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MEASURE-VALUED BRANCHING PROCESSES ASSOCIATED WITH RANDOM WALKS ON p-ADICS

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Measure-valued branching random walks (superprocesses) on p-adics are introduced and investigated. The uniqueness and existence of solutions to associated linear and nonlinear heat-type (parabolic) equations are proved, provided some condition on the parameter of the random walks is satisfied. The solutions of these equations are shown to be locally constant if their initial values are. Moreover, the heat-type equations can be identified with a system of ordinary differential equations. Conditions for the measure-valued branching stable random walks to possess the property of quasi-self-similarity are given, as well as a sufficient and necessary condition for these processes to be locally extinct. The latter result is consistent with the Euclidean case in the sense that the critical value for measure-valued branching stable processes to be locally extinct is the Hausdorff dimension of the image of the underlying processes divided by the dimension of the state space.

1. Introduction. There has been much work in the study of stochastic processes on p-adics and local fields [see, e.g., Evans (1998), Albeverio, Karwowski and Zhao (1999), Albeverio and Zhao (1999a, b, c, d) and references therein]. Meanwhile, superprocesses have been extensively studied in a general setting [see Dawson (1993), Dynkin (1994), and Zhao (1994, 1999)]. Recently, Evans and Fleischmann (1996) have constructed a class of measure-valued diffusions from Lévy processes on totally disconnected groups. Their processes, which are not measure-valued branching processes (superprocesses), motivated us to investigate superprocesses on p-adics (which do not seem to have been studied so far). The p-adic field, $\mathbb{Q}_p$, is a totally disconnected, nondiscrete, Abelian group and is very different from the Euclidean space [see, e.g., Koblitz (1984), Gel'fand, Graev and Pyatetskii-Shapiro (1969), Chapter 2]. It is interesting to get insight into the difference in properties of superprocesses on different spaces, like $\mathbb{Q}_p$ and Euclidean spaces.

Since the existence and uniqueness of superprocesses on locally compact, complete metric spaces is well known [see Dawson (1993), Dynkin (1994) and Zhao (1999)], we can work directly with measure-valued branching processes on $\mathbb{Q}_p$. We consider here the measure-valued branching processes built from

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the random walks constructed in Albeverio and Karwowski (1991, 1994) and studied in Yasuda (1996), Albeverio, Karwowski and Zhao (1999) and Albeverio and Zhao (1999a, b, c, d). For the reader's convenience, we first review the construction of random walks in Albeverio and Karwowski (1991, 1994) and Karwowski and Vilela-Mendes (1994). Some new results for these random walks are also presented [the identification of these random walks with other construction in Figà-Talamanca (1994) has been discussed in Hussmann (1997)]. For background and applications in physics see Albeverio (1985), Brekke and Olson (1989).

We then consider the initial valued heat-type equations associated with these random walks. Uniqueness and existence of solutions to these equations are proved. The solutions are locally constant if their initial values are. The equations can be identified with a system of linear ordinary differential equations, which can be solved recursively.

To study the superprocesses associated with symmetric random walks on $\mathbb{Q}_p$, we need first to consider the corresponding nonlinear evolution equations. Uniqueness and existence of nonnegative solutions for the nonlinear heat-type equations are obtained. As in the linear cases, the solutions to these nonlinear evolution equations are also locally constant if their initial values are. In particular, when the initial value is the indicator of a ball the corresponding nonlinear evolution equation is equivalent to a system of nonlinear ordinary differential equations, which provides a way to solve the former equations. Some comparison lemmas for the nonlinear evolution equations are established.

We also consider in this paper some typical questions for superprocesses having the branching characteristic $\lambda^{1+\beta}$, $0 < \beta \leq 1$. We first investigate the absolute continuity of these superprocesses with respect to the Haar measure. We show that the superprocess is absolutely continuous in the case of binary branching (i.e., $\beta = 1$) and the corresponding Radon–Nikodym derivative forms a random field indexed by $\mathbb{R}_+ \times \mathbb{Q}_p$. This motivates us to further consider the "white noise" on $\mathbb{Q}_p$ [first introduced by Evans (1995, 1998)]. Second, if the random walk is stable (see Section 2.2 for the definition) we find that the superprocesses possess a property of "quasi-self-similarity," similar to (but different from) the self-similarity of super α-stable processes on $\mathbb{R}^d$, provided some condition on the parameter determining the random walk is satisfied. Finally, we study the local extinction, the counterpart for super α-stable processes on $\mathbb{R}^d$ first investigated by Dawson and his research groups [see Dawson, (1977), (1993), Dawson, Fleischmann, Foley and Peletier (1986)] and for the superdiffusions recently studied by Pinsky (1996). A natural and interesting question is to find the difference of the properties of superprocesses on $\mathbb{R}^d$and $\mathbb{Q}_p$. For this purpose, applying the idea in Dawson, Fleischmann, Foley and Peletier (1986) we prove that (ξ, β)-superprocesses (see Section 4 below for the definition) are locally extinct when $\beta \leq -\log_p c$, where c is closely connected with the Lévy measure of the underlying process [see Albeverio and Zhao (1999a)]. Conversely, we shall also prove the nonlocal extinction when $\beta > -\log_p c$. To approach this goal we first reduce our question to check the finiteness of an integral on the half line whose integrand is the β-power of the L^1-norm of the square of the transition function. We then prove that the L^1-norm of solutions of the nonlinear evolution equations possesses a lower positive bound. In fact, in the terminology of Evans (1989) we find that the critical point $(-\log_p c) \wedge 1$ for these superstable processes to be locally extinct is the Hausdorff dimension of the image of the stable random walk [see Albeverio and Zhao (1999b)]. This result can be regarded as an analogue in the p-adic case of the corresponding result for the Euclidean case, reflecting typically the structural differences between $\mathbb{Q}_p$ and $\mathbb{R}^d$.

2. Random walks on p-adics. Let $p > 1$ be a prime number. A p-adic number can be defined through the formal power series

$$a = \sum_{i=-m}^{\infty} \alpha_i p^i, \tag{2.1}$$

where m is an integer (i.e., $m \in \mathbb{Z}$), and $\alpha_i = 0, 1, \ldots, p-1$. With addition and multiplication defined in the natural way for formal power series, the set $\mathbb{Q}_p$ of all p-adic numbers becomes a field.

Let a be given by (2.1) and $i_0 = \min\{i \geq -m; \alpha_i \neq 0\}$. We define

$$\|a\|_p = p^{-i_0}. \tag{2.2}$$

It is well known that the map $a \to \|a\|_p$ defines a norm in $\mathbb{Q}_p$. This norm has the non-Archimedean triangle property,

$$\|x+y\|_p \leq \max\{\|x\|_p, \|y\|_p\}, \qquad x, y \in \mathbb{Q}_p \tag{2.3}$$

and $\mathbb{Q}_p$ is a complete separable metric, locally compact, totally disconnected space (with the cardinality of the continuum). The series (2.1), called Hensel expansion of a, is convergent to a in $\|\cdot\|_p$ norm. The rational number field Q is dense in $\mathbb{Q}_p$ and it is a subfield of $\mathbb{Q}_p$ [see, e.g., Koblitz (1984), Taibleson (1975)].

Let $a \in \mathbb{Q}_p$ and $M \in \mathbb{Z}$. The set

$$K(a, p^M) = \{x \in \mathbb{Q}_p; \|x-a\|_p \leq p^M\}. \tag{2.4}$$

is called a ball of radius p^M centered at a. $K(a, p^M)$ is both open and compact. $\mathbb{Q}_p$ can be uniquely represented as a countable union of disjoint balls of radius p^M. Let $\mathscr{Q}_p$ denote the α-algebra generated by the set of all balls in $\mathbb{Q}_p$. Then, the set function defined on balls by

$$\nu\big(K(a, p^M)\big) = p^M \tag{2.5}$$

can be uniquely extended to a measure on $\mathscr{Q}_p$ which we denote by ν. We note that ν is the Haar measure for the additive group in $\mathbb{Q}_p$ [see Albeverio and Karwowski (1994)].

Let ρ be a nonnegative measurable function on $\mathbb{Q}_p$ such that

$$0 < \int_{K(a,\, p^M)} \rho(x)\nu(dx) < \infty$$

for any $a \in \mathbb{Q}_p$ and $M \in \mathbb{Z}$. ρ determines a σ-finite measure $\nu_\rho(dx) := \rho(x)\nu(dx)$ on $\mathbb{Q}_p$.

Let $\{a(M), M \in \mathbb{Z}\}$ be a given sequence of nonnegative numbers satisfying:

(i) $a(M) \geq a(M+1)$;
(ii) $\lim_{M\to\infty} a(M) = 0$.

Put

$$u(M, j) \equiv a(M+j-1) - a(M+j).$$

Albeverio and Karwowski (1994), and Karwowski and Vilela-Mendes (1994) solved the following forward, resp., backward Kolmogorov equations on the balls of $\mathbb{Q}_p$:

$$\dot{P}_{K_i^M K_f^M}(t) = \tilde{a}(K_f^M) P_{K_i^M K_f^M}(t) + \sum_{j\neq f}^{\infty} \tilde{u}(K_j^M, K_f^M) P_{K_i^M K_j^M}(t), \tag{2.6}$$

resp.,

$$\dot{P}_{K_i^M K_f^M}(t) = \tilde{a}(K_i^M) P_{K_i^M K_f^M}(t) + \sum_{j\neq i}^{\infty} \tilde{u}(K_i^M, K_j^M) P_{K_j^M K_f^M}(t). \tag{2.7}$$

with $\tilde{a}(K_i^M) \equiv \sum_{j\neq i}^{\infty} \tilde{u}(K_j^M, K_i^M)$ and the initial conditions $P_{K_i^M K_f^M}(0) = \delta_{if}$ for $t \geq 0, i, f \in \mathbb{N}$. Here $\tilde{u}(K_i^M, K_j^M) := \nu_\rho(K_j^M) u(M, k)$ when $\operatorname{dist}_p(K_i^M, K_j^M) = p^{M+k}$. $\mathscr{K}^M = \{K^M\}_{i=1}^{\infty}$ is the family of disjoint balls of radius P^M such that $\mathbb{Q}_p = \bigcup_{i=1}^{\infty} K_i^M$. Let

$$\mathscr{W}^{j}_{K(a,\, p^M)} := -\sum_{k=j}^{\infty} (u(M, k) - u(M, k+1)) \nu_\rho(K(a, p^{M+k})).$$

These authors then shrink the balls into points and construct a Markov process on $\mathbb{Q}_p$ with the following transition functions: (1) In case $\rho_\infty < \infty$ we normalize the function ρ so that $\rho_\infty = 1$,

$$\begin{aligned} P_t(x, K(a, p^M)) = \nu_\rho(K(a, p^M)) \Bigg\{ 1 + \sum_{i=0}^{\infty} \bigg(& \frac{1}{\nu_\rho(K(a, P^{M+i}))} \\ & - \frac{1}{\nu_\rho(K(a, P^{M+i+1}))} \bigg) e^{t\mathscr{W}^{i+1}_{K(a,\, p^M)}} \Bigg\}, \end{aligned} \tag{2.8}$$

resp.,

$$\begin{aligned} & P_t(x, K(b, p^M)) \\ & \quad = \nu_\rho(K(b, p^M)) \Bigg\{ 1 - \frac{1}{\nu_\rho(K(b, p^{M+j_0}))} e^{t\mathscr{W}^{j_0}_{K(b,\, p^M)}} \\ & \qquad + \sum_{i=0}^{\infty} \bigg(\frac{1}{\nu_\rho(K(b, p^{M+j_0+i}))} \\ & \qquad - \frac{1}{\nu_\rho(K(b, p^{M+j_0+i+1}))} \bigg) e^{t\mathscr{W}^{i+1+j_0}_{K(b,\, p^M)}} \Bigg\}, \end{aligned} \tag{2.9}$$

where $x \in K(a, p^M)$ and $\mathrm{dist}_p(K(b, p^M), K(a, p^M)) = p^{M+j_0}$. (2) If $\rho_\infty = \infty$ we have

$$P_t(x, K(a, p^M)) \tag{2.10}$$
$$= \nu_\rho(K(a, p^M)) \sum_{i=0}^{\infty} \left(\frac{1}{\nu_\rho(K(a, P^{M+i}))} - \frac{1}{\nu_\rho(K(a, P^{M+i+1}))} \right) e^{t\mathscr{W}^{i+1}_{K(a,\, p^M)}}$$

and

$$P_t(x, K(b, p^M)) \tag{2.11}$$
$$= \nu_\rho(K(b, p^M)) \Bigg\{ - \frac{1}{\nu_\rho(K(b, p^{M+j_o}))} e^{t\mathscr{W}^{j_0}_{K(b,\, p^M)}} + \sum_{i=0}^{\infty} \left(\frac{1}{\nu_\rho(K(b, p^{M+j_0+i}))} - \frac{1}{\nu_\rho(K(b, p^{M+j_0+i+1}))} \right) e^{t\mathscr{W}^{i+1+j_0}_{K(b,\, p^M)}} \Bigg\},$$

where $x \in K(a, p^M)$ and $\mathrm{dist}_p(K(b, p^M), K(a, p^M)) = p^{M+j_0}$.

Denote by $\mathscr{B}(\mathbb{Q}_p)$ the set of $\mathscr{Q}_p$-measurable functions on $\mathbb{Q}_p$, and for any $q > 0$, introduce

$$L^q(\mathbb{Q}_p, \nu_\rho) := \left\{ \phi \in \mathscr{B}(\mathbb{Q}_p); \int_{\mathbb{Q}_p} |\phi(x)|^q \nu_\rho(dx) < \infty \right\},$$

and denote $L^q(\mathbb{Q}_p, \nu_\rho)_+$ the set of nonnegative elements in $L^q(\mathbb{Q}_p, \nu_\rho)$. Define

$$S_t\phi(x) := \int_{\mathbb{Q}_p} \phi(y) P_t(x, dy), \qquad t \ge 0, \phi \in L^2(\mathbb{Q}_p, \nu_\rho).$$

It is easy to show that $\{S_t, t \ge 0\}$ is the semigroup associated with the transition function $\{P_t(x, dy), x, y, \in \mathbb{Q}_p, t \ge 0\}$. From Albeverio, Karwowski and Zhao [(1999), Proposition 2.3, 2.4] we know that the following holds.

LEMMA 2.1. *$P_t(x, A), t > 0, x \in \mathbb{Q}_p, A \in \mathscr{Q}_p$ is a Markovian ν_ρ-symmetric transition function (or kernel) on the measurable space $(\mathbb{Q}_p, \mathscr{Q}_p)$. Its semigroup $(S_t, t > 0)$ is a strongly continuous ν_ρ-symmetric Markovian semigroup.*

For $\phi \in \mathscr{B}(\mathbb{Q}_p)$, define

$$D(\mathbb{H}) := \left\{ \phi \in \mathscr{B}(\mathbb{Q}_p); \lim_{t \to 0+} \frac{S_t\phi(x) - \phi(x)}{t} \text{ exists for any } x \in \mathbb{Q}_p \right\}$$

and

$$-\mathbb{H}\phi(x) := \lim_{t \to 0+} \frac{S_t\phi(x) - \phi(x)}{t}, \qquad \phi \in D(\mathbb{H}).$$

From Albeverio, Karwowski and Zhao (1999), we know that $(-\mathbb{H}, D(\mathbb{H}))$ is the generator of the Markovian strongly continuous semigroup $S_t, t > 0$, acting in $L^2(\mathbb{Q}_p, \nu_\rho)$, and we have the following.

LEMMA 2.2. *Let $\chi_{K(a,\, p^M)}$ be the indicator function for the ball $K(a,\, p^M)$, then*

$$(-\mathbb{H}\chi_{K(a,\, p^M)})(x) = \begin{cases} W_{K(a,\, p^M)}, & \text{for } x \in K(a,\, p^M), \\ \nu_\rho(K(a,\, p^M))u(M,\, j), & \text{for } x \text{ such that} \\ & \qquad \operatorname{dist}_p(x, K(a,\ p^M)) = p^{M+j}. \end{cases}$$

Here $W_{K(a,\, p^M)} := -\sum_{i=1}^{\infty}(u(M,i))\nu_\rho(K(a,\, p^{M+j+1}) \setminus K(a,\, p^{M+i}))$.

REMARK. In general, the infinitesimal generator $-\mathbb{H}$ does not possesses the *positive definite property* in the sense that if $\phi \in D(\mathbb{H})$ has a minimal point, that is, there is $x_0 \in \mathbb{Q}_p$ and a neighborhood $U(x_0)$ such that

$$\phi(x) \ge \phi(x_0), \qquad x \in U(x_0),$$

then $-\mathbb{H}\phi(x_0) \ge 0$. This is so because by Lemma 2.2, we have $\lim_{t\to 0+} 1 - P_t(x,\ K(x,\ p^M))/t > 0$ for any $M \in \mathbb{Z}$ such that $a(M+1) > 0$ and

$$\int_{\phi_p \setminus K(x,\, p^M)} \rho(y)\nu(dy) > 0.$$

This fact can also be seen from (2.8)–(2.11). Of course, if $\phi \in D(\mathbb{H})$ and x_0 is a global minimum point of ϕ, that is, $\phi(x) \ge \phi(x_0)$ for all $x \in \mathbb{Q}_p$, then the fact that $S_t 1 \equiv 1$ (the conservative property) implies $-\mathbb{H}\phi(x_0) \ge 0$.

2.1. *The Haar symmetric case.* We now consider the case where ρ is constant. In this case the transition kernel is spatially homogenous and the corresponding process is a Lévy process. It is convenient to work with this process in the terminology of Albeverio and Karwowski (1994). This means that we need to express $\{a(M), M \in \mathbb{Z}\}$ in terms of another sequence of real numbers, say $\{a'(M), M \in \mathbb{Z}\}$ satisfying the condition (i) and (ii) in Section 2 above as follows:

$$a(M) = p^{-M}\left[(p-1)^{-1}a'(M) - \sum_{i=1}^{\infty} a'(M+i)p^{-i}\right].$$

Without loss of generality, we may assume $\rho \equiv 1$, and we still denote $\{a'(M), M \in \mathbb{Z}\}$ by $\{a(M), M \in \mathbb{Z}\}$. According to Evans (1989) and Yasuda (1996), the sequence $\{a(M), M \in \mathbb{Z}\}$ determines the Lévy measure, say κ on $\mathbb{Q}_p$, of the additive Lévy process $(\xi_t, t \ge 0, P_0)$ through the relation $\kappa(\mathbb{Q}_p \setminus K(0, P^M)) = a(M)$, $M \in \mathbb{Z}$ [see Albeverio and Zhao (1999 a, b)]. We know from Albeverio and Karwowski (1994), Albeverio, Karwowski and Zhao (1999) that the transition function can be rewritten as follows:

$$\begin{aligned} &P_t(x, K(a, p^M)) \\ &\quad = \frac{p-1}{p}\sum_{i=0}^{\infty} p^{-i} \exp\bigl\{-(p-1)^{-1}[pa(M+i) - a(M+i+1)]t\bigr\} \end{aligned} \tag{2.12}$$

if $x \in K(a, p^M)$ and

$$
\begin{aligned}
&P_t(x, K(b, p^M)) \\
&\quad = p^{-j}\Big[p^{-1}(p-1)\sum_{i=0}^{\infty} p^{-i} \\
&\qquad\qquad \times \exp\{-(p-1)^{-1}[pa(M+j+i) - a(M+j+i+1)]t\} \\
&\qquad\qquad\qquad - \exp\{-(p-1)^{-1}[pa(M+j-1) - a(M+j)]t\}\Big]
\end{aligned}
\tag{2.13}
$$

if $\operatorname{dist}_p(x, K(b, p^M)) = p^{M+j}$, $j \in \mathbb{N}$. It is worth remarking that after a change of variables the infinitesimal generator is defined by

$$
\begin{aligned}
&(-\mathbb{H}\chi_{K(a,\, p^M)})(x) \\
&\quad = \begin{cases} -a(M), & \text{for } x \in K(a, p^M); \\ p^{-j+1}(p-1)^{-1}u(M, j), & \text{for } x \text{ such that} \\ & \qquad \operatorname{dist}_p(K(a, p^M), x) = p^{M+j}. \end{cases}
\end{aligned}
\tag{2.14}
$$

[see Albeverio and Karwowski (1994), (3.4)]. From (2.12) and (2.13) we have the following symmetry property, resp., rotation invariance, resp., translation invariance for the transition kernels:

$$P_t(x, K(y, p^M)) = P_t(y, K(x, p^M)), \qquad x, y \in \mathbb{Q}_p,\ M \in \mathbb{Z}, \tag{2.15}$$

$$P_t(x, K(y, p^M)) = P_t(x, K(z, p^M)), \qquad x, y, z \in \mathbb{Q}_p,\ M \in \mathbb{Z} \tag{2.16}$$

if $\operatorname{dist}_p(x, y) = \operatorname{dist}_p(x, z)$, and

$$P_t(x, K(y, p^M)) = P_t(x+z, K(y+z, p^M)), \quad x, y, z \in \mathbb{Q}_p,\ M \in \mathbb{Z}. \tag{2.17}$$

PROPOSITION 2.3. *If $x \neq y$, then for any $t \geq 0$ the density function $p_t(x, y) := \lim_{M\to-\infty} p^{-M}P_t(x, K(y, p^M))$ exists and can be expressed by*

$$p_t(x, y) = (p-1)^{-1}p^{-m+1}(P_t(m) - P_t(m-1)) \tag{2.18}$$

if $\operatorname{dist}_p(x, y) = p^m$, where $P_t(m) := P_t(0, K(0, p^m))$. If

$$\lim_{M\to-\infty} a(M) = \infty, \tag{2.19}$$

then for any $t > 0$ and $x \in \mathbb{Q}_p$, $P_t(x, \cdot)$ is absolutely continuous with respect to ν. In particular, if

$$\sum_{i=-\infty}^{\infty} p^{-i} \exp\{-t(p-1)^{-1}[pa(i) - a(i+1)]\} < \infty, \tag{2.20}$$

then

$$p_t(x, x) = p^{-1}(p-1)\sum_{i=-\infty}^{\infty} p^{-i} \exp\{-t(p-1)^{-1}[pa(i) - a(i+1)]\}. \tag{2.21}$$

PROOF. Because of the rotation invariance and the symmetry property, we have for $x, y\colon \mathrm{dist}_p(x, y) = p^m$,

$$\begin{aligned} p_t(x, y) &= \lim_{M\to-\infty} p^{-M} P_t(x, K(y, p^M)) \\ &= \lim_{M\to-\infty} p^{-M} \frac{P_t(x, K(y, p^m)) - P_t(x, K(y, p^{m-1}))}{(p-1)p^{m-M-1}} \\ &= (p-1)^{-1} p^{-m+1}(P_t(m) - P_t(m-1)). \end{aligned}$$

As for the absolute continuity, we only need to check that condition (2.19) implies $\lim_{M\to-\infty} P_T(x, K(y, p^M)) = 0$. This is clear from (2.12). To prove (2.21), we consider

$$\begin{aligned} &p^{-M} P_t(x, K(a, p^M)) \\ &\qquad = p^{-1}(p-1) \sum_{i=M}^{\infty} p^{-i} \exp\{-(p-1)^{-1}[pa(i) - a(i+1)]t\}. \end{aligned}$$

Letting $M \to -\infty$ we see that $p_t(x, x)$ exists if and only if (2.20) holds. □

REMARK. Obviously, (2.20) is equivalent to

$$\sum_{i=-\infty}^{0} p^{-i} \exp\{-t(p-1)^{-1}[pa(i) - a(i+1)]\} < \infty. \tag{2.22}$$

We remark that the validity of (2.22) for all $t > 0$ is then equivalent to

$$\lim_{n\to\infty} \frac{a(-n)}{n} = \infty. \tag{2.23}$$

which is the condition (iii) of Proposition 2 in Evans (1989).

We say a point $y \in \mathbb{Q}_p$ is *recurrent* for a process $\xi_t, t \geq 0$ if for any $t_0 > 0$ there exists almost surely a finite time $t > t_0$ such that $\xi_t = y$. In other words, ξ_t hits the point y an infinite number of times after any given time.

PROPOSITION 2.4. *Let $\{a(M), M \in \mathbb{Z}\}$ be a sequence of positive numbers satisfying the condition* (i) *and* (ii). *Suppose*

$$\sum_{i=0}^{\infty} \frac{p^{-i}}{a(i)} < \infty, \tag{2.24}$$

then the only possible recurrent point in $\mathbb{Q}_p$ for the process $\xi_t, t \geq 0$ with transition semigroup S_t started at x, is x itself. On the contrary, if

$$\sum_{i=0}^{\infty} \frac{p^{-i}}{a(i)} = \infty, \qquad \sum_{i=-\infty}^{0} \frac{p^{-i}}{1 + a(i)} < \infty, \tag{2.25}$$

then the corresponding process is point recurrent (*in the sense that every point is recurrent*).

PROOF. Define

$$V_x := \inf\{t > 0; \xi_t = x\}$$

and

$$g^\lambda(x, y) := \int_0^\infty \exp(-\lambda t) p_t(x, y) dt, \qquad \lambda \geq 0$$

if the density $p_t(x, y)$ exists. Since $(\xi_t, t \geq 0, P_x)$ is a Markovian process, a point y is recurrent iff $P_x(V_y < \infty) = 1$.

Now we suppose that the condition (2.24) holds, and we intend to prove that $P_x(V_y < \infty) < 1$ for any $y \neq x$. For this purpose, we observe that for any $y \neq x$, Yasuda [(1996), Theorem 3.4] implies $P_x(V_y < \infty) < 1$ if $\lim_{M\to-\infty} a(M) < \infty$ or if $\lim_{M\to-\infty} a(M) = \infty$ but $\sum_{i=-\infty}^{0}(p^{-1}/1+a(i)) = \infty$. Therefore, the remaining case is the one where $\sum_{i=-\infty}^{0}(p^{-1}/1+a(i)) < \infty$. In this case and under the condition (2.24), it is easy to see that (2.20) is true; that is, the density exists, and

$$\sum_{i=m}^{\infty} \frac{p^{-i}}{pa(i) - a(i+1)} < \sum_{i=-\infty}^{\infty} \frac{p^{-i}}{pa(i) - a(i+1)} < \infty.$$

From Yasuda [(1996), Theorem 3.4(3)], we have

$$P_x\{V_y < \infty\} = \lim_{\lambda\to 0} \frac{g^\lambda(x, y)}{g^\lambda(x, x)} \tag{2.26}$$

for any $x, y \in \mathbb{Q}_p$. However, if $\|x - y\|_p = p^m$, we have

$$\begin{aligned} g^\lambda(x\ y) &= \sum_{i=m}^{\infty}\left(\frac{p^{-i}}{\lambda + (p-1)^{-1}(pa(i) - a(i+1))}\right. \\ &\qquad \left. - \frac{p^{-i}}{\lambda + (p-1)^{-1}(pa(i-1) - a(i))}\right) \\ &= \frac{p-1}{p}\sum_{i=m}^{\infty}\frac{p^{-i}}{\lambda + (p-1)^{-1}(pa(i) - a(i+1))} \\ &\qquad - \frac{p^{-m}}{\lambda + (p-1)^{-1}(pa(m-1) - a(m))} \end{aligned}$$

and

$$g^\lambda(x, x) = \frac{p-1}{p}\sum_{i=-\infty}^{\infty}\frac{p^{-i}}{\lambda + (p-1)^{-1}(pa(i) - a(i+1))}.$$

It is easy to see that for $x \neq y$,

$$\lim_{\lambda\to 0}\frac{g^\lambda(x, y)}{g^\lambda(x, x)} < \frac{\sum_{i=m}^{\infty}\left(p^{-i}/\left((p-1)^{-1}(pa(i) - a(i+1))\right)\right)}{\sum_{i=-\infty}^{\infty}\left(p^{-i}/\left((p-1)^{-1}(pa(i) - a(i+1))\right)\right)} < 1.$$

That is, $P_x(V_y < \infty) < 1$ as we desired.

Now we turn to prove that $P_x(V_y < \infty) = 1$ for any $x, y \in \mathbb{Q}_p$ under the condition (2.25). We first note that for any $m \in \mathbb{Z}$,

$$\lim_{\lambda \to 0+} \sum_{i=m}^{\infty} \frac{p^{-i}}{\lambda + (p-1)^{-1}(pa(i) - a(i+1))} = \infty,$$

but

$$\lim_{\lambda \to 0+} \sum_{i=-\infty}^{m-1} \frac{p^{-i}}{\lambda + (p-1)^{-1}(pa(i) - a(i+1))} < \infty.$$

We then observe that

$$\begin{aligned}
\frac{g^\lambda(x, y)}{g^\lambda(x, x)} &= \frac{\sum_{i=-m}^{\infty} \big(p^{-i}/\big(\lambda + (p-1)^{-1}(pa(i) - a(i+1))\big)\big)}{\sum_{i=-\infty}^{\infty} \big(p^{-i}/\big(\lambda + (p-1)^{-1}(pa(i) - a(i+1))\big)\big)} \\
&\quad - \frac{\big(p^{-m}/\big(\lambda + (p-1)^{-1}(pa(m-1) - a(m))\big)\big)}{((p-1)/p) \sum_{i=-\infty}^{\infty} \big(p^{-i}/\big(\lambda + (p-1)^{-1}(pa(i) - a(i+1))\big)\big)} \\
&= \left(1 + \frac{\sum_{i=-\infty}^{m-1} \big(p^{-i}/\big(\lambda + (p-1)^{-1}(pa(i) - a(i+1))\big)\big)}{\sum_{i=m}^{\infty} \big(p^{-i}/\big(\lambda + (p-1)^{-1}(pa(i) - a(i+1))\big)\big)}\right)^{-1} \\
&\quad - \frac{\big(p^{-m}/\big(\lambda + (p-1)^{-1}(pa(m-1) - a(m))\big)\big)}{((p-1)/p) \sum_{i=-\infty}^{\infty} \big(p^{-i}/\big(\lambda + (p-1)^{-1}(pa(i) - a(i+1))\big)\big)} \\
&\to 1 \quad \text{as } \lambda \to 0+.
\end{aligned}$$

Therefore, we have $P_x(V_y < \infty) = 1$ for any $x, y \in \mathbb{Q}_p$. This means that every point in $\mathbb{Q}_p$ is recurrent. □

REMARK. In the Haar symmetric case, we know from Albeverio and Karwowski (1994) that $\{(p-1)^{-1}(pa(M) - a(M = 1)), M \in \mathbb{Z}\}$ constitutes the pure point spectrum of the infinitesimal generator $-\mathbb{H}$ given by (2.14).

2.2. *The stable case.* In addition to the Haar symmetry, let us put

$$a(M) = a_0 c^M,\ M \in \mathbb{Z} \tag{2.27}$$

for some $0 < c < 1$ and $a_0 > 0$. Obviously, $\{a(M), M \in \mathbb{Z}\}$ defined in this way satisfies the conditions (i) and (ii). We refer to the associated processes as c-random walks on p-adics. For c-random walks, we have:

PROPOSITION 2.5. *Let $\{a(M), M \in \mathbb{Z}\}$ be given by (2.27); then in addition to the results in Propositions 2.3, 2.4, $P_t(x, A), x \in \mathbb{Q}_p, A \in \mathscr{D}_p$ has the following "scaling properties":*

$$P_{c^m t}(x, K(a, p^M)) = P_t(x, K(x + (a - x)p^{-n}, P^{M+n})), \tag{2.28}$$

$n, M \in \mathbb{Z}, t \geq 0, a \in \mathbb{Q}_p$. Moreover, the density $p_t(x, y), t > 0, x, y \in \mathbb{Q}_p$ exists, and

$$p_{ct}(x, y) = p p_t(p^{-1}x, p^{-1}y). \tag{2.29}$$

PROOF. Equation (2.28) follows easily from (2.12) and (2.13). The existence of the density is given by the fact that condition (2.20) is satisfied; (2.29) follows from (2.18), (2.21) and a direct computation. □

PROPOSITION 2.6. *Assume that* $\{a(M), M \in \mathbb{Z}\}$ *is given by* (2.27), *then*

$$p_t(0, x) \sim \frac{a_0 p(1-c)}{c(p-1)} \|x\|_p^{\log_p cp^{-1}} t, \ (x \to \infty) \tag{2.30}$$

and

$$p_t(0, 0) = t^{\log_c p} \exp\{\psi(\log t)\}, \tag{2.31}$$

where "$f(x) \sim g(x)$" *means* $\lim_{\|x\|_{p\to\infty}} f(x)/g(x) = 1$, *and* ψ *is a continuous periodic function of period* $-\log c$. *Moreover, for* $\|x\|_p \geq \big((p-1)^{-1}(p-c)^t\big)^{-\log_c p}$,

$$p_t(0, x) \leq \frac{a_0 p(2-c)}{2c(p-1)} \|x\|_p^{\log_p cp^{-1}} t. \tag{2.32}$$

PROOF. For $\|x\|_p = p^m$, we have proved in Proposition 2.3 that

$$\begin{aligned} p_t(0, x) &= (p-1)^{-1} p^{-m+1}(P_t(m) - P_t(m-1)) \\ &= p^{-m}\Bigg(\sum_{i=0}^{\infty} p^{-i}\{\exp[-a_0(p-1)^{-1}(p-c)c^{m+i}t] \\ &\qquad\qquad - \exp[-a_0(p-1)^{-1}(p-c)c^{m-1+i}t]\}\Bigg). \end{aligned} \tag{2.33}$$

Since

$$\|x\|_p^{-1} = p^{-m} = p^{-\log \|x\|_p/\log p}$$

and

$$c^m = c^{-\log \|x\|_p/\log p} = \|x\|_p^{\log c/\log p},$$

we shall write for $x \neq 0$,

$$p_t(0, x) = \|x\|_p^{-1} f(h\|x\|_p^{\log c/\log p} t), \tag{2.34}$$

where $h := a_0(p-1)^{-1}(p-c)$, and $f(y) := \sum_{i=0}^{\infty} p^{-i}(\exp(-c^i y) - \exp(-c^{i-1} y))$. Since $\log c/\log p < 0$, we have

$$\begin{aligned} p_t(0, x) &\sim \|x\|_p^{-1} f'(0) h \|x\|_p^{\log c/\log p} t \\ &= \|x\|_p^{-1} h \|x\|_p^{\log c/\log p} t \sum_{i=0}^{\infty} \left(\frac{c}{p}\right)^i (c^{-1} - 1) \\ &= \frac{a_0 p(1-c)}{c(p-1)} \|x\|_p^{\log_p cp^{-1}} t, \end{aligned}$$

as $\|x\|_p \to \infty$. We now turn to prove (2.31). If we write $g(u) := p_{e^u}(0,0)$, then from (2.28) we know $p_{ct}(0,0) = pp_t(0,0)$, and then

$$g(u - \log c) = p_{c^{-1}e^u}(0,0) = p^{-1}g(u).$$

Let $\psi(u) = \log g(u) - (\log p/\log c)u$. It is easy to check that $\psi(u + \log c^{-1}) = \psi(u)$, and

$$\begin{aligned} p_t(0,0) &= g(\log t) \\ &= \exp[\psi(\log t) + \log_c p \log t] \\ &= t^{\log_c p}\exp\{\psi(\log t)\}. \end{aligned}$$

As for (2.33), we notice that $1 - x \le e^{-x} \le 1 - x + x^2/2$, $x \in [0,\infty)$. By the definition of f and (2.34), we know that if $\|x\|_p \ge \big((p-1)^{-1}(p-c)t\big)^{-\log_c p}$,

$$\begin{aligned} p_t(0,x) &\le \frac{2-c}{2c} h\|x\|_p^{\log_p cp^{-1}} t \sum_{i=0}^{\infty}\left(\frac{c}{p}\right)^i \\ &= \frac{a_0 p(2-c)}{2c(p-1)}\|x\|_p^{\log_p cp^{-1}} t. \end{aligned}$$

This completes the proof. $\square$

REMARK 1. Proposition 2.3 and the first part of Proposition 2.6 can be found in Yasuda (1996), but the proof is slightly different. A particular case of Proposition 2.5 is given in Yasuda (1996).

REMARK 2. If we set

$$a_0 = \frac{p-1}{p(1-p^{-\alpha-1})}, \qquad c = p^{-\alpha},\ \alpha > 0 \tag{2.35}$$

then the corresponding random walk is associated with the operator D^α, $\alpha > 0$ introduced by Vladimirov (1988) [see Albeverio and Zhao (1999b)]. This operator has been extensively studied [see Vladimirov (1988), Vladimirov, Volovich and Zelenov (1993), Kochubei (1992, 1993a, b) and references therein].

From now on, we consider the symmetric random walks on $\mathbb{Q}_p$; that is, we consider the Haar symmetry case.

3. The heat-type equations on p-adics and the uniqueness of their solutions. In this section we study the Haar symmetry case for any given $a(i)$, $i \in \mathbb{Z}$ satisfying conditions (i), (ii) in Section 2. We first give a definition of locally constant functions.

DEFINITION 3.1. We say a function ϕ on $\mathbb{Q}_p$ is *locally constant* if there exists an integer n (depending on ϕ) such that ϕ is constant on any ball whose radius is not larger than p^n.

From this it follows automatically that ϕ is a continuous function on $\mathbb{Q}_p$. Clearly, the function family of locally constant functions on $\mathbb{Q}_p$ forms a linear space.

DEFINITION 3.2. Let $\{g(t), t \geq 0\}$ be an increasing positive functions on $[0, \infty)$ with $\lim_{t\to\infty} g(t) = \infty$. Set

$$\theta := \inf\left\{c > 0 \sum_{i=1}^{\infty}[a(i) - a(i+1)][g(p^{i+1})]^c = \infty\right\} \tag{3.1}$$

with the convention $\inf \varnothing = \infty$. We call θ the *converging index* of the sequence $\{a(i), i \in \mathbb{N}\}$ with respect to the function $g(t)$.

EXAMPLE. If $a(i) = a(0)c^i$ for some $0 < c < 1, a(0) > 0$ and $g(t) = t$, then $\theta = -\log_p c > 0$.

We now fix a given function g as in Definition 3.2 and consider the following heat equation:

$$\dot{u}(t, x) = -\mathbb{H}u(t, x) \tag{3.2}$$

with initial value $u(0, x) = \phi(x) \in C_c(\mathbb{Q}_p)$, where $\mathbb{H}$ is given in (2.14), and $C_c(\mathbb{Q}_p)$ is the set of continuous functions with compact support in $\mathbb{Q}_p$. It is easy to verify that

$$u(t, x) = S_t\phi(x) := \int_{\mathbb{Q}_p} P_t(x, dy)\phi(y) \tag{3.3}$$

satisfies (3.2) with initial value $\phi(x)$.

In general, $\theta \geq 0$. In fact, for any given $a(i), i \in \mathbb{Z}$ satisfying conditions (i), (ii) in Section 2 we can always find g such that $\theta > 0$, which will be proved later. We restrict here our consideration to the case $\theta > 0$, where we have

$$\limsup_{M\to\infty} a(M)(g(p^{M+1}))^\gamma = 0 \tag{3.4}$$

for any $0 < \gamma < \theta$. Now for any given $0 < \gamma < \theta$, we introduce

$$\begin{aligned}\mathscr{H}_\gamma := \big\{\phi \in \mathscr{B}(\mathbb{Q}_p); |\phi(x)| \leq C_\phi(1 + [g(\|x\|_p)]^\gamma), x \in \mathbb{Q}_p \\ \text{for some constant } C_\phi > 0\big\}.\end{aligned} \tag{3.5}$$

THEOREM 3.3. *Suppose $\theta > 0$, then the solution to* (3.2) *is unique in the function class $\mathscr{H}_\gamma, 0 < \gamma < \theta$.*

PROOF. It is sufficient to prove that a solution $u \in \mathscr{H}_\gamma$ to (3.2) with initial value 0 must be equal to 0. To this end, we set

$$w_a(s) := \begin{cases} 1, & \text{if } 0 \leq s \leq a, \\ 0, & \text{otherwise,} \end{cases}$$

and, for an integer $n > 0$,

$$\delta_n(x) = p^n w_{p^{-n}}(\|x\|_p).$$

Clearly, $\delta_n(x)$ has the compact support $K(0, p^{-n})$ and satisfies $\int_{\mathbb{Q}_p} \delta_n(x) \times \nu(dx) = 1$.

For any $\gamma < \eta < \theta$, let

$$\psi(x) := \int_{\mathbb{Q}_p} \delta_n(x-y)[g(\|y\|_p)]^\eta \nu(dy).$$

It is easy to see that ψ is locally constant and is equal to $[g(\|x\|_p)]^\eta$ if $\|x\|_p \geq p^{-n}$. By an elementary computation we have

$$-\mathbb{H}\psi(x) = a_0 \chi_{K(0,P^{-n})}(x) + \sum_{k=1}^\infty a_k \chi_{\{\|y\|_p = p^{-n+k}\}}(x), \tag{3.6}$$

where

$$\begin{aligned}
a_0 &= -a(-n)G_n + \sum_{i=1}^\infty [g(p^{-n+i})]^\eta [a(-n+i-1) - a(-n+i)],\\
a_1 &= \frac{1}{p-1}[a(-n) - a(-n+1)]G_n\\
&\quad + [g(p^{-n+1})]^\eta \left[-a(-n) + \frac{p-2}{p-1}(a(-n) - a(-n+1))\right]\\
&\quad + \sum_{i=-n+2}^\infty [g(p^{-n+i})]^\eta [a(-n+i-1) - a(-n+i)],\\
a_k &= \frac{p^{-k+1}}{p-1}[a(-n+k-1) - a(-n+k)]\left[G_n + \sum_{i=1}^{k-1}[g(p^{-n+i})]^\eta p^{i-1}(p-1)\right]\\
&\quad + [g(p^{-n+k})]^\eta \left[-a(-n+k-1) + \frac{p-2}{p-1}(a(-n+k-1) - a(-n+k))\right]\\
&\quad + \sum_{i=-n+k+1}^\infty [g(p^{-n+i})]^\eta [a(-n+i-1) - a(-n+i)], \qquad k \geq 2,\\
G_n &:= \int_{K(0,\, p^{-n})} [g(\|y\|_p)]^\eta \nu(dy).
\end{aligned}$$

Clearly, G_n is finite. Set

$$A_n := \sup_{i \geq -n} a(i)[g(p^{i+1})]^\eta,$$

$$B_n := \sum_{i=1}^\infty [g(p^{-n+i})]^\eta [a(-n+i-1) - a(-n+i)].$$

Because of $\eta < \theta$, A_n, B_n are finite and,

$$a_0 \le B_n,$$
$$a_1 \le \frac{1}{p-1} a(-n) G_n + B_n$$

and

$$a_k \le \frac{1}{p-1} a(-n) G_n + \frac{1}{p-1} A_n + B_n$$

for $k \ge 2$. Therefore,

$$O := \max\{a_k, k = 0, 1, \dots, \} < a(-n) G_n + A_n + B_n < \infty.$$

Now we turn to prove $u(t, x) \ge 0$. Suppose there exist $x' \in \mathbb{Q}_p$ and $t' \in (0, \infty)$ such that $u(t', x') = \zeta < 0$. We can find a $T > t'$ and some constants $a > 0$ and $b > 0$ such that

$$\zeta + Ta + b\psi(x') < 0 \tag{3.7}$$

and

$$a - Ob > 0. \tag{3.8}$$

We consider the function

$$v(t, x) = u(t, x) + at + b\psi(x).$$

From (3.7) we know that $v(t', x') < 0$ and then

$$\inf_{x \in \mathbb{Q}_p,\, 0 \le t \le T} v(t, x) < 0.$$

Assume $v(0, x) = b\psi(x) \ge 0$. Since $u(t, \cdot) \in \mathscr{H}_\gamma$, $u(t, x) = o(\psi(x))$ as $\|x\|_p \to \infty$. This implies that $v(t, x) > 0$ for sufficiently large $\|x\|_p$ and the point x making $v(t, x)$ negative must be located in a finite open-compact ball. By the completeness of any bounded open-compact subset of $\mathbb{Q}_p$, there exist $x_0 \in \mathbb{Q}_p$ and $t_0 \in (0, T]$ such that

$$\inf_{x \in \mathbb{Q}_p,\, 0 \le t \le T} v(t, x) = v(t_0, x_0) < 0; \tag{3.9}$$

that is, (t_0, x_0) is one of global minimum points of $v(t, x)$ in $(0, T] \times \mathbb{Q}_p$. Thus, $\dot{v}(t_0, x_0) \le 0$ and $-\mathbb{H} v(t_0, x_0) \ge 0$; that is,

$$\dot{v}(t_0, x_0) + \mathbb{H} v(t_0, x_0) \le 0. \tag{3.10}$$

However, on the other hand,

$$\dot{v}(t_0, x_0) + \mathbb{H} v(t_0, x_0) = a + b\mathbb{H}\psi(x_0) \ge a - bO > 0. \tag{3.11}$$

Thus, we have a contradiction. Therefore, $u(t, x) \ge 0$.

We can prove $u(t, x) \le 0$ by considering $-u(t, x)$ instead of $u(t, x)$ in the above argument. That is, $u(t, x) \equiv 0$. □

PROPOSITION 3.4. *There is an increasing sequence* $\{b(i), i \geq 1\}$ *with* $\lim_{i\to\infty} b_i = \infty$ *such that*

$$\sum_{i=1}^{\infty}[a(i) - a(i+1)]b(i) < \infty, \tag{3.12}$$

That is, $\theta \geq 1$ *for a* g *satisfying the condition in Definition* 3.2. *Therefore*, *the bounded solution to* (3.2) *is unique*.

PROOF. Since the series $\sum_{i=1}^{\infty}[a(i) - a(i+1)]$ converges and each term in this series is nonnegative, we can find a sequence of strictly increasing integers $\{k_i, i \in \mathbb{N}\}$ such that $k_1 = 1$, and $k_i > k_{i-1} + 1$ such that

$$\sum_{j=k_i}^{\infty}[a(j) - a(j+1)] < \frac{1}{2^i}, \qquad i \geq 2.$$

For $i \in \mathbb{N}$, set

$$b(i) := 1.5^j, \qquad k_j \leq i \leq k_{j+1}.$$

Obviously $\{b(i), i \in \mathbb{N}\}$ is strictly increasing and $\lim_{i\to\infty} b_i = \infty$. Moreover,

$$\begin{aligned}\sum_{i=1}^{\infty}[a(i) - a(i+1)]b(i) &= \sum_{i=1}^{\infty}\sum_{j=k_i}^{k_{i+1-1}}[a(j) - a(j+1)]b(j)\\ &< \sum_{i=1}^{\infty} 0.75^i < \infty.\end{aligned}$$

This proves the first part.

Now we construct a nonnegative function $g(t), t \geq 0$ which is increasing and satisfies $\lim_{t\to\infty} g(t) = \infty$. Let $b_0 = 0$ and define $g(t), t \geq 0$,

$$g(t) = b_1 \frac{t}{p}, \qquad 0 \leq t \leq p$$

and

$$g(t) = b_n \frac{p^{n+2} - t}{p^{n+1}(p-1)} + b_{n+1}\frac{t - p^{n+1}}{p^{n+1}(p-1)}, \qquad p^{n+1} \leq t < p^{n+2},$$

for $n \geq 0$. That is, $g(t)$ is the polygonal line derived from $\{b_n, n \geq 0\}$. We easily know that $\theta \geq 1 > 0$ with respect to this $g(t)$ and the uniqueness of bounded solutions is immediate from Theorem 3.3. This completes the proof. □

The unique solution to (3.2) can be expressed by

$$\begin{aligned}u(t,x) &= e^{-t\lim_{M\to-\infty} a(M)}\phi(x)\\ &\quad + \sum_{M=-\infty}^{\infty}\int_{\|x-y\|_p = p^M}\phi(y)\frac{p^{-M+1}}{p-1}(P_t(M) - P_t(M-1))\nu(dy).\end{aligned} \tag{3.13}$$

In this case, (3.2) can be reduced to a system of linear ODEs, which can be solved recursively.

PROPOSITION 3.5. *Suppose* $\phi(x) = \chi_{K(a,\, p^M)}(x)$ *for some* $a \in \mathbb{Q}_p$ *and* $M \in \mathbb{Z}$, *then the solution* $u(t, x)$ *satisfies*

$$u(t, x) = u(t, y) \text{ if } \|x - a\|_p = \|y - a\|_p > p^M. \tag{3.14}$$

Therefore if we let

$$u(t, x) = w_{M,0}(t)\chi_{K(a,\, p^M)}(x) + \sum_{i=1}^{\infty} w_{M,i}(t)\chi_{K(a,\, p^{M+i})\setminus K(a,\, p^{M+i-1})}(x), \tag{3.15}$$

then $\{w_{M,i}(t), i = 0, 1, \ldots,\}$ *satisfies*

$$\dot{w}_{M,0}(t) = -a(M)w_{M,0}(t) + \sum_{i=1}^{\infty}(a(M+i-1) - a(M+i))w_{M,i}(t) \tag{3.16}$$

and

$$\begin{aligned}\dot{w}_{M,i}(t) = {} & \left[-\frac{1}{p-1}a(M+i-1) - \frac{p-2}{p-1}a(M+i)\right]w_{M,i}(t) \\ & + \sum_{j=i+1}^{\infty}(a(M+j-1) - a(M+j))w_{M,j}(t) \\ & + p^{-i+1}(p-1)^{-1}(a(M+i-1) - a(M+i)) \\ & \times \left[\sum_{j=1}^{i-1} p^{j-1}(p-1)w_{M,j}(t) + w_{M,0}(t)\right]\end{aligned} \tag{3.17}$$

with initial condition $w_{M,i}(0) = 0$, $i \in \mathbb{N}$, $w_{M,0}(0) = 1$.

PROOF. Equation (3.5) is easily seen from (2.15)–(2.17), (3.16) and (3.17) can be derived from (2.6) and (2.7) by classifying the balls of p^M in terms of their distances from $K(a, p^M)$ and taking into account the form of the solution (3.13). □

The function family $C_c(\mathbb{Q}_p)$ is not easy to deal with and it is useful to make the following remark concerning its dense subset of locally constant functions.

PROPOSITION 3.6. *Suppose the initial value* $\phi \in C_c(\mathbb{Q}_p)$ *is locally constant, then the solution of* (3.2) *is also locally constant, but is not of compact support.*

The proof can be seen from (3.13) and the spatial homogenity of the transition kernels of the random walks involved.

REMARK. We remark that Kochubei (1992) considers the heat-type equation (3.2) for the case where

$$a(M) = \frac{(p-1)}{p(1-p^{-\alpha-1})} p^{-\alpha M}, \qquad M \in \mathbb{Z},$$

which corresponds to the operator D^α, $\alpha > 0$, introduced by Vladimirov (1988).

4. Measure-valued branching processes. Since $\mathbb{Q}_p$ is a locally compact, complete separable metric space and the process $\xi = (\xi_t)_{t \in \mathbb{R}+}$ constructed (e.g., by Kolmogorov's procedure) from the transition semigroup S_t of Section 2 [cf. Albeverio and Karwowski (1994)] is a strongly continuous Markovian process, the theory of superprocesses can naturally be applied to the measure-valued branching processes on $\mathbb{Q}_p$.

Let $\mathscr{M}(\mathbb{Q}_p)$ be a cone set of (positive) measures under the ordinary addition operator and scale multiplication. We recall from Dawson (1993), Dynkin (1994) or Zhao (1999) that the measure-valued branching process $\{X_t, t \geq 0, P^\mu\}_{\mu \in \mathscr{M}(\mathbb{Q}_p)}$ with underlying process $\{\xi_t, t \geq 0\}$ is determined by the following Laplace functional:

$$P^\mu \exp\{-\langle X_t, \phi\rangle\} = \exp\{-\langle \mu, V_t\phi\rangle\}, \phi \in C_c(\mathbb{Q}_p)_+, \mu \in \mathscr{M}(\mathbb{Q}_p), \tag{4.1}$$

where $C_c(\mathbb{Q}_p)_+$ stands for the family of nonnegative continuous functions with compact support, and $V_t\phi$ satisfies the nonlinear integral equation [i.e., log-Laplace equation; see, e.g., Dawson (1993), Section 4.3]

$$V_t\phi(x) + E_x \int_0^t \Psi(\xi_s, V_{t-s}\phi(\xi_s))\, ds = E_x\phi(\xi_t) \tag{4.2}$$

and

$$\begin{aligned} \Psi(x, \lambda) = b(x)\lambda + c(x)\lambda^2 + \int_0^\infty (e^{-\lambda u} - 1 + u\lambda) n(x, du), \\ x \in \mathbb{Q}_p,\ \lambda \geq 0 \end{aligned} \tag{4.3}$$

for some bounded $\mathscr{Q}_p$-measurable function b, a bounded nonnegative $\mathscr{Q}_p$-measurable function c and a positive measure $n(x, \cdot)$ on the positive half-line such that

$$\int_0^\infty u n(x, du) < \infty \tag{4.4}$$

is a bounded $\mathscr{Q}_p$-measurable function in $x \in \mathbb{Q}_p$. Here $\langle \mu, \phi\rangle$ means the integral of ϕ with respect to μ. We note that (4.2) has a *unique* solution [see, e.g., Dawson (1993), Lemma 4.3.3, Dynkin (1994), Chapters 3, 5 and Zhao (1999), Chapter 3] and the corresponding superprocess X has finite first-order moments under condition (4.4). Heuristically, the measure-valued branching processes can be regarded as a highly dense limit of branching particle systems [see Dynkin (1994) and Zhao (1999)].

For simplicity, from now on we assume $\Psi(x,\lambda) = \lambda^{1+\beta}, 0 < \beta \le 1$ and $\mathscr{M}(\mathbb{Q}_p) = M_q(\mathbb{Q}_p)$, where

$$M_q(\mathbb{Q}_p) := \Big\{\mu, \text{(positive) Radon measure on } \mathbb{Q}_p; \int_{\mathbb{Q}_p} (1+\|x\|_p)^{-q}\mu(dx) < \infty\Big\}.$$

In order to make the Haar measure ν belong to $M_q(\mathbb{Q}_p)$ we assume $q > 1$. In fact,

$$\begin{aligned}\int_{\mathbb{Q}_p} (1+\|x\|_p)^{-q}\nu(dx) &\le 1 + \sum_{i=1}^{\infty}(1+p^i)^{-q}(p^i - p^{i-1})\\ &\le 1 + \sum_{i=1}^{\infty} p^{-(q-1)i} < \infty.\end{aligned}$$

We call the corresponding superprocesses (ξ,β)-superprocesses.

We refer to $M_q(\mathbb{Q}_p)$ as the q-tempered measured space on $\mathbb{Q}_p$ (this is in analogy with the corresponding known definition in the case of $\mathbb{R}^d$ instead of $\mathbb{Q}_p$). Under this setting we can rewrite (4.2) as

$$V_t\phi(x) + \int_0^t S_s[V_{t-s}\phi(x)]^{1+\beta}\, ds = S_t\phi(x), \tag{4.5}$$

and its strong version is

$$\dot{u}(t,x) = -\mathbb{H}u(t,x) - [u(t,x)]^{1+\beta} \tag{4.6}$$

with the initial condition $u(0,x) = \phi(x) \in C_c(\mathbb{Q}_p)_+$.

Uniqueness of solutions to (4.5) can be proved in a similar manner to, for example, Dawson [(1993), Lemma 4.3.3], and its unique solution satisfies $V_t\phi(x) \ge 0$ if $\phi(x) \ge 0$. As for (4.6), we have the following theorem.

THEOREM 4.1. *With the notations in Section* 3, *the nonnegative solution to* (4.6) *is unique in the function class* $\mathscr{H}_\gamma$, $0 < \gamma < \theta$. *In particular*, *the bounded nonnegative solution is unique*; *that is*, *the unique* (*nonnegative*) *solution to* (4.6) *is* $V_t\phi(x)$ *given by* (4.5).

PROOF. Suppose that there exist two solutions $u_1(t,x)$ and $u_2(t,x)$. We shall prove $u_1(t,x) = u_2(t.x)$. Let $u(t,x) = u_1(t,x) - u_2(t,x)$, then $u(t,x)$ satisfies

$$\dot{u}(t,x) = -\mathbb{H}u(t,x) - [u_1(t,x)]^{1+\beta} + [u_2(t,x)]^{1+\beta} \tag{4.7}$$

with the initial value $u(0,x) = 0$. As in the proof of Theorem 3.3 we then prove $u(t,x) \ge 0$ [by noticing that $u(t_0,x_0) < 0$ means

$$[u_2(t_0,x_0)]^{1+\beta} - [u_1(t_0,x_0)]^{1+\beta} > 0$$

because $u_1(t,x)$ and $u_1(t,x)$ are nonnegative].

In the same manner we can also prove $u(t, x) \leq 0$. That is, (4.6) has a unique (nonnegative) solution in the class of $\mathscr{H}_\gamma$, $0 < \gamma < \theta$. The remaining statement is clear. □

Similarly to Proposition 3.5, concerning nonnegative solutions, (4.6) can also be reduced to a system of nonlinear ODEs, which can be solved recursively.

PROPOSITION 4.2. *For any initial condition of the form* $\phi(x) = \chi_{K(a,\, p^M)}(x)$ *the nonnegative solution to* (4.6) *can be expressed by*

$$u(t, x) = v_{M,0}(t)\chi_{K(a,\, p^M)}(x) + \sum_{i=1}^{\infty} v_{M,i}(t)\chi_{K(a,\, p^{M+i})\setminus K(a,\, p^{M+i-1})(x)}, \tag{4.8}$$

in which $\{v_{M,i}(t),\ i = 0, 1, \dots,\}$ *satisfy*

$$\begin{aligned}\dot{v}_{M,0}(t) = {}&-a(M)v_{M,0}(t) + \sum_{i=1}^{\infty}(a(M+i-1)\\ &- a(M+i))v_{M,i}(t) - v_{M,0}^{1+\beta}(t)\end{aligned} \tag{4.9}$$

and

$$\begin{aligned}\dot{v}_{M,i}(t) = {}&\left[-\frac{1}{p-1}a(M+i-1) - \frac{p-2}{p-1}a(M+i)\right]v_{M,i}(t)\\ &+ \sum_{j=i+1}^{\infty}(a(M+j-1) - a(M+j))v_{M,j}(t)\\ &+ p^{-i+1}(p-1)^{-1}(a(M+i-1) - a(M+i))\\ &\times\left[\sum_{j=1}^{i-1} p^{j-1}(p-1)v_{M,j}(t) + v_{M,0}(t)\right] - v_{M,i}^{1+\beta}(t)\end{aligned} \tag{4.10}$$

with initial condition $u_{M,i}(0) = 0$, $i \in \mathbb{N}$, $u_{M,0}(t) = 1$. *In general,* (4.5) *and* (4.6) *are equivalent if the initial condition* ϕ *is locally constant.*

The proof is similar to the one of Proposition 3.5 using the uniqueness of solutions of (4.8).

REMARK. Further study of the heat equation in both linear and nonlinear cases is an interesting topic, depending on the chosen initial conditions. On one hand, we have failed to prove that (4.6) has no negative solutions. On the other hand, concerning the heat-type equations associated with the generators of random walks, our knowledge is very limited so far. For example, if $a(M) \equiv 1$, $M \geq 0$ and $a(M) \equiv 0$, $M < 0$, we know that the corresponding random walk jumps among the p number of p^{-1}-balls within the integer ball $K(0, 1)$. It is not reasonable to consider the corresponding heat equation for some initial value, for example, which is not constant on p^{-1}-balls.

LEMMA 4.3 (Maximum principle). *Let $v_i(t,x)$ be the nonnegative solutions of* (4.6) *with $\phi(x) = \phi_i(x) \in C_c(\mathbb{Q}_p)_+$, $i = 1, 2$. If $\phi_1 \geq \phi_2$ on $\mathbb{Q}_p$, then $v_1(t,x) \geq v_2(t,x)$ on $[0,\infty) \times \mathbb{Q}_p$.*

PROOF. Suppose that the assertion is not true, that is, there exists $(t', x') \in (0,\infty) \times \mathbb{Q}_p$ such that $v_1(t',x') < v_2(t',x')$. Let $u(t,x) = v_1(t,x) - v_2(t,x)$. Because $v_i(t,x)$ is nonnegative we have $v_i(t,x) \leq S_t\phi_i(x)$ by Theorem 4.1, and this implies $\lim_{\|x\|p\to\infty} v_i(t,x) = 0$ for $i = 1$ or 2. Therefore, for any fixed $T > 0$, we can find a minimum point of $u(t,x)$, say $(t_0,x_0) \in (0,T] \in K(0,p^M)$ for some $M \in \mathbb{N}$ such that

$$\inf_{0<t\leq T,\, x\in\mathbb{Q}_p} u(t,x) = u(t_0,x_0) = v_1(t_0,x_0) - v_2(t_0,x_0) < 0.$$

Noticing that (t_0,x_0) is actually a global minimum point of $u(t,x)$ in $(0,T] \times \mathbb{Q}_p$, we know that $-\mathbb{H}u(t_0,x_0) \geq 0$ and $\dot{u}(t_0,x_0) \leq 0$. However,

$$0 \leq -\mathbb{H}u(t_0,x_0) - \dot{u}(t_0,x_0) = v_1^{1+\beta}(t_0,x_0) - v_2^{1+\beta}(t_0,x_0) < 0,$$

a contradiction, which proves the lemma. □

From the classical theory of superprocesses in Dawson (1993), Dynkin (1994) and Zhao (1999), we know that the superprocesses defined above are Markovian processes having the càdlàg path property. Naturally, the questions investigated for the super α-stable processes and the superdiffusions on $\mathbb{R}^d$ are also interesting for the superprocesses on $\mathbb{Q}_p$. We first discuss the absolute continuity of these processes with respect to the Haar measure ν.

THEOREM 4.4. *Assume $\{\xi_t, t \geq 0\}$ is Haar symmetric and $p_t(x,\cdot)$ has a density $p_t(x,y)$ with respect to the Haar measure ν. Then for any measure $\mu \in M_q(\mathbb{Q}_p)$ such that $\mu(dx) \leq \alpha\nu(dx)$ for some constant $\alpha > 0$, the corresponding $(\xi,1)$-superprocess X_t, $t \geq 0$ is absolutely continuous with respect to Haar measure ν, with Radon–Nikodym derivative $X(t,x)$, which is a random field on $[0,\infty) \times \mathbb{Q}_p$ and is not identically zero.*

PROOF. By a similar discussion to that in Konno and Shiga (1988), we need to check that for any fixed $T > 0$,

$$\sup_{h>0} \int_{\mathbb{Q}_p} \int_0^T P^\mu\biggl(\int_{\mathbb{Q}_p} p_h(x,y)X_t(dy)\biggr)^2 (1+\|x\|_p)^{-q}\, dt\nu(dx) < \infty \tag{4.11}$$

and

$$\begin{aligned}\lim_{h,h'\to 0} \int_{\mathbb{Q}_p} \int_0^T P^\mu\biggl(\int_{\mathbb{Q}_p} (p_h(x,y) - p'_h(x,y))X_t(dy)\biggr)^2 \\ \times (1+\|y\|_p)^{-q}\, dt\nu(dx) = 0\end{aligned} \tag{4.12}$$

hold. This follows from the estimates in Section 2 and the moments formulas [cf. Dawson (1993), Lemma 4.7.1 and Zhao (1999), Theorem 3.4.2]. If we set

$X^h(t, x) := \langle X_t, p_h(x, \cdot)\rangle$, we conclude that $\{X^h(t, x), t \geq 0\}$ is a Cauchy sequence in the Hilbert space $L^2(\Omega \times [0, T] \times \mathbb{Q}_p, P^u \times dt \times \nu(dx))$, where Ω is the probability space for X_t [and dt the Lebesgue measure on $[0, \infty)$]. Therefore, there exists a random field $\{X(t, x), t \geq 0, x \in \mathbb{Q}_p\}$ such that

$$\lim_{h\to 0+} X_t^h(x) \overset{L^2}{=} X(t, x),$$

where $L^2 := L^2(\Omega \times [0, T] \times \mathbb{Q}_p, P^u \times ds \times \nu(dx))$. It is easy to check that for any $\phi \in C_c(\mathbb{Q}_p)_+$,

$$\int_{\mathbb{Q}_p} \phi(x) X_t(dx) \overset{L^2}{=} \int_{\mathbb{Q}_p} \phi(x) X(t, x)\nu(dx). \tag{4.13}$$

Moreover, by the arbitrariness of $T > 0$, we can define $X(t, x)$ for $t \geq 0$, $x \in \mathbb{Q}_p$. That is, the corresponding superprocess $\{X_t, t \geq 0, P^u\}$ is absolute continuous with respect to the Haar measure ν. The nontriviality of the corresponding Radon–Nikodym derivative $X(t, x)$ can be seen easily from (4.13).

REMARK 1. The random field $X(t, x)$ can be further studied; for example, it can be characterized by a stochastic partial differential equation [cf. Dawson (1993), Theorem 8.3.2 and Konno and Shiga (1988)]. Following the manner of previous authors [see, e.g., Walsh (1986)] we need to define a "white noise" on $\mathbb{Q}_p$ [in fact, "white noise" has been introduced in Evans (1995), even though it is not precisely what we want here]. Nevertheless, it is an interesting question to study stochastic fields, differential equations and stochastic partial differential equations on $\mathbb{Q}_p$ and local fields.

REMARK 2. For the case $\beta < 1$ we know that the superprocesses have only finite first-order moments but infinite higher-order moments. This means the method we use above is invalid for this case. However, it is possible to study this question following Fleischmann (1988).

5. The quasi-self-similarity of (ξ, β)-superprocesses. In this and next sections, we consider

$$L^1(\mathbb{Q}_p, \nu) := \{f \in \mathscr{B}(\mathbb{Q}_p); \int_{\mathbb{Q}_p} |f(x)|\nu(dx) < \infty\},$$

and denote by $L^1(\mathbb{Q}_p, v)_+$ the set of nonnegative elements in $L^1(\mathbb{Q}_p, \nu)$.

We consider (ξ, β)-superprocesses where ξ is a c-random walks as defined in Section 2.2. We call the corresponding superprocess a (c, β)-superprocess.

We first state a fundamental result [see Vladimirov, Volovich and Zelenov (1993)].

LEMMA 5.1 (Formula for change of variable). *For any $\phi \in L^1(\mathbb{Q}_p, \nu)$,*

$$\int_{\mathbb{Q}_p} \phi(x)\nu(dx) = p^{-1} \int_{\mathbb{Q}_p} \phi(px)\nu(dx). \tag{5.1}$$

PROOF. By standard arguments (method of monotone classes), it suffices to check the formula for indicator functions. Let $\phi(x) = \chi_{K(a,\, p^M)}(x)$. The left-hand side is clearly equal to p^M, and for the right-hand side, we have

$$\int_{\mathbb{Q}_p} \phi(px)\nu(dx) = \int_{\mathbb{Q}_p} \chi_{K(p^{-1}a,\, p^{M+1})}(x)\nu(dx) = p^{M+1}.$$

Therefore, (5.1) holds. □

LEMMA 5.2. *For any* $\phi \in L^1(\mathbb{Q}_p, \nu)$,

$$S_{ct}\phi(x) = p^2[S_t\phi(p\cdot)](p^{-1}x). \tag{5.2}$$

PROOF. This follows from

$$\begin{aligned} S_{ct}\phi(x) &= \int_{\mathbb{Q}_p} p_{ct}(x, y)\phi(y)\nu(dx) \\ &= \int_{\mathbb{Q}_p} pp_t(p^{-1}x, p^{-1}y)\phi(y)\nu(dx) \quad [\text{by } (2.29)] \\ &= p^2 \int_{\mathbb{Q}_p} p_t(p^{-1}x, y)\phi(py)\nu(dx) \quad [\text{by } (5.1)] \\ &= p^2[S_t\phi(p\cdot)](p^{-1}x). \end{aligned}$$

□

Let us consider (4.5), where V_t is the nonlinear semigroup entering (4.1). We have the following lemma.

LEMMA 5.3. *If* $c = p^{-2(1+\beta)}$, *then*

$$[V_{ct}\phi](px) = p^2[V_t\phi(p\cdot)](x), \qquad \phi \in L^1(\mathbb{Q}_p, \nu)_+. \tag{5.3}$$

PROOF. Consider

$$\begin{aligned} V_{ct}\phi(x) &= [S_{ct}\phi](x) - \int_0^{ct} S_{ct-s}[V_s\phi]^{1+\beta}(x)\,ds \\ &= p^2[S_t\phi(p\cdot)](p^{-1}) - p^2 c\int_0^t ds\Big[S_{t-s}[V_{cs}\phi]^{1+\beta}(p\cdot)\Big](p^{-1}x). \end{aligned}$$

we have

$$p^{-2}[V_{ct}\phi](px) = [S_t\phi(p\cdot)](x) - \int_0^t \Big[S_{t-s}[C^{1/(1+\beta)}V_{cs}\phi]^{1+\beta}(p\cdot)\Big](x).$$

when $c = p^{-2(1+\beta)}$, the above formula becomes

$$[p^{-2}V_{ct}\phi](px) = [S_t\phi(p\cdot)](x) - \int_0^t ds\Big[S_{t-s}[p^{-2}V_{cs}\phi]^{1+\beta}(p\cdot)\Big](x). \tag{5.4}$$

However, $[V_t\phi(p\cdot)](x)$ also satisfies

$$[V_t\phi(p\cdot)](x) = [S_t\phi(p\cdot)](x) - \int_0^t ds\Big[S_{t-s}[V_s\phi(p\cdot)]^{1+\beta}\Big](x). \tag{5.5}$$

Since the solution of (5.5) is unique, we have the desired conclusion. □

THEOREM 5.4 (Quasi-self-similarity). *Let* $(X_t, t \geq 0, P^{\mu})_{\mu \in M_q(\mathbb{Q}_p)}$ *be the* (c, β)*-superprocess on* $\mathbb{Q}_p$ *associated with the process* $\xi_t, t \geq 0$ *on* $\mathbb{Q}_p$. *Suppose* $c = p^{-2(1+\beta)}$; *then for* $t > 0, (X_{ct}(K(a, p^M)), P^{\mu})$ *is equal to* $(X_t(K(p^{-1}a, p^{M+1})), P^{\mu'})$ *in distribution for any* $a \in \mathbb{Q}_p, M \in \mathbb{Z}, \mu \in M_q(\mathbb{Q}_p)$. *Here* μ' *is given by*

$$\int_{\mathbb{Q}_p} \phi(x)\mu'(dx) = p^2 \int_{\mathbb{Q}_p} \phi(p^{-1}x)\mu(dx), \qquad \phi \in L^1(\mathbb{Q}_p, \nu)_+.$$

PROOF. It is sufficient to prove that

$$P^{\mu} \exp\{-\lambda X_{ct}(K(a, p^M))\} = P^{\mu'} \exp\{-\lambda X_t(K(p^{-1}a, p^{M+1}))\}, \qquad \lambda \geq 0.$$

In fact, from (4.1) and (4.5), we have

$$\begin{aligned} P^{\mu} \exp\{-\lambda X_{ct}(K(a, p^M))\} &= \exp\{-\langle \mu, V_{ct}\lambda\chi_{K(a, p^M)}\rangle\} \\ &= \exp\{-\langle \mu, p^2[V_t\lambda\chi_{K(p^{-1}a, p^{M+1})}](p^{-1}\cdot)\rangle\} \\ &= P^{\mu'} \exp\{-\lambda X_t(K(p^{-1}a, p^{M+1}))\}, \end{aligned}$$

which completes the proof. □

REMARK. The property of quasi-similarity is analogous to the one of self-similarity for super-α-stable process on $\mathbb{R}^d$ [cf. Dawson (1993), Lemma 4.5.1]. This is the reason for calling it "quasi-self-similarity."

6. The local extinction. In this section we again consider the (c, β)-superprocesses. The main topic of investigation will be the local extinction of these superprocesses. Without loss of generality we assume $a_0 = 1$.

We say the superprocess $\{X_t, t \geq 0\}$ is *locally extinct* if

$$\lim_{t \to \infty} X_t(K) = 0 \tag{6.1}$$

in probability for any compact set K. For any $f \in L^1(\mathbb{Q}_p, \nu)$, let

$$\|f\|_1 := \int_{\mathbb{Q}_p} |f(x)|\nu(dx).$$

Clearly, $L^1(\mathbb{Q}_p, \nu)$ with this norm forms a Banach space.

It is easy to see from (2.18) the symmetry of the transition density,

$$p_t(x, y) = p_t(y, x), \qquad t > 0,\ x, y \in \mathbb{Q}_p. \tag{6.2}$$

This implies that

$$\|S_t f\|_1 = \|f\|_1,\ f \in L^1(\mathbb{Q}_p, \nu)_+. \tag{6.3}$$

Using (6.1) we easily deduce the following.

PROPOSITION 6.1. *Suppose* $v_M(t,x)$ *is the solution to* (4.5) *with* $\phi = \chi_{K(0,p^M)}$, $M \in \mathbb{N}$, $(X_t, t \geq 0, P^\nu)$ *is locally extinct* iff *for any* $M \in \mathbb{N}$,

$$\lim_{t\to\infty} \|v_M(t\cdot)\|_1 = 0. \tag{6.4}$$

According to Zhao (1995), if μ is a finite measure then X_t is P^μ-a.s. extinct; that is,

$$\lim_{t\to\infty} X_t(\mathbb{Q}_p) = 0, \quad P^\mu\text{-a.s.} \tag{6.5}$$

If μ is an infinite measure we cannot expect such a strong result. From now on we assume that the initial measure μ is the Haar measure ν.

We first have the theorem.

THEOREM 6.2. *If* $\beta \leq -\log_p c$, *then the* (ξ, β)*-superprocess is locally extinct.*

PROOF. For any integer $k \in \mathbb{N}$, let

$$B_t^k := \left\{x \in \mathbb{Q}_p; \|x\|_p \leq p^{k+[-\log_c t]}\right\}, \tag{6.6}$$

where $[\cdot]$ stands for the integer part of the number. We can prove that

$$\int_0^\infty (\nu(B_t^k))^{-\beta}\, dt = \infty. \tag{6.7}$$

In fact, by noticing $p^{-\beta[-\log_c t]} \geq 1$ for $0 < t \leq 1$, we have

$$\begin{aligned}\int_0^\infty (\nu(B_t^k))^{-\beta}\, dt &\geq p^{-\beta k} + p^{-\beta k} \sum_{i=1}^\infty \int_{(c^{-1})^{i-1}}^{(c^{-1})^i} p^{-\beta[-\log_c t]}\, dt \\ &\geq p^{-\beta k} + p^{-\beta k}(1-c) \sum_{i=1}^\infty (p^\beta c)^{-1} = \infty\end{aligned}$$

for $\beta \leq -\log_p c$.

From Propositions 2.3 and 2.6 we know that $p_1(0,x)$ exists and is bounded, continuous and positive on $\mathbb{Q}_p$. Therefore, by Lemma 4.3 it is sufficient to prove

$$\lim_{t\to\infty} \|v_t\|_1 \equiv \lim_{t\to\infty} \int_{\mathbb{Q}_p} v_t(x)\nu(dx) = 0, \tag{6.8}$$

where v_t satisfies

$$v_t + \int_0^t ds S_{t-s} v_s^{1+\beta}(x) = S_t p_1(0,x). \tag{6.9}$$

From (6.3) and (6.9) we have

$$\|v_t\|_1 + \int_0^t ds \left\|v_s^{1+\beta}\right\|_1 = 1. \tag{6.10}$$

This tells us that $\|v_t\|_1$ is decreasing in t and is bounded by 1. Thus,

$$\begin{aligned}1 &\geq \int_0^t ds \|v_s^{1+\beta}\|_1 \\ &\geq \int_0^t ds \left(\int_{B_s^k} dx\right)^{-\beta} g^{1+\beta}(s) \quad \text{by the Hölder inequality,}\end{aligned}$$

where $g(t) := \int_{B_t^k} v_t(x)\,dx$. Noticing (6.7), we have

$$\liminf_{t\to\infty} g(t) = 0. \tag{6.11}$$

On the other hand,

$$\begin{aligned}\int_{\mathbb{Q}_p\setminus B_t^k} v_t(x)\nu(dx) &\leq \int_{\mathbb{Q}_p\setminus B_t^k} p_{t+1}(0,x)\nu(dx) \\ &= 1 - (P_{t+1}(0, B_t^k)) \\ &\leq 1 - \frac{p-1}{p}\sum_{i=0}^{\infty} p^{-i} \exp\left[-\frac{p-c}{p-1}c^{i-1+k-\log_c t}t\right] \\ &= 1 - \frac{p-1}{p}\sum_{i=0}^{\infty} p^{-i} \exp\left[-\frac{p-c}{p-1}c^{i-1+k}\right] \to 0\end{aligned} \tag{6.12}$$

as $k\to\infty$. Combining (6.11) and (6.12), it is easy to see that

$$\liminf_{t\to\infty} \int_{\mathbb{Q}_p} v_t(x)\nu(dx) = 0; \tag{6.13}$$

therefore, by the monotonicity of $\|v_t\|_1$, $\lim_{t\to\infty}\|v_t\|_1 = 0$. $\square$

Let us consider the complementary case where $\beta > -\log_p c$. It is natural to guess that X_t is not locally extinct. To see this we first establish some lemmas.

LEMMA 6.3. *If $\beta > -\log_p c$, then*

$$\int_0^{\infty} \|P_t^2(\cdot, U)\|_1^{\beta} dt < +\infty, \tag{6.14}$$

where $U := \{x \in \mathbb{Q}_p; \|x\|_p \leq 1\}$, then unit ball of $\mathbb{Q}_p$.

PROOF. We first notice that for any sequence $\{c(i), i \in \mathbb{N}\cup\{0\}\}$, we have

$$\begin{aligned}\left(\sum_{i=0}^{\infty} p^{-1}c(i)\right)^2 &= \left(\sum_{i=0}^{\infty} p^{-1/2}[p^{-i/2}c(i)]\right)^2 \\ &\leq \sum_{i=0}^{\infty} p^{-i}\sum_{i=0}^{\infty} p^{-i}[c(i)]^2 \quad \text{(by the Hölder inequality)} \\ &= \frac{p}{p-1}\sum_{i=0}^{\infty} p^{-i}[c(i)]^2.\end{aligned} \tag{6.15}$$

From (2.12) and (2.18), we have

$$
\begin{aligned}
\|P_t^2(\cdot,U)\|_1 &= \left(\frac{p-1}{p}\sum_{i=0}^{\infty} p^{-i}\exp\left\{-\frac{p-c}{p-1}c^i t\right\}\right)^2 \\
&\quad + \sum_{m=1}^{\infty} p^{m-1}(p-1)\biggl(p^{-m}\biggl[\frac{p-1}{p}\sum_{i=0}^{\infty} p^{-i}\exp\left\{-\frac{p-c}{p-1}c^{m+i}t\right\} \\
&\qquad\qquad - \exp\left\{-\frac{p-c}{p-1}c^{m-1}t\right\}\biggr]\biggr)^2 \\
&\le \left(\frac{p-1}{p}\right)\sum_{i=0}^{\infty} p^{-i}\exp\left\{-2\frac{p-c}{p-1}c^i t\right\} \\
&\quad + \left(\frac{p-1}{p}\right)^2\sum_{m=0}^{\infty} p^{-m}\sum_{i=0}^{\infty} p^{-i}\exp\left\{-2\frac{p-c}{p-1}c^{m+i}t\right\},
\end{aligned}
$$

where the latter inequality is obtained using (6.15). Noticing that for $0 < \beta \le 1$,

$$(a+b)^\beta \le a^\beta + b^\beta, \qquad a, b \ge 0,$$

we then have

$$
\begin{aligned}
&\int_0^\infty dt \|P_t^2(\cdot,U)\|_1^\beta \\
&\quad \le \int_0^\infty dt\biggl[\bigl(p^{-1}(p-1)\bigr)^\beta \sum_{i=0}^{\infty} p^{-\beta i}\exp\left\{-2\beta\frac{p-c}{p-1}c^i t\right\} \\
&\qquad\qquad + \bigl(p^{-1}(p-1)\bigr)^{2\beta}\sum_{m=0}^{\infty} p^{-\beta m}\sum_{i=0}^{\infty} p^{-\beta i}\exp\left\{-2\beta\frac{p-c}{p-1}c^{m+i}t\right\}\biggr] \\
&\quad \le \frac{\bigl(p^{-1}(p-1)\bigr)^\beta}{2\beta(p-1)^{-1}(p-c)}\sum_{i=0}^{\infty}(cp^\beta)^{-i} \\
&\qquad + \frac{\bigl(p^{-1}(p-1)\bigr)^{2\beta}}{2\beta(p-1)^{-1}(p-c)}\sum_{m=0}^{\infty}(cp^\beta)^{-m}\sum_{i=0}^{\infty}(cp^\beta)^{-i}.
\end{aligned}
$$

Obviously, the series in the above formulas converge iff $cp^\beta > 1$; that is, $\beta > -\log_p c$. This completes the proof. □

Let $I := \int_0^\infty \|P_t^2(\cdot,U)\|_1^\beta \, dt$.

LEMMA 6.4. *Assume $\beta > -\log_p c$. Let v_t be the unique solution of* (4.5) *with $\phi(x) = \chi_U(x)$. Then there exists a positive constant c_0 such that*

$$\|v_t\|_1 \ge c_0 \quad \textit{for any } t \ge 0. \tag{6.16}$$

PROOF. By Lemma 4.3, it is sufficient to prove (6.16) for v_t which is the solution of (4.5) with a smaller initial value $\phi_0(x) = \frac{1}{b}\chi_U(x)$ for some $b > 1$. To this end, from (4.5) and (6.3) we have

$$\begin{aligned}\|v_t\|_1 &= \|\phi\|_1 - \int_0^t \|v_s^{1+\beta}\|_1\, ds \\ &\geq \|\phi\|_1 - \int_0^t \|v_s^2\|_1^{\beta}\|v_s\|_1^{1-\beta}\, ds \quad \text{(by the Hölder inequality)} \\ &\geq \|\phi\|_1 - \int_0^t \|(S_s\phi)^2\|_1^{\beta}\|\phi\|_1^{1-\beta}\, ds.\end{aligned} \tag{6.17}$$

Putting $\phi(x) = \frac{1}{b}\chi_U(x)$ into above formula, we get

$$\begin{aligned}\|v_t\|_1 &\geq \frac{1}{b} - \frac{1}{(b)^{1+\beta}}\int_0^t \|P_s^2(\cdot, U)\|_1^{\beta}\, ds \\ &\geq \frac{1}{b} - \frac{I}{(b)^{1+\beta}} = \frac{1}{b}\left(1 - \frac{I}{b^{\beta}}\right),\end{aligned}$$

which is positive when $b > I^{1/\beta} \vee 1$. For this b, we choose $c_0 = \frac{1}{b}\left(1 - \frac{I}{b^{\beta}}\right) > 0$ and this completes the proof. □

With the help of Lemmas 6.3, 6.4 and 4.3 we actually have proved the following.

THEOREM 6.5. *If $\beta > -\log_p c$, then the (c, β)-superprocess is not locally extinct.*

Combining Theorems 6.2 and 6.5 we have the following criteria.

THEOREM 6.6. *The (ξ, β)-superprocess is locally extinct if and only if $\beta \leq (-\log_p c) \wedge 1$. That is, $(-\log_p c) \wedge 1$ is the critical value of local extinction.*

REMARK. (a) The results of local extinction can be generalized to all measures μ dominated by $C\nu$ for some constant $C > 0$ [see Dawson (1977) and Zhao (1999)].

(b) Pinsky (1996) studied the local extinction for measure-valued L-diffusions on $\mathbb{R}^d$ with $\beta\lambda - \alpha(\lambda)^2$ branching and proved that local extinction (defined in a slightly more restrictive sense than here) occurs iff the principal (i.e., highest) eigenvalue of the generator L (corresponding to the motion process) is not larger than β. This is analogous to Theorem 6.6 in the sense that it compares a parameter of the branching (mass production) to a "transience" parameter releasing the strength of the transience of the motion process.

Since the range of β is the interval $(0, 1]$, we then have the following result.

COROLLARY 6.7. *If $c \leq p^{-1}$, then the corresponding (ξ, β)-superprocess is locally extinct for any $0 < \beta \leq 1$.*

REMARK. We know from Albeverio and Zhao (1999b) that whereas the Hausdorff dimension of $\mathbb{Q}_p$ is 1, the Hausdorff dimension of the image of the stable random walks is $1 \wedge (-\log_p c)$, just the critical point for the (ξ, β)-superprocess to be locally extinct. On the other hand, it is well known that the corresponding critical point for super α-stable processes on $\mathbb{R}_d$ is $(\alpha/d) \wedge 1$ and the Hausdorff dimension of the image of α-stable processes is $\alpha \wedge d$ [see Dawson (1977), Dawson, Iscoe and Perkins (1989) and Perkins (1988)]. Therefore, our result is consistent with the Euclidean case. However, we can not expect the result for general Lévy processes because extinction–nonextinction is a question about the relation between large-scale behavior of the spatial motion, whereas the Hausdorff dimension of the image is a question about the local behavior.

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