - z_i^{(\varepsilon_i)})}.$$

Factoring a ε_i out of the ith row and ith column for each i, the matrix has the same determinant as the matrix with ij entry

$$\frac{2}{\pi(z_j^{(\varepsilon_j)} - z_i^{(\varepsilon_i)})}.$$

Such a matrix has a simple determinant.

LEMMA 3.1. *Let M be the $k \times k$ matrix $M = (m_{ij})$ with $m_{ii} = 0$ and $m_{ij} = \frac{1}{x_j - x_i}$ when $i \neq j$. Then for k odd* $\det M = 0$; *for k even we have*

$$\det(M) = \sum \frac{1}{(x_{\sigma(1)} - x_{\sigma(2)})^2 (x_{\sigma(3)} - x_{\sigma(4)})^2 \cdots (x_{\sigma(k-1)} - x_{\sigma(k)})^2}, \tag{6}$$

where the sum is over all $(k-1)!!$ possible pairings $\{\{\sigma(1), \sigma(2)\}, \ldots, \{\sigma(k-1), \sigma(k)\}\}$ of $\{1, \ldots, k\}$.

This lemma also appears in [4].

PROOF. Since M is antisymmetric, $\det M = 0$ when k is odd. We may therefore assume k is even. The proof is by induction on k. The formula clearly holds when $k = 2$. For $k > 2$ and even, the determinant is a rational function of x_1 with a double pole at $x_1 = x_2$; we can write

$$\det(M) = \frac{c_{-2}}{(x_1 - x_2)^2} + \frac{c_{-1}}{(x_1 - x_2)} + c_0 + O(x_1 - x_2).$$

The coefficient c_{-1} is zero since the determinant is even under the exchange of x_1 and x_2 (exchange the first two rows and exchange the first two columns). The coefficient c_{-2} is the determinant of M_{12}, the matrix obtained from M by deleting the first two rows and columns. Therefore the right- and left-hand sides of (6) both represent rational functions (in each variable) with the same poles and residues; hence they differ by a constant. This constant is zero by homogeneity: replacing x_i with λx_i for each i multiplies the determinant by λ^{-k}. □

Let $p, q \in U$. From Proposition 2.2 we have $\lim_{\varepsilon \to 0} \mathbb{E}(h_0(p) h_0(q))$ equal to

$$\begin{aligned}
-\int_{\gamma_1,\gamma_2} &\begin{vmatrix} 0 & F_+(z_1,z_2) \\ F_+(z_2,z_1) & 0 \end{vmatrix} dz_1 dz_2 + \int_{\gamma_1,\gamma_2} \begin{vmatrix} 0 & F_-(z_1,z_2) \\ \overline{F_-(z_2,z_1)} & 0 \end{vmatrix} d\overline{z_1} dz_2 \\
&+ \int_{\gamma_1,\gamma_2} \begin{vmatrix} 0 & \overline{F_-(z_1,z_2)} \\ F_-(z_2,z_1) & 0 \end{vmatrix} dz_1 d\overline{z_2} \\
&- \int_{\gamma_1,\gamma_2} \begin{vmatrix} 0 & \overline{F_+(z_1,z_2)} \\ \overline{F_+(z_2,z_1)} & 0 \end{vmatrix} d\overline{z_1} d\overline{z_2}.
\end{aligned}$$

Plugging in for $F_\pm$ gives

$$\begin{aligned}\lim_{\varepsilon\to 0}\mathbb{E}(h_0(p)h_0(q)) &= -\frac{4}{\pi^2}\int_{\gamma_1}\int_{\gamma_2}\frac{1}{(z_2-z_1)^2}dz_1dz_2\\ &\quad+\frac{4}{\pi^2}\int_{\gamma_1}\int_{\gamma_2}\frac{1}{(z_2-\overline{z_1})^2}d\overline{z_1}dz_2\\ &\quad+\frac{4}{\pi^2}\int_{\gamma_1}\int_{\gamma_2}\frac{1}{(\overline{z_2}-z_1)^2}dz_1d\overline{z_2}\\ &\quad-\frac{4}{\pi^2}\int_{\gamma_1}\int_{\gamma_2}\frac{1}{(\overline{z_2}-\overline{z_1})^2}d\overline{z_1}d\overline{z_2}=\frac{8}{\pi^2}\text{Re }\log\left(\frac{\bar{p}-q}{p-q}\right).\end{aligned}\tag{7}$$

Note that this is $-\frac{16}{\pi}g(p,q)$ where g is the Green's function on U [see (2)]. Let $p_1,\dots,p_k$ be distinct points in the upper half-plane U. Combining the lemma with Proposition 2.2 gives the following.

PROPOSITION 3.2. *Let U be a Jordan domain with smooth boundary. Let $p_1,\dots,p_k\in U$ be distinct points. If k is odd we have* $\lim_{\varepsilon\to 0}\mathbb{E}(h_0(p_1)\cdots h_0(p_k))=0$. *If k is even we have*

$$\lim_{\varepsilon\to 0}\mathbb{E}(h_0(p_1)\cdots h_0(p_k)) = \left(-\frac{16}{\pi}\right)^{k/2}\sum_{\text{pairings }\sigma} g(p_{\sigma(1)},p_{\sigma(2)})\cdots g(p_{\sigma(k-1)},p_{\sigma(k)}).$$

PROOF. When U is the upper half plane this follows by combining Proposition 2.2 with Lemma 3.1 and the calculation (7), in an easy but notationally cumbersome computation which we leave to the reader (one simply inverts the order of the summations over the ε_i and the pairings σ). For arbitrary U, equation (7) shows that $\mathbb{E}(h_0(p_1)h_0(p_2))=-\frac{16}{\pi}g^U(p_1,p_2)$ (where g^U is the Green's function on U) by conformal invariance of the height moments and of g. This completes the proof. □

The proof of Theorem 1.1 is completed as follows. Let $f_{n_1},\dots,f_{n_k}$ be (not necessarily distinct) eigenvectors of Δ with Dirichlet boundary conditions. Let $C_{n_j}^{(\varepsilon)}$ be the real-valued random variable $C_{n_j}^{(\varepsilon)}=\varepsilon^2\sum_{x\in V_\varepsilon}h_0(x)f_{n_j}(x)$, where the sum is over the vertices V_ε of P_ε, and $f_{n_j}(x)$ is f_{n_j} evaluated at the vertex x. We have

$$\begin{aligned}\lim_{\varepsilon\to 0}\mathbb{E}(C_{n_1}^{(\varepsilon)}\cdots C_{n_k}^{(\varepsilon)}) &= \lim_{\varepsilon\to 0}\mathbb{E}\left(\sum_{x_1\in V_\varepsilon}\varepsilon^2h_0(x_1)f_{n_1}(x_1)\cdots\sum_{x_k\in V_\varepsilon}\varepsilon^2h_0(x_k)f_{n_k}(x_k)\right)\\ &= \lim_{\varepsilon\to 0}\sum_{x_1\in V_\varepsilon}\cdots\sum_{x_k\in V_\varepsilon}\varepsilon^2f_{n_1}(x_1)\cdots\varepsilon^2f_{n_k}(x_k)\mathbb{E}(h_0(x_1)\cdots h_0(x_k))\end{aligned}$$

$$= \left(-\frac{16}{\pi}\right)^{k/2} \int_U \cdots \int_U f_{n_1}(x_1) \cdots f_{n_k}(x_k)$$
$$\times \sum_\sigma g(x_{\sigma(1)}, x_{\sigma(2)}) \cdots g(x_{\sigma(k-1)}, x_{\sigma(k)})$$
$$= \left(-\frac{16}{\pi}\right)^{k/2} \sum_\sigma \int_U \cdots \int_U f_{n_1}(x_1) \cdots f_{n_k}(x_k)$$
$$\times \sum_{m_1,\dots,m_{k/2}} \frac{f_{m_1}(x_{\sigma(1)}) f_{m_1}(x_{\sigma(2)})}{\lambda_{m_1}} \cdots \frac{f_{m_{k/2}}(x_{\sigma(k-1)}) f_{m_{k/2}}(x_{\sigma(k)})}{\lambda_{m_{k/2}}}$$
$$= \left(\frac{16}{\pi}\right)^{k/2} \sum_\sigma \frac{\delta_{n_{\sigma(1)}, n_{\sigma(2)}}}{(-\lambda_{n_{\sigma(1)}})} \cdots \frac{\delta_{n_{\sigma(k-1)}, n_{\sigma(k)}}}{(-\lambda_{n_{\sigma(k-1)}})}.$$

By Wick's theorem (see, e.g., [13]), these are exactly the moments for a set of independent Gaussians of mean zero and variances $-\frac{16}{\pi\lambda_i}$. Now to conclude we invoke the following standard probability lemma.

LEMMA 3.3 [2]. *A sequence of* (*multidimensional*) *random variables whose moments converge to the moments of a Gaussian*, *converges itself to a Gaussian*.

This completes the proof. □

4. Proof of Proposition 1.2. Since X and Y are Gaussians (each being the sum of Gaussians), and have mean 0, it suffices to compute their variances. But

$$\begin{aligned}\mathbb{E}(X^2) &= \int_U \int_U \omega(z_1)\omega(z_2)\mathbb{E}(F(z_1)F(z_2)) \\ &= \int_U \int_U \omega(z_1)\omega(z_2) g^U(z_1, z_2) \\ &= \int_V \int_V f^*\omega(y_1) f^*\omega(y_2) g^U(f(y_1), f(y_2)) \\ &= \int_V \int_V f^*\omega(y_1) f^*\omega(y_2) g^V(y_1, y_2) \\ &= \mathbb{E}(Y^2),\end{aligned}$$

where we used the conformal invariance of the Green's function, $g^U(f(y_1), f(y_2)) = g^V(y_1, y_2)$. This completes the proof. □

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