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CONFORMAL INVARIANCE OF DOMINO TILING[1]

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Let U be a multiply connected region in $\mathbf{R}^2$ with smooth boundary. Let P_ϵ be a polyomino in $\epsilon\mathbf{Z}^2$ approximating U as $\epsilon \to 0$. We show that, for certain boundary conditions on P_ϵ, the height distribution on a random domino tiling (dimer covering) of P_ϵ is conformally invariant in the limit as ϵ tends to 0, in the sense that the distribution of heights of boundary components (or rather, the difference of the heights from their mean values) only depends on the conformal type of U. The mean height is not strictly conformally invariant but transforms analytically under conformal mappings in a simple way. The mean height and all the moments are explicitly evaluated.

1. Introduction. Conformal invariance of a lattice-based statistical mechanical system is a symmetry property of the system at large scales. It says that, in the limit as the lattice spacing ϵ tends to 0, macroscopic quantities associated with the system transform covariantly under conformal maps of the domain.

Conformal invariance for statistical mechanical lattice models is a physical principle which until now has not been proved except in certain models which were tailored to be conformally invariant [6] (recently in [2] Benjamini and Schramm prove conformal invariance in a discrete, but nonlattice, percolation model). Nonetheless, conformal invariance is an extremely powerful principle: in the plane, conformally invariant models are classified, in a sense, by representations of the Virasoro algebra [1]. Physicists have used this theory fruitfully to compute exact "critical exponents" and other physical quantities associated to critical lattice models [6]. For example, the cycle in Figure 1 is believed to have Hausdorff dimension $\frac{3}{2}$ in the limit (see, e.g., [15]) and the path in Figure 8 is believed to have dimension $\frac{5}{4}$ [11]. Although many well-known models are believed to be conformally invariant at their critical point, no rigorous techniques were known to prove conformal invariance in these models.

In this paper we deal with the two-dimensional lattice dimer model, or domino tiling model (a domino tiling is a tiling with 2×1 and 1×2 rectangles). We prove that in the limit as the lattice spacing ϵ tends to zero, certain macroscopic properties of the tiling are conformally invariant.

The *height function h* on a domino tiling is an integer-valued function on the vertices in a tiling. It is defined below in Section 2.2; see also [4, 19]. One

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can think of a domino tiling of U as a map h from U to $\mathbb{Z}$, where for each unit lattice square, the images of the four vertices under h are four consecutive integers v, $v+1$, $v+2$, $v+3$. Furthermore, each boundary edge of U must have image of length 1 and not 3. The map h defines and is defined by the tiling: the edges crossed by a domino are those whose image under h has length 3. Our main result is the conformal invariance of h for a random tiling:

THEOREM 1. *Let U be a bounded, multiply connected domain in $\mathbb{C} = \mathbb{R}^2$ with $k+1$ smooth boundary components, each with a marked point $d_0, d_1, \dots, d_k$. Let $\{P_\epsilon\}_{\epsilon>0}$ be a sequence of polyominos, with $P_\epsilon \in \epsilon\mathbb{Z}^2$, approximating U as described in Section* 5.3. *Let $d_j^{(\epsilon)}$ be a vertex of P_ϵ within $O(\epsilon)$ of d_j. Let μ_ϵ be the uniform measure on domino tilings of P_ϵ. Then the joint distribution of the height variations of the points $d_j^{(\epsilon)}$ (that is, the difference of the heights from their mean value) tends to a finite limit which is conformally invariant.*

By conformal invariance we mean that if $f: U \to U'$ is a conformal isomorphism then the distribution of the height variations of $f(d_j)$ is the same as the distribution of the height variations of the d_j themselves.

The mean height of a point of P_ϵ is not strictly conformally invariant in the limit: there is an extra term coming from the heights on the boundary (Theorem 23). We prove there that the limiting mean height is a harmonic function on U whose boundary values depend on the tangent direction of the boundary.

The picture of the height function is completed by understanding the distribution of heights at interior points of U. For an interior point x of P_ϵ, Theorem 2 below and [13] show that the height $h(x)$ tends to a Gaussian with variance $c\log(1/\epsilon)$ for a constant c (which can be shown to be $8/\pi^2$ by a computation similar to that in [13]). See below. This variance diverges as $\epsilon \to 0$. On the other hand, the proof of Theorem 1 shows that the moments

$$\mathbb{E}\left(\left(h(x_1) - \overline{h(x_1)}\right)\left(h(x_2) - \overline{h(x_2)}\right)\cdots\left(h(x_m) - \overline{h(x_m)}\right)\right)$$

for distinct x_i tend to a finite and conformally invariant limit.

Theorem 1 can be extended to regions U with piecewise smooth boundary, on condition that at each corner the boundary tangents have one-sided limits. See below.

Figure 1 illustrates one consequence of Theorem 1. In that figure we took two random domino tilings of an annular region (a square with a square hole). A domino tiling corresponds to a dimer covering, or perfect matching, of the underlying graph (a perfect matching is a collection of edges covering each vertex exactly once). Two perfect matchings form a union of closed cycles and doubled edges in the graph. One can ask about the distribution of the number of cycles separating the inner and outer boundaries of the annulus (there is just one such cycle in the figure). The argument of [13] shows that the

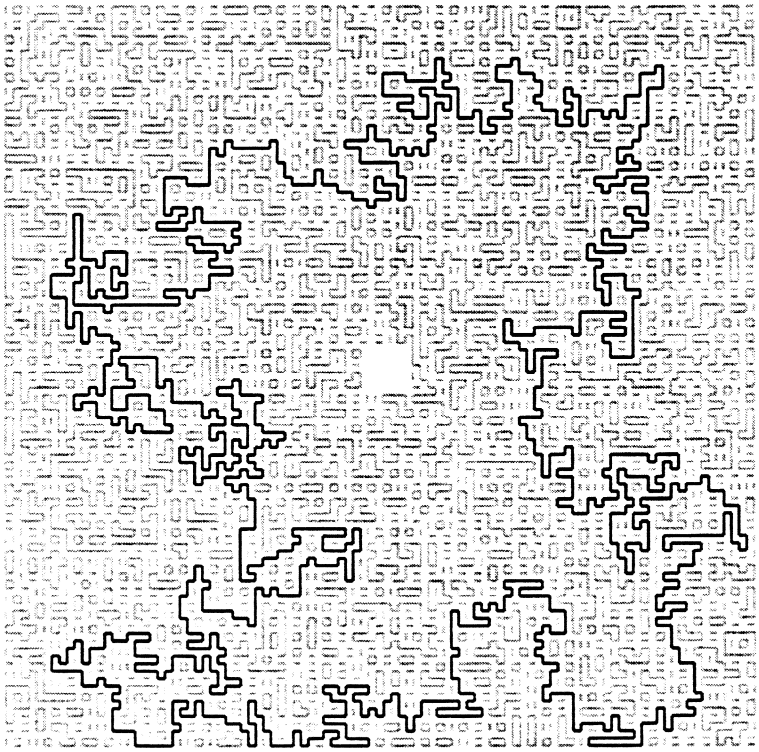

FIG. 1. *A cycle in a union of two random domino tilings of an annulus.*

distribution of the height difference between two boundary components for a single domino tiling is directly related to the distribution of the number of cycles separating those two components in a union of two tilings. Indeed, the expected number of cycles is $1/16$ times the variance of the height difference. Theorem 1 therefore implies that the distribution of the number of cycles separating the boundary components from each other is conformally invariant.

Another interpretation of the height function uses the connection between domino tilings and spanning trees on $\mathbb{Z}^2$ [5]. In Section 7 we relate the height function to the "winding number" of arcs in the corresponding spanning tree.

Theorem 1 follows from a more fundamental result. The *coupling function* on P_ϵ is a function C: $P_\epsilon \times P_\epsilon \to \mathbb{C}$ which determines the measure μ_ϵ (the uniform measure on the set of all tilings of P_ϵ) in the sense that subdeterminants of the coupling function matrix give probabilities of finite configurations of dominos occurring in a tiling [13]. The coupling function is closely related to the Green's function. The following is a loose statement of the result.

THEOREM 2. *Let U and $\{P_\epsilon\}_{\epsilon>0}$ be defined as in Theorem* 1. *Let $v \neq w$ be points in the interior of U and $v^{(\epsilon)}$, $w^{(\epsilon)}$ be vertices of P_ϵ within $O(\epsilon)$ of v, w,* *respectively. The coupling function C for domino tilings of P_ϵ satisfies*

$$C(v^{(\epsilon)}, w^{(\epsilon)}) = \epsilon F_j(v, w) + o(\epsilon),$$

where $j = 0$ or 1 depending on a parity condition, where F_0 and F_1 are analytic in the second variable and depend only on the conformal type of U.

For a precise statement see Theorem 13. This result has an immediate corollary regarding densities of local configurations.

COROLLARY 3. *In a random tiling of P_ϵ, the expected density of occurrences of a local configuration E of dominos at a point v in the interior of U is of the form $c(E) + \epsilon W_E(v) + o(\epsilon)$, where $c(E)$ equals the density of E in a random tiling of the whole plane $\epsilon\mathbb{Z}^2$, and W_E is a function depending only on the conformal type of U.*

The proofs of the above results are given for polyominos with somewhat special boundary conditions. We discuss in Section 8 alternate boundary conditions for which it may be possible, using similar methods, to prove similar results. We remark that certain restrictions on the boundary are definitely necessary, however: in [7] Cohn, Kenyon and Propp compute the mean height when the height function on the boundary is of order $1/\epsilon$. In this case the mean height satisfies a much more complicated nonlinear elliptic PDE and does not appear to have any simple conformal invariance properties.

The paper is organized as follows. In Section 2 we define the polyominos, graphs and notations we will be using. We also define the height function. In Section 3 we define discrete analytic functions and show that the coupling function is one. In Section 4 we discuss boundary values of the coupling function. In Section 5 we prove Theorem 2. In Section 6 we prove Theorem 1 using Theorem 2. In Section 6.2 we compute explicitly the average height function on a region. In Section 7 we discuss the connection with spanning trees, and in Section 8 we discuss other boundary conditions and give some concluding remarks.

2. Definitions.

2.1. *Polyominos and their dual graphs.* Let T be the checkerboard tiling of $\mathbb{R}^2$ with unit squares, each square centered at a lattice point of $\mathbb{Z}^2$ and where the square centered at the origin is white. Let W_0 be the set of white squares both of whose coordinates (the coordinates of the center of the square) are even; let W_1 be the set of white squares both of whose coordinates are odd. Let B_0 be the set of black squares whose coordinates are $(1, 0) \bmod 2$ and B_1 the set of black squares whose coordinates are $(0, 1) \bmod 2$.

A *polyomino* is a finite union of unit squares of T bounded by disjoint simple closed lattice paths. (Later we will consider some special infinite polyominos.) A corner of (the boundary of) a polyomino is *convex* if the interior angle is $\pi/2$; a corner is *concave* if the interior angle is $3\pi/2$. In either case the *corner* *lattice square* is the lattice square adjacent to the corner, which contains the angle bisector of interior angle. An *even polyomino* is a polyomino P in which all corner squares are of type B_1. Note that this implies that any boundary edge of P whose two corners are both convex or both concave has odd length; any boundary edge of P with a convex and a concave corner has even length. A polyomino is *simply connected* if it has only one boundary component.

LEMMA 4. *A simply connected even polyomino contains one more black square than white square.*

PROOF. This is easily proved by induction on the number of corners, starting from the case of a rectangle. □

A *Temperleyan polyomino* is a polyomino which is obtained from an even polyomino P as follows. *Remove* from P a black lattice square d_0 adjacent to an edge or corner of the outer boundary of P. For each interior boundary component D_j of P, *add* a black lattice square d_j adjacent to an edge of that boundary. We assume that d_j only borders on a single square of P. See Figure 2. These added squares will be called *exposed squares*. Note that d_0 must be in B_1 and d_j must be in B_0 for $j > 0$. From the lemma it follows that a Temperleyan polyomino, even if not simply connected, contains the same number of black squares as white squares.

Let P be an even polyomino, and let $\mathbf{B}_1(P)$ be the graph whose vertices are the squares B_1 in P, with edges connecting all squares at distance 2. Then to each horizontal edge of $\mathbf{B}_1(P)$ corresponds a square W_1 of P (the square it crosses) and to each vertical edge of $\mathbf{B}_1(P)$ corresponds a square of type W_0 of P. To each face of $\mathbf{B}_1(P)$ which is not a boundary component of P corresponds a square of P of type B_0. The planar graph $\mathbf{B}_1(P)$ has a planar dual $\mathbf{B}_0(P)$, whose vertices are faces of $\mathbf{B}_0(P)$ (squares of type B_0), as well as a vertex for each boundary component of P. For a Temperleyan polyomino constructed from P, we can still associate the same graphs $\mathbf{B}_1(P)$ and $\mathbf{B}_0(P)$, but we mark the special vertex d_0 of $\mathbf{B}_1(P)$ and mark in $\mathbf{B}_0(P)$ the special edges adjacent to the d_i for $i \geq 0$.

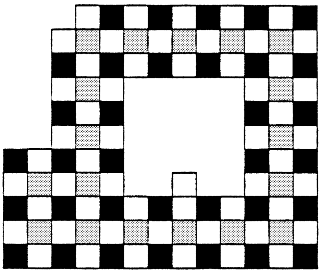

FIG. 2. *A Temperleyan polyomino. The black squares are in B_1, the gray are in B_0.*

Temperley [17] gave a bijection between spanning trees on an $m \times n$ grid and domino tilings of a $(2m-1) \times (2n-1)$ polyomino with a corner removed. A Temperleyan polyomino is a polyomino which arises from a subgraph of the grid by a generalization of his construction, as above, where $\mathbf{B}_1(P)$ is the subgraph one starts with (see [14]).

The *interior dual graph* M of a Temperleyan polyomino P is the graph with a vertex for each lattice square in P, with edges joining pairs of vertices whose corresponding squares are at distance 1 (in other words, it is the dual graph without the boundary vertices). Domino tilings of P are in bijection with perfect matchings of its interior dual graph (a perfect matching of a graph is a set of edges such that each vertex is an endpoint of exactly one edge). The exposed squares of P are called *exposed vertices* of M.

The interior dual graph M of a polyomino P is a subgraph of $\mathbb{Z}^2$ and its vertices inherit a coloring from the checkerboard coloring of the lattice squares: (x, y) is in W_0 if and only if $(x, y) \equiv (0, 0) \bmod 2$ and so on. We will usually denote a vertex $(x, y) \in \mathbb{Z}^2$ by the complex number $x + iy$.

2.2. *The height function* Thurston [19] defines the height function on a domino tiling as follows. The height function is a $\mathbb{Z}$-valued function on the vertices of the tiling, defined only up to an additive constant. Start at an arbitrary vertex v_0 of some domino and define the height there to be 0. For every other vertex v in the tiling, take an edge-path γ from v_0 to v which follows the boundaries of the dominos. The height along γ changes by ± 1 along each edge of γ: if the edge traversed has a black square on its left (which may be exterior to the region) then the height increases by 1; if it has a white square on its left then it decreases by 1. This defines a height at v. If the tiled region is simply connected, the height is independent of the choice of γ since the height change going around a domino is 0. If the tiled region is not simply connected the height is still well defined as long as each hole contains the same number of black and white squares [19]. See Figure 3.

Let M be the interior dual graph of a Temperleyan polyomino P, and take a perfect matching of M. A height function on the tiling determines a height function defined on the (nonboundary) faces of M. The height function may be defined by assigning an arbitrary value to some face and then applying the following rules: for each unmatched edge of M, when following the edge from its black vertex to its white vertex, the height of the face on the left minus the height of the face on its right is 1. For matched edges this difference is -3.

2.2.1. *Heights of boundary components* Let P be a Temperleyan polyomino with boundary components $D_0, \dots, D_k$ where D_0 is the outer component. Since each D_j encloses the same number of black squares as white squares the net height change around each D_j is zero, so the height is well defined for any tiling of P.

Given a tiling of P, the height function along D_j depends only on the height of any single point on D_j. That is, given two points x_0, x_1 of D_j, let γ be the path running along D_j from x_0 to x_1. The height difference $h(x_1) - h(x_0)$ is independent of the tiling since γ crosses no dominos. Since the height of D_j

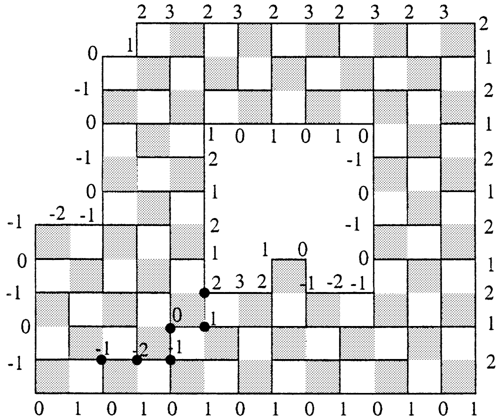

FIG. 3. *Heights in a domino tiling.*

depends only on a single integer value, it makes sense to talk about the height of D_j as a single $\mathbb{Z}$-valued random variable.

Note how the height changes as you go around a boundary component with the interior of P on your left (see Figure 3). Along a straight edge the height alternates between two successive values. Except at the exposed vertex, after a right turn the alternating pair decreases by 1, and after a left turn it increases by 1 (this follows since all corners are black). This means that the height of two points on the same boundary component is related in a simple way to the amount of winding of the boundary component between them (i.e., the number of left turns minus the number of right turns).

2.3. *Tilability of big Temperleyan polyominos.* The Temperleyan polyominos we will be using are those with small lattice spacing which approximate a region U with smooth boundary (or piecewise smooth with one-sided limits of tangents at each corner). Tilability of such a polyomino can be shown using the following result of Fournier.

PROPOSITION 5 [10]. *A simply connected polyomino with the same number of black and white squares can be domino tiled unless there are two boundary vertices x, y whose distance in the L^1-metric (length of the shortest lattice path from x to y in P) is less than their height difference.*

Actually Fournier's condition is stronger than this (he uses a modified metric) but this will suffice for our needs. Also, Fournier only considered simply connected regions but his argument generalizes to regions with many boundary components, as long as a height has been assigned to each component (and one is interested in tilings whose height function extends the function already defined on the boundary).

Since the region U has a piecewise smooth boundary as defined above, the winding number of the boundary path between two points on the same boundary component of U is bounded. As a consequence if P_ϵ is a Temperleyan polyomino in $\epsilon\mathbb{Z}^2$ approximating U (and if locally the boundary of P_ϵ follows that of U in the sense that they are always directed into the same approximate quadrant), the height difference between two points on the same boundary component of P_ϵ is approximately the same as the winding number of the boundary of U between those two points. Therefore the height function on the boundary of P_ϵ varies by, at most, a constant.

In particular, if ϵ is sufficiently small, Proposition 5 and Lemma 4 show that P_ϵ is tilable.

A more elementary proof of tilability using spanning trees is sketched in Section 7.

3. Discrete analytic functions. The important discrete functions appearing in this article are examples of discrete analytic functions (also called monodiffric functions); see [9]. This section reviews the relevant definitions. Our definition is slightly different from the classical definition in [9] but is equivalent.

3.1. *The $\partial_{\bar{z}}$ operator.* We define several operators on $\mathbb{Z}^2$. The operator ∂_x: $\mathbb{C}^{\mathbb{Z}^2} \to \mathbb{C}^{\mathbb{Z}^2}$ is defined by

$$\partial_x f(v) = f(v+1) - f(v-1).$$

Similarly define

$$\partial_y f(v) = f(v+i) - f(v-i).$$

We define operators

$$\partial_z = \partial_x - i\partial_y$$

and

$$\partial_{\bar{z}} = \partial_x + i\partial_y.$$

These operators restrict to operators from $\mathbb{C}^B$ to $\mathbb{C}^W$: if $f \in \mathbb{C}^B$, that is, if f is zero on white vertices, then $\partial_x f$, $\partial_y f \in \mathbb{C}^W$. Similarly ∂_x, ∂_y map $\mathbb{C}^W$ to $\mathbb{C}^B$. A *discrete analytic function* is a function $F \in \mathbb{C}^B$ which is real on B_0 and pure imaginary on B_1 and satisfies $\partial_{\bar{z}} F = 0$. If $F = f + ig$ where $f \in \mathbb{R}^{B_0}$ and $g \in \mathbb{R}^{B_1}$, then F being discrete analytic is equivalent to f and g satisfying the *discrete Cauchy–Riemann equations*

$$\partial_x f(v) = \partial_y g(v) \quad \text{for } v \in W_0, \tag{1}$$

$$\partial_y f(v) = -\partial_x g(v) \quad \text{for } v \in W_1. \tag{2}$$

(Note that when $f \in \mathbb{R}^{B_0}$ and $g \in \mathbb{R}^{B_1}$, we have $\partial_x f, \partial_y g \in \mathbb{R}^{W_0}$ and $\partial_y f, \partial_x g \in \mathbb{R}^{W_1}$.)

The function f is called the real part of $f + ig$, and g is called the imaginary part of $f + ig$.

If $f + ig$ satisfies the discrete CR-equations at all but a finite number of (white) vertices, we say that $f + ig$ is discrete analytic with **poles** at those vertices.

The operators $\partial_x, \partial_y, \partial_z, \partial_{\bar{z}}$ restrict to operators on subgraphs M of $\mathbb{Z}^2$ in a natural way: we consider $\mathbb{C}^M$ to be the subset of $\mathbb{C}^{\mathbb{Z}^2}$ which consists of functions zero outside of M. We apply the operator and then project back to $\mathbb{C}^M$.

3.2. *Laplacian.* A simple calculation shows that, if $f \in \mathbb{R}^{B_0}$, then $\partial_z \partial_{\bar{z}} f \in \mathbb{R}^{B_0}$ and $-\partial_z \partial_{\bar{z}} f$ is the Laplacian of f on the graph $\mathbf{B}_0(\mathbb{Z}^2)$. That is,

$$-\partial_z \partial_{\bar{z}} f(v) = \Delta f(v) = 4f(v) - f(v+2) - f(v+2i) - f(v-2) - f(v-2i).$$

Note that this is four times the usual Laplacian since we left out factors of $\frac{1}{2}$ in the definition of $\partial_{\bar{z}}$ and ∂_z. Often when discussing the discrete Laplacian there is a disagreement about the choice of sign. Here we chose the positive (semi)definite Laplacian, which corresponds in the continuous limit to $-(\partial^2/\partial x^2) - (\partial^2/\partial y^2)$.

In a similar fashion if $g \in \mathbb{R}^{B_1}$ then $-\partial_z \partial_{\bar{z}} g$ is the Laplacian of g on the graph $\mathbf{B}_1(\mathbb{Z}^2)$.

In particular if $f + ig$ is discrete analytic on $\mathbb{Z}^2$ we have $\partial_z \partial_{\bar{z}}(f + ig) = \partial_z(0) = 0$ and so $\Delta f = 0$ and $\Delta g = 0$, where the first Δ is the Laplacian on $\mathbf{B}_0(\mathbb{Z}^2)$ and the second is the Laplacian on $\mathbf{B}_1(\mathbb{Z}^2)$.

For a discussion of the boundary behavior of the Laplacian on $\mathbf{B}_0(P)$, see Section 4.1.

3.3. *Weighting the graph.* An alternative way to define discrete analytic functions, which relates more closely with domino tilings, is as follows. On the graph $\mathbb{Z}^2$ put *weights* on the edges; at each white vertex the four edge weights going counterclockwise from the right-going edge are 1, i, -1, $-i$, respectively. See Figure 4.

Now for a pair of real-valued functions $f \in \mathbb{R}^{B_0}$ and $g \in \mathbb{R}^{B_1}$, the function $f + ig$ is discrete analytic if and only if it satisfies $K(f + ig) = 0$, where K is the adjacency matrix of $\mathbb{Z}^2$ with these weights. The matrix K is called the *Kasteleyn matrix* of $\mathbb{Z}^2$. Kasteleyn proved that for a finite region the absolute

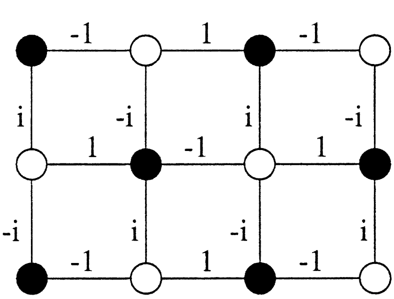

FIG. 4. *Weights of the Kasteleyn matrix.*

value of the determinant of the Kasteleyn matrix is the square of the number of perfect matchings. (Usually the Kasteleyn matrix is defined with different weights [12]; but in fact any choice of complex weights of modulus 1 satisfying $ac = -bd$ for the four weights a, b, c, d around a square gives rise to a Kasteleyn-like matrix whose determinant counts tilings.)

When considered as an operator on $\mathbb{C}^B$, the operator K is the operator $\partial_{\bar{z}}$. When considered as an operator on $\mathbb{C}^W$, however, it is $-\partial_{\bar{z}}$. Let K^* be the Hermitian conjugate of K. Then the operator K^*K is acting as the Laplacian on both $\mathbf{B}_0$ and $\mathbf{B}_1$.

LEMMA 6. *A discrete analytic function on a simply connected Temperleyan region P is determined up to an additive* (*imaginary*) *constant by its real part.*

PROOF. Note first that $\mathbf{B}_1(P)$ is connected. Let $f \in \mathbb{R}^{B_0}$ be harmonic on $\mathbf{B}_0(P)$. Given the value of the imaginary part g at one vertex $v \in B_1$, the value $g(w)$ for any other vertex w in B_1 is uniquely determined as follows. Take a path in $\mathbf{B}_1(P)$ from v to w. Each edge of the path crosses an edge of $\mathbf{B}_0(P)$. One of the Cauchy–Riemann equations [(1) or (2)] at the crossing point determines the difference in values of g at the endpoints of this edge. The value $g(w)$ is obtained by summing this difference along the path. The harmonicity of f implies that the value $g(w)$ obtained is independent of the path chosen. □

When the region is not simply connected, in general the conjugate function of a harmonic function $f \in \mathbb{R}^{B_0}$ is not single valued: the "integral" in the above lemma along a path surrounding a hole may not be zero.

4. The coupling function. Let M be the interior dual graph of a Temperleyan polyomino P. Let K be the corresponding Kasteleyn matrix and let E be a finite collection of disjoint edges of M. Let $b_1, \dots, b_k$ and $w_1, \dots, w_k$ be the black vertices (respectively, white vertices) covered by E. Let μ be the uniform probability measure on perfect matchings of M.

THEOREM 7 [13]. *The μ-probability that E occurs in a perfect matching is given by $|\det(K_E^{-1})|$, where K_E^{-1} is the submatrix of K^{-1} whose rows are indexed by $b_1, \dots, b_k$ and columns are indexed by $w_1, \dots, w_k$. More precisely, the probability is $(-1)^{\sum p_i + q_i} a_E \det(K_E^{-1})c$, where p_i, q_i is the index of b_i, respectively, w_i, in a fixed ordering of the vertices, $c = \pm 1$ is a constant depending only on that ordering, and a_E is the product of the edge weights of the edges E.*

Thus the μ-measures of cylinder sets for perfect matchings on M are determined by this function K^{-1}: $M \times M \to \mathbb{C}$, called the *coupling function*. For historical reasons we denote the coupling function with a C.

Actually this theorem holds for arbitrary bipartite planar graphs, not just those arising from the square grid; see [13].

In all of our applications of this theorem we will use only a small number of edges out of the total number of edges of M; in this case we can choose the ordering of vertices so that all the relevant indices p_i and q_i are even, and $c = 1$. Then we can use the simpler form $|\det(K_E^{-1})| = a_E \det(K_E^{-1})$.

The defining property of $C(v_1, v_2)$ is that it satisfies $KC(v_1, v_2) = \delta_{v_1}(v_2)$. Here δ_{v_1} is the delta function

$$\delta_{v_1}(v_2) = \begin{cases} 1, & \text{if } v_2 = v_1, \\ 0, & \text{otherwise.} \end{cases}$$

We have the following.

LEMMA 8. *The function C is symmetric*: $C(v_1, v_2) = C(v_2, v_1)$. *We have $C(v_1, v_2) = 0$ whenever v_1 and v_2 are both black or both white. If v_1 is white, the coupling function $C(v_1, v_2)$ is discrete analytic as a function of v_2, with a pole at v_1.*

PROOF. Since we already have $KC(v_1, v_2) = \delta_{v_1}(v_2)$, it suffices to show that $C(v_1, v_2)$ is real when $v_2 - v_1 \equiv (1, 0) \bmod 2$, purely imaginary when $v_2 - v_1 \equiv (0, 1) \bmod 2$ and zero in the remaining cases.

If we order the vertices of M in such a way that all the W_0 are first, then W_1 then B_0 and then B_1, then the matrix K in this basis has the form

$$K = \begin{pmatrix} 0 & 0 & K_1 & iK_2 \\ 0 & 0 & iK_3 & K_4 \\ K_1^t & iK_3^t & 0 & 0 \\ iK_2^t & K_4^t & 0 & 0 \end{pmatrix},$$

where K_1, K_2, K_3, K_4 are real matrices. The conjugate of the above matrix by the matrix

$$\begin{pmatrix} I & 0 & 0 & 0 \\ 0 & iI & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & iI \end{pmatrix}$$

is real. Hence the inverse of K has the same form as K. This completes the proof. □

See Figure 6 for (part of) an example.

Since $C(v_1, v_2) = 0$ when v_1, v_2 are both black or both white, and $C(v_1, v_2) = C(v_1, v_2)$, we will almost always take the first argument of C to be a white vertex and the second to be black.

4.1. *Boundary conditions for the coupling function.* A discrete analytic function is determined by its boundary values, since its real and imaginary parts are harmonic. In this section we describe the behavior of $C(v_1, v_2)$ for v_2 on the boundary of M.

Assume that $v_1 \in W_0$. By Lemma 8, $C(v_1, v_2)$ is real when $v_2 \in \mathbf{B}_0(P)$ and purly imaginary when $v_2 \in \mathbf{B}_1(P)$ (and zero when $v_2 \in W_0 \cup W_1$). Let Y be the set of vertices in B_0 adjacent to (a white vertex of) M but not in M (that is, at distance 1 from a vertex of M). Let $\mathbf{B}_0'(P)$ be the graph whose vertices are $\mathbf{B}_0(P) \cup Y$, and whose edges connect every pair of vertices of distance 2, provided that the white vertex lying between these two is in M. The set Y is the set of *boundary vertices* of $\mathbf{B}_0'(P)$. Let V be the set of exposed vertices $d_1, \dots, d_k$ (recall that they are all in B_0). See Figure 5 for an example of a graph $\mathbf{B}_0'(P)$.

LEMMA 9. *For a fixed $v_1 \in W_0$, consider $C(v_1, v_2)$ as a function of v_2. The real part of $C(v_1, v_2)$, extended to be zero on Y and considered as a function on the graph $\mathbf{B}_0'(P)$, has the following properties*:

(i) *It is harmonic at all vertices in $\mathbf{B}_0(P) \setminus (V \cup \{v_1 + 1, v_1 - 1\})$.*
(ii) $\Delta \operatorname{Re} C(v_1, v_1 \pm 1) = \pm 1$.
(iii) *Its harmonic conjugate is single valued.*

If rather $v_1 \in W_1$ then the imaginary part of $C(v_1, v_2)$, extended to be zero on Y and considered as a function on $\mathbf{B}_0'(P)$, has the following properties:

(a) *It is harmonic at all vertices in $\mathbf{B}_0(P) \setminus (V \cup \{v_1 + i, v_1 - i\})$.*
(b) $\Delta \operatorname{Im} C(v_1, v_1 \pm i) = \mp 1$.
(c) *Its harmonic conjugate is single valued.*

PROOF. The first two properties in both cases follow from

$$\Delta C(v_1, \cdot) = K^* K C(v_1, \cdot) = K^* \delta_{v_1} = \delta_{v_1+1} - \delta_{v_1-1} - i\delta_{v_1+i} + i\delta_{v_1-i}.$$

This equation is valid at every vertex of $\mathbf{B}_0(P)$ except the exposed vertices (which do not have four neighbors). The third property in each case follows

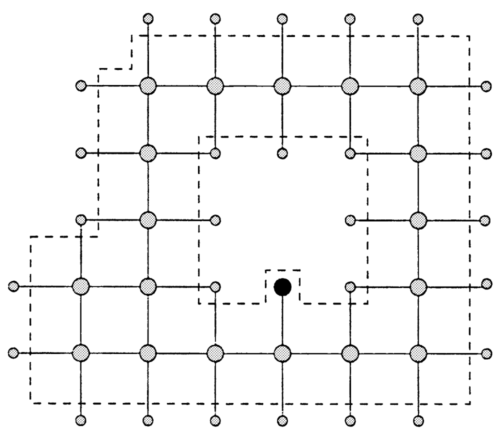

FIG. 5. *Example of the graph $\mathbf{B}_0'(P)$ for the polyomino P of Figure 2 (P is in dashed lines). The smaller gray dots are vertices in Y; the black vertex is the exposed vertex.*

by definition, since $\operatorname{Im} C(v_1, \cdot)$ is the harmonic conjuate of $\operatorname{Re} C(v_1, \cdot)$ and $-\operatorname{Re} C(v_1, \cdot)$ is the harmonic conjugate of $\operatorname{Im} C(v_1, \cdot)$. □

We will see later that $\operatorname{Re} C(v_1, v_2)$, $\operatorname{Im} C(v_1, v_2)$ are, respectively, the unique functions with the above properties. As a consequence we will be able to use some general theorems about harmonic functions to reach conclusions about the coupling function.

The conditions in Lemma 9 are particularly simple because we started with a Temperleyan polyomino. For a polyomino with different boundary conditions, the corresponding boundary conditions for the coupling function can be quite complicated; see Section 8.

5. Asymptotic values of the coupling function. Here we will show that, as ϵ tends to 0, the scaled discrete analytic function $(1/\epsilon)C(v_1, \cdot)$ converges to a pair of complex-analytic functions F_0, F_1 (F_0 when $v_1 \in W_0$ and F_1 when $v_1 \in W_1$) which transform analytically (see Proposition 15) under conformal mappings of the domain U.

We first study what happens when the polyomino P is the whole plane, since as we will see, for any region U the leading term in $C(v_1, v_2)$ equals $C_0(v_1, v_2)$, the coupling function on the plane (as long as v_1 is not too close to the boundary of U).

5.1. *On the plane.* In [13] we gave an explicit formula for the coupling function on $\mathbb{Z}^2$. This was shown to be the limit as $n \to \infty$ of the coupling function on the $2n \times 2n$ square, centered at the origin. In that paper we used different weights for the Kasteleyn matrix: 1 on all horizontal edges and i on all vertical edges. The present calculation is straightforward using the same methods (in fact the result is identical after changing the sign on alternating vertices of B_0 and B_1) and yields the following.

PROPOSITION 10 [13]. *Let C_0 denote the coupling function for the whole plane $\mathbb{Z}^2$. Then*

$$C_0(0, x+iy) = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \frac{\exp(i(x\theta - y\phi))}{2i\sin(\theta) + 2\sin(\phi)} \, d\theta \, d\phi.$$

By translation invariance, $C_0(v_1, v_2) = C_0(0, v_2 - v_1)$ so this theorem describes the entire coupling function. In [13] it is shown how to evaluate this integral explicitly. Figure 6 shows the first few values of $C_0(0, x+iy)$ when $x + iy$ is in the positive quadrant. The values in the other quadrants are obtained by the symmetry $C_0(0, iz) = -iC_0(0, z)$, which arises from the corresponding symmetry of the edge weights.

Recall that the origin in $\mathbb{Z}^2$ is a vertex of type W_0.

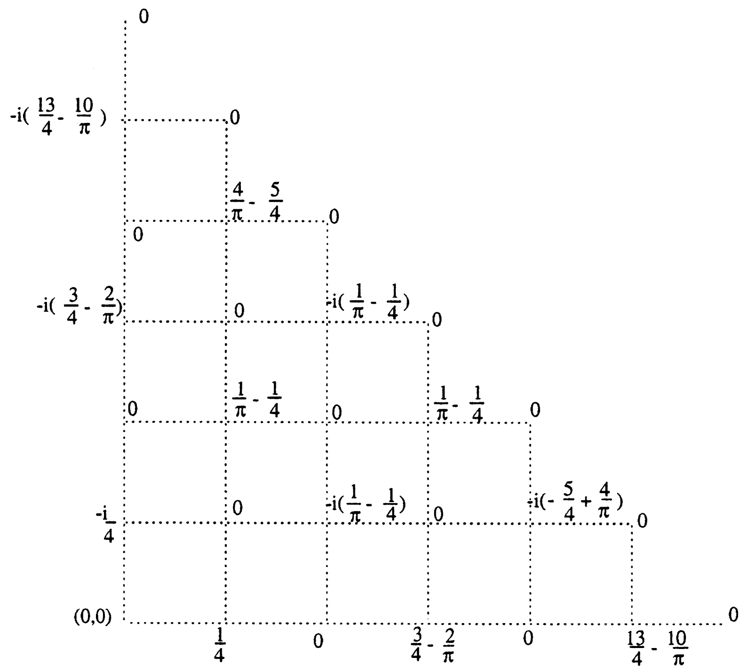

FIG. 6. *The function $C_0(0, x+iy)$, the coupling function for $\mathbb{Z}^2$.*

THEOREM 11. *As $|z| \to \infty$, the coupling function on $\mathbb{Z}^2$ is asymptotically equal to $\frac{1}{\pi z}$, that is,*

$$C_0(0, z) = \begin{cases} \operatorname{Re} \dfrac{1}{\pi z} + O\left(\dfrac{1}{|z|^2}\right), & z \in B_0, \\ i \operatorname{Im} \dfrac{1}{\pi z} + O\left(\dfrac{1}{|z|^2}\right), & z \in B_1. \end{cases}$$

PROOF. There is the following relation between C_0 and the Green's function for the plane. The real part of C_0 is the unique function on $\mathbf{B}_0(\mathbb{Z}^2)$ satisfying $\Delta \operatorname{Re} C_0 = \delta_1 - \delta_{-1}$ and tending to 0 at infinity (see Lemma 9, and recall that C_0 is the limit of C on square regions centered at the origin).

Now the classical Green's function $G_0(v, w)$ on $\mathbb{Z}^2$ satisfies $\Delta G_0(0, w) = \delta_0(w)$ and for any fixed v, $G_0(0, w) - G_0(v, w) \to 0$ as $w \to \infty$ (see Lemma 12). As a consequence we have

$$\operatorname{Re} C_0(0, w) = G_0\left(0, \frac{w-1}{2}\right) - G_0\left(0, \frac{w+1}{2}\right),$$

where on the right we used coordinates on $\mathbf{B}_0(\mathbb{Z}^2)$ which has index 4 in $\mathbb{Z}^2$.

Using Lemma 12 we have

$$\begin{aligned}\operatorname{Re} C_0(0,w) &= G_0\left(0, \frac{w-1}{2}\right) - G_0\left(0, \frac{w+1}{2}\right)\\ &= \frac{1}{2\pi}\left(\log\left|\frac{w+1}{2}\right| - \log\left|\frac{w-1}{2}\right|\right) + O\left(\frac{1}{|w|^2}\right)\\ &= \frac{1}{2\pi}\operatorname{Re}\log\left(\frac{w+1}{w-1}\right) + O\left(\frac{1}{|w|^2}\right)\\ &= \frac{1}{2\pi}\operatorname{Re}\frac{2}{w-1} + O\left(\frac{1}{|w|^2}\right)\\ &= \operatorname{Re}\frac{1}{\pi w} + O\left(\frac{1}{|w|^2}\right)\end{aligned}$$

where we used $\log(1+z) = z + O(|z|^2)$. A similar argument holds for the imaginary part. □

LEMMA 12 [16]. *For the Green's function G_0 on $\mathbf{B}_0(\mathbb{Z}^2)$ we have*

$$G_0(0,v) = -\frac{1}{2\pi}\log|v| + c_0 + O\left(\frac{1}{|v|^2}\right) \tag{3}$$

for a constant c_0.

Note that Stöhr's Laplacian is $-1/4$ times ours, so his Green's function is -4 times that in (3).

5.2. *The half-plane.* For later use we will need to compute the coupling function on a half-plane. Let $\{P_n\}$ be a sequence of Temperleyan polyominos in the upper half-plane $H = \{x+iy \in \mathbb{Z}^2 \mid y > 0\}$, such that P_n contains the rectangle $[-n,n]\times[1,n]$, and the base point d_0 of P_n is outside this rectangle. Then (as we will show in the proof of Theorem 14), for fixed v_1, v_2 the coupling function $C^{(n)}(v_1,v_2)$ on P_n converges to a limit $C_H(v_1,v_2)$ satisfying the properties below. In particular the uniform measures on the $P^{(n)}$ converge to a unique measure μ_H.

Suppose $v_1 \in W_0$. The real part of $C_H(v_1,v_2)$ satisfies the conditions of Lemma 9: $\Delta\operatorname{Re} C_H(v_1,\cdot) = \delta_{v_1+1} - \delta_{v_1-1}$, $\operatorname{Re} C_H(v_1, x+iy) = 0$ when $y=0$, and $\operatorname{Re} C_H$ tends to zero at infinity. There is a unique harmonic function with these three properties: the real part of $C_0(v_1,v_2) - C_0(\overline{v_1},v_2)$ (note that $v_1 \in W_0$ implies $\overline{v_1} \in W_0$). The conjugate harmonic function $\operatorname{Im} C_H$ is single valued, and uniquely defined by the condition that it tends to zero at infinity; as a consequence we have

$$C_H(v_1,v_2) = C_0(v_1,v_2) - C_0(\overline{v_1},v_2) \quad \text{when } v_1 \in W_0. \tag{4}$$

If $v_1 \in W_1$, on the other hand, it is the *imaginary part* of $C_H(v_1, x+iy)$ which is zero when $y=0$. In this case there is again a unique harmonic function satisfying the requisite properties: $\operatorname{Re} C_H(v_1, v_2) = \operatorname{Re}(C_0(v_1, v_2)+C_0(\overline{v_1}, v_2))$. So then

$$C_H(v_1, v_2) = C_0(v_1, v_2) + C_0(\overline{v_1}, v_2) \quad \text{when } v_1 \in W_1. \tag{5}$$

There is a big difference between these two cases: from Theorem 11, in the case $v_1 \in W_0$ we have

$$\begin{aligned} C_H(v_1, v_2) &= \frac{1}{\pi}\left(\frac{1}{v_2 - v_1} - \frac{1}{v_2 - \overline{v_1}}\right) + O\left(\frac{1}{|v_2 - v_1|^2}\right) \\ &= \frac{v_1 - \overline{v_1}}{\pi(v_2 - v_1)(v_2 - \overline{v_1})} + O\left(\frac{1}{|v_2 - v_1|^2}\right) \end{aligned}$$

which is $O(d)$, where d is the distance from v_1 to the boundary. In the case $v_1 \in W_1$, rather, we have

$$\begin{aligned} C_H(v_1, v_2) &= \frac{1}{\pi}\left(\frac{1}{v_2 - v_1} + \frac{1}{v_2 - \overline{v_1}}\right) + O\left(\frac{1}{|v_2 - v_1|^2}\right) \\ &= \frac{2v_2 - v_1 - \overline{v_1}}{\pi(v_2 - v_1)(v_2 - \overline{v_1})} + O\left(\frac{1}{|v_2 - v_1|^2}\right), \end{aligned}$$

which does not go to zero as v_1 approaches the boundary.

There are similar formulas for the other half-planes with horizontal or vertical boundary.

5.3. *Bounded regions.* One of the main results in this paper is to show that the coupling function on a finite region converges, as ϵ tends to zero, to a pair of analytic functions which transform analytically under conformal maps of the region. For a fixed region U we cannot prove this for all Temperleyan polyominos P_ϵ approximating U: we require that the approximating P_ϵ have a nice behavior in a neighborhood of their exposed vertices. This shortcoming is due to our lack of understanding of the asymptotics of the discrete Green's function near the boundary of a polyomino. It seems nonetheless reasonable to suspect that this flaw can and will be overcome in the near future.

We will begin at this point to use the metric on $\epsilon\mathbb{Z}^2$ rather than $\mathbb{Z}^2$. That is, we work on polyominos in $\epsilon\mathbb{Z}^2$ with interior dual graphs having edges of length ϵ. The graphs $\mathbf{B}'_0(P)$ have edges of length 2ϵ.

Let U be a region in $\mathbb{C}$ with smooth boundary (or piecewise smooth as previously defined). Let $D_0, \dots, D_k$ be the boundary components of U, with D_0 being the outer component. Let d'_j be a marked point of D_j. Let z_1 be a point in the interior of U and z_2 be any point of U.

We define two functions $F_0(z_1, z_2)$ and $F_1(z_1, z_2)$, whose existence and uniqueness will be shown in the proof of Theorem 13, below. For fixed z_1, the function $F_0(z_1, z_2)$ is analytic as a function of z_2, has a simple pole of residue $1/\pi$ at $z_2 = z_1$ and no other poles on $\overline{U}$ except possibly simple poles at the d'_j, $j > 0$. Furthermore, it is zero at d'_0 and has real part 0 on the boundary of U. For fixed z_1, the function $F_1(z_1, z_2)$ is analytic as a function of z_2, has a simple pole of residue $1/\pi$ at $z_2 = z_1$ and no other poles on $\overline{U}$ except possibly simple poles at the d'_j, $j > 0$. Furthermore, it is zero at d'_0 and has imaginary part 0 on the boundary of U.

For each $\epsilon > 0$ sufficiently small, let P_ϵ be a Temperleyan polyomino in $\epsilon\mathbb{Z}^2$ approximating U in the following sense. The boundaries of P_ϵ are within $O(\epsilon)$ of the boundaries of U, and except near a corner of ∂U the tangent vector to ∂U points into the same half-space as the direction of the corresponding edges of ∂P_ϵ. Furthermore, assume that the exposed vertices d_j of P_ϵ are within $O(\epsilon)$ of the d'_j. Suppose further that for a certain $\delta = \delta(\epsilon) > 0$ tending to zero sufficiently slowly (see below), in a δ-neighborhood of each d_j, the boundary of P_ϵ is straight (horizontal or vertical). Let M_ϵ be the interior dual of P_ϵ. Let v_1 be a white vertex and v_2 a black vertex of M_ϵ. We then have the following result.

THEOREM 13. *Fix any real $\xi > 0$. The coupling function $C(v_1, v_2)$ on the graph M_ϵ satisfies: for $v_1 \in W_0$ and v_1, v_2 not within ξ of the boundary of M_ϵ,*

$$\frac{1}{\epsilon}C(v_1, v_2) = \frac{1}{\epsilon}C_0(v_1, v_2) + F_0^*(v_1, v_2) + o(1),$$

where F_0^* is defined by the condition that $F_0(z_1, z_2) = 1/(\pi(z_2 - z_1)) + F_0^*(z_1, z_2)$, with F_0 as above, and C_0 is the coupling function on $\epsilon\mathbb{Z}^2$.

If $v_1 \in W_1$, rather, then

$$\frac{1}{\epsilon}C(v_1, v_2) = \frac{1}{\epsilon}C_0(v_1, v_2) + F_1^*(v_1, v_2) + o(1),$$

where F_1^* is defined by the condition that $F_1(z_1, z_2) = 1/(\pi(z_2 - z_1)) + F_1^*(z_1, z_2)$, with F_1 as above.

The equality in the theorem should be interpreted as saying: when $v_1 \in W_0$ and $v_2 \in B_0$ then $C(v_1, v_2)$ equals the real part of the right-hand side; and when $v_1 \in W_0$ and $v_2 \in B_1$ then $C(v_1, v_2)$ equals i times the imaginary part of the right-hand side. Similarly for $v_1 \in W_1$ and $v_2 \in B_0$, then $C(v_1, v_2)$ equals i times the imaginary part of the right-hand side; when $v_1 \in W_1$ and $v_2 \in B_1$ then $C(v_1, v_2)$ equals the real part of the right-hand side.

When v_1 and v_2 are far apart [not within $o(1)$] then we can replace $(1/\epsilon) \cdot C_0(v_1, v_2)$ with $1/(\pi(v_2 - v_1)) + o(1)$ and so the statement is simply

$$\frac{1}{\epsilon}C(v_1, v_2) = F_j(v_1, v_2) + o(1),$$

where $j = 0$ or 1 as the case may be.

PROOF. Let U_δ be equal to U except in a 2δ-neighborhood of the d'_j, and such that ∂U_δ is flat and horizontal or vertical in a δ-neighborhood of the d'_j. We will first prove the theorem for U_δ for any *fixed* $\delta > 0$.

We will do only the case $v_1 \in W_0$. The case $v_1 \in W_1$ is identical using the imaginary part of C rather than the real part of C below.

Let $G(w_1, w_2)$ be the Green's function on $\mathbf{B}'_0(P_\epsilon)$ [recall the construction of $\mathbf{B}'_0(P_\epsilon)$ from Section 4.1]; that is, the function which satisfies $\Delta G(w_1, w_2) = \delta_{w_1}(w_2)$ and $G(w_1, w_2) = 0$ when $w_2 \in Y \cup V \setminus \{w_1\}$.

The function $\operatorname{Re} C(v_1, v_2)$, considered as a function of v_2, is a linear combination of the Green's functions $G(v_1 \pm \epsilon, v_2)$ and $G(d_j, v_2)$ for $j = 1, \dots, k$ since it is harmonic off of these vertices. In fact since

$$\Delta \operatorname{Re} C(v_1, \cdot) = \delta_{v_1+\epsilon} - \delta_{v_1-\epsilon} + \sum_{j=1}^{k} \alpha_j \delta_{d_j}$$

for some constants α_j, we have

$$\operatorname{Re} C(v_1, v_2) = G(v_1 + \epsilon, v_2) - G(v_1 - \epsilon, v_2) + \sum_{j=1}^{k} \alpha_j G(d_j, v_2). \tag{6}$$

By Corollary 19 below, the rescaled Green's function $(1/\epsilon)G(d_j, v_2)$ (considered as a function of v_2) converges away from d_j to a continuous harmonic function with a logarithmic singularity at d_j and boundary values 0. (This is the place where we need U_δ rather than U.) Similarly by Lemma 17 the difference $(1/\epsilon)(G(v_1+\epsilon, v_2) - G(v_1 - \epsilon, v_2))$ converges. It remains to show that the coefficients α_j in (6) converge as $\epsilon \to 0$. Note that if U is simply connected then $k = 0$ and we are done.

For general U, the right-hand side of (6) automatically satisfies the conditions (i) and (ii) of Lemma 9 defining the coupling function, but the Green's functions $G(v_1, v_2)$ do not in general have a single-valued harmonic conjugate. It is necessary to choose the α_j so that the harmonic conjugate of the right-hand side of (6) is single valued. We show that, in fact, the α_j are *uniquely determined* by this property.

We will use the language of electrical networks, see, e.g., [8]. Consider the graph $\mathbf{B}'_0(P)_\epsilon$ to be a resistor network with resistances 1 on each edge. The function $G(v_1, v_2)$ is the potential at v_2 when one unit of current flows into the network at v_1 and the boundary $Y \cup V$ is held at potential 0. The α_j must be chosen so that, when currents α_j flow into the network at d_j, and current ± 1 flows into the network at $v_1 \pm \epsilon$, and the boundary is held at potential 0, then the net amount of current exiting each boundary component D_j is zero. For the harmonic conjugate is the integral of the current flow: the integral of the current crossing a closed curve surrounding D_j is 0 if and only if the harmonic conjugate is single valued around that curve.

We claim that given any $k+1$ real numbers $c_0, c_1, \dots, c_k$ such that $c_0 + \cdots + c_k = 0$, there exists a unique choice of reals $\alpha_1, \dots, \alpha_k$ such that, when currents α_j flow into the network at d_j, and the boundary is held at potential 0, the net current flow out of each boundary component D_j is c_j. This will then determine the α_j, because letting $c_0, \ldots, c_k$ be the current flow out of the boundaries from the function $G(v_1+\epsilon, v_2) - G(v_1-\epsilon, v_2)$ (we mean, when 1 unit of current flows in at $v_1+\epsilon$ and 1 flows out at $v_1-\epsilon$), we must choose the unique α_j to exactly cancel this flow.

To prove the claim, note that the map $\Phi: \mathbb{R}^k \to \mathbb{R}^k$ which gives the outgoing currents $c_1, \ldots, c_k$ (and therefore $c_0 = -c_1 - \cdots - c_k$ as well) as a function of $\alpha_1, \ldots, \alpha_k$ is linear (this is the principle of superposition). It suffices to show that the determinant of Φ is nonzero.

However, on each column of the matrix of Φ (in the basis $\{c_1, \ldots, c_k\}$ and $\{\alpha_1, \ldots, \alpha_k\}$) the diagonal entry is the only negative entry: $G(d_j, v_2)$ induces a positive net current flow out of each boundary component except the component D_j which contains d_j, since $G(d_j, v_2)$ is a *positive* harmonic function. Furthermore, the diagonal entry in Φ is larger than the absolute value of the sum of the other entries in that column, since a nonzero amount of current flows out of D_0; that is, $c_0 > 0$ (and the total inflowing current equals the total outflowing current). This implies that $\det \Phi \neq 0$ (see Lemma 16 below).

Now as ϵ tends to 0, the rescaled Green's function $(1/\epsilon)G(d_j, v_2)$ converges (Corollary 19). This implies that the entries of the matrix of Φ converge: the pointwise convergence of a sequence of harmonic functions implies convergence of their derivatives (even in the discrete case), due to Poisson's formula: the derivative at a point is determined by integrating the values of the function on a neighborhood of that point against (the derivative of) the Poisson kernel. By integrating the derivative we get convergence of the net current flow out of each boundary. Furthermore, the amount of current out of D_0 due to $(1/\epsilon)G(d_j, \cdot)$ is bounded from below. This implies that $\det \Phi$ is bounded away from 0 (Lemma 16). Since the difference in Green's functions $(1/\epsilon)G(v_1+\epsilon, \cdot) - (1/\epsilon)G(v_1-\epsilon, \cdot)$ also converges (Lemma 17), the net current out of D_j from $(1/\epsilon)G(v_1+\epsilon, \cdot) - (1/\epsilon)G(v_1-\epsilon, \cdot)$ converges. Therefore the α_j converge as well. We conclude that $\operatorname{Re} C$ converges.

The C^0-convergence of $\operatorname{Re} C$ implies convergence of its derivatives and so by integrating we get local convergence of $\operatorname{Im} C$ as well. By uniqueness of the harmonic conjugate (up to an additive constant) we have that $\operatorname{Im} C$ converges (the constant is determined by the fact that it is zero at d_0).

In conclusion, when $v_1 \in W_0$ $(1/\epsilon)C(v_1, v_2)$ converges to an analytic function (of v_2) with all the properties of the function F_0. Furthermore the proof shows that there is a unique function with these properties. When $v_1 \in W_1$ then $C(v_1, v_2)$ converges to F_1 which is also unique.

When $|v_2 - v_1| = o(1)$, the main contribution to $C(v_1, v_2)$ is from $G(v_1 + \epsilon, v_2) - G(v_1 - \epsilon, v_2)$; the unrescaled Green's functions $\alpha_j G(d_j, v_2)$ contribute at most $o(1)$. Since $G(v_1+\epsilon, v_2) - G(v_1-\epsilon, v_2) = G_0(v_1+\epsilon, v_2) - G_0(v_1-\epsilon, v_2) + o(1)$ (see the proof of Lemma 17), we conclude that $C(v_1, v_2) = C_0(v_1, v_2) + o(1)$. This gives the "local" term in the statement.

The above holds for U_δ for any $\delta > 0$. It remains to see that when $\delta \to 0$ the functions $F_0^{(\delta)}, F_1^{(\delta)}$ on U_δ converge to F_0, F_1 on U. This follows from Proposition 15 below, and the fact that the Riemann map from U_δ to U converges (if appropriately normalized) to the identity mapping. Therefore the result holds for U as long as $\delta \to 0$ sufficiently slowly. □

A similar result holds when v_1 is close to a flat boundary of P_ϵ. Here is the statement when it is close to a flat horizontal boundary. This is the only case we will need later.

THEOREM 14. *Fix $\delta > 0$. Let z_1 be a point on the boundary of U such that the boundary is flat and horizontal in a δ-neighborhood of z_1. Let $v_1 \in W_0$ be a point within $O(\epsilon)$ of z_1 and v_2 a black vertex. The coupling function $C(v_1, v_2)$ satisfies*

$$\frac{1}{\epsilon}C(v_1, v_2) = \frac{1}{\epsilon}C_H(v_1, v_2) + o(1),$$

where C_H is the coupling function defined in (4) *for the appropriate half-plane in $\epsilon\mathbb{Z}^2$. If rather $v_1 \in W_1$ then*

$$\frac{1}{\epsilon}C(v_1, v_2) = \frac{1}{\epsilon}C_H(v_1, v_2) + F_1^{**}(z_1, v_2) + o(1),$$

where F_1^{**} is defined by the condition that

$$F_1(z_1, z_2) = 2/(\pi(z_2 - z_1)) + F_1^{**}(z_1, z_2)$$

and F_1 is as before.

PROOF. We use the notation of the previous proof. If $v_1 \in W_0$, then by (4), the function $(1/\epsilon)(G_0(v_1 + \epsilon, v_2) - G_0(v_1 - \epsilon, v_2))$ is already $o(1)$ for v_2 near the boundary of U except at the point z_1. Therefore the α_j will all tend to 0 as well. The result follows if we define $C_H(v_1, v_2) = G_0(v_1+\epsilon, v_2) - G_0(v_1-\epsilon, v_2)$.

On the other hand if $v_1 \in W_1$, then by (5), the function $(1/\epsilon)(G_0(v_1 + i\epsilon, v_2) + G_0(v_1 - i\epsilon, v_2))$ has two poles (each of residue $1/\pi$) within $o(1)$ of v_1. The remainder of the proof is similar to that of the previous theorem. □

Again note that when v_2 and v_1 are not close, in case $v_1 \in W_0$ we have $(1/\epsilon)\cdot C(v_1, v_2) = F_0(v_1, v_2) + o(1) = o(1)$ and when $v_1 \in W_1$ we have $(1/\epsilon)C(v_1, v_2) = F_1(v_1, v_2) + o(1)$.

The functions F_0, F_1 depend only on the conformal type of the domain U in the following sense. Let $F_+ = F_0 + F_1$ and $F_- = F_0 - F_1$.

PROPOSITION 15. *The function $F_+(z_1, z_2)$ is analytic as a function of both variables. The function $F_-(z_1, z_2)$ is analytic as a function of z_2 and antianalytic as a function of z_1. If V is another domain with smooth boundary and if $f: U \to V$ is a bijective complex analytic map sending the marked points on U to those of V, and if F_+^V, F_-^V are the functions defined as above for the region V then*

$$F_+^U(v, w) = f'(v)F_+^V(f(v), f(w)), \tag{7}$$

$$F_-^U(v, w) = \overline{f'(v)}F_-^V(f(v), f(w)). \tag{8}$$

PROOF. We already know that F_+, F_- are analytic in the second variable. Going back to the coupling function, for a fixed black vertex v_2 not adjacent to v_1 we have

$$-C(v_1+\epsilon, v_2) + C(v_1-\epsilon, v_2) - iC(v_1+i\epsilon, v_2) + iC(v_1-i\epsilon, v_2) = 0.$$

If $v_2 \in B_0$ and $v_1+\epsilon \in W_0$, this gives in the limit (using Theorem 13)

$$-\partial_{x_1} \operatorname{Re} F_0(v_1, v_2) + \partial_{y_1} \operatorname{Im} F_1(v_1, v_2) = 0$$

and if $v_2 \in B_1$ and $v_1+\epsilon \in W_0$, this gives

$$-\partial_{x_1} \operatorname{Im} F_0(v_1, v_2) - \partial_{y_1} \operatorname{Re} F_1(v_1, v_2) = 0.$$

These can be combined into a single complex equation

$$-\partial_{x_1} F_0(v_1, v_2) - i\partial_{y_1} F_1(v_1, v_2) = 0.$$

Similarly if $v_1+\epsilon \in B_1$ this gives

$$-\partial_{x_1} F_1(v_1, v_2) - i\partial_{y_1} F_0(v_1, v_2) = 0.$$

Summing these gives $\partial_{\overline{z_1}}(F_0+F_1) = 0$ and taking their difference and conjugating gives $\partial_{z_1}(F_0-F_1) = 0$. This proves the first two statements.

As a function of z_2, the function $F_0^V(f(z_1), f(z_2))$ has all the properties of F_0^U except that the residue at $z_2 = z_1$ is $1/(\pi f'(z_1))$. Similarly, the function $F_1^V(f(z_1), f(z_2))$ has all the properties of F_1^U except that the residue at $z_2 = z_1$ is $1/(\pi f'(z_1))$. So letting α, β be the real and imaginary parts of $f'(z_1)$ we have that

$$\alpha(z_1) F_0^V\big(f(z_1), f(z_2)\big) + i\beta(z_1) F_1^V\big(f(z_1), f(z_2)\big)$$

has residue $(\alpha(z_1)+i\beta(z_1))/\pi f'(z_1) = 1/\pi$ at $z_2 = z_1$, and all the other properties of F_0^U, and so must equal F_0^U since F_0^U is unique. A similar argument shows that

$$i\beta(z_1) F_0^V(f(z_1), f(z_2)) + \alpha(z_1) F_1^V(f(z_1), f(z_2)) = F_1^U.$$

The equations for F_+ and F_- follow. □

As an example, on the upper half-plane we have from (4) and (5) that

$$F_0(z_1, z_2) = \frac{1}{\pi(z_2 - z_1)} - \frac{1}{\pi(z_2 - \overline{z_1})}$$

and

$$F_1(z_1, z_2) = \frac{1}{\pi(z_2 - z_1)} + \frac{1}{\pi(z_2 - \overline{z_1})}.$$

These functions vanish at ∞, which can be thought of as the location of d_0. In particular $F_+(z_1, z_2) = 2/(\pi(z_2 - z_1))$, which is analytic in both variables, and $F_-(z_1, z_2) = -2/(\pi(z_2 - \overline{z_1}))$, which is analytic in z_2 and antianalytic in z_1.

Let U be the upper half-plane with d_0 located at 0 (that is, a square of type B_1 is removed near the origin). We can compute F_0^U, F_1^U for this new region U

by using the above transformation rules. A conformal isomorphism from the upper half-plane to itself which takes 0 to ∞ is $f(z) = -1/z$.

Since $f'(z_1) = z_1^{-2}$ we have

$$F_+^U(z_1, z_2) = \frac{1}{z_1^2} \frac{2}{\pi(f(z_2) - f(z_1))} = \frac{2z_2}{\pi z_1(z_2 - z_1)}.$$

Any other choice of $f(z)$ would give the same result. The function F_-^U is obtained similarly.

LEMMA 16. *Suppose $\delta > 0$. If Q is an $n \times n$ matrix $Q = (q_{ij})$ and for all i,*

$$q_{ii} - \delta > \sum_{j,\ j \neq i} |q_{ji}|$$

then $\det Q > \delta^n > 0$.

PROOF. Gaussian elimination using rows preserves this property: if for each j we multiply the first row by q_{j1}/q_{11} and subtract it from the jth row, the first column of the new matrix is all 0 except for the first entry q_{11}, and the remaining $n-1 \times n-1$ submatrix still has the property in the statement. For example, the first column of the submatrix is,

$$\left(q_{22} - \frac{q_{12}}{q_{11}} q_{21}, q_{32} - \frac{q_{12}}{q_{11}} q_{31}, \ldots, q_{n2} - \frac{q_{12}}{q_{11}} q_{n1}\right)$$

and

$$\begin{aligned} q_{22} - \frac{q_{12}}{q_{11}} q_{21} - \delta &> |q_{12}| + |q_{32}| + \cdots + |q_{n2}| - \frac{q_{12}}{q_{11}} q_{21} \\ &\geq |q_{32}| + \cdots + |q_{n2}| + \left(|q_{12}| - \frac{|q_{12}| \cdot |q_{21}|}{q_{11}}\right) \\ &= |q_{32}| + \cdots + |q_{n2}| + |q_{12}| \left(\frac{q_{11} - |q_{21}|}{q_{11}}\right) \\ &> |q_{32}| + \cdots + |q_{n2}| + \frac{|q_{12}|}{q_{11}} (\delta + |q_{31}| + \cdots + |q_{n1}|) \\ &\geq \left|q_{32} - \frac{q_{12}}{q_{11}} q_{31}\right| + \cdots + \left|q_{n2} - \frac{q_{12}}{q_{11}} q_{n1}\right|. \end{aligned}$$

□

Recall that the continuous Green's function on a region U is the real-valued function g_U satisfying $\Delta g_U(z_1, z_2) = \delta_{z_1}(z_2)$, and which is zero when z_2 is on the domain boundary (here δ_{z_1} is the Dirac delta-function, and $\Delta = -(\partial^2/\partial x^2) - (\partial^2/\partial y^2)$).

LEMMA 17. *Let $z_1 = x_1 + iy_1$ be a point in the interior of U, and let $z_2 \in U$, $z_2 \neq z_1$. Let v_1 be a vertex of $\mathbf{B}'_0(P_\epsilon)$ within $O(\epsilon)$ of z_1, and let v_2 be a vertex of $\mathbf{B}'_0(P_\epsilon)$ within $O(\epsilon)$ of z_2. Then the difference of (rescaled) Green's functions $(1/\epsilon)G(v_1+\epsilon, v_2) - (1/\epsilon)G(v_1-\epsilon, v_2)$ converges to $2\partial_{x_1} g_U(z_1, z_2)$.*

PROOF. Let $H(v_1, v_2) = (1/\epsilon)(G(v_1+\epsilon, v_2) - G(v_1-\epsilon, v_2))$. From Theorem 11, on the plane $\epsilon\mathbb{Z}^2$ we have

$$H_0(v_1, v_2) \stackrel{\text{def}}{=} \frac{1}{\epsilon}(G_0(v_1+\epsilon, v_2) - G_0(v_1-\epsilon, v_2))$$
$$= \operatorname{Re}\frac{1}{\pi(v_2 - v_1)} + O\left(\frac{1}{|v_2-v_1|^2}\right).$$

The function $H(v_1, v_2) - H_0(v_1, v_2)$ is harmonic (as a function of v_2) on all of $\mathbf{B}'_0(P_\epsilon)$ (including $v_1 \pm \epsilon$) and has bounded boundary values, since $H_0(v_1, v_2)$ is $O(1)$ on the boundary of $\mathbf{B}'_0(P_\epsilon)$ and $H(v_1, v_2)$ is zero there. Let g be the continuous harmonic function which has boundary values equal to the boundary values of the limit

$$\lim_{\epsilon\to 0} H(v_1, v_2) - H_0(v_1, v_2).$$

Since these boundary values are continuous in the limit, g exists and is unique. Note that the boundary values of $H - H_0$ are within $O(\epsilon)$ of the limiting values (Theorem 11).

Restrict g to a function on the vertices of $\mathbf{B}'_0(P_\epsilon)$. The discrete Laplacian of g at a vertex $v \in \mathbf{B}'_0(P_\epsilon)$ is

$$\Delta_\epsilon g(v_1, v) = 4g(v) - g(v+\epsilon) - g(v-\epsilon) - g(v-i\epsilon) - g(v+i\epsilon)$$

and when ϵ is small we can approximate this using the Taylor expansion of the smooth function g, yielding

$$\Delta_\epsilon g(v_1, v) = -\frac{\epsilon^4}{24}\left(\frac{\partial^4 g(v)}{\partial x^4} + \frac{\partial^4 g(v)}{\partial y^4}\right) + O(\epsilon^5).$$

Therefore, $H(v_1, v_2) - H_0(v_1, v_2) - g(v_1, v_2)$ has discrete Laplacian which is $O(\epsilon^4)$ on $\mathbf{B}'_0(P_\epsilon)$, and the boundary values are $O(\epsilon)$. A standard argument now shows that $H - H_0$ is close to g: the function $x + iy \mapsto x^2$ has discrete Laplacian which is a constant; choose constants B_2, B_3 sufficiently large so that

$$\Delta_\epsilon\left(B_2\epsilon^4(\operatorname{Re}(v_2))^2 + H(v_1, v_2) - H_0(v_1, v_2) - g(v_1, v_2)\right) \geq 0$$

and

$$\Delta_\epsilon\left(B_3\epsilon^4(\operatorname{Re}(v_2))^2 - H(v_1, v_2) + H_0(v_1, v_2) + g(v_1, v_2)\right) \geq 0$$

on $\mathbf{B}'_0(P_\epsilon)$. By the maximum principle for superharmonic functions, these functions must take their maximum value on the boundary of the domain $\mathbf{B}'_0(P_\epsilon)$.

Since $H(v_1, v_2) - H_0(v_1, v_2) - g(v_1, v_2) = O(\epsilon)$ on the boundary of $\mathbf{B}_0'(P_\epsilon)$, we conclude that

$$\left|H(v_1, v_2) - H_0(v_1, v_2) - g(v_1, v_2)\right| = O(\epsilon).$$

Therefore $H(v_1, v_2)$ converges to the function $\mathrm{Re}(1/\pi(v_2 - v_1)) + g(v_1, v_2)$ which has boundary values 0 and a single "pole" of residue $1/\pi$ at v_1. This two 2 times the x_1-derivative of the continuous Green's function. □

A similar result holds for the y_1-derivative of g_U, yielding the following corollary.

COROLLARY 18. *Recall the definitions of the functions* F_0, F_1, F_+, F_- *from Theorem* 13 *and Proposition* 15. *Letting* $z_1 = x_1 + iy_1$, *we have*

$$\begin{aligned} 2\, dg_U(z_1, z_2) &= F_0(z_1, z_2)\, dx_1 + F_1(z_1, z_2)\, dy_1 \\ &= \tfrac{1}{2} F_+(z_1, z_2)\, dz_1 + \tfrac{1}{2} F_-(z_1, z_2)\, \overline{dz_1} \end{aligned}$$

where the exterior differentiation dg_U *is with respect to the first variable.*

When $z_1 \in \partial U$ the proof of Lemma 17 implies the convergence of the Green's function as well.

COROLLARY 19. *Let* $\delta > 0$. *If* z_1 *is on the boundary of* U, *and the boundary of both* U *and* P_ϵ *is straight and horizontal in a* δ*-neighborhood of* z_1, *then for* v_1 *within* $O(\epsilon)$ *of* z_1,

$$\frac{1}{\epsilon} G(v_1, v_2) = g_U(z_1, z_2) + o(1).$$

PROOF. Reflect $\mathbf{B}_0'(P_\epsilon)$ across the boundary edge near z_1 (the edge consisting of vertices in Y) to get a graph $\mathbf{B}_0''(P_\epsilon)$. Glue $\mathbf{B}_0'(P_\epsilon)$ and $\mathbf{B}_0''(P_\epsilon)$ along their common edge in a δ-neighborhood of v_1. A harmonic function f on $\mathbf{B}_0'(P_\epsilon)$ which is zero on the boundary extends to a harmonic function on this glued graph by setting $f(v') = -f(v)$ when v' is the reflection of v. In other words, the Green's function $G(v_1, v_2)$ on $\mathbf{B}_0'(P_\epsilon)$ is the difference of two Green's functions on $\mathbf{B}_0'(P_\epsilon) \cup \mathbf{B}_0''(P_\epsilon)$: one centered at v_1 and one centered at v_1'.

On the glued graph $\mathbf{B}_0'(P_\epsilon) \cup \mathbf{B}_0''(P_\epsilon)$, the vertices v_1, v_1' are at distance at least δ from the boundary $\partial(\mathbf{B}_0'(P_\epsilon) \cup \mathbf{B}_0''(P_\epsilon))$, but only distance $O(\epsilon)$ from each other. The argument of Lemma 17 can then be applied in this case, replacing $H(v_1, v_2)$ by $(1/\epsilon)(G(v_1, v_2) - G(\overline{v_1}, v_2))$. □

A similar result holds when the boundary is vertical.

6. Conformal invariance of heights.

6.1. *Proof of Theorem* 1. Let U be a region in $\mathbb{C}$ with boundary which is piecewise smooth as previously defined. Let d'_j be a point on the jth boundary component D_j of U. Let $e'_j \neq d'_j$ be another point of D_j, which is not at a corner of the boundary.

Let P_ϵ be a Temperleyan polyomino approximating U in the sense of Section 5.3, with the additional constraint of having horizontal boundary in a neighborhood of each e'_j, and so that the interior of U is locally below each e_j. We show that the distribution of the heights of the boundary components of P_ϵ is conformally invariant.

Let e_j be a vertex on the boundary of P_ϵ near e'_j. We assume for simplicity that each e_j has the same parity (its coordinates have the same parity) as e_0. For definiteness we suppose the lattice square whose lower left corner is e_j is of type B_1 for each j.

Let h_j be the random variable giving the height of e_j for a random tiling of P_ϵ assuming the height of e_0 is zero. Let $\bar{h}_j$ be the mean value of h_j.

We will show that for integers $n_1, n_2, \dots, n_k \geq 0$, the moment

$$\mathbb{E}\big((h_1-\bar{h}_1)^{n_1}(h_2-\bar{h}_2)^{n_2}\cdots(h_k-\bar{h}_k)^{n_k}\big) \tag{9}$$

is conformally invariant. Let $K = n_1+\cdots+n_k$. The precise value of the moment (9) is as follows.

PROPOSITION 20. *Let* $\{\gamma_i\}_{i\in[1,K]}$ *be a collection of pairwise disjoint paths in* U, *such that for each* $j \in [1,k]$ *there are* n_j *paths runnning from the outer boundary to the* j*th boundary component. Then as* $\epsilon \to 0$ *the moment* (9) *converges to*

$$(-i)^K \sum_{\varepsilon_1,\dots,\varepsilon_K\in\{\pm1\}} \varepsilon_1\cdots\varepsilon_K \int_{\gamma_1}\cdots\int_{\gamma_K} \det_{i,j\in[1,K]}\big(F_{\varepsilon_i,\varepsilon_j}(z_i,z_j)\big)\, dz_1^{(\varepsilon_1)}\cdots dz_K^{(\varepsilon_k)}, \tag{10}$$

where $dz_j^{(1)} = dz_j$ *and* $dz_j^{(-1)} = d\overline{z_j}$ *and*

$$F_{\varepsilon_i,\varepsilon_j}(z_i,z_j) = \begin{cases} 0, & \text{if } i=j,\\ F_+(z_i,z_j), & \text{if } (\varepsilon_i,\varepsilon_j)=(1,1),\\ F_-(z_i,z_j), & \text{if } (\varepsilon_i,\varepsilon_j)=(-1,1),\\ \overline{F_-(z_i,z_j)}, & \text{if } (\varepsilon_i,\varepsilon_j)=(1,-1),\\ \overline{F_+(z_i,z_j)}, & \text{if } (\varepsilon_i,\varepsilon_j)=(-1,-1). \end{cases}$$

Note that in each of the 2^K multiple integrals in (10), the integrand I is conformally invariant, in the sense that

$$\int_\gamma I^U(\mathbf{z})\,d\mathbf{z} = \int_{f(\gamma)} I^V(f(\mathbf{z}))\,d\mathbf{z}.$$

This follows because of the transformation rules (7) and (8) and the fact that each integrand is analytic or antianalytic in z_i according to $\varepsilon_i = \pm 1$. Therefore the moment (9) is conformally invariant.

An example calculation is done in Section 6.3.

By [3], Section 30, there is a unique probability distribution with these moments on condition that the moment generating function

$$H(t_1, \dots, t_k) = \sum_{n_1, \dots, n_k \geq 0} \frac{m(n_1, \dots, n_k) t_1^{n_1} \cdots t_k^{n_k}}{n_1! \cdots n_k!}$$

has nonzero radius of convergence around the origin [here $m(n_1, \dots, n_k)$ is a shorthand for (9)]. This convergence is shown in Lemma 22, below. We can then conclude that the probability distribution with these moments is conformally invariant, and by [3], Theorem 30.2, that this distribution is the limit of the distributions for finite ϵ. This will complete the proof of Theorem 1. □

PROOF OF PROPOSITION 20. For each ϵ sufficiently small and for each $j \in [1, k]$ let $\gamma_{j1}^{(\epsilon)}, \dots, \gamma_{jn_j}^{(\epsilon)}$ be pairwise disjoint lattice paths (which are also disjoint for distinct j's) in P_ϵ which start on the flat boundary near e_0 and end on the flat boundary near e_j. We require that each straight edge of $\gamma_{js}^{(\epsilon)}$ have even length (by this we mean, a length which is an even multiple of ϵ). This is possible by our choice of parities for e_0 and e_j.

In a given tiling, the height change on $\gamma_{js}^{(\epsilon)}$ equals $4(A_{js} - B_{js})$, where A_{js} is the number of dominos crossing $\gamma_{js}^{(\epsilon)}$ with the black square on the right and B_{js} is the number of dominos crossing $\gamma_{js}^{(\epsilon)}$ with the black square on the left. To see this, note that if $\gamma_{js}^{(\epsilon)}$ does not cross any dominos, the height change is 0: the straight edges have even length so the height change along them is zero. Then, for each domino crossed by $\gamma_{js}^{(\epsilon)}$, the height difference changes along that edge from -1 to $+3$ if the domino has black square on the right, and from $+1$ to -3 if the black square is on the left.

Since $h_j = 4(A_{js} - B_{js})$ for each s, the moment (9) is equal to

$$4^K \mathbb{E}\big((A_{11} - B_{11} - \bar{A}_{11} + \bar{B}_{11}) \cdots (A_{kn_k} - B_{kn_k} - \bar{A}_{kn_k} + \bar{B}_{kn_k})\big), \tag{11}$$

where $K = n_1 + \cdots + n_k$.

The remainder of the proof involves expanding this out, cancelling various terms and then recombining in the right way.

For notational simplicity we renumber the paths $\gamma_{js}^{(\epsilon)}$ from 1 to K. Similarly, change indices of A_{js}, B_{js} to values in $[1, K]$. For $j \in [1, K]$ let α_{jt} be the tth possible domino of $\gamma_j^{(\epsilon)}$ crossing $\gamma_j^{(\epsilon)}$ whose black square is right of $\gamma_j^{(\epsilon)}$. Similarly let β_{jt} be the tth possible domino crossing $\gamma_j^{(\epsilon)}$ whose black square is on the left. Let α_{jt}, β_{jt} also denote the indicator functions of the presence of these edges/dominos. Then

$$A_j - B_j = \sum_t \alpha_{jt} - \sum_{t'} \beta_{jt'}. \tag{12}$$

Let (w_{js}, b_{js}) be the white and black squares, respectively, of the domino α_{js} and (w'_{js}, b'_{js}) be the white and black squares of the domino β_{js}.

Since the straight edges in the path $\gamma_j^{(\epsilon)}$ have even length, we can pair the α_{jt} dominos with adjacent $\beta_{jt'}$ dominos which are parallel to α_{jt}. It is then convenient to write

$$A_j - B_j - \bar{A}_j + \bar{B}_j = \sum_t \big(\alpha_{jt} - \bar{\alpha}_{jt} - \beta_{jt} + \bar{\beta}_{jt}\big),$$

where α_{jt} and β_{jt} are paired. Equation (11) is now

$$4^K \sum_{t_1,\dots,t_\ell} \mathbb{E}\big((\alpha_{1t_1} - \bar{\alpha}_{1t_1} - \beta_{1t_1} + \bar{\beta}_{1t_1}) \cdots (\alpha_{Kt_K} - \bar{\alpha}_{Kt_K} - \beta_{Kt_K} + \bar{\beta}_{Kt_K})\big), \tag{13}$$

where the sums are over all pairs $(\alpha_{1t_1}, \beta_{1t_1})$ of $\gamma_1^{(\epsilon)}$, $(\alpha_{2t_2}, \beta_{2t_2})$ of $\gamma_2^{(\epsilon)}$ and so on.

LEMMA 21. *Let $e_i = (w_i, b_i)$ for $i = 1, \dots, n$ be a set of n disjoint edges; then*

$$\mathbb{E}\big((e_1 - \bar{e}_1) \cdots (e_n - \bar{e}_n)\big) = a_E \det \begin{pmatrix} 0 & C(w_1, b_2) & \cdots & C(w_1, b_n) \\ C(w_2, b_1) & 0 & & \vdots \\ \vdots & & & C(w_{n-1}, b_n) \\ C(w_n, b_1) & \cdots & C(w_n, b_{n-1}) & 0 \end{pmatrix},$$

where (using the convention after Theorem 7) a_E is the product of the edge weights of the e_i.

PROOF. This follows from Theorem 7, induction on n and the fact that

$$\begin{vmatrix} 0 & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & \\ \vdots & & \ddots & \\ a_{n1} & & & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & \\ \vdots & & \ddots & \\ a_{n1} & & & a_{nn} \end{vmatrix} - \begin{vmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & a_{n2} & \cdots & a_{nn} \end{vmatrix}. \qquad \square$$

Now expand the summand of (13) into 2^K terms,

$$\mathbb{E}\big((\alpha_{1t_1} - \bar{\alpha}_{1t_1}) \cdots (\alpha_{Kt_K} - \bar{\alpha}_{Kt_K})\big) + \cdots + (-1)^K \mathbb{E}\big((\beta_{1t_1} - \bar{\beta}_{1t_1}) \cdots (\beta_{Kt_K} - \bar{\beta}_{Kt_K})\big). \tag{14}$$

By Lemma 21, each term is a certain quantity a_E times the determinant of a $K \times K$ matrix whose entries are given by the coupling function connecting black squares of the dominos α_{st_s}, β_{st_s} with white squares of the other dominos. Since each "β" edge has weight of the opposite sign as the "α" edge to which it is paired, the signs in (14) cancel with the sign changes in the a_E and so (14) is equal to the *sum* of all 2^K determinants times the product a_E of the edge weights of the *first* determinant.

Consider the first term in (14):

$$\mathbb{E}\big((\alpha_{1t_1} - \bar{\alpha}_{1t_1}) \cdots (\alpha_{Kt_K} - \bar{\alpha}_{Kt_K})\big). \tag{15}$$

Recall that $(w_{js}, b_{js}) = \alpha_{js}$ and $(w'_{js}, b'_{js}) = \beta_{js}$. Fix a choice of indices $s = s_j$ for the moment so we can drop the second subscripts. By Lemma 21, (15) is then equal to

$$a_E \begin{vmatrix} 0 & C(w_2, b_1) & \cdots & C(w_K, b_1) \\ C(w_1, b_2) & 0 & & \vdots \\ \vdots & & \ddots & C(w_K, b_{K-1}) \\ C(w_1, b_K) & \cdots & C(w_{K-1}, b_K) & 0 \end{vmatrix}. \tag{16}$$

A typical term in the expansion of (16) is

$$a_E \operatorname{sgn}(\sigma) C(w_1, b_{\sigma(1)}) C(w_2, b_{\sigma(2)}) \cdots C(w_K, b_{\sigma(K)}), \tag{17}$$

where σ has no fixed points.

Let us first assume that σ is a K-cycle; reorder the indices so that (17) becomes

$$a_E \operatorname{sgn}(\sigma) C(w_1, b_2) C(w_2, b_3) \cdots C(w_K, b_1). \tag{18}$$

To expand this out, define variables $r_i = \pm 1$ according to whether $w_i \in W_0$ or $w_i \in W_1$, and $s_i = \pm 1$ according to whether $b_i \in B_0$ or $b_i \in B_1$. If we assume that neither w_1 or b_2 is close to the boundary, we can then write (see Theorem 13 and the remarks immediately after its statement)

$$\begin{aligned} C(w_1, b_2) &= \epsilon\Big(\frac{1 - r_1 s_2}{2} i \operatorname{Im} + \frac{1 + r_1 s_2}{2} \operatorname{Re}\Big)\Big(\frac{1 + r_1}{2} F_0(w_1, b_2) + \frac{1 - r_1}{2} F_1(w_1, b_2)\Big) \\ &\qquad + o(\epsilon) \\ &= \frac{\epsilon}{4}\big(F_+(w_1, b_2) + r_1 F_-(w_1, b_2) + s_2 \overline{F_-}(w_1, b_2) + r_1 s_2 \overline{F_+}(w_1, b_2)\big) + o(\epsilon). \end{aligned}$$

For each fixed $\xi > 0$, when neither of w_1, b_2 are within ξ of the boundary, this approximation holds for sufficiently small ϵ. When one or both of w_1, b_2 are within ξ of the boundary, we only need to know that $(1/\epsilon)C(w_1, b_2)$ is bounded by some constant independent of ϵ and ξ. Then in the sum (13) [and in the integral (10)] we can ignore all terms in which some w_i or b_j is within

ξ of the boundary, as these will contribute at most $O(\xi)$. The boundedness of $(1/\epsilon)C(w_1, b_2)$ follows from the convergence of the discrete Green's function as in Theorem 13.

We can now write (18) as

$$
\begin{aligned}
4^{-K}\epsilon^K a_E \operatorname{sgn}(\sigma)\Big(\big(&F_+(w_1,b_2)+r_1F_-(w_1,b_2)\\
&+s_2\overline{F_-}(w_1,b_2)+r_1s_2\overline{F_+}(w_1,b_2)\big)\times\cdots\times\\
&\times\big(F_+(w_K,b_1)+r_KF_-(w_K,b_1)\\
&+s_1\overline{F_-}(w_K,b_1)+r_Ks_1\overline{F_+}(w_K,b_1)\big)\Big)+o(\epsilon^K).
\end{aligned}
\tag{19}
$$

We obtain a similar expression if we replace (w_1, b_1) by (w_1', b_1'), except that the signs of r_1 and s_1 are reversed. In particular if we sum up over all 2^K choices of α_j and β_j [as we need to do to obtain (14)], we get 2^K times the sum of those terms in (19) which have r_i to the same power (1 or 0) as s_i, for each i. This sum can therefore be written as an error $o(\epsilon^K)$ plus

$$
\begin{aligned}
2^{-K}\epsilon^K \operatorname{sgn}(\sigma) a_E \sum_{\varepsilon_1,\dots,\varepsilon_K\in\{-1,1\}} &(r_1s_1)^{(1-\varepsilon_1)/2}\cdots(r_Ks_K)^{(1-\varepsilon_K)/2}\\
&\times F_{\varepsilon_1,\varepsilon_2}(z_1,z_2)F_{\varepsilon_2,\varepsilon_3}(z_2,z_3)\cdots F_{\varepsilon_K,\varepsilon_1}(z_K,z_1),
\end{aligned}
\tag{20}
$$

where $F_{\varepsilon_i,\varepsilon_j}(z_i, z_j)$ is as defined in Proposition 20.

Now in view of replacing the sum (14) by an integral when ϵ is small, we can replace ϵ by a certain phase time $\frac{1}{2}\,dz_j$ or $\frac{1}{2}\,d\bar{z}_j$. When the path γ_j is going east (horizontal and to the right), we have $2\epsilon = dx_j = dz_j = d\bar{z}_j$, and the edge of type α has weight $-i$, because its upper vertex is white and lower vertex black (recall that edges of type α have black vertices on their right). Furthermore, $r_js_j = -1$ on an east-going path. When the path γ_j is going west, $2\epsilon = -dx_j = -dz_j = -d\bar{z}_j$, the edge of type α has weight i, and $r_js_j = -1$. When the path γ_j is going north, $2\epsilon = dy_j = -idz_j = id\bar{z}_j$, the edge α has weight 1, and $r_js_j = 1$. When the path γ_j is going south, $2\epsilon = -dy_j = idz_j = -id\bar{z}_j$, the edge α has weight -1, and $r_js_j = 1$. Notice that in each case 2ϵ times the edge weight, times $(r_js_j)^{(1-\varepsilon_j)/2}$ is $-\varepsilon_j i dz_j^{(\varepsilon_j)}$ (recall the definition of $dz_i^{(\varepsilon_i)}$ from Proposition 20). Recalling that a_E is the product of the edge weights (of the α-type edges), for any choices of the ε_j we have

$$
a_E(2\epsilon)^K(r_1s_1)^{(1-\varepsilon_1)/2}\cdots(r_Ks_K)^{(1-\varepsilon_K)/2} = (-i)^K\varepsilon_1\cdots\varepsilon_K dz_1^{(\varepsilon_1)}\cdots dz_K^{(\varepsilon_K)}.
$$

The sum (20) is therefore

$$
\begin{aligned}
4^{-K}(-i)^K \operatorname{sgn}(\sigma) \sum_{\varepsilon_1,\dots,\varepsilon_K\in\{-1,1\}} &\varepsilon_1\cdots\varepsilon_K\cdot F_{\varepsilon_1,\varepsilon_2}(z_1,z_2)F_{\varepsilon_2,\varepsilon_3}(z_2,z_3)\\
&\cdots F_{\varepsilon_K,\varepsilon_1}(z_K,z_1)\,dz_1^{(\varepsilon_1)}\cdots dz_K^{(\varepsilon_K)}.
\end{aligned}
\tag{21}
$$

When σ is a product of disjoint cycles we can treat each cycle separately and the result is the product of terms like (21) involving disjoint sets of indices.

Thus when we sum over all (fixed-point free) permutations we obtain the formula of the proposition, but without the integral. The factor of 4^{-K} cancels with the factor of 4^K in (13), and summing over all pairs gives the integral in (10). This completes the proof. □

LEMMA 22. *The moment generating function for the moments* (10) *has positive radius of convergence.*

PROOF. Letting $K = n_1 + \cdots + n_k$ denote the "size" of the moment, it suffices to show that a moment of size K is smaller than $(cK)^K$ for a constant c. Let $\gamma_1, \ldots, \gamma_K$ be the paths of integration in (10). We can choose the γ_i so that no two are closer than c_1/K for some constant c_1; indeed, we can choose the paths so that the distance between γ_i and γ_j is at least $c_1|i-j|/K$. Since $F_0(z_1, z_2)$ and $F_1(z_1, z_2)$ are $O(1/|z_1 - z_2|)$, in the determinant in (10) the ij-entry is at most $c_2K/|i-j|$ in absolute value. The determinant of a matrix is bounded by the product of the ℓ_2-norms of its rows, and each row of the determinant in (10) has ℓ_2-norm bounded by $K(2+2/2^2+\cdots+2/(K/2)^2)^{1/2} = c_3K$ for another constant c_3. Therefore the sum of the integrals in (10) is bounded by $c_4^K K^K$ for a constant c_4. This completes the proof. □

6.2. *The average height.* Let $U \subset \mathbb{C}$ be a region with piecewise smooth boundary as previously defined. Let b be a point on the outer boundary of U, $b \neq d_0$. For each $\epsilon \ll \delta$ let P_ϵ approximate U as in Section 5.3, but with the additional constraint of having horizontal boundary in a δ-neighborhood of b. (We also assume that the interior of P_ϵ is locally above the boundary at b.) Let z be a point in the interior of U. Let $z' \in P_\epsilon$ be within $O(\epsilon)$ of z and let $b' \in \partial P_\epsilon$ be within $O(\epsilon)$ of b. We assume that b' and z' are the lower left corners of lattice squares of type B_1. Let $\gamma^{(\epsilon)}$ be a lattice path from b' to z' such that all edges of $\gamma^{(\epsilon)}$ have even length, and which starts straight and north-going for a distance at least $c\delta$ for some constant c. In the notation of the previous section, we have $\mathbb{E}(h(z)) = 4\sum \mathbb{E}(\alpha_s) - \mathbb{E}(\beta_s)$ where α_s, β_s are pairs of potential dominos crossing the path $\gamma^{(\epsilon)}$.

Near the boundary, $\gamma^{(\epsilon)}$ is north-going. When α_s, β_s are within $o(1)$ of the boundary we have

$$\mathbb{E}(\alpha_s) = C(w_s, b_s) = C_H(w_s, b_s) + O(\epsilon)$$

and

$$\mathbb{E}(\beta_s) = -C(v_s', w_s') = -C_H(w_s', b_s') + O(\epsilon)$$

(note that α_s has weight 1 and β_s has weight -1 when $\gamma^{(\epsilon)}$ is north-going). Therefore, using Theorem 14,

$$\begin{aligned}\mathbb{E}(\alpha_s - \beta_s) &= C_H(w_s, b_s) + C_H(w_s', b_s') + O(\epsilon)\\ &= \frac{1}{4} + C_0(\overline{w_s}, b_s) + \frac{-1}{4} + C_0(\overline{w_s'}, b_s') + O(\epsilon)\\ &= C_0(0, b_s - \overline{w_s}) + C_0(0, b_s' - \overline{w_s'}) + O(\epsilon).\end{aligned}$$

Near the boundary, $b_s - \overline{w_s}$ takes successively values $1+2i, 1+6i, \dots, 1+(2+4k)i \cdots$ and $b'_s - \overline{w'_s}$ takes successively values $1+4i, 1+8i, \dots, 1+4ki \cdots$. When we sum over all pairs $(w_s, b_s), (w'_s, b'_s)$ on the path $\gamma^{(\epsilon)}$ which are within $o(1)$ of the boundary, the contribution is $o(\epsilon)$ plus

$$C_0(0, 1+2i) + C_0(0, 1+4i) + \cdots + C_0(0, 1+2ki) + \cdots = \tfrac{1}{2}.$$

(This formula can be proved analytically from Proposition 10 or more simply by symmetry, noting that the average height on the upper half-plane is $\frac{1}{2}$ given that the height on the boundary alternates between 0 and 1.)

For the terms not near the boundary we have, by Theorem 13, when $\gamma^{(\epsilon)}$ is north-going,

$$\begin{aligned} C(w_s, b_s) + C(w'_s, b'_s) &= \frac{1}{4} + \epsilon \operatorname{Re} F_1^*(z_s, z_s) + \frac{-1}{4} + \epsilon \operatorname{Re} F_0^*(z_s, z_s) + o(\epsilon) \\ &= \operatorname{Re}(F_+^*(z_s, z_s)\epsilon) + o(\epsilon), \end{aligned}$$

where z_s is the coordinate of w_s and $F_+^* = F_0^* + F_1^*$. Similarly for the other directions of γ we have

$$C(w_s, b_s) + C(w'_s, b'_s) = \begin{cases} \operatorname{Re}\big(-F_+^*(z_s, z_s)\epsilon\big) + o(\epsilon), & \text{when } \gamma \text{ is south-going,} \\ \operatorname{Im}\big(F_+^*(z_s, z_s)\epsilon\big) + o(\epsilon), & \text{when } \gamma \text{ is east-going,} \\ \operatorname{Im}\big(-F_+^*(z_s, z_s)\epsilon\big) + o(\epsilon), & \text{when } \gamma \text{ is west-going.} \end{cases}$$

We can replace ϵ by $\frac{1}{2}\,dz_s$, $-\frac{i}{2}\,dz_s$, $-\frac{1}{2}\,dz_s$, $\frac{i}{2}\,dz_s$, respectively, according to whether γ is east-, north-, west- or south-going. Then all four cases become

$$C(w_s, b_s) + C(w'_s, b'_s) = \tfrac{1}{2} \operatorname{Im}\big(F_+^*(z_s, z_s)\, dz_s\big) + o(\epsilon).$$

The average height is then given by the imaginary part of the integral of $2F_+^*(z, z)dz$ from b to z (recall the factor of 4 from the first paragraph of this section), plus $\frac{1}{2}$, the constant coming from the boundary. This expression does not depend on δ.

For another region V conformally equivalent to U we have the following. Let $f\colon V \to U$ be a conformal isomorphism. Then $F_+^V(z_1, z_2) = f'(z_1)F_+^U(f(z_1), f(z_2))$ from Proposition 15. Therefore,

$$\begin{aligned} (F_+^V)^*(z_1, z_2) &= F_+^V(z_1, z_2) - \frac{2}{\pi(z_2 - z_1)} \\ &= -\frac{2}{\pi(z_2 - z_1)} \\ &\quad + f'(z_1)\left((F_+^U)^*(f(z_1), f(z_2)) + \frac{2}{\pi(f(z_2) - f(z_1))}\right) \end{aligned}$$

and in the limit as $z_2 \to z_1$ this is (simplifying using the Taylor expansion of f)

$$f'(z_1)(F_+^U)^*(f(z_1), f(z_1)) - \frac{f''(z_1)}{\pi f'(z_1)}.$$

So the average height of $z \in V$ equals the average height of $f(z)$ in U, plus a term

$$-\frac{2}{\pi}\int_{f(\gamma)} (\log f'(z))'\, dz.$$

This term is $-2/\pi$ times the change in total turning (in radians) of the path $f(\gamma)$ from the path γ.

This implies that if the path γ starts at the outer boundary of U, at a point where the tangent vector (chosen in the counterclockwise direction) has angle θ with respect to the horizontal axis (where $\theta \in [0, 2\pi)$), then the average height of a point $z \in U$ is

$$\frac{1}{2} + \frac{2\theta}{\pi} + 2\,\mathrm{Im}\int_{\gamma} F_{+}^{*}(z_1, z_1)\, dz_1.$$

Therefore we have the following theorem.

THEOREM 23. *Up to an additive constant, the average height of a point z not within $o(1)$ of the boundary of U is given by the harmonic function whose boundary values are $2\theta(x)/\pi$, where $\theta(x)$ is the total turning (in radians) of the tangent vector to the boundary on the boundary path going counterclockwise from d_0' to x.*

Note that the boundary values are discontinuous at the point d_0'.

For example, as noted earlier on the upper half-plane when $d_0 = \infty$ the average height of every point is $\frac{1}{2}$. When $d_0 = 0$, rather, then recall that $F_{+}(z_1, z_2) = 2z_2/\pi z_1(z_2 - z_1)$. So $F_{+}^{*}(z_1, z_1) = 2/\pi z_1$. The average height at a point z is [integrating from $x = \mathrm{Re}(z)$]

$$\begin{aligned}\mathbb{E}(h(z)) &= \frac{1}{2} + 2\,\mathrm{Im}\int_x^z \frac{2dz_1}{\pi z_1}\\ &= \frac{1}{2} + \frac{4}{\pi}\,\mathrm{Im}\log(z/x)\\ &= \frac{1}{2} + \frac{4}{\pi}\arg(z).\end{aligned}$$

This is the harmonic function with boundary values (on the axis) $\frac{1}{2}$ to the right of the origin and $\frac{9}{2} = \frac{1}{2} + 4$ to the left of the origin. Note that on the boundary of the polyomino P_ϵ, the height alternates between 0 and 1 to the right of the origin and between 4 and 5 to the left of the origin.

6.3. *Example: a second moment computation.* For a random tiling of the upper half-plane with $d_0 = \infty$ we compute the moment $\mathbb{E}((h(p) - \bar{h}(p))(h(q) - \bar{h}(q))$ for two points p, q. Since $\bar{h}(p) = \bar{h}(q) = \frac{1}{2}$, this will also give $\mathbb{E}(h(p)h(q))$.

Let r, s be the vertical projections of p, q, respectively, to the x-axis. Let γ_1 and γ_2 be disjoint paths running straight from the boundary to p, q, respectively. From Theorem 10, we have

$$\begin{aligned}\mathbb{E}((h(p)-\bar{h}(p))(h(q)-\bar{h}(q)) = &-\int_{\gamma_1,\gamma_2}\begin{vmatrix} 0 & F_+(z_1,z_2) \\ F_+(z_2,z_1) & 0\end{vmatrix}\,dz_1\,dz_2 \\ &+\int_{\gamma_1,\gamma_2}\begin{vmatrix} 0 & F_-(z_1,z_2) \\ \overline{F_-(z_2,z_1)} & 0\end{vmatrix}\,d\overline{z_1}\,dz_2 \\ &+\int_{\gamma_1,\gamma_2}\begin{vmatrix} 0 & \overline{F_-(z_1,z_2)} \\ F_-(z_2,z_1) & 0\end{vmatrix}\,dz_1\,d\overline{z_2} \\ &-\int_{\gamma_1,\gamma_2}\begin{vmatrix} 0 & \overline{F_+(z_1,z_2)} \\ \overline{F_+(z_2,z_1)} & 0\end{vmatrix}\,d\overline{z_1}\,d\overline{z_2}.\end{aligned}$$

For the upper half-plane we have $F_+(z_1,z_2) = 2/\pi(z_2-z_1)$ and $F_-(z_1,z_2) = 2/\pi(z_2-\overline{z_1})$. Plugging these in gives

$$\begin{aligned}&-\frac{4}{\pi^2}\int_{\gamma_1}\int_{\gamma_2}\frac{1}{(z_2-z_1)^2}\,dz_1\,dz_2 + \frac{4}{\pi^2}\int_{\gamma_1}\int_{\gamma_2}\frac{1}{(z_2-\overline{z_1})^2}\,d\overline{z_1}\,dz_2 \\ &+\frac{4}{\pi^2}\int_{\gamma_1}\int_{\gamma_2}\frac{1}{(\overline{z_2}-z_1)^2}\,dz_1\,d\overline{z_2} - \frac{4}{\pi^2}\int_{\gamma_1}\int_{\gamma_2}\frac{1}{(\overline{z_2}-\overline{z_1})^2}\,d\overline{z_1}\,d\overline{z_2}.\end{aligned}$$

The first of these integrals gives

$$-\frac{4}{\pi^2}\log\frac{(p-q)(r-s)}{(p-s)(r-q)}.$$

Therefore,

$$\begin{aligned}&\mathbb{E}\big((h(p)-\bar{h}(p))(h(q)-\bar{h}(q))\big) \\ &\quad=\frac{4}{\pi^2}\left(-2\,\mathrm{Re}\log\frac{(p-q)(r-s)}{(p-s)(r-q)} + 2\,\mathrm{Re}\log\frac{(\overline{p}-q)(r-s)}{(\overline{p}-s)(r-q)}\right) \\ &\quad=\frac{8}{\pi^2}\,\mathrm{Re}\log\left(\frac{\overline{p}-q}{p-q}\right).\end{aligned}$$

7. Trees and winding number. A *directed spanning tree* on a (undirected) graph G is a connected contractible (acyclic) collection of edges of G, where each edge has a chosen direction such that each vertex but one has exactly one outgoing edge. The single vertex with no outgoing edge is called the *root* of the tree. If G is a graph with boundary, (that is, there is a subset of vertices called the *boundary* of G), then a *directed essential spanning forest* is a collection of edges of G, each component of which is contractible, where each edge has a chosen direction, such that each nonboundary vertex has exactly one outgoing edge, and no boundary vertex has an outgoing edge.

"Temperley's trick" (see [5]) is a mapping between domino tilings of certain polyominos and directed essential spanning forests of associated graphs. In the case P is a Temperleyan polyomino, the directed essential spanning forest is on the graph $\mathbf{B}_0'(P)$ of Section 4.1 and the boundary consists of the set Y. The forest is defined from a tiling as follows. Each square v in $B_0 \cap P$ is covered by a domino. The white square of this domino lies over an edge of $\mathbf{B}_0'(P)$. This edge is chosen to be the outgoing edge of v on the tree on $\mathbf{B}_0'(P)$. See Figure 7 for the directed essential spanning forest associated to the domino tiling of Figure 3. To see that the essential spanning forest constructed from a tiling has no cycles, it suffices to construct the planar dual forest, which is constructed in a similar way from the graph $\mathbf{B}_1(P) \cup \{d_0\}$. In the case P is a Temperleyan polyomino, the dual forest is a tree rooted at d_0 (since d_0 is the only possible root). Since the dual tree is connected, the primal tree has no cycles.

Conversely, any essential spanning forest on $\mathbf{B}_0'(P)$ gives a domino tiling of P, so these systems are in bijection.

The height function of a domino tiling has a nice interpretation for the directed paths in the associated spanning tree. To a vertex v in $\mathbf{B}_0'(P)$ associate a height which is the average of the heights of the four vertices of P adjacent to v. If the outgoing edge of the tree at v points to an adjacent vertex v', and the outgoing edge at v' points to a vertex v'', then the height at v' equals the height at v if the three vertices v, v', v'' are aligned; if the path turns left at v' then the height at v' is one less than the height at v; if the path turns right at v' then the height at v' is one more than the height at v.

Therefore the height function along the directed path measures the net turning of the path.

PROPOSITION 24. *Let P be a Temperleyan polyomino with a tiling and let T be the associated essential spanning forest. The height change along a directed*

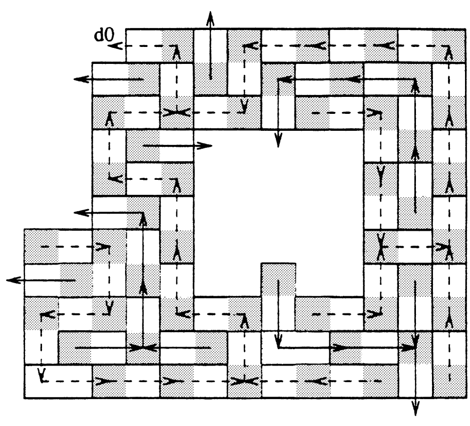

FIG. 7. *The directed essential spanning forest (solid arrows) associated to the tiling of Figure* 3. *The dual tree is shown in dotted arrows.*

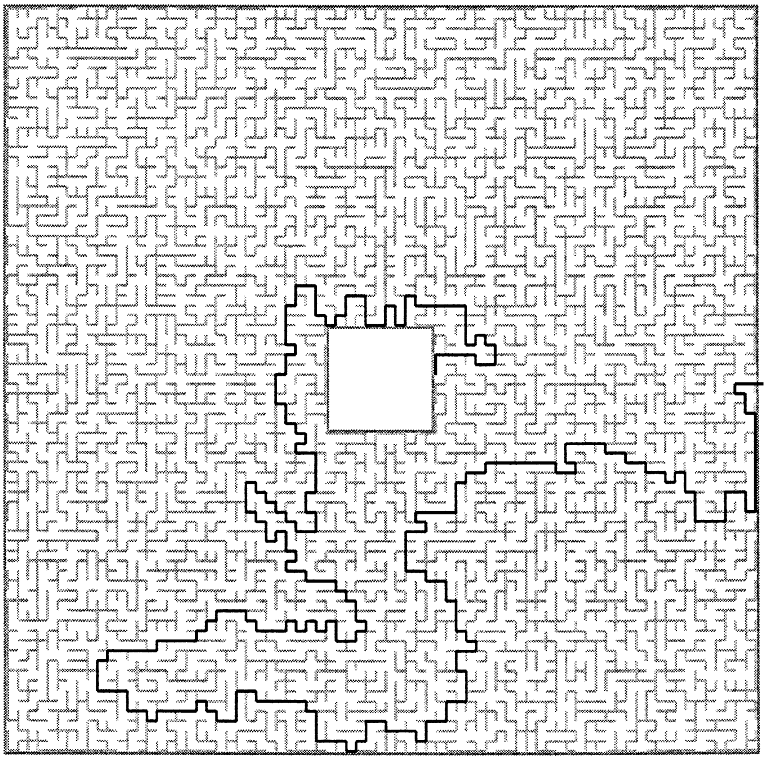

FIG. 8. *The spanning forest associated to a tiling of an annulus.*

path γ in T equals the net turning of the path, that is, the number of right turns minus the number of left turns.

In particular if γ is a directed path in T running between $d_j \in D_j$ and the outer boundary, the height difference between D_j and D_0 is exactly measured by the winding number of the path γ (around D_j).

In Figure 8 we show the spanning tree associated to a tiling of a Temperleyan annulus in which the height difference between the boundaries is 4. The directed path from a vertex adjacent to d_1 to d_0 is highlighted.

8. Other boundary conditions. There are a number of intuitive ideas in the proof of Theorem 1 which are worthwhile exploring. Foremost is the interesting link between the height function along a boundary component and the singularities of the coupling function. When we introduced the exposed vertices in our polyominos (in order to make it tilable) we "created" poles in the coupling function at those points. There are a number of other, equally simple, boundary conditions which give different boundary behavior for the coupling function. The most natural seems to be to have all boundary edges have even length. This is natural from the point of view of tilings since it is trivial to show that such a region has a tiling. Furthermore the height function along such a boundary is particularly simple in this case. However the boundary conditions for the coupling function are more difficult: on some boundary edges the real part will be zero and on others the imaginary part will be zero. The coupling function will have poles at certain corners and zeros at the remaining corners. It seems more difficult to prove the convergence of the coupling function when $\epsilon \to 0$ in this case.

Another potential improvement in the proof would be a more general result (more general than Corollary 19) concerning the convergence of the discrete Green's function centered near the boundary of a domain. Surprisingly, this problem does not seem to have been considered in the literature.

Another direction to be explored is the case of regions without boundary. In [13] we computed a formula for the coupling function on a torus. By a recent result of Tesler [18] higher-genus surfaces can be handled by similar methods.

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