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ON THE SPEED OF RANDOM WALKS ON GRAPHS[1]

BY BÁLINT VIRÁG

University of California, Berkeley

Lyons, Pemantle and Peres asked whether the asymptotic lower speed in an infinite tree is bounded by the asymptotic speed in the regular tree with the same average number of branches. In the more general setting of random walks on graphs, we establish a bound on the expected value of the exit time from a vertex set in terms of the size and distance from the origin of its boundary, and prove this conjecture. We give sharp bounds for limiting speed (or, when applicable, sublinear rate of escape) in terms of growth properties of the graph. For trees, we get a bound for the speed in terms of the Hausdorff dimension of the harmonic measure on the boundary. As a consequence, two conjectures of Lyons, Pemantle and Peres are resolved, and a new bound is given for the dimension of the harmonic measure defined by the biased random walk on a Galton–Watson tree.

1. Introduction. Once the transience of a random walk on a graph is determined, it is natural to ask questions about its rate of escape from the starting point. This paper studies how linear rate of escape (speed) is related to the structure of the graph.

Let $G = (V, E, w)$ be a countable, connected, undirected graph with a positive edge weight function $w\colon\ E \to \mathbb{R}$. Heuristically, $w(e)$ is the multiplicity of the edge e in G. The random walk on the weighted graph G starts at a distinguished vertex (o, the root), and at each step moves to one of the neighbors of its current position with odds given by the edge weights.

The size of graphs with exponential growth can be measured in many ways. A simple measure of growth is the *lower growth*, $\underline{\mathrm{gr}}(G)$, given by the lim inf of the nth root of the total weight of edges at distance n from a fixed vertex, which we will call the root, o. The same quantity can be expressed using the following size measure for edge sets. For an edge or vertex, let $|\cdot|$ denote its graph distance from o. For a subset of edges K, we can define a size measure which exponentially punishes edges that are far from the root:

$$\|K\|_\beta = c^{-1} \sum_K w(e)\beta^{-|e|}, \tag{1.1}$$

where the normalizing constant c equals the total weight on edges adjacent to o. Let ∂W denote the edge boundary vertex set W. Then the expression

$$\sup\left\{\beta \mid \inf_W \|\partial W\|_\beta > 0\right\},$$

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where W ranges over all balls about o, equals $\underline{\mathsf{gr}}$. If we let $W \ni o$ range over all finite sets, we get the *branching number* br of the graph; if we let $W \ni o$ range over all sets from which the walk started from o exits almost surely, we get a new quantity called the *essential branching number*, eb. It is easy to see that br and eb do not depend on the choice of o. For more intuition behind br in trees, see Lyons and Peres (1999).

The speed of a random walk $\{X_k\}$ on a graph is given by the process $\{|X_k|/k\}$, where $|\cdot|$ denotes graph distance from the root, which is also the starting point of the walk. The lower speed $\underline{S}$ of the random walk can be defined as the lim inf of the speed process; when this a.s. coincides with the lim sup, we say that asymptotic speed exists. Our main result is the following.

THEOREM 1.1. $\underline{S} \le (\mathsf{eb}-1)/(\mathsf{eb}+1) \vee 0$ *a.s.*

Variants of this theorem have been conjectured by Benjamini and Peres [see Peres (1997)], Lyons, Pemantle and Peres (1997). Specifically, our results solve two of the questions raised in Lyons, Pemantle and Peres (1997).

An elementary argument using the strong law of large numbers shows that equality holds for regular trees. The intuitive meaning of this bound is that among graphs with the same essential branching number, none admits a simple random walk that is with positive probability faster than the one on the regular tree.

As a by-product of our results, we get an upper bound for the lim inf sublinear rate of escape for graphs of subexponential growth; the following statement is a simple corollary of a more comprehensive result.

THEOREM 1.2. *Let G be a weighted graph satisfying*

$$\log \|\partial W\|_1 \le 2c \cdot \operatorname{dist}(o, \partial W)^\gamma$$

for $c > 0$, $\gamma \in (0,1)$ fixed, and infinitely many balls W about o. Then the random walk $\{X_k\}$ on G satisfies

$$\liminf |X_k|/k^{1/(2-\gamma)} \le c^{1/(2-\gamma)}.$$

The proof of these results rely on some finitistic lemmas about the lifetime of the random walk killed when it exits a vertex set W. Random walks on finite graphs are used in the theory of randomized algorithms, where the lifetime of a walk (or running time of an algorithm) is of natural interest [see Sinclair and Jerrum (1989)]. In Sections 2 and 4 we prove bounds on the lifetime and exit speed in terms of the size of ∂W and $\operatorname{dist}(o, \partial W)$; the following theorem contains a summary of some of these results.

THEOREM 1. *For $\beta > \beta' > 1$ there exist N, $p > 0$ so that the following holds. Let τ be the lifetime of the random walk on a locally finite weighted graph G started at a vertex o and killed when it exits a set W with $n := \mathrm{dist}(o, \partial W) \geq N$. Suppose that $\tau < \infty$ a.s.*:

(i) *If $\|\partial W\|_1^{1/n} \leq \beta'$, then $\mathbf{E}\tau/n > \alpha(\beta)$, and $\mathbf{P}(\tau/n > \alpha(\beta)) > p$.*
(ii) *If $\|\partial W\|_{\beta'} \leq 1$, then $\mathbf{P}(\tau/|X_\tau| > \alpha(\beta)) > p$.*

Proposition 2.1 gives a bound on the expected lifetime; the proof uses a deterministic flow construction on the graph which records the "expected path" of the random walk. It turns out that the expected lifetime does not always reflect the typical behavior of the lifetime of the walk. To go from expected value results to positive probability ones, we will need some known tools such as the second moment method and Doob transforms, which are introduced in Section 3. As a high expected lifetime might come from atypical parts of the graph, we need to use an argument which, roughly, keeps track of which parts of the graph are visited. This gives the positive probability lifetime bounds of Proposition 4.1; finally, conditioning on the distance of the exit point from the root, we gain a positive probability bound on the exit speed (Lemma 4.2).

As expected, these bounds are nearly sharp for balls about o in regular trees. We thus get an answer to a question of Benjamini and Peres, who asked which, among all trees with k^l vertices at level l, minimizes the expected hitting time of level l. Lee (1994) considered a special case of this problem for random walks on spherically symmetric trees and found that the answer is close to the regular tree, different only because of a slight asymmetry introduced by starting and stopping the walk. Our results for finite graphs show that, even in the more general setting, the regular tree is not far from optimal.

Previously, Peres gave a rough upper bound for the lower speed in trees in terms of the branching number, using a percolation argument [see Häggström (1997), Peres (1997)]. It follows from this bound that positive lower speed implies positive branching number. Takacs (1997, 1998) and Takacs and Takacs (1998) calculated the asymptotic speed for special classes of trees, using walk-invariant measures on tree-space. Lyons, Pemantle, and Peres (1995) computed the speed explicitly for the simple random walk on Galton–Watson family trees. For biased random walks on Galton–Watson trees, a tree-space and random walk average version of the analogue of Theorem 1.1 was proved by Chen (1997).

For a tree T, one can define the distance for two rays (infinite self-avoiding paths starting at the root) with n common edges as e^{-n}. Under this distance, the set of rays, ∂T, is a compact metric space with Hausdorff dimension $\log \mathsf{br}$. Consider the λ-biased random walk on T. This moves to a neighbor of its current position with odds λ for the parent and one for each child. If this walk is transient, then erasing cycles from its path gives us a random ray; the corresponding probability measure on ∂T is called harmonic measure. The dimension $d(\lambda)$ of this measure is related to what portion of the tree the random walk could potentially explore; of course, $d(\lambda) \le \log \mathrm{br}$. In Section 6 we prove that

$$\underline{S} \le \frac{e^{d(\lambda)} - \lambda}{e^{d(\lambda)} + \lambda} \qquad \text{a.s.} \tag{1.2}$$

This was believed to be false for general trees [Lyons, Pemantle, Peres (1997)]; the same paper, as well as Benjamini and Peres (1992), show through several counterexamples that many other properties, although intuitive, do not hold for general trees.

The inequality (1.2) can also be thought of as a lower bound for the dimension of the harmonic measure. From this perspective, our result is related to, albeit not a proof of, a conjecture of Lyons, Pemantle and Peres (1997) that the dimension of the harmonic measure on the family tree of a Galton–Watson branching process is a.s. greater than the dimension of the corresponding measure for greedier random walk that moves to a uniformly chosen random offspring of its current position; in short, the greedy walker sees less of the tree. Such implications of our results to Galton–Watson trees are discussed in Section 7.

2. A bound on the expected lifetime. This section gives a bound on the expected lifetime of a random walk killed when it exits a set W in terms of the size and distance from o of its boundary ∂W. The walk on G killed when it exits W is defined as the usual random walk on the weighted graph up to time $\tau + 1$, where the *lifetime* τ is defined as the last time before the first exit from W. After time $\tau + 1$, the walk is undefined.

PROPOSITION 2.1. *Let W be a vertex set in a weighted graph $G = (V, E, w)$, and let τ be the lifetime of the random walk started at $o \in W$ and killed when it exits W. Let g_v denote the expected number of visits to the vertex v, and let*

$$\alpha(x) := (x+1)/(x-1). \tag{2.1}$$

(i) *We have*

$$\mathbf{E}\tau/\mathrm{dist}(o, \partial W) \ge \alpha\left([g_o \|\partial W\|_1]^{1/\mathrm{dist}(o, \partial W)}\right), \tag{2.2}$$

(ii) *For $\beta > \beta' > 1$, there exists N so that for all G, W satisfying $\mathrm{dist}(o, \partial W) > N$ and $\|\partial W\|_1^{1/\mathrm{dist}(o, \partial W)} \le \beta'$, we have*

$$\mathbf{E}\tau/\mathrm{dist}(o, \partial W) \ge \alpha(\beta).$$

PROOF. Without loss of generality we may identify all vertices of G outside W as a single vertex δ, and may assume that G and W are connected.

The difficulty in studying the relation between $\mathrm{dist}(o, \partial W)$, $\|\partial W\|_1$ and $\mathbf{E}\tau$ comes from the fact that the latter quantity is a complicated function of the weights on the graph. To avoid this problem, a deterministic flow will be introduced, which allows for a relatively simple expression of all quantities we want to compare.

Let $E^* = \{(u, v)\colon \ \{u, v\} \in E(G)\}$ be the set of directed edges of G. The random walk on G defines an E^*-valued process $\{Y_k\} = \{(X_k, X_{k+1})\}$. The random edge function

$$F(e) := \sum_{k=0}^{\tau} \mathbf{1}(Y_k = e)$$

can be thought of as a random flow with source o and sink δ. A flow here means an edge function for which $\sum_u F((v,u)) - \sum_u F((u,v))$, where the sums range over all neighbors u of v, is nonnegative if v is the source, nonpositive if v is the sink, and 0 otherwise.

The main idea is to study the deterministic flow

$$f(e) := \mathbf{E}F(e),$$

which helps relate $\mathbf{E}\tau$ to $\|\partial W\|_1$. Indeed, it is clear that $\sum_e f(e) = \mathbf{E}(\tau+1)$. Expressing $\|\partial W\|_1$ is slightly more difficult; towards this end, pick an orientation $\vec{e}$ for each edge $e \in E$, and consider the flow $\tilde{f}$ on this set of directed edges: $\tilde{f}(\vec{e}) = f(\vec{e}) - f(\overleftarrow{e})$. Then, redirect the edges so that $\tilde{f}$ is a nonnegative function. Denote the set of these directed edges by $\vec{E}$, and the set of their reversal by $\overleftarrow{E}$. For a directed or undirected edge e we will use the notation $\vec{e}$, $\overleftarrow{e}$ for the corresponding edge in $\vec{E}$, $\overleftarrow{E}$, respectively. By the connection between electrical networks and random walks, $\tilde{f}$ can be thought of as the current flow on the edges of G when voltage ν and 0 is attached to o and δ, respectively. Here ν must be equal to the effective resistance between o and δ so that the strength of the flow indeed equals 1.

It is possible to decompose the current flow as a sum of constant flows along paths from o to δ such that for all $\vec{e} \in \vec{E}$,

$$\tilde{f}(\vec{e}) = \sum_{\varphi \in \Phi} \tilde{f}_\varphi(\vec{e}).$$

Here $\tilde{f}_\varphi$ is a positive flow on $\vec{E}$ with source o and sink δ. It is supported on the path φ. Denote its strength by s_φ. We can now decompose the flow f as well. For $\varphi \in \Phi$, define

$$f_\varphi(e) := \frac{\tilde{f}_\varphi(\vec{e})}{\tilde{f}(\vec{e})} \times f(e),$$

and then for all edges $e \in E^*$ for which $\tilde{f}(\vec{e}) \neq 0$ we have $\sum_{\varphi\in\Phi} f_\varphi(e) = f(e)$.

Let φ^* denote the path φ truncated after its first $n := \operatorname{dist}(o, \partial W)$ edges. The following inequalities, to be proved later, give the relatively simple connection between $\mathbf{E}\tau$, $\|\partial W\|_1$ and the flow f and its decomposition,

$$\mathbf{E}\tau \geq \alpha\left(\frac{\sum f_\varphi(\vec{e})}{\sum f_\varphi(\overleftarrow{e})}\right) \times n, \tag{2.3}$$

where the sums on the right range over $\varphi \in \Phi$ and edges $e \in \varphi^*$. Also,

$$g_o\|\partial W\|_1 \geq \sum_{\varphi\in\Phi} s_\varphi \prod_{e\in\varphi^*} \frac{f_\varphi(\vec{e})}{f_\varphi(\overleftarrow{e})}. \tag{2.4}$$

The above two inequalities reduce our problem to an inequality about numbers. For simple reference, the result we need is stated in the language of flows, but flows are not used in its proof.

LEMMA 2.2. *Let Φ be a countable collection of "paths" each containing n "edges," and let s be a positive function on Φ which sums to 1. For each $\varphi \in \Phi$, let $e \mapsto f_\varphi(\vec{e})$, $e \mapsto f_\varphi(\overleftarrow{e})$ be positive functions of the edges $e \in \varphi$ satisfying $f_\varphi(\vec{e}) - f_\varphi(\overleftarrow{e}) = s_\varphi$. Then*

$$\sum_{\varphi\in\Phi} s_\varphi \prod_{e\in\varphi} \frac{f(\vec{e})}{f(\overleftarrow{e})} \geq \left(\frac{\sum f_\varphi(\vec{e})}{\sum f_\varphi(\overleftarrow{e})}\right)^n, \tag{2.5}$$

where the sums on the right range over paths $\varphi \in \Phi$ and edges $e \in \varphi$.

Combining the formulas (2.3), (2.4) and Lemma 2.2 applied to the collection of paths $\{\varphi^*\colon\ \varphi \in \Phi\}$ yields the first claim (2.2) of the proposition.

For the second claim, observe that if it were false, then there would be a sequence of graphs, vertex sets W_i, and increasing $n_i = \operatorname{dist}(o_i, \partial W_i)$, with $\|\partial W_i\|_1^{1/n_i} < \beta'$ so that $g_{o,i} \leq \mathbf{E}\tau_i + 1 \leq \mathbf{E}\tau_i < \alpha(\beta)n_i + 1$. Thus by (2.2),

$$\mathbf{E}\tau_i/n_i \geq \alpha\left([n_i\alpha(\beta)+1]^{1/n_i}(\beta')\right),$$

and the right-hand side converges to $\alpha(\beta') > \alpha(\beta)$ as $i \to \infty$, a contradiction.

PROOF OF (2.3). It follows from the definitions that

$$\begin{aligned}\mathbf{E}\tau + 1 &= \sum_{e\in E} (f(\vec{e}) + f(\overleftarrow{e})) \geq \sum_{\varphi\in\Phi,\, e\in\varphi} \left(f_\varphi(\vec{e}) + f_\varphi(\overleftarrow{e})\right) \\ &\geq \sum_{\varphi\in\Phi,\, e\in\varphi^*} \left(f_\varphi(\vec{e}) + f_\varphi(\overleftarrow{e})\right) + 1.\end{aligned} \tag{2.6}$$

The constant 1 appears because summing over the truncated flows means leaving out the sum of the values of the flow on ∂W, which equals 1. Since the total strength of the current flows is 1, and each φ^* is of length n,

$$\sum_{\varphi\in\Phi,\, e\in\varphi^*} f_\varphi(\vec{e}) - f_\varphi(\overleftarrow{e}) = n.$$

The claim (2.3) follows from this and (2.6).

PROOF OF (2.4). Let $\varphi \in \Phi$, and let $\varphi_i, \vec{\varphi}_i, \overleftarrow{\varphi}_i$ denote the ith edge in the path undirected, directed forward (in the path or, equivalently, in $\vec{E}$), or directed backward, respectively. Let l denote the number of edges (the length) of the path φ, and let $w(E_o)$ denote the total weight on edges adjacent to o. Then φ_l is in ∂W, and we can write

$$\frac{w(\varphi_l)}{w(E_o)} = \frac{w(\varphi_1)}{w(E_o)} \prod_{i=1}^{l-1} \frac{w(\varphi_{i+1})}{w(\varphi_i)}, \tag{2.7}$$

since the product telescopes. Let $p((u,v))$ denote the transition kernel of the walk, and recall the notation g_v for the expected number of hits to a vertex v up to time τ. Then by the definition of p, (2.7) equals

$$p(\vec{\varphi}_1)\prod_{i=1}^{l-1}\frac{p(\vec{\varphi}_{i+1})}{p(\overleftarrow{\varphi}_i)} = \frac{f(\vec{\varphi}_1)}{g_o}\prod_{i=1}^{l-1}\frac{f(\vec{\varphi}_{i+1})}{f(\overleftarrow{\varphi}_i)} = \frac{f(\vec{\varphi}_l)}{g_o}\prod_{i=1}^{l-1}\frac{f(\vec{\varphi}_i)}{f(\overleftarrow{\varphi}_i)}. \tag{2.8}$$

For the first equality, note that $f((u,v)) = p((u,v))g_u$ for all $(u,v) \in E^*$, and the second one is just a rearrangement. Since $f(\vec{e})/f(\overleftarrow{e}) = f_\varphi(\vec{e})/f_\varphi(\overleftarrow{e})$, we can replace the former by the latter in the last formula. It follows that $\prod_{i=1}^{l(\varphi)-1} f_\varphi(\vec{\varphi}_i)/f_\varphi(\overleftarrow{\varphi}_i)$ is constant over paths φ with the same final edge, and so for $e \in \partial W$,

$$\frac{w(e)}{w(E_o)} = \sum_{\varphi\ni e}\frac{s_\varphi}{g_o}\prod_{i=1}^{l(\varphi)-1}\frac{f_\varphi(\vec{\varphi}_i)}{f_\varphi(\overleftarrow{\varphi}_i)}.$$

Note that each factor in the product is greater than 1. Leaving out all but the first n factors and summing over $e \in \partial W$ yields (2.4). □

PROOF OF LEMMA 2.2. Inequality (2.5) can be reduced to two simple cases.

The parallel inequality. Suppose that $n = 1$. Let $a_\varphi := f_\varphi(\overleftarrow{\varphi}_1)$ and let $A := \sum a_\varphi$, then (2.5) reduces to

$$\sum_{\varphi\in\Phi} s_\varphi \frac{a_\varphi + s_\varphi}{a_\varphi} \geq \frac{A+1}{A}.$$

This simplifies to $A\sum s_\varphi^2/a_\varphi \geq 1$, which follows from Cauchy–Schwarz. Equality holds iff a_φ/s_φ is constant, that is, $f_\varphi(\vec{\varphi}_1)/f_\varphi(\overleftarrow{\varphi}_1)$ is the same for all φ.

The series inequality. Consider a graph consisting of a single path. Dropping the path index we get

$$\begin{aligned}\prod_e \frac{f(\vec{e})}{f(\overleftarrow{e})} &= \exp\sum_e \log\frac{s + f(\overleftarrow{e})}{f(\overleftarrow{e})}\\ &\geq \exp\left(n\log\frac{s+\sum_e f(\overleftarrow{e})/n}{\sum_e f(\overleftarrow{e})/n}\right) = \left(\frac{\sum_e f(\vec{e})}{\sum_e f(\overleftarrow{e})}\right)^n.\end{aligned}$$

The inequality uses that the function $x \mapsto \log((s+x)/x)$ is convex. Equality holds iff $f(\vec{e})/f(\overleftarrow{e})$ is constant over the edges.

By the series inequality the left-hand side of (2.5) is at least

$$\sum_{\varphi\in\Phi} s_\varphi\left(\frac{\sum_{e\in\varphi} f_\varphi(\vec{e})}{\sum_{e\in\varphi} f_\varphi(\overleftarrow{e})}\right)^n \geq \left(\sum_{\varphi\in\Phi} s_\varphi \frac{\sum_{e\in\varphi} f_\varphi(\vec{e})}{\sum_{e\in\varphi} f_\varphi(\overleftarrow{e})}\right)^n \geq \left(\frac{\sum f_\varphi(\vec{e})}{\sum f_\varphi(\overleftarrow{e})}\right)^n,$$

where the sums on the right-hand side range over $\varphi \in \Phi$ and edges $e \in \varphi$. The first inequality follows from the convexity of $x \mapsto x^n$, the second, from the parallel inequality. □

3. Tools for probability bounds. The tools introduced in this section will be useful in converting the expected value bounds of the previous section to positive probability ones.

LEMMA 3.1 (Second moment method). *Let W be a subset of vertices in a weighted graph $G = (V, E, w)$, and let $\mathbf{P}_v$, τ denote the distribution and the lifetime of the random walk started at a vertex $v \in W$ and killed when it exits W. Suppose there is a vertex $v \in W$ where $v \mapsto \mathbf{E}_v\tau$ takes its maximum. Then for $p \in (0, 1)$ we have*

$$\mathbf{P}_v(\tau > (1-p)\mathbf{E}_v\tau - p) \geq p^2/2. \tag{3.1}$$

PROOF. The Markov property of $\{X_k\}$ can be used to show that the second moment of $\tau' := \tau + 1$ is bounded by twice the first moment squared,

$$\begin{aligned}\mathbf{E}_v\tau'^2 &= \mathbf{E}_v \sum_{\substack{u,\,w\in W\\ k,\,l\geq 0}} \mathbf{1}(X_k = u)\mathbf{1}(X_l = w) \leq 2\mathbf{E}_v \sum_{\substack{u,\,w\in W\\ k,\,d\geq 0}} \mathbf{1}(X_k = u)\mathbf{1}(X_{k+d} = w)\\ &\leq 2\mathbf{E}_v\tau' \sup_{u\in W}\mathbf{E}_u\tau' = 2(\mathbf{E}_v\tau')^2.\end{aligned}$$

For any random variable Z with $\mathbf{E}Z \geq 0$ and $\mathbf{E}Z^2 \leq 1$ and for $0 < p \leq 1$ we have

$$\beta := \mathbf{P}(Z > -p) \geq \frac{1}{1+p^{-2}} \geq p^2/2, \tag{3.2}$$

and with the choice $Z := (\tau' - \mathbf{E}_v\tau')/\mathbf{E}_v\tau'$, the inequality (3.1) follows. To see (3.2), let Z_- and Z_+ be random variables distributed as Z given that it is at most $-p$, and greater than $-p$, respectively. The assumptions can be written as

$$(1-\beta)\mathbf{E}Z_- + \beta\mathbf{E}Z_+ \geq 0, \qquad (1-\beta)\mathbf{E}Z_-^2 + \beta\mathbf{E}Z_+^2 \leq 1. \tag{3.3}$$

If we replace Z_- by $-p$ and Z_+ by its expectation γ, (3.3) still holds, and it becomes a system of inequalities in two unknowns β, γ. Solving these for β, one gets (3.2). □

The second tool, Doob transform from random walks on graphs, describes how the law of random walk changes when we condition on simple future events [Doob (1959)]. It is valid for a larger class of events than stated here.

LEMMA 3.2 (Doob transform). *Let W be a subset of vertices in a weighted graph $G = (V, E, w)$, and let τ be the lifetime of the random walk $\{X_k\}$ started at $o \in W$ and killed when it exits W. Let A be an event that G never* (*not even at the time of its death*) *hits another set of vertices V. Let G' be the reweighting of G determined by the edge weights*

$$w'(\{u, v\}) = w(\{u, v\})\mathbf{P}_u A \mathbf{P}_v A,$$

and let $\{X'_k\}$ be random walk on G' killed when it exits the set $W' := W$ in G'. Then $\{X_k\}$ given A and $\{X'_k\}$ have the same distribution. Moreover, we have $\|\partial W'\|_\beta \le (\mathbf{P}_o A)^{-2}\|\partial W\|_\beta$.

PROOF. By summing over paths it is easily checked that $\{X_k\}$ conditioned on A is a Markov chain with stationary transition probabilities given by

$$\mathbf{P}(X_k = v | X_{k-1} = u, A) = P_u(X_1 = v)\frac{\mathbf{P}_v A}{\mathbf{P}_u A}. \tag{3.4}$$

But $\{X'_k\}$ has exactly these transition probabilities. If E_o is the set of vertices adjacent to o, then $w'(E_o)$ can be written as

$$\sum_{v\sim o} w(\{o, v\})\mathbf{P}_o A \mathbf{P}_v A = \mathbf{P}_o A \sum_{v\sim o} w(E_o)\mathbf{P}_o(X_1 = v)\mathbf{P}_v A = (\mathbf{P}_o A)^2 w(E_o).$$

The second statement follows from substituting this into (1.1) as a normalizing constant. □

4. Probability bounds for lifetime and speed. We now have all the ingredients to prove a positive probability bound on the lifetime of a walk killed when it exits a set.

PROPOSITION 4.1. *Let W be a subset of vertices in a locally finite weighted graph $G = (V, E, w)$, and let τ be the lifetime of the random walk started at $o \in W$ and killed when it exits W. Let g_v denote the expected number of visits to the vertex v, and let $\alpha(x) := (x + 1)/(x - 1)$.*

(i) *For all $p \in (0, 1)$ the event*

$$(\tau + p)/\operatorname{dist}(o, \partial W) \ge \alpha\left([8g_o\|\partial W\|_1]^{1/\operatorname{dist}(o,\partial W)}\right)(1 - p), \tag{4.1}$$

has probability at least $p^2/8$.

(ii) *For $\beta > \beta' > 1$, there exists $N, p > 0$ so that for all G, W satisfying $\operatorname{dist}(o, \partial W) > N$ and $\|\partial W\|_1^{1/\operatorname{dist}(o,\partial W)} \le \beta'$, we have*

$$\mathbf{P}(\tau/\operatorname{dist}(o, \partial W) \ge \alpha(\beta)) > p. \tag{4.2}$$

PROOF. We will define recursively a finite sequence of vertex sets $\{V_i\}$, which exclude more and more of the atypically slow parts of the graph. Here B_i will denote the event that the walk on G stays in the set V_i. To reduce the analysis to finite graphs, let V_0 be a finite connected subset of W so that $\mathbf{P}B_0 \ge 3/4$.

If $\mathbf{P}_o B_i > 0$, then define recursively v_i as a vertex V_i where $v \mapsto \mathbf{E}_v(\tau|B_i)$ takes its maximum. Set V_{i+1} to be the (possibly empty) connected component of o in $V_i \setminus \{v_i\}$.

This recursive definition terminates when $\mathbf{P}_o B_i = 0$. The events B_i are decreasing in i, so for some m we have $\mathbf{P}_o B_m \ge 1/2 > \mathbf{P}_o B_{m+1}$. Let t be some constant, and let A denote the event $\{\tau \ge t\}$. Set $C_i := B_i \setminus B_{i+1}$; roughly speaking, the event C_i means that the walk reaches certain slow parts of the graph but avoids others,

$$\begin{aligned}\mathbf{P}_o A &\geq \sum_{i=0}^{m} \mathbf{P}_o(A|C_i)\mathbf{P}_o C_i \\ &\geq \inf_{i=0,\dots,m} \mathbf{P}_o(A|C_i) \sum_{i=0}^{m} \mathbf{P}_o C_i \geq \inf_{i=0,\dots,m} \mathbf{P}_{v_i}(A|B_i)/4.\end{aligned} \tag{4.3}$$

For the last inequality, note that last sum equals $\mathbf{P}_o(B_0 \setminus B_{m+1}) \geq 1/4$, and that

$$\mathbf{P}_o(A|C_i) \geq \mathbf{P}_{v_i}(A|C_i) = \mathbf{P}_{v_i}(A|B_i).$$

Lemma 3.2 implies that the Doob transform of G, V_i corresponding to B_i is G^i, V^i with

$$\|\partial V^i\|_1 \leq \|\partial V_i\|_1 (\mathbf{P}_o B_i)^{-2} \leq 4(\beta')^{\operatorname{dist}(o,\partial V)}. \tag{4.4}$$

If we have bounds $\mathbf{E}_o^i \tau \geq t$ in G^i, then necessarily $\mathbf{E}_v^i \tau \geq t$ for $v = v_i$, where this expected value takes its maximum. Lemma 3.1 applied to G' gives $\mathbf{P}_{v_i}(\tau \geq t(1-p)|B_i) \geq p^2/2$, and this together with (4.3) implies

$$\mathbf{P}_o(\tau \geq t(1-p)) \geq p^2/8. \tag{4.5}$$

Since $\mathbf{P}B_i \geq 1/2$, we have that the expected number of hits to the root given B_i satisfies $g_{o,i} \leq 2g_o$. Using this, (4.4), and Proposition 2.1, we get a possible value for the bound t on $\mathbf{E}^i\tau$. Claim (4.1) then follows from (4.5).

For the second claim, pick $\beta''' < \beta''$ in (β', β), and let N be large so that $4\beta'^N < \beta'''^N$. For large N, (4.4) and Proposition 2.1 then allow us to set $t = \alpha(\beta'')\operatorname{dist}(o, \partial W)$ and, for a small enough p, (4.2) follows from (4.5). □

If the edges in ∂W have about equal distance from the root, then the above result gives a reasonable bound on the exit speed. If this is not the case, we need to condition on where the walk exits from W, as in the proof of the following proposition.

PROPOSITION 4.2. *Let $1 < \beta' < \beta$, then there exist N, $p > 0$ so that the following holds. Let τ be the lifetime of the random walk on a weighted graph G started at a vertex o and killed when it exits a set W with* $\operatorname{dist}(o, \partial W) \geq N$. *If $\|\partial W\|_{\beta'} \leq 1$, and $\tau < \infty$ a.s., then the event*

$$A := \{\tau/|X_\tau| > \alpha(\beta)\}$$

has probability at least p.

PROOF. Assume that $\|\partial V\|_{\beta'} \leq 1$. In order to get a bound on the exit speed, we will have to condition on the distance of the exit point from the root o. Let

$$B_l := \{|X_\tau| = l\}.$$

Consider those events B_l which have relatively high probability: for $\gamma > 1$, let $L = \{l\colon\ \mathbf{P}B_l > \gamma^{-l}\}$. Decomposing A we get

$$\mathbf{P}A \geq \sum_{l\in L} \mathbf{P}(A|B_l)\mathbf{P}(B_l) \geq \inf_{l\in L} \mathbf{P}(A|B_l) \sum_{l\in L} \mathbf{P}(B_l). \tag{4.6}$$

Let G^l denote the Doob transform of G with respect to the event B_l. Given B_l, each edge adjacent to X_τ has graph distance at most l from o, so for $l \in L$, Lemma 3.2 implies that

$$\gamma^{-2l}(\beta')^{-l}\|\partial V^l\|_1 \leq \gamma^{-2l}\|\partial V^l\|_{\beta'} \leq \|\partial V\|_{\beta'} \leq 1.$$

For a small choice of γ we have $\beta'' := \beta'\gamma^2 < \beta$. Then $\|\partial V^l\|_1 \leq \beta''^l$, and the second part of Proposition 4.1 in G^l implies that there is $N_0 = N_0(\beta'', \beta)$, so that for $l > N_0$ we have $\mathbf{P}(A|B_l) > 2p$. Now note that

$$\sum_{l\in L} \mathbf{P}B_l = 1 - \sum_{l\notin L} \mathbf{P}B_l > 1 - \sum_{l=N}^{\infty} \gamma^{-l}.$$

If $N > N_0$ is large, then this is more than $\frac{1}{2}$, hence by (4.6) $\mathbf{P}A > p$, as claimed. □

5. Rate of escape. Our main results, presented here, are relatively simple consequences of the positive probability bounds of the previous section.

THEOREM 1.1. *The lower speed* $\underline{S} = \liminf |X_k|/k$ *of the random walk on an infinite weighted graph and the essential branching number* eb *satisfy the inequality* $\underline{S} \leq (\mathsf{eb}-1)/(\mathsf{eb}+1) \vee 0$ *a.s.*

REMARK. Since $\mathsf{eb} \leq \mathsf{br} \leq \underline{\mathsf{gr}}$, we can replace eb in Theorem 1.1 by any of these quantities. Note that when $\mathsf{br} \leq 1$, this statement means that $\underline{S} = 0$. A nice proof of $\mathsf{br} = 1 \Rightarrow \underline{S} = 0$ in trees using percolation is due to Peres; see Häggström (1997) and Peres (1997).

PROOF OF THEOREM 1.1. Let $\beta > \beta' > \mathsf{eb}$. Then there are sets W from which the walk exits almost surely with arbitrary small size $\|\partial W\|_{\beta'}$. In particular, given i, we can find such a set W_i satisfying $\|\partial W_i\|_{\beta'} \leq 1$ and $\operatorname{dist}(o, \partial W_i) > i$. Let τ_i be the time spent in W_i before the first exit. Each time is a first exit time for only finitely many of these sets, and therefore the lim inf speed is bounded above by the lim inf of the speed at the times τ_i.

By Proposition 4.2 there is a positive p so that with $\alpha(x) := (x+1)/(x-1)$ we have

$$\mathbf{P}\left[\tau_i/|X_{\tau_i}| > \alpha(\beta)\right] > p,$$

and, taking limits, we get

$$\mathbf{P}\left[\limsup\ \tau_i/|X_{\tau_i}| \geq \alpha(\beta)\right] \geq \limsup\ \mathbf{P}\left[\tau_i/|X_{\tau_i}| > \alpha(\beta)\right] \geq p.$$

Using that $\underline{S}^{-1} \geq \limsup \ \tau_i/|X_{\tau_i}|$, we get

$$\mathbf{P}\left[\underline{S}^{-1} \geq \alpha(\beta)\right] \geq p. \tag{5.1}$$

Here p depends only on β and β', but not on the graph G. Since labeling another vertex v as root does not change $\limsup k/|X_k|$ nor the essential branching number, (5.1) holds for the random walk started at any vertex. Hence by the Markov property we have

$$\mathbf{P}\left[\underline{S}^{-1} \geq \alpha(\beta)|X_1, X_2, \dots, X_k\right] \geq p.$$

The σ-field generated by $X_1, X_2, \dots, X_k$ increases to one containing the event on the left. Thus the probability above converges to the indicator of this event by Lévy's 0–1 law. Therefore $\limsup \ k/|X_k| \geq \alpha(\beta)$ a.s., and since $\beta > \mathsf{eb}$ is arbitrary, the theorem follows. $\square$

The following proposition bounds sublinear lim inf rate of escape of the random walks in graphs of subexponential growth. A more concrete bound is given in Theorem 1.2.

PROPOSITION 5.1. *Suppose that G is a weighted graph with subexponential growth, in the sense that for some increasing sequence of balls W_i about o with radii l_i we have $\lim \|\partial W_i\|_1^{1/l_i} \leq 1$. Let $r\colon \ \mathbb{R}_{\geq 0} \to \mathbb{R}$ be a concave increasing function satisfying*

$$r(2l_i^2/\log \|\partial W_i\|_1) \geq l_i \tag{5.2}$$

for all i. Then the random walk $\{X_k\}$ on G satisfies $\liminf |X_k|/r(k) \leq 1$.

PROOF. Let $a_i := \|\partial W_i\|_1$. We can assume that $\lim \ a_i = \infty$, since otherwise $\{X_k\}$ is recurrent by standard arguments and the claim follows trivially. Thus $\lim \ a_i^{1/l_i} = 1$. Similarly, we can assume that the expected number of hits to each vertex is finite.

We will prove that for each $p \in (0,1)$ and vertex $v \in V$,

$$\mathbf{P}_v \left[\limsup \ r(k)/|X_k| \geq 1 - 2p\right] \geq p^2/8. \tag{5.3}$$

This implies the proposition by a 0–1 argument identical to the one in the previous proof.

Let $n_i = \operatorname{dist}(v, \partial W_i)$. Note that $l_i - |v| \leq n_i \leq l_i + |v|$. For large i, we have $v \in W_i$; let τ_i denote the time the walk started at v spends in W_i before the first exit. By Proposition 4.1, there exists a constant c so that, for all $p \in (0,1)$, each event

$$\tau_i + p \geq \alpha\left([ca_i]^{1/n_i}\right) n_i(1-p)$$

has probability at least $p^2/8$. Dividing both sides by $2l_i^2/\log \ a_i$ we get

$$\frac{\tau_i + p}{2l_i^2/\log \ a_i} \geq \frac{[ca_i]^{1/n_i} + 1}{([ca_i]^{1/n_i} - 1)\, 2l_i/\log \ a_i} \frac{n_i}{l_i}(1-p). \tag{5.4}$$

Since

$$(ca_i)^{1/n_i} - 1 = \exp\left(\log(ca_i)/n_i\right) - 1 \approx \log(ca_i)/n_i \approx \log\ a_i/l_i,$$

the right-hand side of (5.4) converges to $1 - p$, and therefore with probability at least $p^2/8$ the following hold. First,

$$\tau_i \geq 2(l_i^2/\log a_i)(1-2p)$$

for infinitely many i, and so, since r is increasing and concave,

$$\begin{aligned} r(\tau_i) &\geq r\left(2(l_i^2/\log\ a_i)(1-2p)\right) \\ &\geq r(0)\cdot 2p + r(2l_i^2/\log\ a_i)(1-2p) \geq 2pr(0) + l_i(1-2p) \end{aligned}$$

for infinitely many i, giving $1 - 2p \leq \limsup\ r(\tau_i)/l_i \leq \limsup\ r(k)/|X_k|$. This proves (5.3). □

As a corollary of this result we get the following proof.

PROOF OF THEOREM 1.2. Set $r(k) = (ck)^{1/(2-\gamma)}$ and apply the proposition. □

6. Trees and harmonic measure. Let T be a weighted infinite tree satisfying

$$w(e)/w(e^*) \leq \gamma \tag{6.1}$$

for some positive γ and all edges, where e^* is the edge adjacent to e closest to the root. A *ray* in T is an infinite self-avoiding path starting from the root. The quantities br, eb are related to dimension properties of the set of rays of T called the boundary, ∂T. For two rays φ, ψ, we denote the edge farthest from the root in their intersection by $\varphi \wedge \psi$. For trees satisfying (6.1), the distance

$$\text{distance}(\varphi, \psi) = w(\varphi \wedge \psi)\gamma^{-|\varphi\wedge\psi|} \tag{6.2}$$

can be easily checked to satisfy the triangle inequality. Moreover, the boundary ∂T under this distance is compact. Also, any open or closed ball about a ray φ is given by the set of all rays that eventually stay in a descendant subtree of a vertex in φ. Using this fact, it is easy to check that the Hausdorff dimension of the boundary satisfies $\gamma^{\dim_\gamma \partial T} = \mathsf{br}(T)$.

If the random walk is transient on T, then its loop-erased path is a random element of ∂T. The corresponding measure μ is called the harmonic measure on ∂T. The Hausdorff dimension $\dim_\gamma\ \mu$ of μ is defined as the infimum of Hausdorff dimensions of Borel sets with full μ-measure. It satisfies

$$\gamma^{\dim_\gamma \mu} \geq \mathsf{eb}(T). \tag{6.3}$$

To check this, note that if $d > \dim_\gamma\ \mu$, then there is a set of full measure with dimension less than d; there are covers where the sum of the dth powers of the diameters is arbitrarily small, so there are sets W which the walk exits almost surely with $\|\partial W\|_{\gamma^d}$ arbitrarily small.

Inequality (6.3) can be strict. Consider an infinite ternary tree, with binary trees of depth $h(|v|)$ attached to each vertex for some rapidly growing function h. Consider the vertex sets W_i whose boundary of all the leaves of the binary trees starting at level $n_1, n_2, \dots$ for some infinite sequence. For any $\beta > 2$, we can find a sequence $n_1, n_2, \dots$ so that the corresponding set has arbitrarily small boundary size $\|\partial W\|_\beta$. Hence such a tree has $3 = \gamma^{\dim_\gamma \mu} > \mathsf{eb}(T) = 2$. The harmonic measure ignores detours of the walk—the essential branching number takes them into account.

The λ-*biased random walk* in a tree moves to a random one of the neighbors of its current position, with odds λ for the parent and 1 for each child. We then have the following corollary to Theorem 1.1.

COROLLARY 6.1. *Consider a transient λ-biased random walk on an infinite unweighted tree T. Let $d(\lambda)$ denote the dimension of the harmonic measure on ∂T with respect to this walk and the metric $e^{-|a\wedge b|}$, and let $\underline{S}_\lambda$ denote the lower speed. Then*

$$\underline{S}_\lambda \le \frac{e^{d(\lambda)} - \lambda}{e^{d(\lambda)} + \lambda} \vee 0 \qquad a.s.$$

PROOF. The λ-biased random walk on T has the same law as the one in the weighted tree T' that has the same graph structure as T, but with edge weights $w(e) = \lambda^{-|e|}$. Denote the harmonic measure by μ, and let $\gamma > \lambda^{-1}$. Denote $\dim_\gamma$, $\dim'_\gamma$ Hausdorff dimension with respect to the distance (6.2) in T and T', respectively. Then from the definitions we get $\dim'_\gamma\ \mu = \log_\gamma\ \lambda^{-1} + \dim_\gamma\ \mu$, and that $d(\lambda) = \dim_e\ \mu = \ln\ \gamma\ \dim_\gamma\ \mu$. Combining these with (6.3), we get $\mathsf{eb}(T') \le e^{d(\lambda)}/\lambda$. The result now follows from Theorem 1.1 applied to T'. □

REMARK. Lyons, Pemantle and Peres (1997) show that, for transient biased walks, $e^{d(\lambda)} \ge \lambda$, so we can remove the "$\vee 0$" from the statement of Corollary 6.1.

7. Galton–Watson trees. Let T be the family tree of a Galton–Watson branching process with offspring distribution Z, and suppose that $Z \ge 1$, that is, each parent has at least one child, and that Z is nonconstant. Consider a transient λ-biased random walk on T. It is known that the asymptotic speed s_λ exists and is constant a.s.

Let $\dim\ \partial T$ denote the Hausdorff dimension of the boundary. Hawkes (1981) and Lyons (1990) showed that $\dim\ \partial T = \log\ \mathbf{E}Z$ a.s. The dimension $d(\lambda)$ of the harmonic measure is also known to be constant for each λ a.s.

Transience implies $\log\ \lambda \le \dim\ \partial T$ [Lyons (1990)]. Lyons, Pemantle and Peres (1996) proved that when $\log\ \lambda \ne \dim\ \partial T$, the dimension $d(\lambda)$ is strictly less than $\dim\ \partial T$ if $\mathbf{E}Z\ \log\ Z < \infty$. A conjecture of the same authors (1997) follows from Corollary 6.1 (note the strict inequality).

COROLLARY 7.1. *For the asymptotic speed s_λ of the λ-biased random walk on Galton–Watson trees with nonconstant offspring distribution Z satisfying*

$\mathbf{E}Z > \lambda$ *we have*

$$s_\lambda < \frac{\mathbf{E}Z - \lambda}{\mathbf{E}Z + \lambda}.$$

The dimension of the harmonic measure $d(\lambda)$ gauges the size of the subtree of T the random walk can potentially explore. Heuristically, as λ increases, the random walk tends to backtrack more, and thus explores more of the tree, so one expects that $d(\lambda)$ is an increasing function of λ. Counterexamples to this heuristic are given for general deterministic trees and for family trees of multi-type branching processes in Lyons, Pemantle, Peres (1997). However, for simple Galton–Watson trees it is still unknown whether $d(\lambda)$ is monotone. In the case $\lambda = 0$ the walk always moves away from the root; the resulting harmonic measure is called visibility measure. Lyons, Pemantle and Peres (1995) showed that $d(0) = \mathbf{E} \log Z$ a.s.

Corollary 6.1 can be used to give a lower bound for general $d(\lambda)$. We have

$$\frac{1/\mathbf{E}Z^{-1} - \lambda}{1/\mathbf{E}Z^{-1} + \lambda} < s_\lambda \le \frac{e^{d(\lambda)} - \lambda}{e^{d(\lambda)} + \lambda},$$

where the first inequality is due to Chen (1997). For $\lambda = 1$, Lyons, Pemantle and Peres (1996) proved that $s_1 = \mathbf{E}[(Z-1)/(Z+1)]$ a.s. Putting these results together, we get the corollary.

COROLLARY 7.2. *With α as in* (2.1), *the dimension $d(\lambda)$ of the harmonic measure of the biased random walk on a Galton–Watson tree satisfies*

$$d(\lambda) > -\log \mathbf{E}(Z^{-1}), \qquad d(1) \ge \log \alpha\left(\mathbf{E}(\alpha(Z)^{-1})\right).$$

To summarize what is known about the dimension $d(\lambda)$, note the following consequence of Jensen's inequality:

$$-\log \mathbf{E}(Z^{-1}) \le \log \alpha\left(\mathbf{E}(\alpha(Z)^{-1})\right) \le \mathbf{E} \log Z \le \log \mathbf{E}Z. \tag{7.1}$$

The first and second expressions are the lower bounds for $d(\lambda)$ for general λ and $\lambda = 1$. The third is the exact value of $d(0)$, the last is the dimension of the boundary. Our corollary is not sufficient to establish monotonicity properties; however, it provides the first known lower bound for $d(\lambda)$ [apart from the simple inequality $d(\lambda) \ge \log \lambda$; see the remark to Corollary 6.1].

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DEPARTMENT OF STATISTICS
UNIVERSITY OF CALIFORNIA
BERKELEY, CALIFORNIA 94720
E-mail: balint@stat.berkeley.edu