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FATOU THEOREM OF *p*-HARMONIC FUNCTIONS ON TREES

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We study bounded p-harmonic functions u defined on a directed tree T with branching order κ ($1 < p < \infty$ and $\kappa = 2, 3, \ldots$). Denote by $BV(u)$ the set of paths on which u has finite variation and $\mathscr{F}(u)$ the set of paths on which u has a finite limit. Then the infimum of $\dim BV(u)$ and the infimum of $\dim \mathscr{F}(u)$ are equal over all bounded p-harmonic functions on T (with p and κ fixed); the infimum $d(\kappa, p)$ is attained and is strictly between 0 and 1 expect when $p = 2$ or $\kappa = 2$.

1. Introduction. In this note, we study the asymptotic behavior of p-harmonic functions on directed trees, in particular, the set of branches where a function has bounded variation or finite limit.

Let $\kappa > 1$ be an integer and T a directed tree with regular κ-branching. That is, T consists of the empty set ϕ and all finite sequences $(b_1, b_2 \cdots b_r)$ of lengths $r = 1, 2, 3, \ldots,$ whose coordinates are chosen from $\{1, 2, \ldots, \kappa\}$. The elements in T are called vertices. Each vertex v has κ successors, obtained by adding another coordinate; these are abbreviated by $(v, 1), \ldots, (v, \kappa)$ and have length one more than the length of v. A branch b of T is an infinite sequence $(b_1, b_2, \ldots, b_r, \ldots)$ with coordinates in $\{1, 2, \ldots, \kappa\}$. Then b can be regarded as an infinite sequence of vertices $(b_1), (b_1, b_2), \ldots, (b_1, b_2 \cdots b_r), \ldots,$ each followed by an immediate successor. Metric concepts can be introduced into the set of branches as follows. The distance between two branches $b = (b_1, b_2, \ldots, b_r, \ldots)$ and $b' = (b'_1, b'_2, \ldots, b'_r, \ldots)$ is κ^{-N+1} where N is the first index n such that $b_n \neq b'_n$. The set of branches then has diameter 1. Hausdorff measure and Hausdorff dimension are defined using this metric.

The *p-Laplacian* ($1 < p < \infty$) of a vector $\langle x_1, \ldots, x_\kappa \rangle$ in $\mathbb{R}^\kappa$ is defined to be

$$\kappa^{-1} \sum x_j |x_j|^{p-2}$$

and a vector is called p-harmonic if the p-Laplacian is zero. When $p = 2$, this is the familiar mean-value property; otherwise p-harmonic need not be 2-harmonic. Let u be a real function on T and $v \in T$; the *gradient* of u at v is defined by its *increments* at v:

$$\big\langle u(v, 1) - u(v), u(v, 2) - u(v), \ldots, u(v, \kappa) - u(v) \big\rangle,$$

and the p-Laplacian of u at v is defined to be the p-Laplacian of the gradient at v. Then u is said to be *p-harmonic* on T if its p-Laplacian is 0 at each vertex.

A real function u on T defines a sequence u_r of functions on branches b: $u_r(b) = u(b_1, b_2 \cdots b_r)$. The *Fatou set* $\mathscr{F}(u)$ is the set of branches b such that

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$\lim u_r(b)$ exists and is finite, and $BV(u)$ is the set of branches b such that $\sum |u_{r+1}(b) - u_r(b)| < +\infty$. Clearly $BV(u) \subseteq \mathscr{F}(u)$.

THEOREM 1. *Let T be a directed tree with regular κ-branching and $H_{\kappa,p}$ be the class of bounded p-harmonic $(1 < p < \infty)$ functions on T. Then there is a number $d(\kappa, p) \in (0, 1]$ (strictly less than 1 when $p \neq 2$ and $\kappa \neq 2$) such that $\dim BV(u) \geq d(\kappa, p)$ for all $u \in H_{\kappa,p}$ and $\dim \mathscr{F}(u) = d(\kappa, p)$ for some $u \in H_{\kappa,p}$; consequently,*

$$\min_{H_{\kappa,p}} \dim \mathscr{F}(u) = \min_{H_{\kappa,p}} \dim BV(u) = d(\kappa, p).$$

The critical dimension $d(\kappa, p)$ takes the form $\log m(\kappa, p)/\log \kappa$, with

$$m(\kappa, p) = \min \left\{ \sum_1^{\kappa} e^{x_j} \colon \sum_1^{\kappa} x_j |x_j|^{p-2} = 0 \right\}.$$

THEOREM 2. *For each fixed $p \in (1, \infty)$, $\lim d(\kappa, p) = 1$ as $k \to +\infty$. However,*

$$\lim(\inf{}_p d(\kappa, p)) = \tfrac{1}{2} \quad \text{as } \kappa \to \infty.$$

Our work is motivated by a recent paper of Cantón, Fernández, Pestana, and Rodríguez [2], and an analogous question on the unit disk. The following is proved in [2].

THEOREM A. *Let T be a directed tree with regular κ-branching. Then for each $p \in (1, \infty)$, there exists a number $\phi(\kappa, p) > 0$ such that $1 \geq \dim BV(u) \geq \phi(\kappa, p)$ for every bounded p-harmonic function u on T; furthermore, there exists a bounded p-harmonic function u so that $BV(u)$ has zero 1-dimensional measure and $\phi(\kappa, 2) = 1$.*

Because the Hausdorff dimension of the entire tree is one and $BV(u) \subseteq \mathscr{F}(u)$, Theorem A is sharp in the case $p = 2$, and gives

$$\min_{H_{\kappa,2}} \dim \mathscr{F}(u) = \min_{H_{\kappa,2}} \dim BV(u) = 1$$

in Theorem 1.

The problem for p-harmonic functions on the unit disk D is much deeper. Rudin [7] proved in 1955 that there exists a bounded harmonic function u on D such that $|BV(u)| = 0$, and Bourgain [1] proved in 1993 that $\dim BV(u) = 1$ for all bounded harmonic functions on D. When $p \neq 2$, Wolff [9, 10] and Lewis [4] gave examples of bounded p-harmonic functions on D for which $|\mathscr{F}(u)| = 0$. Results on lower bound for $\dim \mathscr{F}(u)$ are given in [5], [3] and [6]. The precise values of $\inf \dim \mathscr{F}(u)$ and $\inf \dim BV(u)$ over all bounded p-harmonic functions on the unit disk remain unknown.

Our strategy for proving Theorem 1 is as follows. To obtain a lower bound on $\dim \mathscr{F}(u)$, u being fixed, we find a probability measure λ on the set of branches so that $(u_1, u_2, u_3 \cdots)$ is a martingale with respect to λ and that λ has the most symmetry. The symmetry is measured by the notion of entropy and estimated by Lagrange multipliers. The multiplicative property of entropy is then used to calculate the Hausdorff dimension of an exceptional set of combinatoric nature. A small variation yields the same lower bound for $\dim BV(u)$. The bounded p-harmonic function which realizes the infimum of $\dim \mathscr{F}(u)$ is a stochastic process modelled on [2], incorporating the device of moving barriers and the idea of stopping times to show that large oscillations take place except on a small set. To make the Hausdorff dimension as small as possible, we minimize the partition function $\sum_1^\kappa e^{x_j}$ over all p-harmonic vectors $\langle x_1, x_2, \dots, x_\kappa \rangle$, a concept adapted from [8].

The link between the upper bound and lower bound, accounting for the equality between the two extrema, is furnished by the identity (to be explained later)

$$\sup\left\{ -\sum_1^\kappa \lambda_j \log \lambda_j \colon \lambda_j \geq 0,\ \sum \lambda_j = 1,\ \sum \lambda_j x_j = 0 \right\}$$
$$= \inf\left\{ \log \sum_1^\kappa e^{t x_j} \colon t \in \mathbb{R} \right\}.$$

We are grateful to Cantón, Fernández, Pestana and Rodríguez for an advance copy of their paper; many ideas in their work are freely adapted here.

2. Examples. Theorem 1 follows from Theorem A when $p = 2$ or $\kappa = 2$, because all p-harmonic functions are 2-harmonic when $\kappa = 2$.

Suppose that $\kappa > 2$, $p \in (1,2) \cup (2, \infty)$ and T is a directed tree with regular κ-branching.

We consider a probability measure P on the set of branches through the mapping $g(b) = \sum_1^\infty \kappa^{-r}(b_r - 1)$ onto [0,1]. The P-measure of a set E is the Lebesgue measure of $g(E)$; this is the infinite product of uniform distributions on each factor $\{1, 2, 3, \dots, \kappa\}$ and is the 1-dimensional Hausdorff measure of E under the metric mentioned earlier.

Let $\langle x_1, x_2, \dots, x_\kappa \rangle$ be a p-harmonic vector with negative sum, that is, $\sum x_j |x_j|^{p-2} = 0$ and $\sum x_j < 0$. It defines a random variable in the usual way with $P(X = x_j) = \kappa^{-1}$ for $1 \leq j \leq \kappa$. Let $\phi(t) = E(e^{tX})$ and note that $\phi(0) = 1$ and $\phi'(0) = E(X) < 0$. Since $\phi(t)$ is convex and X has values of both signs, $\phi(t)$ attains a minimum $\beta(X) < 1$ at some $t(X)$.

Let S_r be the sum $X_1 + X_2 + \cdots + X_r$ of independent identically distributed random variables with the same law as X. Then for any c,

$$P(S_r > c) = P(\exp(S_r t(X)) > \exp(ct(X)))$$
$$\leq \exp(-ct(X)) E(\exp(S_r t(X))) \leq \exp(-ct(X)) \beta(X)^r.$$

PROPOSITION. *For each random variable X chosen above, there is a bounded p-harmonic function u on T such that*

$$\dim \mathscr{F}(u) \le 1 + \log \beta(X)/\log \kappa < 1.$$

We arrive at the example in Theorem 1 by a special choice of X in the proposition.

Let $m(\kappa, p)$ be the minimum of $\sum_1^\kappa e^{x_j}$ over all p-harmonic vectors and observe that the minimum is attained because $x_j < \log \kappa$ as soon as the sum is less than κ and that only vectors with negative sum are of interest. Fix such an extremal $\langle a_1, a_2, \dots, a_\kappa \rangle$, that is, $\sum a_j |a_j|^{p-2} = 0$, $\sum a_j < 0$ and

$$\sum e^{a_j} = m(\kappa, p).$$

Let X be the random variable derived from $\langle a_1, \dots, a_\kappa \rangle$. Then $P(X = a_j) = \kappa^{-1}$ and $E(e^{tX}) = \kappa^{-1} \sum e^{t a_j}$ attains its minimum $\kappa^{-1} m(\kappa, p)$ at $t = 1$; furthermore,

$$\sum a_j e^{a_j} = 0.$$

In view of the proposition, there is a bounded p-harmonic function u on T such that

$$\dim \mathscr{F}(u) \le \log m(\kappa, p)/\log \kappa.$$

This gives the example in Theorem 1.

PROOF. We follow the stopping time argument in [2] with some modifications designed to yield an exceptional set of dimension less than 1 rather than P-measure zero.

We define sequences $(n_k)_1^\infty$, $(\alpha_k)_1^\infty$ and $(h_k)_1^\infty$ by the formulas $n_1 = 0$, $\alpha_1 = \frac{1}{4}$, $h_1 = 1$ and $n_k = (4k)!$, $\alpha_k = (k^3 n_k)^{-1}$ and $h_k = h_{k-1} + (k-1)^{-2}$ for $k \ge 2$. We choose $a > 0$ so small that $|aX| < 1$ and write $Z = aX$.

The definition of the p-harmonic function u on T begins at time $n_1 = 0$ with $u(\phi) = 0$ and continues in succession for time $r = 0, 1, 2, \dots$ with the following rules. (Time at a vertex v is its length.) At time $r < n_2$ and at a vertex v, if $|u(v)| < h_1$, u has increments whose distribution is determined by variable $\alpha_1 Z$ so that u is p-harmonic at v; if $|u(v)| \ge h_1$, u stops and takes the value $u(v)$ at all successors $(v, 1), (v, 2) \cdots (v, \kappa)$ of v. This defines u up to and including time $r = n_2$ and $|u| < h_1 + \alpha_1 = \frac{5}{4} < h_2$.

At time $t = n_2$, if $u(v) \ge 0$, then u evolves with increments whose distribution is $\alpha_2 Z$ on branches containing v; if $u(v) < 0$, then u evolves according to $-\alpha_2 Z$ on branches containing v. As before u stops if $|u(v)| \ge h_2$ at some time $r < n_3$. Continue in the obvious fashion for $n_k \le r < n_{k+1}$, $k = 3, 4, \dots$. Since $h_k + \alpha_k < h_{k+1}$ and $|Z| < 1$, we see that at time n_{k+1}, $|u| < h_{k+1}$. Therefore u is bounded on the tree T.

Next we prove that $\dim \mathscr{F}(u) \le 1 + \log \beta(\kappa, p)/\log \kappa$. Before embarking on this, we explain the reason for rapidly increasing numbers n_k and rapidly decreasing α_k. Along certain branches u increases at every vertex where a choice is allowed; these branches belong to $BV(u)$. Clearly these branches have P-measure 0, but more work is needed to estimate the dimension of this set and other sets as well. Denote by M_k the first time after n_k at which $|u| \geq h_k$ is possible. Clearly $M_k \geq n_k + \alpha_k^{-1}(h_k - h_{k-1} - \alpha_{k-1}) \geq n_k + k^3 n_k((k-1)^{-2} - k^{-3}) \geq (k+1)n_k$ for $k \geq 2$, and thus $M_k + n_k \cong M_k$ for large k. This is necessary in estimating Hausdorff dimension.

We shall prove that except on a set of branches of dimension at most $1 + \log\beta(\kappa, p)/\log\kappa$, u eventually exits every barrier $|u| = h_k$ in time interval $[n_k + 1, n_{k+1}]$ with alternating signs; thus on these branches, $\limsup u > 1$ and $\liminf u < -1$ as $r \to \infty$. Denote by σ_k the sign appearing in the law chosen at time n_k, that is, $+1$ for $\alpha_k Z$ and -1 for $-\alpha_k Z$. Now $E(Z) < 0$, and thus the process drifts downward when $\sigma_k = +1$, and upward when $\sigma_k = -1$. Let B_k be the set of branches on which $\sigma_k u_{n_{k+1}} > -h_k$. [Recall that $u_r(b) = u(b_1, b_2, \ldots, b_r)$.] Then

$$\mathscr{F}(u) \subseteq \limsup B_k.$$

On the set B_k, one of several large deviations occurs; we estimate the probability of each large deviation and then are in a position to estimate the dimension of $\mathscr{F}(u)$.

In $[n_k, n_{k+1}]$, let τ be the first time $|u| \geq h_k$ if it happens and $\tau = n_{k+1}$ otherwise; let w be the process beginning with the same law as u at time n_k and continuing to n_{k+1} with no stopping. Thus $u_r = w_{\tau\wedge r}$.

On $b \in B_k$, if $\tau = n_{k+1}$ then $|w_{n_{k+1}}| < 3$, and if $\tau < n_{k+1}$ then $\sigma_k(w_\tau - w_{n_k}) > 0$; we have seen that $\tau \geq kn_k$ when $k \geq 2$.

We examine first the event $|w_{n_{k+1}}| < 3$. Now $w_{n_{k+1}} - w_{n_k}$ has the distribution of a sum $\sigma_k\alpha_k a(X_1 + \cdots + X_J)$ where $J = n_{k+1} - n_k$ and X'_js are i.i.d. with the same law as X. Since $|w_{n_{k+1}}| < 3$, we have $|S_J| < 6a^{-1}\alpha_k^{-1}$ with $S_J = X_1 + X_2 + \cdots + X_J$. Suppose that $\sigma_k = 1$. (The case $\sigma_k = -1$ follows by the same method with changing of signs.) Since $E(X) < 0$ and $c_k \equiv 6a^{-1}\alpha_k^{-1} = o(J)$, it follows from the comments before the proposition that the event $|w_{n_{k+1}}| < 3$ has probability at most

$$P(|S_J| < c_k) \leq P(S_J > -c_k) \leq \beta(X)^J \exp(c_k t(X)) = \beta(X)^{J(1+o(1))}.$$

Next we consider those branches $b \in B_k$ on which $\sigma_k u \geq h_k$ at some $\tau < n_{k+1}$. At the first time τ, we have $\sigma_k(w_\tau - w_{n_k}) > 0$ and $\tau \geq M_k \geq (k+1)n_k$. For each $r \in [(k+1)n_k, n_{k+1})$, the event $\sigma_k(w_r - w_{n_k}) > 0$ has probability at most $\beta(X)^{r-n_k}$ by an argument similar to that in the last paragraph.

Let $\mathscr{S}_r$ be the class of subsets of branches defined by the first r coordinates. This is a finite field whose atoms are sometimes called cylinders of rank r. Now $\mathscr{S}_r$ contains κ^r cylinders of P-measure κ^{-r} each and the same diameter. Suppose that $\{A_r\}$ is a sequence of sets with $A_r \in \mathscr{S}_r$ and $P(A_r) \leq \rho^{r+o(r)}$ for some ρ in $(0, 1)$. Then A_r is a union of atoms of $\mathscr{S}_r$, their number is at most $\kappa^r\rho^{r+o(r)}$ and $\dim(\limsup A_r) \leq 1 + \log\rho/\log\kappa$.

Therefore B_k is contained in $\bigcup_{r=(k+1)n_k}^{n_{k+1}} A_r$, where $A_{n_{k+1}}$ consists of at most $\beta(X)^{(n_{k+1}-n_k)(1+o(1))}\kappa^{n_{k+1}}$ atoms in $\mathscr{S}_{n_{k+1}}$, and A_r consists of at most $\beta(X)^{r-n_k}\kappa^r$

atoms in $\mathscr{S}_r$ when $(k+1)n_k \le r < n_{k+1}$. From the discussion above, we conclude that

$$\dim \mathscr{F}(u) \le 1 + \log \beta(X)/\log \kappa.$$

By a small variation of the method, we can prove a stronger result: except on a set of dimension $1 + \log \beta(X)/\log \kappa$, $\limsup |u(b_{r+1}) - u(b_r)| > 0$. To obtain a function u with this property we introduce a p-harmonic vector $X_0 = \langle a_1, \ldots, a_\kappa \rangle$, such that $0 < |a_j| < 1/16$. We adjust the formula for u as follows, when $|\alpha_k| < 1/2$. At the first time σ in the interval $[n_k, n_{k+1})$ at which $|u| \le 1/8$, if this occurs, $u_{\sigma+1} - u_\sigma$ has the distribution of X_0; at all other times, the formula is as before. If this new rule is invoked at a time σ, then $|u_{\sigma+1}| < 5/8$, so the analysis proceeds substantially as before. □

3. Entropies and dimensions. We recall that the *entropy* of a probability measure λ on $\{1, 2, \ldots, \kappa\}$ is

$$H(\lambda) = -\sum \lambda_j \log \lambda_j,$$

where $\lambda_j = \lambda(\{j\})$. See [8] for basic properties.

LEMMA 1. *Let $\langle x_1 \cdots x_\kappa \rangle$ be a p-harmonic vector. Then there is a probability measure μ on $\{1, \ldots, \kappa\}$ such that*

(i) $\sum \mu_j x_j = 0$;
(ii) $H(\mu) \ge \log m(\kappa, p)$;
(iii) $\min \mu_j \ge c(\kappa, p) > 0$.

PROOF. The lemma is obvious if $\sum x_j = 0$; in this case we choose all $\mu_j = \kappa^{-1}$. We can suppose that $\sum x_j < 0$, and recall that $\phi(t) = \sum e^{tx_j}$ attains a minimum at some value $\tau > 0$. Then $\phi(\tau) \ge m(\kappa, p)$ and $\phi'(\tau) = \sum x_j \exp(\tau x_j) = 0$. Define $\mu_j = \exp(\tau x_j)/\phi(\tau)$ and observe that $\sum \mu_j = 1$, $\sum \mu_j x_j = 0$ and $H(\mu) = \log \phi(\tau) \ge \log m(\kappa, p)$. To prove (iii), we use first the trivial facts $\phi(\tau) \le \phi(0) = \kappa$ and $\tau x_j \le \log \kappa$ for all j. Since $\langle x_1, \ldots, x_\kappa \rangle$ is p-harmonic, $|\tau x_j| \le (\kappa - 1)^{1/(p-1)} \log \kappa$. Therefore $\mu_j \ge c(\kappa, p) > 0$.

Our choice of μ ensures

$$H(\mu) = \inf_t \log \sum \exp(t x_j);$$

and $H(\mu)$ maximizes $H(\lambda)$ subject to the constraint $\sum \lambda_j x_j = 0$, a fact which can be seen by Lagrange multipliers; this proves the identity in the Introduction. □

LEMMA 2. *Let $\langle x_1, \ldots, x_\kappa \rangle$ be a p-harmonic vector and $\eta > 0$. Then there is a probability measure ν on $\{1, 2, \ldots, \kappa\}$ such that*:

(i) $\sum \nu_j x_j \ge 0$*;*
(ii) $H(\nu) \ge \log m(\kappa, p) - \eta$*;*
(iii) $\min \nu_j \ge \frac{1}{2} c(\kappa, p)$*;*
(iv) $\sum \nu_j |x_j| \le c'(\kappa, p, \eta) \sum \nu_j x_j$.

PROOF. By arguments similar to those used in Lemma 1, we see that $\max |x_j| \le (\kappa - 1)^{1/(p-1)} \max x_j$. For simplicity we can assume $x_1 = \max x_j$. We set $\nu = s\mu + (1-s)\delta_1$, where $0 < s < 1$, μ is the measure in Lemma 1 and δ_1 is the unit mass at 1. Since $x_1 \ge 0$, we have

$$\sum \nu_j x_j = (1-s)x_1 \ge (1-s)(\kappa-1)^{-1/(p-1)} \sum \nu_j |x_j|.$$

By the continuity of H, there exists $s \in (\frac{1}{2}, 1)$ depending only on κ and η and not on μ such that $H(\nu) \ge H(\mu) - \eta \ge \log m(\kappa, p) - \eta$. Using this value of s, we obtain (iv) with $c' = (\kappa-1)^{1/(p-1)}(1-s)^{-1}$, and (iii) because $s > \frac{1}{2}$. □

Let b be a branch and $r \ge 1$. Denote by $C_r(b)$ the cylinder of rank r consisting of branches whose first r coordinates are $(b_1, b_2, \dots, b_r)$, by C_r any of the κ^r cylinders of rank r and by $C_0 = B$ the collection of all branches.

For completeness, we record two lemmas on a dimension associated with a measure, which have been discovered and used by many authors in various forms, as early as Besicovitch and Eggleston.

Let λ be a probability measure on B, and suppose for simplicity that all cylinders have positive measure. Let

$$L(b) = \liminf -\log \lambda(C_r(b))/r \log \kappa.$$

LEMMA 3. *If $L \ge \alpha > 0$ on a set $S \subseteq B$ and S has dimension less than α then it has zero λ-measure. Conversely, the infimum of* $\dim S$ *over all sets of positive λ-measure is the essential infimum of L.*

PROOF. Suppose that $\dim S < \alpha' < \alpha$ and S_i is the set of branches b in S with $\lambda_r(C_r(b)) \le \kappa^{-r\alpha'}$ for all $r \ge i$. Since $L \ge \alpha$ on S, $S = \cup S_i$. The λ-measure of a cylinder C_r meeting S_i is at most $\kappa^{-r\alpha'}$ and the diameter of C_r is κ^{-r} whenever $r \ge i$. Since $\dim S_i < \alpha'$ we have $\lambda(S_i) = 0$; therefore $\lambda(S) = 0$. For the converse, note that the set on which $L \le \beta$ has Hausdorff dimension at most β by a Vitali-type argument. □

Recall that the *conditional entropy* on a cylinder C_r is

$$H(\lambda \mid C_r) = -\sum_{C_{r+1} \subseteq C_r} \frac{\lambda(C_{r+1})}{\lambda(C_r)} \log \frac{\lambda(C_{r+1})}{\lambda(C_r)}.$$

LEMMA 4. *Let λ be a probability measure on B satisfying $\lambda(C_{r+1}) > \delta\lambda(C_r)$ for some $\delta > 0$, whenever $C_{r+1} \subseteq C_r$, and*

$$h = \inf H(\lambda \mid C_r) \textit{ over all cylinders.}$$

Then λ is zero on any subset of dimension less than $h/\log \kappa$.

PROOF. For each $r \ge 0$, define on B a function

$$f_r(b) = -\log \frac{\lambda(C_{r+1}(b))}{\lambda(C_r(b))} - H(\lambda \mid C_r(b)).$$

Note that f_r is constant on each C_{r+1} and has mean value 0 with respect to λ on each C_r. The functions $f_0, f_1, f_2, \ldots$ therefore are orthogonal in $L^2(\lambda)$. Since $\lambda(C_{r+1}) > \delta\lambda(C_r)$ whenever $C_{r+1} \subseteq C_r$, the sequence $(f_r)_0^\infty$ is uniformly bounded and therefore has partial sums $f_1 + f_2 + \cdots + f_r = o(r)$ λ-a.e. Consequently, $-\log \lambda(C_r(b)) \geq rh + o(r)$ λ-a.e. The conclusion follows from Lemma 3. □

4. Lower estimates in Theorem 1. Let u be a bounded p-harmonic function on T.

By applying Lemma 1 to the gradient of u at each vertex, we may find a probability measure μ on B such that u is a martingale with respect to μ, and hence converges μ-almost surely. The size of the support of μ then gives a lower bound for $\dim \mathscr{F}(u)$.

More care is needed in finding a lower bound for $\dim BV(u)$; in fact, u is a submartingale with respect to the measure λ below. Recall that the variation of u on a branch b is

$$\operatorname{var}(u, b) = \sum_1^\infty |u_r(b) - u_{r-1}(b)|,$$

where $u_r(b) = u(b_1, b_2, \ldots, b_r)$ and $u_0(b) = u(\phi)$.

For each $\eta > 0$, we shall define a probability measure λ on B such that $\lambda(C_{r+1}) \geq \delta\lambda(C_r)$ for some $\delta > 0$, whenever $C_{r+1} \subseteq C_r$,

$$\int_B \operatorname{var}(u, b)\, d\lambda < +\infty$$

and the conditional entropy

$$H(\lambda|C_r) > \log m(\kappa, p) - \eta$$

for all cylinders. The set $BV(u) = \{b\colon \operatorname{var}(u, b) < +\infty\}$ then has λ-measure one and its Hausdorff dimension is at least $(\log m(\kappa, p) - \eta)/\log \kappa$ for any $\eta > 0$; hence at least $\log m(\kappa, p)/\log \kappa$. This completes the lower estimates in Theorem 1.

It remains to construct λ; we follow [2] in all but minor details and define λ inductively. Of course $\lambda(C_0) = 1$. Assume that λ has been defined on all cylinders of rank less than or equal to r. Let C_r be a cylinder of rank r represented by $(b_1, b_2, \ldots, b_r)$. The *gradient* $\langle x_1, x_2, \ldots, x_\kappa\rangle$ of u at the vertex $v = (b_1, b_2, \ldots, b_r)$ forms a p-harmonic vector. Let ν be the probability measure on $\{1, 2, \ldots, \kappa\}$ associated with the present $\langle x_1, x_2, \ldots, x_\kappa\rangle$ defined in Lemma 2, and define λ on the κ cylinders C_{r+1} of rank $r+1$ contained in C_r by

$$\lambda(C_{r+1}) = \nu_j \lambda(C_r)$$

if C_{r+1} is represented by $(b_1, b_2, \ldots, b_r, j)$. This way we define λ on all cylinders of rank $r+1$ and then on B, by σ-additivity.

Properties of ν in Lemma 2 yield that there exists $\delta = \delta(\kappa, p) > 0$, $\lambda(C_{r+1}) > \delta\lambda(C_r)$ whenever $C_{r+1} \subseteq C_r$,

$$H(\lambda \mid C_r) > \log m(\kappa, p) - \eta$$

and

$$\int_{C_r} |u_{r+1} - u_r| \, d\lambda \le C'(\kappa, p, \eta) \int_{C_r} u_{r+1} - u_r \, d\lambda.$$

The fact $\int_B \operatorname{var}(u, b) \, d\lambda < +\infty$ follows by first summing over all C_r of rank r, and then over r. This completes the proof of Theorem 1. □

5. The critical dimension and Theorem 2. Some information about $m(\kappa, p)$ can be obtained by Lagrange multipliers. At an extremum $\langle a_1, \dots, a_\kappa \rangle$, we have

$$e^{a_j} = \Lambda |a_j|^{p-2}, \qquad j = 1, 2, \dots, \kappa.$$

When $2 < p < \infty$, for a fixed $\Lambda > 0$, the equation $e^a = \Lambda |a|^{p-2}$ has only one negative solution, but has one or two positive solutions depending on whether $\Lambda = (e/(p-2))^{p-2}$ or not. In the case of two positive solutions, they are separated by $p-2$. When $1 < p < 2$, the roles of positive and negative are reversed. It appears to be difficult to determine whether $\{a_1, \dots, a_\kappa\}$ assumes two or three distinct values at the minimum $\sum e^{a_j}$, not to mention the exact values.

To obtain upper bounds for $d(\kappa, p)$, we consider a specific p-harmonic vector $\langle y, -1, -1, \dots, -1 \rangle$ for $p > 2$ or $\langle -y, 1, 1, \dots, 1 \rangle$ for $1 < p < 2$ with $0 < y = (\kappa - 1)^{1/(p-1)}$. The corresponding random variable Y has $E(Y) < 0$. When $p > 2$,

$$\beta(Y) = \min_t \kappa^{-1} e^{ty} + (1 - \kappa^{-1}) \, e^{-t}$$

is attained at some t satisfying $ye^{ty} = (\kappa - 1) \, e^{-t}$. Calculation shows that for large κ, (i) $1 + \log \beta(Y) / \log \kappa \le 1 - \delta(c)$ when $p > c \log \kappa$, and (ii) $1 + \log \beta(Y) / \log \kappa \le 1/2 + (\log 2 / \log \kappa)(1 + \varepsilon)$ when $p(\log \kappa)^{-2}$ is sufficiently large. Since $m(\kappa, p) \le \kappa \beta(Y)$, these estimates give upper bounds for $d(\kappa, p)$.

(A) *If $p > c \log \kappa$ for some $c > 0$, we have $d(\kappa, p) \le 1 - \delta(c)$ for large κ. Conversely, in order that $d(\kappa, p) \le 1 - \delta$ with some $\delta > 0$ and κ large, p must be at least $c(\delta) \log \kappa$.*

To see the second statement, we suppose that $m(\kappa, p) \le \kappa^{1-\delta} < \kappa/4$; therefore $p > 2$ (since $m(\kappa, p) \ge \kappa/4$ when $p < 2$ as we shall see in (B). In the extremal p-harmonic vector $\langle a_1, a_2, \dots, a_\kappa \rangle$, at least half of the coordinates are negative and have a common value a satisfying $e^a < 2\kappa^{-\delta} < \frac{1}{2}$ or $a < \log 2 - \delta \log \kappa$; all positive coordinates a' satisfy $0 < a' < \log \kappa$ so that $a' < 2|a|/\delta$. The Lagrange equation for extremals yields $e^a |a|^{2-p} = e^{a'} (a')^{2-p}$. Since $p > 2$, we find that $(p-2) \log(2\delta^{-1}) > -a > \delta \log \kappa - \log 2$. We conclude that $p > c(\delta) \log \kappa$.

(B) *For* $p \in (1, q]$ *we have* $m(\kappa, p) > c(q)\kappa$ *for some* $c(q) \in (0, 1)$*, thus*

$$d(\kappa, p) \geq 1 + \log c(q)/\log \kappa$$

and $c(2) = \frac{1}{4}$.

Suppose that $m(\kappa, p) < \kappa/4$, so the number of positive coordinates is at most $\kappa/4$. At the extremal $\langle a_1, \ldots, a_\kappa \rangle$, there are at least $\kappa/2$ coordinates less than $-\log 2$. Since $\sum a_j |a_j|^{p-2} = 0$, the sum $\sum' a_j^{p-1}$ over the positive coordinates is at least $\frac{\kappa}{2}(\log 2)^{p-1}$. The positive coordinates assume at most two values. One of them, call it x, occurs r times and contributes at least half of the sum $\sum' a_j^{p-1}$. Hence $rx^{p-1} > (\kappa/4)(\log 2)^{p-1}$ or $x > (\log 2)(\kappa/4r)^{1/(q-1)}$. Using calculus, we obtain a positive lower bound $c(q)$ for $(r/\kappa)e^x$. Thus $m(\kappa, p) > re^x > c(q)\kappa$. Calculation also shows that we may choose $c(2) = \frac{1}{4}$; we recall that the positive coordinates are equal when $1 < p < 2$.

(C) *As* $\kappa \to \infty$,

$$\frac{1}{2} + \frac{\log 2}{\log \kappa}(1 - o(1)) \leq \inf_p d(\kappa, p) \leq \frac{1}{2} + \frac{\log 2}{\log \kappa}(1 + o(1)).$$

The upper bound has been proved earlier using Y. The lower bound follows from (B) and the paragraph below.

We claim that $m(\kappa, p) \geq 2(\kappa - 1)^{1/2}$ when $p > 2$ and $\kappa > 15$. Otherwise $m(\kappa, p) < 2(\kappa - 1)^{1/2} < \kappa/2$. Then the number of positive coordinates in $\langle a_1, \ldots, a_\kappa \rangle$ is less than $\kappa/2$; call this number r. Consequently, the maximum value A of a_j's exceeds $-a$, a being the common value for all negative coordinates. Then $m(\kappa, p) = \sum e^{a_j} \geq e^a(\kappa - r) + e^A + (r - 1) \geq r - 1 + 2(\kappa - r)^{1/2} \geq 2(\kappa - 1)^{1/2}$. Therefore, $m(\kappa, p) < 2(\kappa - 1)^{1/2}$ does not happen and $d(\kappa, p) \geq (\log \kappa)^{-1}(\log 2 + \frac{1}{2}\log(\kappa - 1))$.

Theorem 2 follows from (B) and (C). □

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