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THE FIRST EXIT TIME OF PLANAR BROWNIAN MOTION FROM THE INTERIOR OF A PARABOLA

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Let D be the interior of a parabola in $\mathbb{R}^2$ and τ_D the first exit time of Brownian motion from D. We show $-\log P(\tau_D > t)$ behaves like $t^{1/3}$ as $t \to \infty$.

1. Introduction. For $x \in \mathbb{R}^n \backslash \{0\}$, we let $\theta(x)$ be the angle between x and the point $(1, 0, \dots, 0)$. The right circular cone of angle $0 < \xi < \pi$ is the domain $\Gamma_\xi = \{x \in \mathbb{R}^n \colon \theta(x) < \xi\}$. Let $\{B_t \colon t \geq 0\}$ be the n-dimensional Brownian motion and denote by E_x and P_x the expectation and probability associated with this motion starting at x and denote by $\tau_\xi = \inf\{t > 0 \colon B_t \notin \Gamma_\xi\}$ its first exit time from Γ_ξ. The following result was proved by Burkholder (1977).

THEOREM A. *There is a number $p(\xi, n)$, defined in terms of the smallest zero of a certain hypergeometric function, such that*

$$E_x(\tau_\xi^p) < \infty, \qquad x \in \Gamma_\xi, \tag{1.1}$$

if and only if $p < p(\xi, n)$.

For $n = 2$ the result reduces to

$$E_x \tau_\xi^p < \infty \tag{1.2}$$

if and only if $p < \pi/2\xi$.

We should mention here that in $\mathbb{R}^2$, formulas for $P_x\{\tau_\xi > t\}$ have existed for many years. Indeed, Spitzer (1958) in his study of the winding of two-dimensional Brownian motion derives an expression for $P_x\{\tau_\xi > t\}$ from which the two-dimensional case (1.2) follows. In DeBlassie (1987), Burkholder's result and techniques from partial differential equations are used to find an exact formula for $P_x\{\tau_\xi > t\}$ as an infinite series involving confluent hypergeometric functions. From this formula the exact asymptotics in t for $P_x\{\tau_\xi > t\}$ follow. Furthermore, his result is also true for more general cones in $\mathbb{R}^n$. Recently, Davis and Zhang (1994) proved an analogue of Burkholder's result for conditioned Brownian motion in Γ_ξ. A uniform treatment of all the above results, with several extensions, is presented in Bañuelos and Smits (1997) where explicit formulas are found for the distributions and expectations of both the conditioned and unconditioned Brownian motion in very

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general cones. Such formulas are derived from the skew product decomposition of Brownian motion and some formulas of Yor (1980) related to the Hartman–Watson distribution. The particular geometric structure of the cone (scale invariance) is essential for these results. The cone (in two dimensions) can be thought of as the domain above the graph of a function of the form $y = \alpha|x|$. The question then arises: are there other unbounded domains above graphs of functions for which one can determine the exact order of integrability of the exit time? To our surprise, this seems to be a nontrivial question. Here we shall deal with the case of parabolas and find information on the asymptotics for the distribution of their exit time. The techniques, based on elementary principles of large deviations, are completely different from those used in the study of cones. Also, the information we obtained is not as precise as that given in DeBlassie (1987) or Bañuelos and Smits (1997) for cones. Finally, before we state our result precisely we note that the tail distribution for one piece of a hyperbola is the same as that of the smallest cone containing it since this cone can be translated into the interior of the hyperbola. Thus, we have a characterization for the tail distribution for all conic sections.

Set $D = \{(x_1, x_2): x_2 > \alpha x_1^2\}$ and denote the exit time of B_t from D by τ_D. Since for each $\xi \in (0, \pi)$ there is some cone Γ_ξ with $D \subseteq \Gamma_\xi$,

$$E_x \tau_D^p < \infty \quad \text{for all } p > 0.$$

On the other hand, since D contains arbitrarily large squares, it is clear that D has no exponential moments:

$$E_x e^{\lambda \tau_D} = \infty \quad \text{for all } \lambda > 0.$$

It seems then natural to conjecture that as $t \to \infty$, $-\log P_x(\tau_D > t)$ is of the form $t^p f(t)$ for some $p > 0$ and $f(t) = o(t^p)$. We will show $p = 1/3$. Our main result is the next theorem.

THEOREM 1.1. *Fix $x \in D$. There are two positive constants A_1 and A_2 such that*

$$-A_1 < \liminf_{t\to\infty} t^{-1/3} \log P_x(\tau_D > t) \le \limsup_{t\to\infty} t^{-1/3} \log P_x(\tau_D > t) < -A_2.$$

Here is an outline of the article. Using a conformal transformation, in Section 2, the problem is changed to the study of the exit time of a diffusion, with singular generator from a strip. We state upper and lower bounds on the tail distribution of the exit time in terms of infinite series involving certain Feynman–Kac functionals, deferring the proof to Section 3. Taking for granted the long-time asymptotics of the functionals, the first term of each series is shown to dominate and the desired bounds follow. In Section 4, bounds on the Feynman–Kac functionals are given in terms of Bessel process expectations. In Section 5, we derive asymptotics of the Bessel expectations. For upper bounds we use the theory of large deviations and for lower bounds we make an ad hoc argument inspired by techniques from the theory of large deviations. This will yield the long-time asymptotics of the Feynman–Kac functionals used in Section 2.

2. Change of coordinates and bounds via infinite series. In this section, we change the problem to the study of the exit time of a degenerate diffusion from an infinite strip. Then we obtain upper and lower bounds by infinite series.

For simplicity, we will assume the starting point x is not the focus of the parabola. Rotate and translate D so its focus is at the origin, its axis of symmetry is along the x_1-axis and it opens to the left. Next cut along the x_1-axis and keep the upper half H of D. By symmetry, it is enough to study the first exit time of Brownian motion from H, with normal reflection at the x_1-axis. Change coordinates $u + iv = \sqrt{x_1 + ix_2}$ (parabolic coordinates), so that for some $k > 0$, H gets transformed into the strip

$$\widetilde{S} = \{(u, v)\colon 0 < u < k,\ v > 0\}.$$

Moreover, the part of the positive x_1-axis bounding H gets mapped to $\partial_1 \widetilde{S} := \{(u, v)\colon 0 < u < k, v = 0\}$ and the negative x_1-axis is mapped to $\partial_2 \widetilde{S} = \{(u, v)\colon u = 0,\ v > 0\}$. By conformality, the normal reflection at the x_1-axis becomes normal reflection at $\partial_1 \widetilde{S}$ and $\partial_2 \widetilde{S}$. Half the Laplacian in the (u, v)-coordinates is

$$L = \frac{1}{8} \frac{1}{u^2 + v^2} \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right). \tag{2.1}$$

Thus it is enough to study the first exit time from $\widetilde{S}$ of the diffusion corresponding to L, with normal reflection at $\partial_1 \widetilde{S}$ and $\partial_2 \widetilde{S}$. By symmetry, this is equivalent to studying the first exit time from

$$S = \{(u, v)\colon -k < u < k\}$$

of the diffusion W_t corresponding to L. Note that since we are assuming the original Brownian motion does not start at the focus of $x_2 = ax_1^2$,

$$W_0 \neq 0. \tag{2.2}$$

Then it is clear from the form of L in (2.1) that W_t never hits 0.

Denoting

$$\tau_S = \inf\{t > 0\colon W_t \notin S\},$$

Theorem 1.1 immediately follows from the next theorem.

THEOREM 2.1. *For each $w = (u, v) \in S \backslash \{0\}$ with $u, v \geq 0$,*

$$-A_1 < \liminf_{t \to \infty} t^{-1/3} \log P_w(\tau_S > t) \leq \limsup_{t \to \infty} t^{-1/3} \log P_w(\tau_S > t) < -A_2,$$

for some positive constants A_1 and A_2.

The first step in the proof is the next lemma, which gives bounds in terms of infinite series. Let Z be the diffusion in $\mathbb{R}^2$ corresponding to

$$L_Z = \frac{1}{8}\frac{1}{v^2+\xi^2+k^2}\left[\frac{\partial^2}{\partial v^2}+\frac{\partial^2}{\partial \xi^2}\right], \qquad (v,\xi)\in\mathbb{R}^2 \tag{2.3}$$

and let R be the diffusion in $\mathbb{R}$, defined up to the first time it hits zero, corresponding to

$$L_R = \frac{1}{8}\frac{1}{v^2}\frac{\partial^2}{\partial v^2}, \qquad v>0. \tag{2.4}$$

Define

$$\begin{aligned} c_n &= \frac{4}{\pi(2n+1)},\\ H_n(u) &= \sin\left(\frac{(2n+1)\pi}{2k}(u+k)\right),\\ \lambda_n &= \frac{(2n+1)^2\pi^2}{4k^2}. \end{aligned} \tag{2.5}$$

LEMMA 2.2. *For any $w=(u,v)\in S\backslash\{0\}$ with $v>0$ and $t>0$,*

$$\begin{aligned} \sum_{n=0}^{\infty} c_n H_n(u) E_v\left[I\big(\tau_0(R)>t\big)\exp\left(-\frac{\lambda_n}{8}\int_0^t\frac{ds}{R_s^2}\right)\right] &\le P_w(\tau_S>t)\\ \le \sum_{n=0}^{\infty} c_n H_n(u) E_{(v,0)}\left[\exp\left(-\frac{\lambda_n}{8}\int_0^t\frac{ds}{|Z_s|^2+k^2}\right)\right], \end{aligned}$$

where $\tau_0(R)=\inf\{t>0: R_t=0\}$.

We will give the proof in Section 3. We need asymptotics as $t\to\infty$ for the Feynman–Kac functionals appearing in the terms of the series in Lemma 2.2. The results are in the next theorem, whose proof is given in Sections 4 and 5.

THEOREM 2.3. *For each $v>0$ there exist positive A_1 and A_2 such that for each $\lambda>0$,*

$$\begin{aligned} -A_1 &\le \liminf_{t\to\infty}\lambda^{-2/3}t^{-1/3}\log E_v\left[I\big(\tau_0(R)>t\big)\exp\left(-\frac{\lambda}{8}\int_0^t\frac{ds}{R_s^2}\right)\right]\\ &\le \limsup_{t\to\infty}\lambda^{-2/3}t^{-1/3}\log E_v\left[I\big(\tau_0(R)>t\big)\exp\left(-\frac{\lambda}{8}\int_0^t\frac{ds}{R_s^2}\right)\right] \le -A_2 \end{aligned}$$

and the same inequalities hold for

$$E_v\left[I\big(\tau_0(R)>t\big)\exp\left(-\frac{\lambda}{8}\int_0^t\frac{ds}{R_s^2}\right)\right]$$

replaced by

$$E_{(v,0)}\left[\exp\left(-\frac{\lambda}{8}\int_0^t \frac{ds}{|Z_s|^2+k^2}\right)\right].$$

PROOF OF THEOREM 2.1. The rough idea is that each series appearing in Lemma 2.2 is asymptotic to its first term and, by Theorem 2.3, the first terms have asymptotics of the desired form. We concentrate on the series bounding $P_w(\tau_S > t)$ from above, the argument for the other series being similar.

For notational convenience set

$$F_n(z,t) = \frac{4}{\pi(2n+1)} E_z\left[\exp\left(-\frac{\lambda_n}{8}\int_0^t \frac{ds}{|Z_s|^2+k^2}\right)\right], \qquad z=(v,\xi)\in\mathbb{R}^2.$$

Then by Lemma 2.2, for $w=(u,v)\in S\backslash\{0\}$ with $v>0$,

$$\begin{aligned} P_w(\tau_S > t) &\le \sum_{n=0}^{\infty} F_n\big((v,0),t\big)H_n(u) \\ &= F_0\big((v,0),t\big)H_0(u)\left[1+\left\{\sum_{n=1}^{\infty} c_n H_n(u)\frac{F_n((v,0),t)/c_n}{F_0((v,0),t)}\right\}\Big/ H_0(u)\right]. \end{aligned}$$

Note that $|u|<k$, $H_0(u)\neq 0$, so the division is allowed. We show that the last summation converges to 0 as $t\to\infty$, by Theorem 2.3,

$$\limsup_{t\to\infty} t^{-1/3}\log P_w(\tau_S > t) < -A_2\lambda_0^{2/3}, \tag{2.6}$$

giving the desired upper bound in Theorem 2.1. Since $\sum_{n=1}^{\infty} c_n H_n(u)$ converges for each $u\in(-k,k)$, by Abel's test, it is enough to prove for each $v>0$, for some $T>0$,

$$\sup_{\substack{n\ge 1\\ t>T}}\left|\frac{F_n((v,0),t)/c_n}{F_0((v,0),t)}\right| < \infty; \tag{2.7}$$

for each $t>T$, the sequence

$$\left\{\frac{F_n((v,0),t)/c_n}{F_0((v,0),t)} : n\ge 1\right\} \tag{2.8}$$

is decreasing (in n), and for each $n\ge 1$ and $v>0$,

$$\lim_{t\to\infty}\frac{F_n((v,0),t)/c_n}{F_0((v,0),t)} = 0. \tag{2.9}$$

To this end, choose $p>1$ so close to 1 that for $\frac{1}{p}+\frac{1}{q}=1$ and A_1, A_2 from Theorem 2.3,

$$\frac{A_2}{2}\left[\left(\lambda_1-\frac{\lambda_0}{p}\right)q\right]^{2/3} > \left(A_1+\frac{A_2}{2}\right)\lambda_0^{2/3}. \tag{2.10}$$

By Theorem 2.3, there exists $T > 0$ such that for $\lambda \in \{(\lambda_1 - \lambda_0/p)q, \lambda_0\}$,

$$\begin{aligned}\exp\left(-\left[A_1 + \frac{A_2}{2}\right]\lambda^{2/3}t^{2/3}\right) &\le E_{(v,0)}\left[\exp\left(-\frac{\lambda}{8}\int_0^t \frac{ds}{|Z_s|^2 + k^2}\right)\right] \\ &\le \exp\left(-\frac{A_2}{2}\lambda^{2/3}t^{1/3}\right), \qquad t \ge T.\end{aligned} \tag{2.11}$$

Then for $n \ge 1$ and $t \ge T$,

$$\begin{aligned}F_n((v,0),t)/c_n &= E_{(v,0)}\left[\exp\left(-\frac{\lambda_n}{8}\int_0^t \frac{ds}{|Z_s|^2+k^2}\right)\right] \\ &\le E_{(v,0)}\left[\exp\left(-\frac{\lambda_1}{8}\int_0^t \frac{ds}{|Z_s|^2+k^2}\right)\right] \\ &= E_{(v,0)}\left[\exp\left(-\frac{\lambda_0}{8p}\int_0^t \frac{ds}{|Z_s|^2+k^2} - \frac{1}{8}\left(\lambda_1 - \frac{\lambda_0}{p}\right)\int_0^t \frac{ds}{|Z_s|^2+k^2}\right)\right] \\ &\le \left\{E_{(v,0)}\left[\exp\left(-\frac{\lambda_0}{8}\int_0^t \frac{ds}{|Z_s|^2+k^2}\right)\right]\right\}^{1/p} \\ &\quad\times\left\{E_{(v,0)}\left[\exp\left(-\frac{q}{8}\left(\lambda_1 - \frac{\lambda_0}{p}\right)\int_0^t \frac{ds}{|Z_s|^2+k^2}\right)\right]\right\}^{1/q} \\ &= \left\{\frac{E_{(v,0)}[\exp(-\frac{q}{8}(\lambda_1 - \frac{\lambda_0}{p})\int_0^t \frac{ds}{|Z_s|^2+k^2})]}{E_{(v,0)}[\exp(-\frac{\lambda_0}{8}\int_0^t \frac{ds}{|Z_s|^2+k^2})]}\right\}^{1/q} \\ &\quad\times E_{(v,0)}\left[\exp\left(-\frac{\lambda_0}{8}\int_0^t \frac{ds}{|Z_s|^2+k^2}\right)\right] \\ &\le \left\{\frac{\exp(-\frac{A_2}{2}(q(\lambda_1 - \frac{\lambda_0}{p}))^{2/3}t^{1/3})}{\exp(-[A_1 + \frac{A_2}{2}]\lambda_0^{2/3}t^{1/3})}\right\}^{1/q}\frac{\pi}{4}F_0((v,0),t) \\ &\le \exp(-A_3 t^{1/3})\frac{\pi}{4}F_0((v,0),t),\end{aligned}$$

where $A_3 > 0$ is independent of $t \ge T$. [The second to the last line follows from (2.11) and the last line from (2.10).] Thus (2.9) holds. Since (2.7) and (2.8) are clear, the proof of (2.6) is complete. □

3. Proof of the infinite series bounds. In this section we prove Lemma 2.2. For $w = (u, v) \in \overline{S}\backslash\{0\}$, define

$$g(w,t) = P_w(\tau_S > t).$$

Then g satisfies [for L from (2.1)]

$$\begin{cases}(L - \frac{\partial}{\partial t})g = 0, & w \in S\backslash\{0\},\ t > 0,\\ g = 0, & u = \pm k,\ v > 0,\ t > 0,\\ g = 1, & t = 0,\ w \in S\backslash\{0\}.\end{cases} \tag{3.1}$$

Consider the operators,

$$L_1 = \frac{1}{8}\frac{1}{v^2+\xi^2+k^2}\left[\frac{\partial^2}{\partial u^2}+\frac{\partial^2}{\partial v^2}+\frac{\partial^2}{\partial \xi^2}\right],$$
$$L_2 = \frac{1}{8}\frac{1}{v^2}\left[\frac{\partial^2}{\partial u^2}+\frac{\partial^2}{\partial v^2}\right]$$

and let $W_1(t) \in \mathbb{R}^3$, $W_2(t) \in \mathbb{R}^2$ denote the diffusions associated with L_1 and L_2, respectively. Of course W_2 is only defined up to the first time it hits the line $v = 0$.

Let

$$\eta_1 = \eta_1(W_1) = \inf\{t > 0\colon W_1(t) \notin S \times \mathbb{R}\},$$

and denote the second coordinate of W_2 by $W_2^{(2)}$. For $\varepsilon > 0$ set

$$\eta_{2,\varepsilon} = \eta_{2,\varepsilon}(W_2) = \inf\{t > 0\colon W_2(t) \notin S \text{ or } W_2^{(2)}(t) = \varepsilon\}.$$

The latter is the first exit time of W_2 from the one-sided strip $\{(u, v)\colon -k < u < k,\ v > \varepsilon\}$. Then for $w = (u, v)$ with $v \geq \varepsilon$, define

$$g_1(w, \xi, t) = P_{(w,\xi)}(\eta_1 > t),$$
$$g_2(w, t) = P_w(\eta_{2,\varepsilon} > t)$$

and observe

$$\text{(3.2)}\qquad \begin{cases} (L_1 - \frac{\partial}{\partial t})g_1 = 0, & w \in S,\ \xi \in \mathbb{R},\ t > 0, \\ g_1 = 0, & u = \pm k,\ t > 0, \\ g_1 = 1, & t = 0,\ w \in S,\ \xi \in \mathbb{R} \end{cases}$$

and

$$\text{(3.3)}\qquad \begin{cases} (L_2 - \frac{\partial}{\partial t})g_2 = 0, & |u| < k,\ v > \varepsilon,\ t > 0, \\ g_2 = 0, & u = \pm k,\ v > \varepsilon,\ t > 0, \\ g_2 = 0, & |u| < k,\ v = \varepsilon,\ t > 0, \\ g_2 = 1, & t = 0,\ |u| < k,\ v > \varepsilon. \end{cases}$$

LEMMA 3.1. *For $w = (u, v) \in \overline{S}\backslash\{0\}$ with $v \geq \varepsilon$ and $t > 0$,*

$$g_2(w, t) \leq g(w, t) \leq g_1(w, 0, t).$$

PROOF. We prove the second inequality, the first being similar. Extend g to $S \times \mathbb{R} \times [0, \infty)$ by

$$g(w, \xi, t) := g(w, t).$$

For $w \in S\backslash\{0\}$, $\xi \in \mathbb{R}$ and $t > 0$,

$$\text{(3.4)}\qquad \begin{aligned} \left(L_1 - \frac{\partial}{\partial t}\right)(g - g_1) &= \left(L_1 - \frac{\partial}{\partial t}\right)g = (L_1 - L)g \\ &= \frac{1}{8}\frac{u^2-\xi^2-k^2}{v^2+\xi^2+k^2}Lg = \frac{1}{8}\frac{u^2-\xi^2-k^2}{v^2+\xi^2+k^2}\frac{\partial g}{\partial t} \geq 0, \end{aligned}$$

since $|u| < k$ and $\frac{\partial g}{\partial t} \le 0$. Moreover,

$$g - g_1 = 0 \quad \text{for } u = \pm k,\ (v, \xi) \in \mathbb{R}^2 \text{ and } t > 0, \tag{3.5}$$

$$g - g_1 = 0 \quad \text{for } t = 0,\ w \in \overline{S}\backslash\{0\} \text{ and } \xi \in \mathbb{R}. \tag{3.6}$$

The next natural step would be to apply the maximum principle to conclude $g \le g_1$. However, L_1 is not uniformly elliptic, and the smoothness of g at $w = 0$ is not known. Thus the maximum principle does not apply, at least not directly. We get around this by using Itô's formula. Write

$$\begin{aligned}
W_1 &= \left(W_1^{(1)}, W_1^{(2)}, W_1^{(3)}\right),\\
\zeta_1 &= \inf\left\{t > 0\colon \left(W_1^{(1)}(t), W_1^{(2)}(t)\right) \notin S\right\},\\
\zeta_2 &= \inf\left\{t > 0\colon \left|\left(W_1^{(1)}(t), W_1^{(2)}(t)\right)\right| = \delta\right\},\\
\zeta_3 &= \inf\left\{t > 0\colon |W_1(t)| = M\right\},\\
\zeta &= \zeta_1 \wedge \zeta_2 \wedge \zeta_3.
\end{aligned}$$

Then for $T > 0$, $w \in S\backslash\{0\}$, $\xi \in \mathbb{R}$ with $|(w, \xi)| < M$ and $|w| > \delta$, by Itô's formula and optional stopping applied to $f(w, \xi, t) = (g - g_1)(w, \xi, T - t)$,

$$\begin{aligned}
&E_{(w,\xi)}\left[f\left(W_1(T \wedge \zeta), T \wedge \zeta\right)\right]\\
&\quad = f(w, \xi, 0) + E_{(w,\xi)}\left[\int_0^{T\wedge\zeta}\left[L_1 f + \frac{\partial f}{\partial s}\right]\left(W_1(s), s\right) ds\right].
\end{aligned} \tag{3.7}$$

Then by (3.4)–(3.6),

$$\begin{aligned}
(g - g_1)(w, \xi, T) &\le (g - g_1)(w, \xi, T)\\
&\quad + E_{(w,\xi)}\left\{\int_0^{T\wedge\zeta}\left[\left(L_1 - \frac{\partial}{\partial s}\right)(g - g_1)\left(W_1(s), T - s\right)\right] ds\right\}\\
&= E_{(w,\xi)}\left[f\left(W_1(T \wedge \zeta)\right), T \wedge \zeta\right] \quad \text{[by (3.7)]}\\
&= E_{(w,\xi)}\left[(g - g_1)\left(W_1(T), 0\right) I(T < \zeta)\right]\\
&\quad + E_{(w,\xi)}\left[(g - g_1)\left(W_1(\zeta_1), T - \zeta_1\right) I(\zeta_1 < T \wedge \zeta_2 \wedge \zeta_3)\right]\\
&\quad + E_{(w,\xi)}\left[(g - g_1)\left(W_1(\zeta_2), T - \zeta_2\right) I(\zeta_2 < T \wedge \zeta_1 \wedge \zeta_3)\right]\\
&\quad + E_{(w,\xi)}\left[(g - g_1)\left(W_1(\zeta_3), T - \zeta_3\right) I(\zeta_3 < T \wedge \zeta_1 \wedge \zeta_2)\right]\\
&\le 0 + 0 + P_{(w,\xi)}(\zeta_2 < T) + P_{(w,\xi)}(\zeta_3 < T).
\end{aligned}$$

These last two quantities also go to zero as $\delta \to 0$ and $M \to \infty$, since W_1 neither explodes nor hits $\{(u, v, \xi) \in \mathbb{R}^3\colon (u, v) = 0\}$. Hence

$$(g - g_1)(w, \xi, T) \le 0,$$

as desired. □

The reason for introducing g_1 and g_2 is that they have eigenfunction expansions. Such expansions follow exactly as in DeBlassie (1987). For $w = (u, v) \in S\backslash\{0\}$,

$$g_1(w, \xi, t) = \sum_{n=0}^{\infty} F_n(v, \xi, t) H_n(u), \tag{3.8}$$

$$g_2(w, t) = \sum_{n=0}^{\infty} G_n(v, t) H_n(u), \tag{3.9}$$

where

$$H_n(u) = \sin\left(\frac{(2n+1)\pi}{2k}(u+k)\right), \qquad |u| < k,$$

$$F_n(v, \xi, t) = \frac{1}{k}\int_{-k}^{k} g_1(w, \xi, t) H_n(u)\, du,$$

$$G_n(v, t) = \frac{1}{k}\int_{-k}^{k} g_2(w, t) H_n(u)\, du.$$

Using (3.2) and (3.3), it is easy to check that for $\lambda_n = (\frac{(2n+1)\pi}{2k})^2$, F_n and G_n satisfy

$$\begin{cases} \dfrac{1}{8}\dfrac{1}{v^2+\xi^2+k^2}\left[\dfrac{\partial^2}{\partial v^2} + \dfrac{\partial^2}{\partial \xi^2} - \lambda_n\right] F_n = \dfrac{\partial F_n}{\partial t}, & \text{for } (v,\xi)\in\mathbb{R}^2 \text{ and } t>0, \\ F_n(v,\xi,0) = \dfrac{4}{\pi(2n+1)}, & \text{for } (v,\xi)\in\mathbb{R}^2 \end{cases} \tag{3.10}$$

and

$$\begin{cases} \dfrac{1}{8}\dfrac{1}{v^2}\left[\dfrac{\partial^2}{\partial v^2} - \lambda_n\right] G_n = \dfrac{\partial G_n}{\partial t}, & \text{for } t > 0,\, v > \varepsilon, \\ G_n(\varepsilon, t) = 0, & \text{for } t > 0, \\ G_n(v, 0) = \dfrac{4}{\pi(2n+1)}, & \text{for } v > \varepsilon. \end{cases} \tag{3.11}$$

With Z and R being the diffusions in $\mathbb{R}^2$ and $\mathbb{R}$ corresponding to the operators L_Z from (2.3) and L_R from (2.4), respectively, (as in Section 2) and define

$$\tau_\varepsilon(R) := \inf\{t > 0\colon R_t = \varepsilon\}.$$

By the Feynman–Kac formula,

$$F_n(v,\xi,t) = \frac{4}{\pi(2n+1)} E_{(v,\xi)}\left[\exp\left(-\frac{\lambda_n}{8}\int_0^t \frac{ds}{|Z_s|^2+k^2}\right)\right], \quad (v,\xi)\in\mathbb{R}^2 \tag{3.12}$$

and

$$G_n(v,t) = \frac{4}{\pi(2n+1)} E_v\left[I\big(\tau_\varepsilon(R) > t\big)\exp\left(-\frac{\lambda_n}{8}\int_0^t \frac{ds}{R_s^2}\right)\right], \quad v > \varepsilon. \tag{3.13}$$

Hence by Lemma 3.1, the conclusion of Lemma 2.2 holds with $I(\tau_0(R) > t)$ replaced by $I(\tau_\varepsilon(R) > t)$. By Abel's test we can let $\varepsilon \to 0$ and replace $\tau_\varepsilon(R)$ by $\tau_0(R)$, giving the conclusion of Lemma 2.2. $\square$

4. Bounds on the Feynman–Kac functionals via Bessel processes. In this section, estimates in terms of Bessel processes are derived for the Feynman–Kac functionals,

$$E_v\left[I\big(\tau_0(R) > t\big)\exp\left(-\frac{\lambda_n}{8}\int_0^t \frac{ds}{R_s^2}\right)\right],$$
$$E_{(v,\,0)}\left[\exp\left(-\frac{\lambda_n}{8}\int_0^t \frac{ds}{|Z_s|^2 + k^2}\right)\right].$$

The first step is a comparison lemma.

LEMMA 4.1. *For $v > 0$ and $t > 0$,*

$$E_v\left[I\big(\tau_0(R) > t\big)\exp\left(-\frac{\lambda_n}{8}\int_0^t \frac{ds}{R_s^2}\right)\right] \le E_{(v,\,0)}\left[\exp\left(-\frac{\lambda_n}{8}\int_0^t \frac{ds}{|Z_s|^2 + k^2}\right)\right].$$

PROOF. By (3.12) and (3.13), it is enough to show

$$G_n(v, t) \le F_n(v, 0, t) \quad \text{for } v > \varepsilon,\, t > 0. \tag{4.1}$$

By (3.10) and (3.11), since G_n is independent of ξ,

$$\begin{aligned}
&\left\{\frac{1}{8}\frac{1}{v^2+\xi^2+k^2}\left[\frac{\partial^2}{\partial v^2} + \frac{\partial^2}{\partial \xi^2} - \lambda_n\right] - \frac{\partial}{\partial t}\right\}\big(G_n(v,t) - F_n(v,\xi,t)\big)\\
&\quad = \left\{\frac{1}{8}\frac{1}{v^2+\xi^2+k^2}\left[\frac{\partial^2}{\partial v^2} + \frac{\partial^2}{\partial \xi^2} - \lambda_n\right] - \frac{\partial}{\partial t}\right\}G_n(v,t)\\
&\quad = \left\{\frac{1}{8}\frac{1}{v^2+\xi^2+k^2}\left[\frac{\partial^2}{\partial v^2} - \lambda_n\right] - \frac{1}{8}\frac{1}{v^2}\left[\frac{\partial^2}{\partial v^2} - \lambda_n\right]\right\}G_n(v,t)\\
&\quad = -\frac{1}{8}\frac{\xi^2+k^2}{v^2+\xi^2+k^2}\frac{1}{v^2}\left[\frac{\partial^2}{\partial v^2} - \lambda_n\right]G_n(v,t)\\
&\quad = -\frac{\xi^2+k^2}{v^2+\xi^2+k^2}\frac{\partial G_n}{\partial t}\\
&\quad \ge 0.
\end{aligned} \tag{4.2}$$

Moreover,

$$\begin{cases} G_n(v,0) - F_n(v,\xi,0) = 0, & \text{for } v > \varepsilon,\, \xi \in \mathbb{R},\\ G_n(\varepsilon,t) - F_n(\varepsilon,\xi,t) \le 0, & \text{for } t > 0,\, \xi \in \mathbb{R}. \end{cases} \tag{4.3}$$

Define for $z = (v, \xi)$,

$$f(z,t) = G_n(v,t) - F_n(v,\xi,t)$$

and set

$$\beta_1 := \inf\{t > 0\colon Z_t^{(1)} = \varepsilon\},$$
$$\beta_2 := \inf\{t > 0\colon |Z_t| = M\} \quad \text{and}$$
$$\beta = \beta_1 \wedge \beta_2.$$

By Itô's formula and optional stopping we have, for $|(v,\xi)| < M$ and $v > \varepsilon$,

$$
\begin{aligned}
&E_{(v,\xi)}\left[f(Z_{T\wedge\beta}, T - T\wedge\beta)\exp\left(-\frac{\lambda_n}{8}\int_0^{T\wedge\beta}\frac{ds}{|Z_s|^2+k^2}\right)\right]\\
&\quad = f(z,T) + E_{(v,\xi)}\left[\int_0^{T\wedge\beta}\left\{\frac{1}{8}\frac{1}{|Z_s|^2+k^2}\left[\frac{\partial^2}{\partial v^2}+\frac{\partial^2}{\partial\xi^2}-\lambda_n\right]f - \frac{\partial f}{\partial t}\right\}(Z_s, T-s)\,ds\right]\\
&\quad \geq f(z,T),
\end{aligned}
$$

by (4.2). Moreover, if $T \leq \beta$,

$$f(Z_{T\wedge\beta}, T - T\wedge\beta) = f(Z_T, 0) = 0 \quad \text{by (4.3).}$$

Also, if $\beta_1 = T\wedge\beta$,

$$f(Z_{T\wedge\beta}, T - T\wedge\beta) = f(Z_{\beta_1}, T-\beta_1) \leq 0$$

and if $\beta_2 = T\wedge\beta$,

$$f(Z_{T\wedge\beta}, T - T\wedge\beta) = f(Z_{\beta_2}, T-\beta_2) \leq \frac{4}{(2n+1)\pi}$$

by (3.13). Hence

$$f(z,T) \leq \frac{4}{(2n+1)\pi}P_{(v,\xi)}(\beta_2 < T).$$

Since $\beta_2 \to \infty$ as $M \to \infty$, this yields $f(z,T) \leq 0$ for $T > 0$ and $v > \varepsilon$. Thus $G_n(v,t) \leq F_n(v,\xi,t)$ for $v > \varepsilon$, $t > 0$ and $\xi \in \mathbb{R}$, giving (4.1). $\square$

REMARK 4.2. The lemma is true for λ_n replaced by any $\lambda > 0$.

Next we bound from below the Feynman–Kac functional involving R.

LEMMA 4.3. *Let B_t be a one-dimensional Brownian motion. Then for $v > 0$ and $\lambda > 0$,*

$$E_{v^2}\left[I\big(\tau_0(B) > t\big)\exp\left(-\lambda\int_0^t\frac{ds}{|B_s|}\right)\right] \leq E_v\left[I\big(\tau_0(R) > t\big)\exp\left(-\lambda\int_0^t\frac{ds}{R_s^2}\right)\right].$$

PROOF. Up to time $\tau_0(R)$, R solves the stochastic differential equation

$$
\begin{aligned}
dR_t &= \frac{1}{2R_t}d\beta_t,\\
R_0 &= v,
\end{aligned}
$$

where β is a one-dimensional Brownian motion. Then by Itô's formula, R_t^4 is a squared Bessel process of dimension 3/2. Since B_t^2 is a squared Bessel process with dimension 1, if $B_0 = v^2$ then by the Ikeda–Watanabe comparison theorem [see Rogers and Williams (1987), Theorem 43, 1 on page 269], the conclusion of the lemma holds. $\square$

Now, we look for an upper bound of the functional involving Z.

THEOREM 4.4. *For each $A > 0$ there is an integer $p = p(A) \geq 3$ such that if γ_t is a squared Bessel process with dimension p, for $\lambda > 0$ and $v \in (0, A^{1/2}]$,*

$$E_{(v,0)}\left[\exp\left(-\lambda \int_0^t \frac{ds}{|Z_s|^2 + k^2}\right)\right] \leq E_{v^4}\left[\exp\left(-\lambda \int_0^t \frac{ds}{\gamma_s^{1/2} + k^2}\right)\right].$$

PROOF. The process Z_t satisfies the stochastic differential equation

$$dZ_t = \tfrac{1}{2}(|Z_t|^2 + k^2)^{-1/2} d\beta_t,$$
$$Z_0 = (v, 0),$$

where now β_t is two-dimensional Brownian motion. Set

$$Y_t = |Z_t|^4.$$

By Itô's formula,

$$\begin{aligned} dY_t &= \frac{2|Z_t|^3}{\sqrt{|Z_t|^2 + k^2}}\left[\frac{Z_t}{|Z_t|} \cdot d\beta_t\right] + 2\frac{|Z_t|^2}{|Z_t|^2 + k^2}\, dt \\ &= \frac{2Y_t^{3/4}}{\sqrt{Y_t^{1/2} + k^2}} dM_t + \frac{2Y_t^{1/2}}{Y_t^{1/2} + k^2}\, dt, \end{aligned}$$

where $dM_t = \frac{Z_t}{|Z_t|} \cdot d\beta_t$ is one-dimensional Brownian motion [note since $Z_0 = (v, 0) \neq 0$, Z_t never hits 0]. Thus,

$$\begin{aligned} dY_t &= \sigma_1(Y_t)\, dM_t + \frac{2Y_t^{1/2}}{Y_t^{1/2} + k^2}\, dt, \\ Y_0 &= v^4, \end{aligned} \tag{4.4}$$

where

$$\sigma_1(y) = 2\left[\frac{y^{3/2}}{y^{1/2} + k^2}\right]^{1/2}. \tag{4.5}$$

Let γ_t be the square of a Bessel process with integer dimension $p \geq 3$ to be chosen later,

$$\begin{aligned} d\gamma_t &= \sigma_2(\gamma_t) dM_t + p\, dt, \\ \gamma_0 &= v^4, \end{aligned} \tag{4.6}$$

where

$$\sigma_2(y) = 2\sqrt{y}. \tag{4.7}$$

We use an idea of O'Brien (1980). Define for $y > 0$,

$$A_1(y) = \frac{2y^{1/2} + k^2}{4y^{1/4}(y^{1/2} + k^2)^{3/2}} \tag{4.8}$$

and

$$A_2(y) = \frac{p-1}{2\sqrt{y}}. \tag{4.9}$$

Then

$$\left[A_1(y) + \frac{1}{2}\sigma_1'(y)\right]\sigma_1(y) = \frac{2y^{1/2}}{y^{1/2}+k^2} \tag{4.10}$$

and

$$\left[A_2(y) + \frac{1}{2}\sigma_2'(y)\right]\sigma_2(y) = p. \tag{4.11}$$

Finally, for $i = 1, 2$ define

$$F_i(y) = \int_{v^4}^{y} \frac{1}{\sigma_i(u)}\,du, \qquad y \ge 0.$$

Note

$$F_2^{-1}(x) = (x+v^2)^2. \tag{4.12}$$

We now need a technical lemma which we prove at the end of this section.

LEMMA 4.5. *There is an integer* $p \ge 3$ *such that*

$$\frac{F_1(y)+v^2}{p-1} \le \frac{2y^{1/4}(y^{1/2}+k^2)^{3/2}}{2y^{1/2}+k^2}, \qquad y > 0.$$

This lemma and (4.12) imply

$$\begin{aligned} A_2 \circ F_2^{-1} \circ F_1(y) &= A_2\big([F_1(y)+v^2]^2\big) = \frac{p-1}{2[F_1(y)+v^2]} \\ &\ge \frac{2y^{1/2}+k^2}{4y^{1/4}(y^{1/2}+k^2)^{3/2}} = A_1(y). \end{aligned} \tag{4.13}$$

Writing

$$\begin{aligned} U_1(t) &= F_1(Y_t), \\ U_2(t) &= F_2(\gamma_t) \end{aligned}$$

by Itô's formula, by (4.4) and (4.6), and by the fact that Y_t and γ_t never hit 0,

$$\begin{aligned} dU_1(t) &= F_1'(Y_t)\left[\sigma_1(Y_t)\,dM_t + \frac{2Y_t^{1/2}}{Y_t^{1/2}+k^2}\,dt\right] + \frac{1}{2}F_1''(Y_t)\sigma_1(Y_t)^2\,dt \\ &= dM_t + \left[\frac{1}{\sigma_1(Y_t)}\cdot\frac{2Y_t^{1/2}}{Y_t^{1/2}+k^2} - \frac{1}{2}\sigma_1'(Y_t)\right]dt \\ &= dM_t + A_1(Y_t)dt \quad \text{[by (4.10)]} \\ &= dM_t + A_1 \circ F_1^{-1}\big(U_1(t)\big)\,dt \end{aligned}$$

and similarly, using (4.11),

$$dU_2(t) = dM_t + A_2 \circ F_2^{-1}\big(U_2(t)\big)\,dt.$$

Since $U_i(0) = F_i(v^4) = 0$, by (4.13) and the Ikeda–Watanabe comparison theorem cited above,

$$U_1(t) \le U_2(t).$$

That is,

$$\int_{v^4}^{Y_t} \sigma_1(u)^{-1} du \le \int_{v^4}^{\gamma_t} \sigma_2(u)^{-1} du. \tag{4.14}$$

From (4.5),

$$\sigma_1(y) = 2y^{1/2}\left[\frac{y^{1/2}}{y^{1/2}+k^2}\right]^{1/2} \le 2y^{1/2} = \sigma_2(y),$$

so we have

$$\int_{v^4}^{y} \sigma_1(u)^{-1}\,du \ge \int_{v^4}^{y} \sigma_2(u)^{-1} du.$$

Combined with (4.14), it follows that

$$Y_t \le \gamma_t$$

(this is O'Brien's idea), and hence

$$\begin{aligned} E_{(v,0)}\left[\exp\left(-\lambda\int_0^t \frac{ds}{|Z_s|^2+k^2}\right)\right] &= E_{v^4}\left[\exp\left(-\lambda\int_0^t \frac{ds}{Y_s^{1/2}+k^2}\right)\right] \\ &\le E_{v^4}\left[\exp\left(-\lambda\int_0^t \frac{ds}{\gamma_s^{1/2}+k^2}\right)\right] \end{aligned}$$

as desired.

We close this section with the proof of Lemma 4.5. Performing the integration (which can easily be done using MAPLE) we find that

$$\begin{aligned} F_1(y) = {} & y^{1/4}[y^{1/2}+k^2]^{1/2} - v[v^2+k^2]^{1/2} \\ & + k^2\ln\big(y^{1/4}+[y^{1/2}+k^2]^{1/2}\big) - k^2\ln\big(v+[v^2+k^2]^{1/2}\big). \end{aligned}$$

Then

$$\begin{aligned} F_1(y)+v^2 &\le y^{1/4}[y^{1/2}+k^2]^{1/2} - v[v^2+k^2]^{1/2} + k^2\ln\big(y^{1/4}+[y^{1/2}+k^2]^{1/2}\big) \\ &\qquad - k^2\ln k + v^2 \\ &\le y^{1/4}[y^{1/2}+k^2]^{1/2} + k^2\ln\frac{y^{1/4}+[y^{1/2}+k^2]^{1/2}}{k}. \end{aligned} \tag{4.15}$$

Now for RHS = right-hand side,

$$\lim_{y\to 0^+} y^{-1/4}\ \text{RHS (4.15)} = 2k$$

and

$$\lim_{y\to 0^+} y^{-1/4}\cdot\frac{2y^{1/4}[y^{1/2}+k^2]^{3/2}}{2y^{1/2}+k^2}=2k.$$

Hence for some $\delta>0$ we have for $0<y<\delta$,

$$\frac{1}{2}\left[y^{1/4}[y^{1/2}+k^2]^{1/2}+k^2\ln\frac{y^{1/4}+[y^{1/2}+k^2]^{1/2}}{k}\right]\le\frac{2y^{1/4}[y^{1/2}+k^2]^{3/2}}{2y^{1/2}+k^2}. \tag{4.16}$$

By (4.15), for $p\ge 3$, this yields

$$\frac{F_1(y)+v^2}{p-1}\le\frac{2y^{1/4}[y^{1/2}+k^2]^{3/2}}{2y^{1/2}+k^2},\qquad 0\le y\le\delta. \tag{4.17}$$

Also,

$$\lim_{y\to\infty} y^{-1/2}\left[y^{1/4}[y^{1/2}+k^2]^{1/2}+k^2\ln\frac{y^{1/4}+[y^{1/2}+k^2]^{1/2}}{k}\right]=1$$

and

$$\lim_{y\to\infty} y^{-1/2}\cdot\frac{2y^{1/4}[y^{1/2}+k^2]^{3/2}}{2y^{1/2}+k^2}=1.$$

So for some $M>0$, (4.16) holds for $y\ge M$, and therefore by (4.15), (4.17) also holds for $y\ge M$ and $p\ge 3$. Notice that both δ and M are independent of $p\ge 3$.

Now choose the integer $p\ge 3$ so large that

$$\inf_{\delta\le y\le M}\frac{2y^{1/4}[y^{1/2}+k^2]^{3/2}}{2y^{1/2}+k^2}\ge\frac{1}{p-1}\sup_{\delta\le y\le M}\quad\text{RHS (4.15)}.$$

Then (4.17) holds for $\delta\le y\le M$. In any event, we have shown for this choice of p, (4.17) holds for all $y\ge 0$. This proves the lemma. □

5. Bounds on the Bessel expectations and proof of Theorem 2.3. By Lemma 4.1, Remark 4.2, Lemma 4.3 and Theorem 4.4, Theorem 2.3 is an immediate consequence of the following theorem.

THEOREM 5.1. *For each $v>0$ there exist positive constants A_1 and A_2 such that for any $\lambda>0$,*

$$-A_1\le\liminf_{t\to\infty}\lambda^{-2/3}t^{-1/3}\log\left(E_v\left[I(\tau_0(B)>t)\exp\left(-\lambda\int_0^t\frac{ds}{|B_s|}\right)\right]\right) \tag{5.1}$$

and

$$\limsup_{t\to\infty}\lambda^{-2/3}t^{-1/3}\log\left(E_v\left[\exp\left(-\lambda\int_0^t\frac{ds}{\gamma_s^{1/2}+k^2}\right)\right]\right)\le -A_2, \tag{5.2}$$

where B_t is one-dimensional Brownian motion and γ_t is the square of a Bessel process with integer dimension $p\ge 3$.

PROOF. For $d = 1$ or p, let β_t be d-dimensional Brownian motion starting from 0. Let P^ε be the law of $\sqrt{\varepsilon}\beta_t$ on $C_0([0,1], \mathbb{R}^d)$, the space of continuous functions on [0,1] which vanish at 0. First we consider (5.1), so take $d = 1$. For typographical simplicity, write $P = P^1$. Then for $v > 0$, changing variables $s = ut$ and using scaling in the third line,

$$
\begin{aligned}
&E_v\left[I(\tau_0(B) > t)\exp\left(-\lambda\int_0^t \frac{ds}{|B_s|}\right)\right]\\
&\quad = E_v\left[\exp\left(-\lambda\int_0^t \frac{1}{|B_s|}ds\right)\right]\\
&\quad = E\left[\exp\left(-\lambda\int_0^t \frac{1}{|\beta_s + v|}\,ds\right)\right]\\
&\quad = E\left[\exp\left(-\lambda\sqrt{t}\int_0^1 \frac{1}{|\beta_u + v/\sqrt{t}|}du\right)\right]\\
&\quad = E\left[\exp\left(-\lambda\sqrt{t\varepsilon}\int_0^1 \frac{1}{|\sqrt{\varepsilon}\,\beta_u + v\sqrt{\varepsilon/t}|}\,du\right)\right].
\end{aligned}
\tag{5.3}
$$

Let $\alpha \in (1/2, 1)$ and define

$$g(u) = u^\alpha.$$

Set

$$\eta = \inf\{u > 0\colon \sqrt{\varepsilon}\beta_u + v\sqrt{\varepsilon/t} \le g(u)\}$$

and

$$\zeta = \inf\{u > 0\colon \beta_u \le -v\sqrt{t}\}.$$

The right-hand side of (5.3) is greater than or equal to

$$E\left[I(\eta > 1)\exp\left(-\lambda\sqrt{t\varepsilon}\int_0^1 \frac{du}{g(u)}\right)\right],$$

and if Q^ε is the law of $\beta(t) - \frac{g(t)}{\sqrt{\varepsilon}}$, then the last expectation is

$$\exp\left(-\lambda\sqrt{t\varepsilon}\int_0^1 \frac{du}{g(u)}\right)Q^\varepsilon(\zeta > 1).$$

Thus (5.3) becomes

$$
\begin{aligned}
&E_v\left[I(\tau_0 > t)\exp\left(-\lambda\int_0^t \frac{ds}{|B_s|}\right)\right]\\
&\qquad \ge \exp\left(-\lambda\sqrt{t\varepsilon}\int_0^1 \frac{du}{g(u)}\right)Q^\varepsilon(\zeta > 1).
\end{aligned}
\tag{5.4}
$$

By the Cameron–Martin–Girsanov formula, Q^ε is absolutely continuous with respect to P and the Radon–Nikodym derivative is

$$\frac{dQ^\varepsilon}{dP} = \exp\left(-\frac{1}{\sqrt{\varepsilon}}\int_0^1 g'(t)\,d\beta_t - \frac{1}{2\varepsilon}\int_0^t \left[g'(t)\right]^2 dt\right).$$

Although g' is singular at 0, $\int_0^1[g'(t)]^2<\infty$, so this is well defined. Hence

$$
\begin{aligned}
Q^{\varepsilon}(\zeta>1) &= E\left[I(\zeta>1)\exp\left(-\frac{1}{\sqrt{\varepsilon}}\int_0^1 g'(u)\,d\beta_u-\frac{1}{2\varepsilon}\int_0^1\left[g'(u)\right]^2du\right)\right]\\
&= \exp\left(-\frac{1}{2\varepsilon}\int_0^1\left[g'(u)\right]^2du\right)E\left[I(\zeta>1)\exp\left(-\frac{1}{\varepsilon}\int_0^1 g'(u)\,d\beta_u\right)\right].
\end{aligned}
\tag{5.5}
$$

For any $\delta>0$,

$$
\begin{aligned}
&E\left[I(\zeta>1)\exp\left(-\frac{1}{\sqrt{\varepsilon}}\int_0^1 g'(u)\,d\beta_u\right)\right]\\
&\quad\ge E\left[I(\zeta>1)e^{-\delta/\varepsilon}I\left(\int_0^1 g'(u)\,d\beta_u\le\delta/\sqrt{\varepsilon}\right)\right]\\
&\quad= e^{-\delta/\varepsilon}\left[P(\zeta>1)-P\left(\zeta>1,\int_0^1 g'(u)\,d\beta_u>\delta/\sqrt{\varepsilon}\right)\right].
\end{aligned}
\tag{5.6}
$$

Since P is the law of one-dimensional Brownian motion started at 0, it is well known that

$$
\begin{aligned}
P(\zeta>1) &= \frac{2}{\sqrt{2\pi}}\int_{-\infty}^{v/\sqrt{t}} e^{-u^2/2}du-1\\
&= \frac{2}{\sqrt{2\pi}}\int_0^{v/\sqrt{t}} e^{-u^2/2}\,du\\
&\sim\sqrt{\frac{2}{\pi}}(v/\sqrt{t})\quad\text{as } t\to\infty
\end{aligned}
\tag{5.7}
$$

(here $f\sim h$ as $t\to\infty$ means $\lim_{t\to\infty}\frac{f}{h}=1$). Also,

$$
P\left(\zeta>1,\int_0^1 g'(u)d\beta_u>\delta/\sqrt{\varepsilon}\right)\le e^{-\delta/\sqrt{\varepsilon}}E\left[\exp\left(\int_0^1 g'(u)\,d\beta_u\right)\right]=Ce^{-\delta/\sqrt{\varepsilon}},
$$

since $\int_0^1 g'(u)\,d\beta_u$ is Gaussian with variance $\int_0^1[g'(u)]^2\,du$ under P. Taking

$$
\varepsilon=\lambda^{-2/3}t^{-1/3},
\tag{5.8}
$$

we get

$$
P\left(\zeta>1,\int_0^1 g'(u)\,d\beta_u>\delta/\sqrt{\varepsilon}\right)=o\big(P(\zeta>1)\big)\quad\text{as } t\to\infty.
\tag{5.9}
$$

Using (5.4)–(5.6) and (5.8),

$$
\begin{aligned}
&\varepsilon\log E_v\left[I(\tau_0>t)\exp\left(-\lambda\int_0^t\frac{ds}{|B_s|}\right)\right]\\
&\quad\ge-\lambda\sqrt{t\varepsilon^3}\int_0^1\frac{du}{g(u)}-\frac{1}{2}\int_0^1[g'(u)]^2\,du\\
&\qquad-\delta+\varepsilon\log\left[P(\zeta>1)-P\left(\zeta>1,\int_0^1 g'(u)\,d\beta_u>\delta/\sqrt{\varepsilon}\right)\right]
\end{aligned}
$$

$$= -\int_0^1 \frac{du}{g(u)} - \frac{1}{2}\int_0^1 \bigl[g'(u)\bigr]^2 du - \delta + \varepsilon \log P(\zeta > 1)$$
$$+ \varepsilon \log\biggl[1 - \frac{P(\zeta > 1, \int_0^1 g'(u)\, d\beta_u > \delta/\sqrt{\varepsilon})}{P(\zeta > 1)}\biggr].$$

By (5.7)–(5.9), this yields

$$\liminf_{t\to\infty} \lambda^{-2/3} t^{-1/3} \log E_v\biggl[I(\tau_0 > t)\exp\biggl(-\lambda\int_0^t \frac{ds}{|B_s|}\biggr)\biggr]$$
$$\ge -\int_0^1 \frac{du}{g(u)} - \frac{1}{2}\int_0^1 \bigl[g'(u)\bigr]^2\, du - \delta.$$

Since $\delta > 0$ was arbitrary, (5.1) holds with

$$A_1 = \int_0^1 \frac{du}{g(u)} + \frac{1}{2}\int_0^1 \bigl[g'(u)\bigr]^2\, du.$$

Now for (5.2). This time $d = p \ge 3$. Changing variables $s = ut$ and using scaling of γ_t,

$$\begin{aligned} E_v\biggl[\exp\biggl(-\lambda\int_0^t \frac{ds}{\gamma_s^{1/2} + k^2}\biggr)\biggr] &= E_v\biggl[\exp\biggl(-\lambda\int_0^1 \frac{t\, du}{\gamma_{ut}^{1/2} + k^2}\biggr)\biggr] \\ &= E_{v/t}\biggl[\exp\biggl(-\lambda\int_0^1 \frac{t\, du}{t^{1/2}\gamma_u^{1/2} + k^2}\biggr)\biggr] \\ &= E_{v/t}\biggl[\exp\biggl(-\lambda t^{1/2}\int_0^1 \frac{du}{\gamma_u^{1/2} + k^2/t^{1/2}}\biggr)\biggr]. \end{aligned}$$

Writing

$$\varepsilon = \lambda^{-2/3} t^{-1/3}$$

and denoting by ω_u the coordinate process on $C_0([0,1], \mathbb{R}^d)$, this becomes

$$\begin{aligned} & E_{v/t}\biggl[\exp\biggl(-\lambda t^{1/2}\varepsilon^{3/2}\frac{1}{\varepsilon}\int_0^1 \frac{du}{(\varepsilon\gamma_u)^{1/2} + k^2(\varepsilon/t)^{1/2}}\biggr)\biggr] \\ &\quad = E_{v/t}\biggl[\exp\biggl(-\frac{1}{\varepsilon}\int_0^1 \frac{du}{(\varepsilon\gamma_u)^{1/2} + k^2(\varepsilon/t)^{1/2}}\biggr)\biggr] \\ &\quad = E\biggl[\exp\biggl(-\frac{1}{\varepsilon}\int_0^1 \frac{du}{\sqrt{\varepsilon}|\beta_u + (v/t)^{1/2}| + k^2(\varepsilon/t)^{1/2}}\biggr)\biggr] \\ &\quad \le E\biggl[\exp\biggl(-\frac{1}{\varepsilon}\int_0^1 \frac{du}{\sqrt{\varepsilon}|\beta_u| + (\varepsilon v/t)^{1/2} + k^2(\varepsilon/t)^{1/2}}\biggr)\biggr] \\ &\quad = E^{P^\varepsilon}\biggl[\exp\biggl(-\frac{1}{\varepsilon}\int_0^1 \frac{du}{|\omega_u| + (\varepsilon v/t)^{1/2} + k^2(\varepsilon/t)^{1/2}}\biggr)\biggr] \\ &\quad = E^{P^\varepsilon}\biggl[\exp\biggl(-\frac{1}{\varepsilon}F_\varepsilon(\omega)\biggr)\biggr], \end{aligned} \tag{5.10}$$

where

$$F_\varepsilon(\omega) = \int_0^1 \frac{du}{|\omega_u| + (v\varepsilon/t)^{1/2} + k^2(\varepsilon/t)^{1/2}}. \tag{5.11}$$

Define

$$F(\omega) = \begin{cases} \int_0^1 \dfrac{du}{|\omega_u|}, & \text{if integral exists,} \\ \infty, & \text{otherwise.} \end{cases}$$

Then F is nonnegative, and by Fatou's lemma it is lower semicontinuous on $C_0([0,1], \mathbb{R}^p)$. Also, as $\omega_n \to \omega$ in $C_0([0,1], \mathbb{R}^p)$ and $\varepsilon \to 0$, by Fatou's lemma,

$$F(\omega) \le \liminf_{\substack{\varepsilon \to 0 \\ n \to \infty}} F_\varepsilon(\omega_n).$$

Application of Theorem 2.3 on page 4 of Varadhan (1984) gives

$$\limsup_{\varepsilon \to 0} \varepsilon \log E^{P^\varepsilon}\left[\exp\left(-\frac{1}{\varepsilon} F_\varepsilon(\omega)\right)\right] \le -\inf_{\omega \in C_0}\left[F(\omega) + I(\omega)\right]$$

where $C_0 = C_0([0,1], \mathbb{R}^p)$ and $I(\omega) = \frac{1}{2}\int_0^1 |\omega'(t)|^2\, dt$ if ω is absolutely continuous with square integrable derivative ω' and $I(\omega) = \infty$ otherwise. Expression (5.2) follows from this and (5.10) with

$$A_2 = \inf_{\omega \in C_0}\left[F(\omega) + I(\omega)\right],$$

once we show the infimum is positive and finite.

To this end, let $\{e_1, \dots, e_p\}$ be the natural basis of $\mathbb{R}^p$ and let $\omega(u) = u^{2/3} e_1$. Clearly, the infimum is finite. To show positivity, assume the contrary. Then there is a sequence $\omega_n \in C_0$ such that

$$F(\omega_n) + I(\omega_n) \to 0 \quad \text{as } n \to \infty.$$

In particular,

$$\int_0^1 |\omega_n'(u)|^2\, du = 2I(\omega_n) \to 0 \quad \text{and} \quad F(\omega_n) \to 0.$$

Then for each n there is a set $I_n \subseteq [0,1]$ with zero Lebesgue measure such that for $t \in [0,1] \backslash I_n$,

$$|\omega_n(t)| = \left|\int_0^t \omega_n'(u)\, du\right| \le \int_0^t |\omega_n'(u)|\, du.$$

Hence for $t \in [0,1] \backslash \bigcup_n I_n$,

$$\omega_n(t) \to 0 \quad \text{as } n \to \infty;$$

that is, $\omega_n \to 0$ almost everywhere as $n \to \infty$. This forces

$$F(\omega_n) = \int_0^1 \frac{1}{|\omega_n(t)|}\, dt \to \infty,$$

contrary to $F(\omega_n) \to 0$. Hence the infimum must be positive as claimed. $\square$

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