The Annals of Probability
2001, Vol. 29, No. 1, 484–503

NO MORE THAN THREE FAVORITE SITES FOR SIMPLE RANDOM WALK[1]

BY BÁLINT TÓTH

Technical University Budapest

We prove that, with probability 1, eventually there are no more than three favorite (i.e., most visited) sites of simple symmetric random walks. This partially answers a relatively longstanding question of Erdös and Révész.

1. Introduction and main result. Let $S(t), t \in \mathbb{Z}_+$, be a simple symmetric random walk on $\mathbb{Z}$ with initial state $S(0) = 0$. Its *upcrossings*, *downcrossings* and (*site*) *local time* are defined for $t \in \mathbb{N}$ and $x \in \mathbb{Z}$ as follows:

$$U(t, x) := \#\big\{0 < s \le t : S(s) = x,\ \ S(s-1) = x - 1\big\}, \tag{1.1}$$

$$D(t, x) := \#\big\{0 < s \le t : S(s) = x,\ \ S(s-1) = x + 1\big\}, \tag{1.2}$$

$$L(t, x) := \#\big\{0 < s \le t : S(s) = x\big\} = U(t, x) + D(t, x). \tag{1.3}$$

The following identities are straightforward:

$$U(t, x) - D(t, x-1) = \mathbb{1}_{\{0<x\le S(t)\}} - \mathbb{1}_{\{S(t)<x\le 0\}}, \tag{1.4}$$

$$D(t, x) - U(t, x+1) = \mathbb{1}_{\{S(t)\le x<0\}} - \mathbb{1}_{\{0\le x<S(t)\}}. \tag{1.5}$$

And from these it follows that

$$\begin{aligned} L(t, x) &= D(t, x) + D(t, x-1) + \mathbb{1}_{\{0<x\le S(t)\}} - \mathbb{1}_{\{S(t)<x\le 0\}} \\ &= U(t, x) + U(t, x+1) + \mathbb{1}_{\{S(t)\le x<0\}} - \mathbb{1}_{\{0\le x<S(t)\}}. \end{aligned} \tag{1.6}$$

The set of *favorite* (*or most visited*) *sites* of the random walk at time $t \in \mathbb{N}$ are those sites where the local time attains its maximum value:

$$\mathscr{K}(t) := \Big\{y \in \mathbb{Z} : L(t, y) = \max_{z\in\mathbb{Z}} L(t, z)\Big\}. \tag{1.7}$$

It is clear that the number of favorite sites changes in time as follows:

$$\#\mathscr{K}(t+1) = \begin{cases} \#\mathscr{K}(t), & \text{if } \ S(t+1) \notin \mathscr{K}(t+1), \\ \#\mathscr{K}(t) + 1, & \text{if } \ \mathscr{K}(t+1) = \mathscr{K}(t) \cup \{S(t+1)\}, \\ 1, & \text{if } \ \mathscr{K}(t+1) = \{S(t+1)\} \subset \mathscr{K}(t). \end{cases} \tag{1.8}$$

In plain words, one of the following three possibilities can occur at each step of the walk: Either the currently occupied site is not favorite and $\mathscr{K}$ remains

Received February 2000; revised July 2000.

[1]Supported in part by the following grants: OTKA-T26176 (Hungarian National Foundation for Scientific Research), FKFP-0638/99 (Ministry of Education) and TKI-Stochastics@TUB (Hungarian Academy of Sciences).

AMS 2000 *subject classifications*. 60J15, 60J55.

Key words and phrases. Random walk, local time, favorite sites, most visited sites.

unchanged. Or the currently occupied site becomes a new favorite *besides* the favorites of the previous stage, and thus the number of favorites increases by one. Or, finally, a favorite site is revisited and so this site becomes now the only new favorite. There are no other possibilities.

Clearly, $\#\mathscr{K}(t) \geq 1$ for all $t \geq 1$, and it is easy to verify that for infinitely many times, $t \geq 1$, there are at least two favorite sites: $\#\mathscr{K}(t) \geq 2$. Erdös and Révész formulated and repeatedly raised the following

QUESTION. Does it happen that $\#\mathscr{K}(t) \geq r$ infinitely often (i.e., almost surely for infinitely many times $t \geq 1$) for $r = 3, 4, \ldots$?

See, for example, Erdös [6], Erdös and Révész [7]–[9] or Révész [14] for an extended list of related questions and problems.

Questions related to the asymptotic behavior of the favorite (or most visited) sites of a random walk have been considered by many authors since the mid-1980s. We quote here a few relevant results, with no claim of exhaustiveness.

- Bass and Griffin [2] prove that almost surely the set of favorites is transient. More exactly, they prove that the distance of the set of favorite sites from the origin increases faster than $\sqrt{n}/(\log n)^{11}$ but slower than $\sqrt{n}/(\log n)$.
- Csáki and Shi [4] prove that the distance between the edge of the range of the random walk and the set of favorite sites increases as fast as $\sqrt{n}/(\log \log n)^{3/2}$.
- Csáki, Révész and Shi [3] prove that the position of a favorite site can have jumps as large as $\sqrt{2n \log \log n}$, that is, comparable with the diameter of the full range of the random walk. They also extend a much earlier result of Kesten [10], identifying the set of joint limit points of the set of favorite sites and the favorite values (i.e., maximum values of local time), both rescaled by $\sqrt{2n \log \log n}$.
- There are many papers dealing with similar questions in the context of symmetric stable processes rather than random walks (or Brownian motion). See, for example, Eisenbaum [5] and Bass, Eisenbaum and Shi [1] and the papers cited therein.

In the present paper we answer in the negative the question of Erdös and Révész quoted above, for $r \geq 4$: we prove that with probability 1, there are at most finitely many times $t \geq 1$ when there are four or more favorite sites of the random walk $S(t)$. In [17] a similar result was proved for the set of *favorite edges* rather than favorite sites. The present paper deals with the original question of Erdös and Révész. The starting general ideas of the present paper (see Sections 1–3) are very close to those of Tóth and Werner [17]. However, the details of the proof require more refined estimates and arguments. On the technical level (see Sections 4–6) this proof is rather different.

For $r \geq 1$ denote by $f(r)$ the (possibly infinite) number of steps, when the currently occupied site is one of the r actual favorites:

$$f(r) := \#\{t \geq 1 : S(t) \in \mathscr{K}(t),\ \#\mathscr{K}(t) = r\}. \tag{1.9}$$

From (1.8) it follows that, for any $r \geq 1$, $f(r+1) \leq f(r)$. (Both sides of the inequality could be infinite.)

The main result of this paper is the following.

THEOREM 1.

$$\mathbf{E}\big(f(4)\big) < \infty. \tag{1.10}$$

REMARKS. 1. From this theorem the negative answer to the question of Erdös and Révész clearly follows, for the cases $r \geq 4$.

2. The case $r = 3$ remains open. From the proof of the above theorem, it becomes clear that $\mathbf{E}(f(3)) = \infty$. Nevertheless we conjecture that $f(3) < \infty$, almost surely.

We, now give a brief overview of the main arguments of the proof: first, we decompose the random variable $f(4)$ in a very natural way, as a doubly infinite sum of indicator variables, indicating when $\#\mathscr{K}(t)$ increased from 3 to 4 [see the second alternative in (1.8)] and which sites were the actual positions of the random walker at these moments of increase of $\#\mathscr{K}$. Inverting the order of summation in this doubly infinite sum, we get an expression of $\mathbf{E}(f(4))$ in terms of the local time process of the random walk, stopped at inverse local times. Next, using the Ray–Knight representation of the local time process stopped at inverse local times, we rewrite these expressions in terms of critical Galton–Watson processes. Eventually, the proof of Theorem 1 will rely on controlling the probability distribution of the number of global maxima of given height of these critical Galton–Watson processes. On the technical level this means controlling various probabilities and expectations arising naturally with the critical Galton–Watson process. As typical examples we mention here a large deviation estimate on the size of the largest jump before hitting level $h \gg 1$ or extinction (Lemma 1), or precise control on the probability that a critical Galton–Watson process hits *exactly* a given level h before extinction (Lemma 2). Other estimates of similar flavor are also involved.

The paper is organized as follows: In Section 2 we perform some straightforward manipulations (essentially rearrangements of sums). In Section 3 we recall the Ray–Knight theorems for the local times of simple random walks. In Section 4 first we express our relevant probabilities and expectations (found in Section 2) in terms of the Galton–Watson processes arising with the Ray–Knight representation. Then we formulate Proposition 1, stating some upper bounds on these probabilities and expectations, and using these bounds we prove Theorem 1. The proof of Proposition 1 is postponed to the end of Section 5. In Section 5 four lemmas and, as their consequence, Proposition 1 are proved. Throughout the technical parts of the proofs, smaller, quite plausible statements are invoked. These are called Side-lemmas (1 to 4). Their proofs are postponed to Section 6.

Throughout the paper, in various upper bounds, multiplicative constants, respectively, constants in exponential rates, will be denoted generically by C, respectively, by γ. The values of these constants may vary even within one proof, but we hope there is no danger of confusion.

2. Preparations.

In the following transformations the *inverse local times*, defined below for $k \in \mathbb{N}$ and $x \in \mathbb{Z}$, will play an essential role:

$$T_U(k,x) := \inf\{t \geq 1 : U(t,x) = k\}, \tag{2.1}$$

$$T_D(k,x) := \inf\{t \geq 1 : D(t,x) = k\}. \tag{2.2}$$

It turns out that questions related to the local time are easier to handle if the random walk is observed at the random stopping times $T_U(k,x)$ and $T_D(k,x)$ rather than at deterministic times $t \geq 1$. This "combinatorial trick" has its origin in Knight [11] and has been successfully applied in various contexts. See, for example, [12], [15] and [16] and the references cited therein.

We express $f(4)$ with the help of some straightforward rearrangements of summations:

$$f(4) = \sum_{x\in\mathbb{Z}} \big(u(x) + d(x)\big), \tag{2.3}$$

where

$$\begin{aligned} u(x) &:= \sum_{t=1}^{\infty} \mathbb{1}_{\{S(t)=x,\, S(t-1)=x-1,\, x\in\mathscr{K}(t),\, \#\mathscr{K}(t)=4\}} \\ &= \sum_{t=1}^{\infty}\sum_{k=1}^{\infty} \mathbb{1}_{\{T_U(k,x)=t,\, x\in\mathscr{K}(t),\, \#\mathscr{K}(t)=4\}} \\ &= \sum_{k=1}^{\infty} \mathbb{1}_{\{x\in\mathscr{K}(T_U(k,x)),\, \#\mathscr{K}(T_U(k,x))=4\}} \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} d(x) &:= \sum_{t=1}^{\infty} \mathbb{1}_{\{S(t)=x,\, S(t-1)=x+1,\, x\in\mathscr{K}(t),\, \#\mathscr{K}(t)=4\}} \\ &= \sum_{t=1}^{\infty}\sum_{k=1}^{\infty} \mathbb{1}_{\{T_D(k,x)=t,\, x\in\mathscr{K}(t),\, \#\mathscr{K}(t)=4\}} \\ &= \sum_{k=1}^{\infty} \mathbb{1}_{\{x\in\mathscr{K}(T_D(k,x)),\, \#\mathscr{K}(T_D(k,x))=4\}}. \end{aligned} \tag{2.5}$$

Clearly,

$$u(x) \stackrel{\text{law}}{=} d(-x) \tag{2.6}$$

and, consequently,

$$\mathbf{E}\big(f(4)\big) = 2\sum_{x=1}^{\infty}\mathbf{E}\big(u(x)\big) + 2\sum_{x=0}^{\infty}\mathbf{E}\big(d(x)\big), \tag{2.7}$$

with

$$\mathbf{E}\big(u(x)\big) = \sum_{k=1}^{\infty}\mathbf{P}\big(x \in \mathscr{K}(T_U(k,x)),\ \#\mathscr{K}(T_U(k,x)) = 4\big), \tag{2.8}$$

$$\mathbf{E}\big(d(x)\big) = \sum_{k=1}^{\infty}\mathbf{P}\big(x \in \mathscr{K}(T_D(k,x)),\ \#\mathscr{K}(T_D(k,x)) = 4\big). \tag{2.9}$$

We shall show in detail that

$$\sum_{x=1}^{\infty}\mathbf{E}\big(u(x)\big) = \sum_{x=1}^{\infty}\sum_{k=1}^{\infty}\mathbf{P}\big(x \in \mathscr{K}(T_U(k,x)),\ \#\mathscr{K}(T_U(k,x)) = 4\big) < \infty. \tag{2.10}$$

The similar statement

$$\sum_{x=0}^{\infty}\mathbf{E}\big(d(x)\big) = \sum_{x=0}^{\infty}\sum_{k=1}^{\infty}\mathbf{P}\big(x \in \mathscr{K}(T_D(k,x)),\ \#\mathscr{K}(T_D(k,x)) = 4\big) < \infty \tag{2.11}$$

can be proved in an identical way.

3. Ray–Knight representation. Throughout this paper we denote by Y_t a critical branching process with geometric offspring distribution (Galton–Watson process) and by Z_t a critical branching process with geometric offspring distribution and one intruder at each generation. Y_t and Z_t are Markov chains with state space $\mathbb{Z}_+$ and transition probabilities:

$$\begin{aligned}\mathbf{P}\big(Y_{t+1} = j \,\big|\, Y_t = i\big) &= \pi(i,j)\\ &:= \begin{cases}\delta_{0,j}, & \text{if } i = 0,\\ 2^{-i-j}\dfrac{(i+j-1)!}{(i-1)!\,j!}, & \text{if } i > 0,\end{cases}\end{aligned} \tag{3.1}$$

$$\mathbf{P}\big(Z_{t+1} = j \,\big|\, Z_t = i\big) = \rho(i,j) := 2^{-i-j-1}\frac{(i+j)!}{i!\,j!}. \tag{3.2}$$

Let $k \ge 0$ and $x \ge 1$ be fixed integers and define the following three processes:

(i) Z_t, $0 \le t \le x-1$, is a Markov chain with transition probabilities $\rho(i,j)$ and initial state $Z_0 = k$;

(ii) Y_t, $-1 \le t < \infty$, is a Markov chain with transition probabilities $\pi(i,j)$ and initial state $Y_{-1} = k$;

(iii) Finally, Y'_t, $0 \le t < \infty$, is another Markov chain with the same transition probabilities $\pi(i,j)$ and initial state $Y'_0 = Z_{x-1}$.

The three chains are independent, except for the fact that Y' starts from the terminal state of Z. Using these three chains, we patch together the process

$$\Delta_{x,k}(y) := \begin{cases} Z_{x-y-1}, & \text{if } 0 \le y \le x-1, \\ Y_{y-x}, & \text{if } x-1 \le y \le \infty, \\ Y'_{-y}, & \text{if } -\infty \le y \le 0. \end{cases} \tag{3.3}$$

We also define

$$\Lambda_{x,k}(y) := \Delta_{x,k}(y) + \Delta_{x,k}(y-1) + \mathbb{1}_{\{0<y\le x\}}. \tag{3.4}$$

From the by now classical Ray–Knight theorems on the local time process of simple symmetric random walks on $\mathbb{Z}$ (cf. [11] and [13]), it follows that, for any integers $x \ge 1$ and $k \ge 0$,

$$\big(D(T_U(k+1, x), y),\ y \in \mathbb{Z}\big) \overset{\text{law}}{=} \big(\Delta_{x,k}(y),\ y \in \mathbb{Z}\big). \tag{3.5}$$

Usually the first, respectively, second, Ray–Knight theorem is stated separately as

$$\big(D(T_U(1, x), x-1-y),\ 0 \le y \le x-1\big) \overset{\text{law}}{=} \big(Z_y,\ 0 \le y \le x-1\big), \tag{3.6}$$

respectively,

$$\big(D(T_U(k, 0), y),\ -1 \le y\big) \overset{\text{law}}{=} \big(Y_y,\ -1 \le y\big). \tag{3.7}$$

These statements directly imply the joint formulation (3.5). See, for example, [17], Proposition 2.1, where it is stated exactly in this present form (with slightly different notation), or [15], Section 2, where it is proved in the more general context of self-interacting random walks.

Using (1.6) and (3.4), from (3.5) we get

$$\big(L(T_U(k+1, x), y),\ y \in \mathbb{Z}\big) \overset{\text{law}}{=} \big(\Lambda_{x,k}(y),\ y \in \mathbb{Z}\big). \tag{3.8}$$

4. Proof of Theorem 1. Given the Markov chains Y_t, Z_t and Y'_t, we define

$$\widetilde{Z}_t := Z_t + Z_{t-1} + 1, \qquad \widetilde{Y}_t := Y_t + Y_{t-1}, \qquad \widetilde{Y}'_t := Y'_t + Y'_{t-1}. \tag{4.1}$$

For $h \in \mathbb{N}$ the following stopping times are introduced:

$$\sigma_h := \inf\{t \ge 0 : Y_t \ge h\}, \tag{4.2}$$

$$\sigma'_h := \inf\{t \ge 0 : Y'_t \ge h\}, \tag{4.3}$$

$$\omega := \inf\{t \ge 0 : Y_t = 0\}, \tag{4.4}$$

$$\omega' := \inf\{t \ge 0 : Y'_t = 0\}, \tag{4.5}$$

$$\tau_h := \inf\{t \ge 0 : Z_t \ge h\}, \tag{4.6}$$

$$\tilde{\sigma}_{h,0} := 0, \qquad \tilde{\sigma}_{h,i+1} := \inf\{t > \tilde{\sigma}_{h,i} : \widetilde{Y}_t \ge h\}, \qquad \tilde{\sigma}_h := \tilde{\sigma}_{h,1}, \tag{4.7}$$

$$\tilde{\sigma}'_{h,0} := 0, \qquad \tilde{\sigma}'_{h,i+1} := \inf\{t > \tilde{\sigma}'_{h,i} : \widetilde{Y}'_t \ge h\}, \qquad \tilde{\sigma}'_h := \tilde{\sigma}'_{h,1}, \tag{4.8}$$

$$\tilde{\tau}_{h,0} := 0, \qquad \tilde{\tau}_{h,i+1} := \inf\{t > \tilde{\tau}_{h,i} : \widetilde{Z}_t \ge h\}, \qquad \tilde{\tau}_h := \tilde{\tau}_{h,1}. \tag{4.9}$$

In plain words, σ_h, σ'_h and τ_h are the first hitting times of the interval $[h, \infty)$ by the processes Y_t, Y'_t, respectively, Z_t, with the convention that $\sigma_h = \infty$ (resp. $\sigma'_h = \infty$) if the process Y_t (resp. Y'_t) never hits this interval. Further on ω, respectively, ω', are the extinction times of the processes Y_t, respectively, Y'_t. Finally, $\tilde{\sigma}_{h,i}$, $\tilde{\sigma}'_{h,i}$ and $\tilde{\tau}_{h,i}$ denote the ith hitting times of the interval $[h, \infty)$ by the processes $\widetilde{Y}_t$, $\widetilde{Y}'_t$, respectively, $\widetilde{Z}_t$.

For $h \geq 1$, $p \geq 0$ and $x \geq 1$ fixed integers we also define the following events:

$$\begin{aligned} A_{h,p} &:= \Big\{ \max_{1 \le t < \infty} \widetilde{Y}_t \le h,\ \#\{1 \le t < \infty : \widetilde{Y}_t = h\} = p \Big\} \\ &= \big\{ \tilde{\sigma}_{h,p} < \infty = \tilde{\sigma}_{h,p+1},\ \widetilde{Y}_{\tilde{\sigma}_{h,1}} = \cdots = \widetilde{Y}_{\tilde{\sigma}_{h,p}} = h \big\}, \end{aligned} \tag{4.10}$$

$$\begin{aligned} A'_{h,p} &:= \Big\{ \max_{1 \le t < \infty} \widetilde{Y}'_t \le h,\ \#\{1 \le t < \infty : \widetilde{Y}'_t = h\} = p \Big\} \\ &= \big\{ \tilde{\sigma}'_{h,p} < \infty = \tilde{\sigma}'_{h,p+1},\ \widetilde{Y}'_{\tilde{\sigma}_{h,1}} = \cdots = \widetilde{Y}'_{\tilde{\sigma}_{h,p}} = h \big\}, \end{aligned} \tag{4.11}$$

$$\begin{aligned} B_{x,h,p} &:= \Big\{ \max_{1 \le t < x} \widetilde{Z}_t \le h,\ \#\{1 \le t < x : \widetilde{Z}_t = h\} = p \Big\} \\ &= \big\{ \tilde{\tau}_{h,p} < x \le \tilde{\tau}_{h,p+1},\ \widetilde{Z}_{\tilde{\tau}_{h,1}} = \cdots = \widetilde{Z}_{\tilde{\tau}_{h,p}} = h \big\}. \end{aligned} \tag{4.12}$$

In plain words, $A_{h,0}$ (resp. $A'_{h,0}$) is the event that the process $\widetilde{Y}_t$ (resp. $\widetilde{Y}'_t$) never hits $[h, \infty)$; $A_{h,p}$ (resp. $A'_{h,p}$), $p \geq 1$, is the event that the process $\widetilde{Y}_t$ (resp. $\widetilde{Y}'_t$), before extinction, hits exactly p times its maximum level h; $B_{x,h,0}$ is the event that the process $\widetilde{Z}_t$ does not hit $[h, \infty)$ in the time interval $1 \le t < x$; finally, $B_{x,h,p}$, $p \geq 1$, is the event that in the time interval $1 \le t < x$, the process $\widetilde{Z}_t$ hits exactly p times its maximum level h.

With the help of the Ray–Knight representation and the events introduced above, we get the expression:

$$\begin{aligned} \mathbf{E}\big(u(x)\big) = \sum_{p+q+r=3} \sum_{h=1}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} &\mathbf{P}\big(A_{h,p} \,\big|\, Y_0 = h-k-1\big) \\ &\times \pi(k,\ h-k-1) \\ &\times \mathbf{P}\big(B_{x,h,q} \cap \{Z_{x-1} = l\} \,\big|\, Z_0 = k\big) \\ &\times \mathbf{P}\big(A'_{h,r} \,\big|\, Y'_0 = l\big), \end{aligned} \tag{4.13}$$

which leads directly to

$$\begin{aligned}\sum_{x=1}^{\infty}\mathbf{E}\big(u(x)\big)\le \sum_{p+q+r=3}\sum_{h=1}^{\infty}\sum_{k=0}^{\infty}\ &\mathbf{P}\big(A_{h,p}\,\big|\,Y_0=h-k-1\big)\\ &\times\pi(k,h-k-1)\\ &\times\Big(\sum_{x=1}^{\infty}\mathbf{P}\big(B_{x,h,q}\,\big|\,Z_0=k\big)\Big)\\ &\times\Big(\sup_{l\ge 0}\mathbf{P}\big(A'_{h,r}\,\big|\,Y'_0=l\big)\Big).\end{aligned}\tag{4.14}$$

The proof of Theorem 1 will follow directly from the bounds provided by the following result.

PROPOSITION 1. *For any $\varepsilon>0$ there exists a finite constant $C<\infty$ such that for any $0\le k\le h$:*

(i) *without any restriction on k or p,*

$$\sum_{x=1}^{\infty}\mathbf{P}\big(B_{x,h,p}\,\big|\,Z_0=k\big)\le Ch;\tag{4.15}$$

(ii) *if either $k\in[(h-h^{1/2+\varepsilon})/2,\ (h+h^{1/2+\varepsilon})/2]$ or $p\ge 1$ holds, then*

$$\mathbf{P}\big(A_{h,p}\,\big|\,Y_0=k\big)\le\big(Ch^{-1/2+\varepsilon}\big)^{p+1},\tag{4.16}$$

$$\sum_{x=1}^{\infty}\mathbf{P}\big(B_{x,h,p}\,\big|\,Z_0=k\big)\le\big(Ch^{-1/2+\varepsilon}\big)^{p+1}h.\tag{4.17}$$

REMARK. For $k\ge h$ the left-hand sides of (4.15), (4.16) and (4.17), of course, vanish.

We postpone the proof of this proposition to the end of the next section and proceed with the proof of Theorem 1. Using the bounds (4.15)–(4.17), we prove (2.10). As we already mentioned, (2.11) is proved in a completely identical way. Theorem 1 follows from (2.10), (2.11) via (2.7).

In the forthcoming argument $\cdots$ will stand as an abbreviation of the summand on the right-hand side of (4.14). On the right-hand side of (4.14) keep $p,\ q,\ r$ and h fixed and decompose the sum over $k\ge 0$ as follows:

$$\sum_k\cdots\ =\sum_{k:|h-2k|\le h^{1/2+\varepsilon}}\cdots\ +\sum_{k:|h-2k|>h^{1/2+\varepsilon}}\cdots.\tag{4.18}$$

Similar decompositions will be applied a few more times throughout the paper.

According to Side-lemma 1(i) (see Section 6),

$$\sum_{k:|h-2k|>h^{1/2+\varepsilon}}\pi(k,h-1-k)<C\exp(-\gamma h^{2\varepsilon}),\tag{4.19*}$$

with some properly chosen constants $C = C(\varepsilon) < \infty$ and $\gamma = \gamma(\varepsilon) > 0$. (The * next to the equation number indicates the fact that this inequality follows from a side-lemma stated and proved in Section 6.)

Using (4.15)–(4.19), we bound the sum over k on the right-hand side of (4.14) as follows:

If $r = 0$ and $p + q = 3$,

$$\sum_k \cdots \le \left(Ch^{-1/2+\varepsilon}\right)^{p+q+2} h + (Ch)\left(C\exp(-\gamma h^{2\varepsilon})\right) \le C' h^{-3/2+5\varepsilon}, \tag{4.20}$$

with some properly chosen $C' < \infty$.

If $r > 0$ and $p + q + r = 3$,

$$\sum_k \cdots \le \left(Ch^{-1/2+\varepsilon}\right)^{p+q+r+3} h + (Ch)\left(C\exp(-\gamma h^{2\varepsilon})\right) \le C' h^{-2+6\varepsilon}, \tag{4.21}$$

with some properly chosen $C' < \infty$.

In both cases the upper bound is summable over $h \ge 1$, if we choose $\varepsilon < 1/10$. Hence (2.10) and the statement of Theorem 1. □

5. Technical lemmas. The present section is divided into five subsections. In Sections 5.1–5.4 we state and prove some lemmas of a more technical nature, needed in the proof of Proposition 1, which is presented in Section 5.5. Within the proofs of the forthcoming lemmas we use the bounds proved in Side-lemmas 1–4 of Section 6. The equation numbers where these "subroutines" are called are indicated by an asterisk. These are (5.6*), (5.10*), (5.17*), (5.22*), (5.30*) and (5.31*). Throughout this section $\varepsilon > 0$ is fixed.

5.1. *The maximal jump.* We prove that the largest jump of the Markov chains Y_t and Z_t, before reaching level h, is less than $h^{1/2+\varepsilon}$, with overwhelming probability. Define the maximal jumps of Y_t, respectively, Z_t, as follows:

$$M_h := \sup\left\{|Y_t - Y_{t-1}| \,:\, 1 \le t \le \sigma_h\right\} = \sup\left\{|Y_t - Y_{t-1}| : 1 \le t \le \sigma_h \wedge \omega\right\}, \tag{5.1}$$

$$N_h := \sup\left\{|Z_t - Z_{t-1}| : 1 \le t \le \tau_h\right\}. \tag{5.2}$$

By definition, $M_h = 0$ if $Y_0 \ge h$ and $N_h = 0$ if $Z_0 \ge h$.

LEMMA 1. *There exist two constants, $C < \infty$ and $\gamma > 0$, such that for any $0 \le k \le h$ the following bounds hold:*

$$\mathbf{P}\left(M_h > h^{1/2+\varepsilon} \,\middle|\, Y_0 = k\right) < C\exp(-\gamma h^{2\varepsilon}), \tag{5.3}$$

$$\mathbf{P}\left(N_h > h^{1/2+\varepsilon} \,\middle|\, Z_0 = k\right) < C\exp(-\gamma h^{2\varepsilon}). \tag{5.4}$$

PROOF. We prove here (5.3) in detail. The proof of (5.4) is essentially the same and it is left for the reader. For the moment let γ be an arbitrary positive number (its value will be fixed at the end of this proof)

$$\begin{aligned}
&\mathbf{P}\big(M_h > h^{1/2+\varepsilon} \,\big|\, Y_0 = k\big)\\
&\quad\le \mathbf{P}\big(\{M_h > h^{1/2+\varepsilon}\} \cap \{\sigma_h \wedge \omega \le h^2 \exp(\gamma h^{2\varepsilon})\} \,\big|\, Y_0 = k\big)\\
&\qquad + \mathbf{P}\big(\sigma_h \wedge \omega > h^2 \exp(\gamma h^{2\varepsilon}) \,\big|\, Y_0 = k\big).
\end{aligned} \tag{5.5}$$

Side-lemma 3 (see Section 6) states that

$$\mathbf{E}\big(\sigma_h \wedge \omega \,\big|\, Y_0 = k\big) \le C h^2, \tag{5.6*}$$

with some finite constant C. Using now Markov's inequality, we get the following upper bound on the second term of the right-hand side of (5.5):

$$\mathbf{P}\big(\sigma_h \wedge \omega > h^2 \exp(\gamma h^{2\varepsilon}) \,\big|\, Y_0 = k\big) \le C \exp(-\gamma h^{2\varepsilon}). \tag{5.7}$$

To bound the first term on the right-hand side of (5.5), we use the following representation of the Markov chain Y_t: let $(\xi_{t,i})_{t\ge 1, i\ge 1}$ be i.i.d. random variables with common geometric distribution $\mathbf{P}(\xi_{t,i} = k) = 2^{-k-1}$. The process Y_t is realized as follows: fix Y_0 and put

$$Y_{t+1} = \sum_{j=1}^{Y_t} \xi_{t+1,j}. \tag{5.8}$$

Using this representation, we note that

$$\begin{aligned}
&\mathbf{P}\big(\{M_h > h^{1/2+\varepsilon}\} \cap \{\sigma_h \wedge \omega \le h^2 \exp(\gamma h^{2\varepsilon})\} \,\big|\, Y_0 = k\big)\\
&\quad\le \mathbf{P}\left(\max\left\{ \max_{1\le j\le h} \left| \sum_{i=1}^{j} (\xi_{t,i} - 1) \right| : 1 \le t \le h^2 \exp(\gamma h^{2\varepsilon}) \right\} > h^{1/2+\varepsilon} \right)\\
&\quad= 1 - \left(1 - \mathbf{P}\left(\max_{1\le j\le h} \left| \sum_{i=1}^{j} (\xi_{1,i} - 1) \right| > h^{1/2+\varepsilon} \right) \right)^{h^2 \exp(\gamma h^{2\varepsilon})}.
\end{aligned} \tag{5.9}$$

From Side-lemma 2 (see Section 6) it follows that

$$\mathbf{P}\left(\max_{1\le j\le h} \left| \sum_{i=1}^{j} (\xi_i - 1) \right| > h^{1/2+\varepsilon} \right) \le 2 \exp(-h^{2\varepsilon}/8). \tag{5.10*}$$

Using this bound, we get

$$\begin{aligned}
&\mathbf{P}\big(\{M_h > h^{1/2+\varepsilon}\} \cap \{\sigma_h \wedge \omega \le h^2 \exp(\gamma h^{2\varepsilon})\} \,\big|\, Y_0 = k\big)\\
&\quad\le 1 - \big(1 - 2\exp(-h^{2\varepsilon}/8)\big)^{h^2 \exp(\gamma h^{2\varepsilon})}\\
&\quad\le 2h^2 \exp\big((\gamma - 8^{-1}) h^{2\varepsilon}\big).
\end{aligned} \tag{5.11}$$

In the last inequality we use the fact that, for $0 < \alpha < 1 < \beta$, $1-\alpha\beta < (1-\alpha)^{\beta}$. We choose $\gamma < 16^{-1}$. From (5.5), (5.7) and (5.11) we get (5.3), with an appropriately chosen constant $C < \infty$. □

5.2. *Hitting exactly h.* We give an upper bound on the probability of the event that the critical Galton–Watson process hits the interval $[h, \infty)$ exactly at level h.

LEMMA 2. *There exists a constant $C < \infty$ such that, for any $0 \le k \le h$,*

$$\mathbf{P}\Big(\{\tilde\sigma_h < \infty\} \cap \{\widetilde{Y}_{\tilde\sigma_h} = h\} \,\Big|\, Y_0 = k\Big) < Ch^{-1/2+\varepsilon}, \tag{5.12}$$

$$\mathbf{P}\Big(\widetilde{Z}_{\tilde\tau_h} = h \,\Big|\, Z_0 = k\Big) < Ch^{-1/2+\varepsilon}. \tag{5.13}$$

PROOF. Again, we give the details of the proof of (5.12), leaving the identical details of (5.13) for the reader:

$$\begin{aligned}&\mathbf{P}\big(\{\tilde\sigma_h < \infty\} \cap \{\widetilde{Y}_{\tilde\sigma_h} = h\} \,\big|\, Y_0 = k\big)\\ &\quad= \sum_{l=0}^{\infty} \mathbf{P}\big(\{\tilde\sigma_h < \infty\} \cap \{Y_{\tilde\sigma_h - 1} = l\} \cap \{Y_{\tilde\sigma_h} = h-l\} \,\big|\, Y_0 = k\big).\end{aligned} \tag{5.14}$$

We divide the sum into two parts, as in (4.18):

$$\begin{aligned}&\sum_{l:|h-2l|>h^{1/2+\varepsilon}} \mathbf{P}\big(\{\tilde\sigma_h < \infty\} \cap \{Y_{\tilde\sigma_h - 1} = l\} \cap \{Y_{\tilde\sigma_h} = h-l\} \,\big|\, Y_0 = k\big)\\ &\qquad\le \mathbf{P}\big(M_h > h^{1/2+\varepsilon} \,\big|\, Y_0 = k\big) < C\exp(-\gamma h^{2\varepsilon})\end{aligned} \tag{5.15}$$

by Lemma 1. On the other hand,

$$\begin{aligned}&\sum_{l:|h-2l|\le h^{1/2+\varepsilon}} \mathbf{P}\big(\{\tilde\sigma_h < \infty\} \cap \{Y_{\tilde\sigma_h - 1} = l\} \cap \{Y_{\tilde\sigma_h} = h-l\} \,\big|\, Y_0 = k\big)\\ &\quad= \sum_{l:|h-2l|\le h^{1/2+\varepsilon}} \mathbf{P}\big(\{\tilde\sigma_h < \infty\} \cap \{Y_{\tilde\sigma_h - 1} = l\} \,\big|\, Y_0 = k\big)\\ &\qquad\quad\times \frac{\pi(l, h-l)}{\sum_{m\ge h-l}\pi(l, m)}.\end{aligned} \tag{5.16}$$

From Side-lemma 1(ii) (see Section 6) we know that, as long as $|2l-h| \le h^{1/2+\varepsilon}$,

$$\frac{\pi(l,\ h-l)}{\sum_{m\ge h-l}\pi(l,\ m)} < Ch^{-1/2+\varepsilon}, \tag{5.17*}$$

with some finite constant C. From (5.16) and (5.17) we get

$$\begin{aligned}&\sum_{l:|h-2l|\le h^{1/2+\varepsilon}} \mathbf{P}\big(\{\tilde\sigma_h < \infty\} \cap \{Y_{\tilde\sigma_h - 1} = l\} \cap \{Y_{\tilde\sigma_h} = h-l\} \,\big|\, Y_0 = k\big)\\ &\qquad\le Ch^{-1/2+\varepsilon}.\end{aligned} \tag{5.18}$$

Finally, (5.15) and (5.18) yield (5.13). □

5.3. *$\widetilde{Y}_t$ does not hit level $\geq h$.* We give an upper bound on the probability that $\widetilde{Y}_t = Y_{t-1} + Y_t$ stays below level h before extinction, provided that Y_0 is close to $h/2$.

LEMMA 3. *There exists a constant $C < \infty$ such that, for any $h \geq 1$ and $k \in [(h - h^{1/2+\varepsilon})/2, (h + h^{1/2+\varepsilon})/2]$,*

$$\mathbf{P}\big(\tilde{\sigma}_h = \infty \,\big|\, Y_0 = k\big) < Ch^{-1/2+\varepsilon}. \tag{5.19}$$

PROOF.

$$\begin{aligned}\mathbf{P}\big(\tilde{\sigma}_h = \infty \,\big|\, Y_0 = k\big) &\leq \mathbf{P}\big(\{\tilde{\sigma}_h = \infty\} \cap \{M_h \leq h^{1/2+\varepsilon}\} \,\big|\, Y_0 = k\big)\\ &\quad + \mathbf{P}\big(M_h > h^{1/2+\varepsilon} \,\big|\, Y_0 = k\big).\end{aligned} \tag{5.20}$$

To bound the first term on the right-hand side, note that

$$\{\tilde{\sigma}_h = \infty\} \cap \{M_h \leq h^{1/2+\varepsilon}\} \subset \{\sigma_{(h+h^{1/2+\varepsilon})/2} = \infty\}. \tag{5.21}$$

According to Side-lemma 3 (see Section 6), there exists a finite constant C such that

$$\mathbf{P}\big(\sigma_h = \infty \,\big|\, Y_0 = k\big) < \frac{h-k}{h} + Ch^{-1/2}. \tag{5.22*}$$

Thus

$$\begin{aligned}&\mathbf{P}\big(\{\tilde{\sigma}_h = \infty\} \cap \{M_h \leq h^{1/2+\varepsilon}\} \,\big|\, Y_0 = k\big)\\ &\quad \leq \mathbf{P}\big(\sigma_{(h+h^{1/2+\varepsilon})/2} = \infty \,\big|\, Y_0 = k\big) \leq Ch^{-1/2+\varepsilon}\end{aligned} \tag{5.23}$$

for $k \in [(h - h^{1/2+\varepsilon})/2,\ (h + h^{1/2+\varepsilon})/2]$. This bound, together with (5.20) and (5.3), yields (5.19). $\square$

5.4. *Expectation of $\tilde{\tau}_h$.* We give upper bounds on the expectation of the hitting times $\tilde{\tau}_h$.

LEMMA 4. *There exists a constant $C < \infty$ such that for any $0 \leq k \leq h$ the following bounds hold:*

(i) *without any further restriction on k,*

$$\mathbf{E}\big(\tilde{\tau}_h \,\big|\, Z_0 = k\big) < Ch; \tag{5.24}$$

(ii) *for $k \in [(h - h^{1/2+\varepsilon})/2,\ (h + h^{1/2+\varepsilon})/2]$,*

$$\mathbf{E}\big(\tilde{\tau}_h \,\big|\, Z_0 = k\big) < Ch^{1/2+\varepsilon}. \tag{5.25}$$

PROOF.

$$\begin{aligned}&\mathbf{E}\big(\tilde{\tau}_h \,\big|\, Z_0 = k\big)\\ &\quad = \mathbf{E}\big(\tilde{\tau}_h \mathbb{1}_{\{N_h \leq h^{1/2+\varepsilon}\}} \,\big|\, Z_0 = k\big) + \mathbf{E}\big(\tilde{\tau}_h \mathbb{1}_{\{N_h > h^{1/2+\varepsilon}\}} \,\big|\, Z_0 = k\big).\end{aligned} \tag{5.26}$$

We bound the first, respectively, the second, term on the right-hand side, by noting

$$\tilde{\tau}_h \mathbb{1}_{\{N_h \le h^{1/2+\varepsilon}\}} \le \tau_{(h+h^{1/2+\varepsilon})/2}, \tag{5.27}$$

respectively,

$$\tilde{\tau}_h \le \tau_h. \tag{5.28}$$

Thus we get

$$\begin{aligned} \mathbf{E}(\tilde{\tau}_h \mid Z_0 = k) &\le \mathbf{E}(\tau_{(h+h^{1/2+\varepsilon})/2} \mid Z_0 = k) \\ &\quad + \sqrt{\mathbf{E}(\tau_h^2 \mid Z_0 = k)}\sqrt{\mathbf{P}(N_h > h^{1/2+\varepsilon} \mid Z_0 = k)}. \end{aligned} \tag{5.29}$$

Side-lemma 4 (see Section 6) provides the required upper bounds:

$$\mathbf{E}(\tau_h \mid Z_0 = k) < (h-k)_+ + Ch^{1/2}, \tag{5.30*}$$

$$\mathbf{E}(\tau_h^2 \mid Z_0 = k) < Ch^2. \tag{5.31*}$$

Putting together (5.29), (5.30) and (5.31), we get (5.24) and (5.25). □

5.5. *Proof of Proposition* 1. First, note that

$$\mathbf{P}(A_{h,0} \mid Y_0 = k) = \mathbf{P}(\tilde{\sigma}_h = \infty \mid Y_0 = k), \tag{5.32}$$

$$\sum_{x=1}^{\infty} \mathbf{P}(B_{x,h,0} \mid Z_0 = k) = \sum_{x=1}^{\infty} \mathbf{P}(\tilde{\tau}_h \ge x \mid Z_0 = k) = \mathbf{E}(\tilde{\tau}_h \mid Z_0 = k), \tag{5.33}$$

and for $p \ge 1$, using the strong Markov property of Y_t, respectively, Z_t,

$$\begin{aligned} \mathbf{P}(A_{h,p} \mid Y_0 = k) = \sum_{l=0}^{\infty} &\mathbf{P}(\{\tilde{\sigma}_h < \infty\} \cap \{Y_{\tilde{\sigma}_h - 1} = h - l\} \\ &\qquad \cap \{Y_{\tilde{\sigma}_h} = l\} \mid Y_0 = k) \\ &\times \mathbf{P}(A_{h,p-1} \mid Y_0 = l), \end{aligned} \tag{5.34}$$

$$\begin{aligned} \sum_{x=1}^{\infty} \mathbf{P}(B_{x,h,p} \mid Z_0 = k) = \sum_{l=0}^{\infty} &\mathbf{P}(\{Z_{\tilde{\tau}_h - 1} = h - l\} \cap \{Z_{\tilde{\sigma}_h} = l\} \mid Z_0 = k) \\ &\times \left(\sum_{x=1}^{\infty} \mathbf{P}(B_{x,h,p-1} \mid Z_0 = l) \right). \end{aligned} \tag{5.35}$$

We prove the bounds of Proposition 1 by induction on p.

According to (5.32), (5.33), for $p = 0$, (4.15), (4.16) and (4.17) are just restatements of (5.24), (5.19) and (5.25), respectively. (See Lemmas 3 and 4.)

Next we consider the case $p = 1$. Again, we divide the sum over l in (5.34) and (5.35) into two parts, as it was done in (4.18). From (5.12) (Lemma 2) and (5.19) (Lemma 3),

$$\begin{aligned}&\sum_{l:|h-2l|\le h^{1/2+\varepsilon}} \mathbf{P}\big(\{\tilde\sigma_h < \infty\}\cap\{Y_{\tilde\sigma_h-1} = h-l\}\cap\{Y_{\tilde\sigma_h} = l\}\,\big|\,Y_0 = k\big)\\ &\qquad\times \mathbf{P}\big(A_{h,0}\,\big|\,Y_0 = l\big)\\ &\qquad\le \big(Ch^{-1/2+\varepsilon}\big)\big(Ch^{-1/2+\varepsilon}\big).\end{aligned}\tag{5.36}$$

From (5.3) (Lemma 1)

$$\begin{aligned}&\sum_{l:|h-2l|> h^{1/2+\varepsilon}} \mathbf{P}\big(\{\tilde\sigma_h < \infty\}\cap\{Y_{\tilde\sigma_h-1} = h-l\}\cap\{Y_{\tilde\sigma_h} = l\}\,\big|\,Y_0 = k\big)\\ &\qquad\times \mathbf{P}\big(A_{h,0}\,\big|\,Y_0 = l\big)\\ &\qquad\le \mathbf{P}\big(M_h > h^{1/2+\varepsilon}\,\big|\,Y_0 = k\big) < C\exp(-\gamma h^{2\varepsilon}).\end{aligned}\tag{5.37}$$

From (5.36) and (5.37) we get (4.16) for $p = 1$.

Applying the same ideas to (5.35): from (5.13) (Lemma 2) and (5.25) (Lemma 4)

$$\begin{aligned}&\sum_{l:|h-2l|\le h^{1/2+\varepsilon}} \mathbf{P}\big(\{Z_{\tilde\tau_h-1} = h-l\}\cap\{Z_{\tilde\sigma_h} = l\}\,\big|\,Z_0 = k\big)\\ &\qquad\times \sum_{x=1}^{\infty}\mathbf{P}\big(B_{x,h,0}\,\big|\,Z_0 = l\big)\\ &\qquad\le \big(Ch^{-1/2+\varepsilon}\big)\big(Ch^{1/2+\varepsilon}\big).\end{aligned}\tag{5.38}$$

From (5.4) (Lemma 1) and (5.24) (Lemma 4)

$$\begin{aligned}&\sum_{l:|h-2l|> h^{1/2+\varepsilon}} \mathbf{P}\big(\{Z_{\tilde\sigma_h-1}=h-l\}\cap\{Z_{\tilde\sigma_h}=l\}\big|Z_0=k\big)\sum_{x=1}^{\infty}\mathbf{P}\big(B_{x,h,0}\big|Z_0=l\big)\\ &\qquad\le \mathbf{P}\big(N_h>h^{1/2+\varepsilon}\big|Z_0=k\big)\bigg(\sup_{l\ge 0}\sum_{x=1}^{\infty}\mathbf{P}\big(B_{x,h,0}\big|Z_0=l\big)\bigg)\\ &\qquad< \big(C\exp(-\gamma h^{2\varepsilon})\big)(Ch).\end{aligned}\tag{5.39}$$

Equations (5.38) and (5.39) yield (4.17) for $p = 1$.

For $p \ge 2$ the induction follows from the same reasoning, only one does not have to split the sum over l as in (5.36), (5.37). After the previous arguments we may ignore these completely straightforward details. □

6. Side-lemmas.

SIDE-LEMMA 1. *For any $\varepsilon > 0$ there exist constants $C < \infty$ and $\gamma > 0$ such that, for any $h \ge 1$,*

(i)

$$\sum_{k:|h-2k|>h^{1/2+\varepsilon}} \pi(k, h-1-k) < C\exp(-\gamma h^{2\varepsilon}).\tag{6.1}$$

(ii) $h \geq 1$ *and* $l \in [(h - h^{1/2+\varepsilon})/2, (h + h^{1/2+\varepsilon})/2]$,

$$\max_{l:|h-2l|\leq h^{1/2+\varepsilon}} \frac{\pi(l, h-l)}{\sum_{m\geq h-l}\pi(l,m)} < Ch^{-1/2+\varepsilon}. \tag{6.2}$$

PROOF. (i) Assume that $h \geq 2$ and denote $h - 2 =: n$, $k - 1 =: l$. Then, using the explicit form (3.1) of $\pi(i, j)$, the right-hand side of (6.1) becomes

$$\sum_{k:|h-2k|>h^{1/2+\varepsilon}} \pi(k, h-1-k) = \tfrac{1}{2}\mathbf{P}\big(|2B_n - n| > (n+2)^{1/2+\varepsilon}\big), \tag{6.3}$$

where B_n is binomially distributed: $\mathbf{P}(B_n = l) = \binom{n}{l}2^{-n}$. Using the fact that, for any $\gamma < 1/2$,

$$\sup_n \mathbf{E}\big(\exp\{\gamma(2B_n - n)^2/n\}\big) = C_\gamma < \infty, \tag{6.4}$$

by Markov's inequality we get (6.1).

(ii) Note first that, for $i \geq 1$ and $j \geq 0$,

$$\frac{\pi(i, j+1)}{\pi(i,j)} = \frac{i+j}{2(1+j)}. \tag{6.5}$$

From this it follows that the distribution $j \mapsto \pi(i, j)$ is unimodular and for $i \geq 2$ fixed

$$\begin{aligned} \pi(i,j) &< \pi(i,j+1) \qquad \text{for } 0 \leq j \leq i-3,\\ \pi(i,i-2) &= \pi(i,i-1),\\ \pi(i,j) &> \pi(i,j+1) \qquad \text{for } i-1 \leq j < \infty. \end{aligned} \tag{6.6}$$

We treat separately the cases $l \in [h/2, (h + h^{1/2+\varepsilon})/2]$ and $l \in [(h - h^{1/2+\varepsilon})/2, h/2]$: for $l \in [h/2, (h + h^{1/2+\varepsilon})/2]$ the following two facts imply (6.2):

1. By (6.6),

$$\begin{aligned} \pi(l, h-l) &\leq \pi(l, l-1)\\ &= \tfrac{1}{2}\binom{2(l-1)}{l-1}2^{-2(l-1)} \leq \tfrac{1}{2}\binom{2(\lfloor h/2\rfloor - 1)}{\lfloor h/2\rfloor - 1}2^{-2(\lfloor h/2\rfloor -1)}\\ &\leq Ch^{-1/2}. \end{aligned} \tag{6.7}$$

2. By the central limit theorem, $\lim_{l\to\infty}\sum_{m\geq l}\pi(l,m) = \frac{1}{2}$ and thus there exists a constant $c > 0$ such that, for any $0 < h/2 \leq l$,

$$\sum_{m\geq h-l}\pi(l,m) \geq \sum_{m\geq l}\pi(l,m) \geq c. \tag{6.8}$$

Let now $l \in [(h-h^{1/2+\varepsilon})/2, h/2]$ and $k := \lfloor h-l+h^{1/2-\varepsilon}\rfloor$ Then

$$
\begin{aligned}
\frac{\pi(l,h-l)}{\sum_{m\ge h-l}\pi(l,m)} &\le (k-h+l+1)^{-1}\frac{\pi(l,h-l)}{\pi(l,k)}\\
&\le (k-h+l+1)^{-1}\left(\frac{\pi(l,k-1)}{\pi(l,k)}\right)^{k-h+l}\\
&= (k-h+l+1)^{-1}\left(\frac{2k}{l+k-1}\right)^{k-h+l} \qquad (6.9)\\
&\le h^{-1/2+\varepsilon}\left(\frac{2(h-l+h^{1/2-\varepsilon})}{h+h^{1/2-\varepsilon}-1}\right)^{h^{1/2-\varepsilon}}\\
&\le h^{-1/2+\varepsilon}\left(\frac{(h+h^{1/2+\varepsilon}+2h^{1/2-\varepsilon})}{h+h^{1/2-\varepsilon}-1}\right)^{h^{1/2-\varepsilon}}\\
&\le h^{-1/2+\varepsilon}\left(1+3h^{-(1/2-\varepsilon)}\right)^{h^{1/2-\varepsilon}}\\
&\le e^3 h^{-1/2+\varepsilon}.
\end{aligned}
$$

In the first inequality we used (6.6). In the second one we exploited the fact that, according to (6.5), for any $i \ge 1$ fixed $\pi(i,j)/\pi(i,j+1)$ is an increasing function of $j \ge 0$. In the next equality (6.5) was explicitly used. In the third inequality we inserted the value of $k = \lfloor h-l+h^{1/2-\varepsilon}\rfloor$. In the fourth inequality $l = \lceil (h-h^{1/2+\varepsilon})/2\rceil$ was inserted to maximize the expression. In the next to the last inequality we used $1 \le h^{1/2-\varepsilon} \le h^{1/2+\varepsilon}$. Finally, in the last inequality we used the fact that $\sup_{\alpha\ge 1}(1+3\alpha^{-1})^{\alpha} \le e^3$. □

SIDE-LEMMA 2. *Let ξ_i be i.i.d. random variables with the common geometric distribution $\mathbf{P}(\xi_i = k) = 2^{-k-1}$. Then there is a constant $\theta_0 > 0$ such that, for any $\lambda > 0$ and $n \in \mathbb{N}$ satisfying $\lambda/(4n) < \theta_0$,*

$$
\mathbf{P}\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j}(\xi_i - 1)\right| > \lambda\right) \le 2\exp\left(-\lambda^2/(8n)\right). \qquad (6.10)
$$

PROOF. The exponential Kolmogorov inequality stated below follows directly from Doob's maximal inequality. For its proof see, for example, page 139 of [18].

EXPONENTIAL KOLMOGOROV INEQUALITY. *Let $\tilde{\xi}_j$, $j \ge 1$, be i.i.d. random variables with $\mathbf{E}(\exp\{\theta|\tilde{\xi}_j|\}) < \infty$ for some $\theta > 0$ and $\mathbf{E}(\tilde{\xi}_j) = 0$. Then, for any $\lambda \in (0,\infty)$ and $n \in \mathbb{N}$,*

$$
\mathbf{P}\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j}\tilde{\xi}_i\right| > \lambda\right) \le \exp(-\lambda\theta)\left\{\mathbf{E}\left(\exp(\theta\tilde{\xi}_i)\right)^n + \mathbf{E}\left(\exp(-\theta\tilde{\xi}_i)\right)^n\right\}. \qquad (6.11)
$$

We apply the exponential Kolmogorov inequality to $\tilde{\xi}_i = \xi_i - 1$, with $\mathbf{P}(\xi_i = k) = 2^{-k-1}$, $k \geq 0$. There exists a constant $\theta_0 > 0$ such that for $0 \leq \theta < \theta_0$ we get

$$\begin{aligned}\mathbf{E}\big(\exp(\theta(\xi_j - 1))\big) &= \exp(-\theta)\big(2 - \exp(\theta)\big)^{-1} \\ &= 1 + \theta^2 + \mathscr{O}\big(\theta^3\big) < \exp(2\theta^2),\end{aligned} \tag{6.12}$$

$$\begin{aligned}\mathbf{E}\big(\exp(-\theta(\xi_j - 1))\big) &= \exp(2\theta)\big(2\exp(\theta) - 1\big)^{-1} \\ &= 1 + \theta^2 + \mathscr{O}\big(\theta^3\big) < \exp(2\theta^2).\end{aligned} \tag{6.13}$$

Inserting these bounds into the right-hand side of (6.8) and choosing $\theta = \lambda/(4n)$, we obtain (6.10). □

Side-lemmas 3 and 4 rely on the forthcoming overshooting lemma and standard optional stopping considerations. The overshooting lemma and its corollary are extended restatements of Lemmas 3.2 and 3.4 of [17].

OVERSHOOTING LEMMA. *For any $0 \leq k \leq h \leq u$ the following overshoot bounds hold:*

$$\begin{aligned}\mathbf{P}\big(Y_{\sigma_h} \geq u \,\big|\, Y_0 = k, \sigma_h < \infty\big) &\leq \mathbf{P}\big(Y_1 \geq u \,\big|\, Y_0 = h, Y_1 \geq h\big) \\ &= \frac{\sum_{v=u}^{\infty} \pi(h, v)}{\sum_{w=h}^{\infty} \pi(h, w)},\end{aligned} \tag{6.14}$$

$$\begin{aligned}\mathbf{P}\big(Z_{\tau_h} \geq u \,\big|\, Z_0 = k\big) &\leq \mathbf{P}\big(Z_1 \geq u \,\big|\, Z_0 = h,\ Z_1 \geq h\big) \\ &= \frac{\sum_{v=u}^{\infty} \rho(h, v)}{\sum_{w=h}^{\infty} \rho(h, w)}.\end{aligned} \tag{6.15}$$

In particular, we have the following result.

COROLLARY. *There exists a constant $C < \infty$ such that, for any $0 \leq k < h$,*

$$\mathbf{E}\big(Y_{\sigma_h} \,\big|\, Y_0 = k, \sigma_h < \infty\big) \leq \frac{\sum_{v=h}^{\infty} \pi(h,\ v)v}{\sum_{w=h}^{\infty} \pi(h,\ w)} \leq h + Ch^{1/2}, \tag{6.16}$$

$$\mathbf{E}\big(Y_{\sigma_h}^2 \,\big|\, Y_0 = k, \sigma_h < \infty\big) \leq \frac{\sum_{v=h}^{\infty} \pi(h,\ v)v^2}{\sum_{w=h}^{\infty} \pi(h,\ w)} \leq h^2 + Ch^{3/2}, \tag{6.17}$$

$$\mathbf{E}\big(Z_{\tau_h} \,\big|\, Z_0 = k\big) \leq \frac{\sum_{v=h}^{\infty} \rho(h,\ v)v}{\sum_{w=h}^{\infty} \rho(h,\ w)} \leq h + Ch^{1/2}, \tag{6.18}$$

$$\mathbf{E}\big(Z_{\tau_h}^2 \,\big|\, Z_0 = k\big) \leq \frac{\sum_{v=h}^{\infty} \rho(h,\ v)v^2}{\sum_{w=h}^{\infty} \rho(h,\ w)} \leq h^2 + Ch^{3/2}. \tag{6.19}$$

The rightmost bounds in (6.16)–(6.19) follow from explicit computations.

PROOF OF THE OVERSHOOTING LEMMA. Straightforward manipulations lead to the following identities for $1 \le h \le v$:

$$\begin{aligned}(6.20)\qquad &\mathbf{P}\big(Y_{\sigma_h} = v \big| Y_0 = k, \sigma_h < \infty\big)\\ &\quad = \sum_{l=0}^{h-1} \mathbf{P}\big(Y_{\sigma_h - 1} = l \,\big|\, Y_0 = k, \sigma_h < \infty\big) \frac{\pi(l, v)}{\sum_{w=h}^{\infty} \pi(l, w)},\end{aligned}$$

$$(6.21)\qquad \mathbf{P}\big(Z_{\tau_h} = v \,\big|\, Z_0 = k\big) = \sum_{l=0}^{h-1} \mathbf{P}\big(Z_{\tau_h - 1} = l \,\big|\, Z_0 = k\big) \frac{\rho(l, v)}{\sum_{w=h}^{\infty} \rho(l, w)}.$$

Using the explicit form (3.1), respectively, (3.2), of the transition probabilities $\pi(i, j)$, respectively, $\rho(i, j)$, it is easy to check the following inequalities, which hold for any $0 < l < h \le v$, respectively, $0 \le l < h \le v$,

$$(6.22)\qquad \frac{\pi(l+1, v)}{\pi(l, v)} = \frac{l+v}{2l} < \frac{l+v+1}{2l} = \frac{\pi(l+1, v+1)}{\pi(l, v+1)},$$

$$(6.23)\qquad \frac{\rho(l+1, v)}{\rho(l, v)} = \frac{l+v+1}{2(l+1)} < \frac{l+v+2}{2(l+1)} = \frac{\rho(l+1, v+1)}{\rho(l, v+1)}.$$

It follows that, for any $0 \le l < h \le v < w$,

$$(6.24)\qquad \pi(l+1, v)\pi(l, w) < \pi(l, v)\pi(l+1, w),$$

$$(6.25)\qquad \rho(l+1, v)\rho(l, w) < \rho(l, v)\rho(l+1, w).$$

Hence, for any $0 \le l < h \le u$,

$$(6.26)\qquad \sum_{v=h}^{\infty} \pi(l+1, v) \sum_{w=u}^{\infty} \pi(l, w) < \sum_{v=h}^{\infty} \pi(l, v) \sum_{w=u}^{\infty} \pi(l+1, w),$$

$$(6.27)\qquad \sum_{v=h}^{\infty} \rho(l+1, v) \sum_{w=u}^{\infty} \rho(l, w) < \sum_{v=h}^{\infty} \rho(l, v) \sum_{w=u}^{\infty} \rho(l+1, w),$$

which directly imply (6.14), respectively, (6.15). $\square$

SIDE-LEMMA 3. *There exists a constant $C < \infty$ such that for any $0 \le k \le h$ the following hold:*

$$(6.28)\qquad \mathbf{P}\big(\sigma_h = \infty \,\big|\, Y_0 = k\big) \le \frac{h-k}{h} + Ch^{-1/2},$$

$$(6.29)\qquad \mathbf{E}\big(\sigma_h \wedge \omega \,\big|\, Y_0 = k\big) \le Ch^2.$$

PROOF. We apply the optional stopping theorem to the martingales Y_t, respectively, $Y_t^2 - 2\sum_{s=0}^{t-1} Y_s$, $t \ge 0$, both stopped at $\sigma_h \wedge \omega$.

PROOF OF (6.28):

$$k = \mathbf{E}\big(Y_{\sigma_h \wedge \omega} \,\big|\, Y_0 = k\big) = \mathbf{E}\big(Y_{\sigma_h} \,\big|\, Y_0 = k,\ \sigma_h < \infty\big) \times \mathbf{P}\big(\sigma_h < \infty \,\big|\, Y_0 = k\big) \le \Big(h + C\sqrt{h}\Big)\mathbf{P}\big(\sigma_h < \infty \,\big|\, Y_0 = k\big), \tag{6.30}$$

where in the last inequality, we applied (6.16). Hence (6.28).

PROOF OF (6.29):

$$k^2 = \mathbf{E}\left(Y^2_{\sigma_h \wedge \omega} - 2\sum_{s=0}^{\sigma_h \wedge \omega - 1} Y_s \,\Big|\, Y_0 = k\right) \le \mathbf{E}\big(Y^2_{\sigma_h \wedge \omega} \,\big|\, Y_0 = k\big) - 2\mathbf{E}\big(\sigma_h \wedge \omega \,\big|\, Y_0 = k\big), \tag{6.31}$$

where in the last inequality we used the fact that $Y_s \ge 1$ for $s < \omega$. Hence

$$2\mathbf{E}\big(\sigma_h \wedge \omega \,\big|\, Y_0 = k\big) \le \mathbf{E}\big(Y^2_{\sigma_h} \,\big|\, Y_0 = k,\ \sigma_h < \infty\big) \times \mathbf{P}\big(\sigma_h < \infty \,\big|\, Y_0 = k\big) - k^2 < Ch^2. \tag{6.32}$$

In the last inequality we used (6.17). □

SIDE-LEMMA 4. *There exists a constant $C < \infty$ such that for any $0 \le k \le h$ the following upper bounds hold:*

$$\mathbf{E}\big(\tau_h \,\big|\, Z_0 = k\big) < (h - k) + Ch^{1/2}, \tag{6.33}$$

$$\mathbf{E}\big(\tau_h^2 \,\big|\, Z_0 = k\big) < Ch^2. \tag{6.34}$$

PROOF. We apply the optional stopping theorem to the martingale $Z_t - t$, respectively, to the supermartingale $t^2 - 2tZ_t$, $t \ge 0$, both stopped at τ_h.

Proof of (6.33):

$$\mathbf{E}\big(\tau_h \,\big|\, Z_0 = k\big) = \mathbf{E}\big(Z_{\tau_h} \,\big|\, Z_0 = k\big) - k \le h - k + C\sqrt{h}. \tag{6.35}$$

In the last inequality (6.18) was used.

Proof of (6.34):

$$\mathbf{E}\big(\tau_h^2 \,\big|\, Z_0 = k\big) \le 2\mathbf{E}\big(\tau_h Z_{\tau_h} \,\big|\, Z_0 = k\big) \le 2\sqrt{\mathbf{E}\big(\tau_h^2 \,\big|\, Z_0 = k\big)}\sqrt{\mathbf{E}\big(Z^2_{\tau_h} \,\big|\, Z_0 = k\big)}. \tag{6.36}$$

Hence, using (6.19), we get

$$\mathbf{E}\big(\tau_h^2 \,\big|\, Z_0 = k\big) \le 4\mathbf{E}\big(Z^2_{\tau_h} \,\big|\, Z_0 = k\big) \le Ch^2. \tag{6.37}$$

□

REFERENCES

[1] BASS, R. F., EISENBAUM, N. and SHI, Z. (2000). The most visited sites of symmetric stable process. *Probab. Theory Related Fields* **116** 391–404.

[2] BASS, R. F. and GRIFFIN, P. S. (1985). The most visited site of Brownian motion and random walk. *Z. Wahrsch. Verw. Gebiete* **70** 417–436.

[3] CSÁKI, E., RÉVÉSZ, P. and SHI, Z. (2000). Favorite sites, favourite values and jumping sizes for random walk and Brownian motion. *Bernoulli*. To appear.

[4] CSÁKI, E. and SHI, Z. (1998). Large favourite sites of simple random walk and the Wiener process. *Electronic J. Probab.* **3** 1–31.

[5] EISENBAUM, N. (1997). On the most visited sites by a symmetric stable process. *Probab. Theory Related Fields* **107** 527–535.

[6] ERDÖS, P. (1994). My work with Pál Révész. Budapest.

[7] ERDÖS, P. and RÉVÉSZ, P. (1984). On the favourite points of random walks. In *Mathematical Structures–Computational Mathematics–Mathematical Modelling* (*Sofia*) **2** 152–157.

[8] ERDÖS, P. and RÉVÉSZ, P. (1987). Problems and results On random walks. In *Mathematical Statistics and Probability Theory* (P. Bauer, F. Koneczny and W. Wertz, eds.) 59–65. Reidel, Dordrecht.

[9] ERDÖS, P. and RÉVÉSZ, P. (1991). Three problems on the random walk in $\mathbb{Z}^d$. *Studia Sci. Math. Hungar*. **26** 309–320.

[10] KESTEN, H. (1965). An iterated logarithm law for the local time. *Duke Math. J.* **32** 447–456.

[11] KNIGHT, F. B. (1963). Random walks and a sojourn density process of Brownian motion. *Trans. Amer. Math. Soc.* **109** 56–86.

[12] KÖNIG, W. (1996). A central limit theorem for a one-dimensional polymer measure. *Ann. Probab.* **24** 1012–1035.

[13] RAY, D. (1963). Sojourn times of a diffusion process. *Illinois J. Math.* **7** 615–630.

[14] RÉVÉSZ, P. (1990). *Random Walk in Random and Non-Random Environment.* World Scientific, Singapore.

[15] TÓTH, B (1995). The true self-avoiding walk with bond repulsion on $\mathbb{Z}$: limit thorems. *Ann. Probab.* **23** 1523–1556.

[16] TÓTH, B (1996). Generalized Ray–Knight theory and limit theorems for self-interacting random walks on $\mathbb{Z}$. *Ann. Probab.* **24** 1324–1367.

[17] TÓTH, B. and WERNER, W. (1997). Tied favourite edges for simple random walk. *Combin. Probab. Comput.* **6** 359–369.

[18] WILLIAMS, D. (1991). *Probability with Martingales.* Cambridge Univ. Press.

INSTITUTE OF MATHEMATICS
TECHNICAL UNIVERSITY BUDAPEST
EGRY JÓZSEF UTCA 1
H-1111 BUDAPEST
HUNGARY
E-MAIL: balint@math.bme.hu