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ISOTROPIC LÉVY PROCESSES ON RIEMANNIAN MANIFOLDS

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Under a natural invariance assumption on the Lévy measure we construct compound Poisson processes and more general isotropic Lévy processes on Riemannian manifolds by projection of a suitable horizontal process in the bundle of orthonormal frames. We characterize such Lévy processes through their infinitesimal generators and we show that they can be realized as the limit of a sequence of Brownian motions which are interlaced with jumps along geodesic segments.

1. Introduction. A Lévy process in $\mathbf{R}^n$ is essentially a stochastic process with stationary and independent increments. There are a number of equivalent characterizations of such processes: Fourier analysis yields the beautiful Lévy–Khintchine formula for the characteristic function, the Markov property gives rise to a Feller semigroup with a canonical infinitesimal generator comprising a second-order differential operator perturbed by an integral operator which expresses an average of translates (see, e.g., [4]) and Itô has shown that the sample paths are Brownian motions with drift interlaced with jumps from a Poisson point process [16].

The generalization of a Lévy process to a Lie group was carried out by Hunt in 1956 [14], where we observe that increments of the process are now defined using the group operation. Since there is, in general, no analogue of the Fourier transform available, Hunt classified the processes through their infinitesimal generators; in fact, the second-order differential operator is now a second-order expression in the Lie algebra of the group and the integral operator expresses an average of group translates. More recently, it was shown in [3] that any such process can be expressed as the solution of a stochastic differential equation (SDE) driven by a Brownian motion in the Lie algebra and a Poisson random measure on the group.

Now we turn our attention to the problem of defining Lévy processes ρ on Riemannian manifolds M. Here of course, the usual definition breaks down as "increment" can have no meaning in this context. Nonetheless, in the case where M is a Riemannian globally symmetric space and the Lévy process is required to be spherically symmetric, the problem was solved by Gangolli in the early 1960s (see [11, 12]). His approach utilized the fact that M is a homogeneous space of a Lie group G and so he was able to effectively define ρ to be the projection of a suitable symmetric Lévy process in G. In this set-up, Fourier methods become available through the spherical transform

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of Harish–Chandra and Gangolli obtains a Lévy–Khintchine type formula for his processes in [11] and shows that their sample paths can be realized as Brownian motions interlaced with isotropic jumps in [12]. An approach to Gangolli's work using stochastic calculus can be found in [2].

When we generalize to arbitrary Riemannian manifolds, we note that one process which has been extensively studied and which is surely a Lévy process under any reasonable definition is Brownian motion. The usual definition of this is that it is a Markov process $\beta = (\beta(t),\ 0 \le t < \eta)$ for which

$$f(\beta(t)) - f(p) - \tfrac{1}{2}\int_0^t \Delta f(\beta(s))\,ds \quad \text{is a martingale}$$

for all $f \in C^\infty(M)$ where $p \in M$ is the starting point for the motion, Δ is the Laplace–Beltrami operator and η is an explosion time (see, e.g., [8]). If the manifold M is not parallelizable, β cannot be obtained as the global solution of an SDE in M. However Eells and Elworthy have shown that it can always be obtained as the projection of the solution of a certain SDE on the bundle of orthonormal frames over M, driven by a Brownian motion and directed by horizontal vector fields (see [6], [7] and [21]).

By analogy with the above, it is natural to define an analogue of a spherically symmetric Lévy process on M in the same way but with $\frac{1}{2}\Delta$ replaced by a more general operator $\mathscr{A}$ which contains an integral to take care of jumps. We propose that $\mathscr{A}$ be of the following form:

$$\mathscr{A}f(p) = \tfrac{1}{2}a\Delta f(p) + \int_{T_p(M)-\{0\}} [f(\mathrm{Exp}(y)(p)) - f(p)]\nu_p(dy)$$

where Δ is the Laplace–Beltrami operator in M, a is a nonnegative constant, Exp denotes the Riemannian exponential and $\{\nu_p,\ p \in M\}$ is a suitable field of Lévy measures on tangent spaces $T_p(M)$ (for further details see Section 3). Note that the jumps of the process at p are along all possible geodesics emanating from p.

In an earlier paper [1], one of us tried without success to imitate the Eells–Elworthy method and construct Lévy processes in M by projection of a suitable horizontal Lévy process in the orthonormal frame bundle. In this paper, we show that this procedure does in fact work but only for an "isotropic" horizontal Lévy process, that is to say if the law of the Lévy process in Euclidean space which drives the SDE is invariant under orthogonal transformations, from which it follows that there is no drift part and the Lévy measures ν_p satisfy a natural invariance property.

The organization of this paper is as follows. In Section 2, we study the prototype isotropic Lévy process in M which is a random walk of geodesics time changed by a standard Poisson process. We show that it is a Markov process and that it can be obtained as the projection of a horizontal compound Poisson process on the orthonormal frame bundle, this being a random walk of motions along integral curves of basic horizontal vector fields which is time changed by the same Poisson process. In Section 3, we turn our attention to more general Lévy processes and show that, with the isotropy property described above, there is a one-to-one correspondence between processes with generator $\mathscr{A}$ and projections of the horizontal Lévy processes constructed in [1]. In Section 4, we then use the interlacing procedure developed in [2] to show that every Lévy process in M is the limit of a sequence of Brownian motions interlaced with the jumps of a Poisson point process along geodesics. In order to avoid complicated arguments involving explosion times, we assume in Sections 3 and 4 that our manifold M is compact; however, we note that our methods will extend to the noncompact case.

A global reference for all the differential geometry used in this paper is [18]. Einstein summation convention will be used throughout.

In this paper, the term "Feller semigroup" will mean a positivity preserving, strongly continuous, identity preserving contraction semigroup on the Banach space $C_b(M)$ of bounded continuous functions on M equipped with the supremum norm.

2. Compound Poisson processes in Riemannian manifolds. Let M be a d-dimensional connected Riemannian manifold with metric g and let $O(M)$ be the bundle of orthonormal frames over M so that there is a canonical surjection $\pi\colon\ O(M) \to M$. Let $r_p \in O(M)$ with $\pi(r_p) = p$; we will find it convenient to regard $r_p = (r_p^1, \dots, r_p^d)$ as a linear isometry $r_p\colon\ \mathbf{R}^d \to T_p(M)$ through the action

$$r_p(x) = x_j r_p^j$$

for each $x \in \mathbf{R}^d$.

We note that M is equipped with its Riemannian connection which enables us to write the Whitney direct sum

$$T(O(M)) = H(M) \oplus V(M),$$

where $H(M)$ and $V(M)$ are the subbundles of horizontal and vertical fibres (respectively). Let $\mathscr{F} = \{F(x),\ x \in \mathbf{R}^d\}$ denote the *basic horizontal vector fields* (sometimes called "standard" or "canonical") on $O(M)$. These are characterized as follows:

1. Each $F(x)(r) \in H_r(M)$.
2. $d\pi_r(F(x)) = r_p(x)$

for each $x \in \mathbf{R}^d,\ r \in O(M)$ with $\pi(r) = p$. Here $d\pi_r$ is the differential of π which maps $T_r(O(M))$ linearly onto $T_p(M)$.

$\mathscr{F}$ will be equipped with the smallest topology for which each of the maps $x \mapsto F(x)$ is continuous.

We will assume that each $F(x)$ is complete in $O(M)$ so that M is *geodesically complete*. We will, as usual, use the exponential map to denote the continuous one-parameter group of diffeomorphisms generated by each $F(x)$. So, for each $u \in \mathbf{R},\ r \in O(M),\ x \in \mathbf{R}^d$,

$$r_x(u) = \exp(uF(x))(r)$$

is the unique solution of the differential equation in $O(M)$,

$$\frac{\partial r_x(u)}{\partial u} = F(x)(r_x(u))$$

with initial condition $r_x(0) = r$.

Now let $(X(n),\ n \in \mathbf{N})$ be a sequence of i.i.d. random variables taking values in $\mathbf{R}^d$ with common law μ and let $N = (N(t),\ t \geq 0)$ be an independent Poisson process with intensity $\alpha > 0$. We denote as $\Pi = (\Pi(t),\ t \geq 0)$, the compound Poisson process in $\mathbf{R}^d$ defined by

$$\Pi(t) = X(1) + X(2) + \cdots + X(N(t))$$

for $t > 0$ with $\Pi(0) = 0$ (a.s.) and Y will denote the associated Poisson random measure on $\mathbf{R}^+ \times (\mathbf{R}^d - \{0\})$ so that for each $t \geq 0,\ A \in \mathscr{B}(\mathbf{R}^d - \{0\})$,

$$Y(t, A) = \#\{s \leq t,\ \Delta(\Pi(s)) \in A\}$$

and $\mathbf{E}(Y(t, A)) = t\nu(A)$ where ν denotes the finite measure $\alpha\mu$.

We define the *horizontal compound Poisson process* starting at $r \in O(M)$ to be the process $R = (R(t),\ t \geq 0)$ given as follows:

$$R(t) = \exp(F(X(N(t))) \circ \cdots \circ \exp(F(X(2))) \circ \exp(F(X(1)))(r)$$

for $t > 0$, where $R(0) = r$ (a.s.). It is easy to check that R is cadlag.

By a standard argument (see, e.g., [2]) we find that for all $f \in C_b(O(M))$, $t \geq 0$,

$$f(R(t)) = f(r) + \int_0^{t+}\int_{\mathbf{R}^d - \{0\}} [f(\exp F(x)(R(s-)) - f(R(s-)]Y(dx, ds) \tag{2.1}$$

so that in particular we see that R is an example of a horizontal Lévy process in the sense of [1]. It is then easy to deduce that R is a Feller process and if we define the associated contraction semigroup $(T(t),\ t \geq 0)$ by

$$T(t)f(r) = \mathbf{E}(f(R(t))|R(0) = r)$$

for $f \in C_b(O(M)),\ r \in O(M),\ t \geq 0$, we again deduce by the argument of [2] that $C_b(O(M)) \subseteq \mathrm{Dom}(\mathscr{L})$ where $\mathscr{L}$ is the infinitesimal generator of the semigroup and

$$\mathscr{L}(f)(r) = \int_{\mathbf{R}^d - \{0\}} [f(\exp(F(x))(r)) - f(r)]\nu(dx)$$

for each $f \in C_b(O(M)),\ r \in O(M)$.

In the sequel we will find a use for the following notation: if $m \in \mathbf{N}$, then r^m will denote the random frame $\exp(F(X(m))) \circ \cdots \circ \exp(F(X(1)))(r)$ and if $z \in (\mathbf{R}^d)^m$ with $z = (z_1, \ldots, z_m)$ then $r(z)$ will denote the frame $\exp(F(z_m)) \circ \cdots \circ \exp(F(z_1))(r)$.

Now we turn our attention to the manifold M. For each $r \in O(M)$, we define a probability measure μ_p^r on $T_p(M)$ [where $p = \pi(r)$] by the prescription,

$$\mu_p^r(A_p) = \mu(r^{-1}(A_p)) \tag{2.2}$$

for all $A_p \in \mathscr{B}(T_p(M))$.

LEMMA 2.1. *For all $p \in M$, the measure μ_p^r on $T_p(M)$ is independent of the choice of $r \in O(M)$ for which $\pi(r) = p$ if and only if the measure μ on $\mathbf{R}^d$ is invariant under orthogonal transformations.*

PROOF. The result follows immediately from the right action of the group $\mathscr{O}(d)$ of all orthogonal transformations in $\mathbf{R}^d$ on the principal fibre bundle $O(M)$ through

$$(O; r, \ 1 \le i \le d) \in \mathscr{O}(d) \times O(M) \mapsto \left(\sum_{1 \le i \le d} O_{ij} r^i, \ 1 \le j \le d \right). \qquad \square$$

From now on, if μ is invariant under orthogonal transformation, we will say that the measure μ is *isotropic* and we will denote as μ_p the unique Borel measure induced by (2.2) on $T_p(M)$, for each $p \in M$.

Now let $X \in T_p(M)$ and let $\{g_X(u), \ u \in \mathbf{R}\}$ be the unique maximal geodesic in M for which

$$g_X(0) = p \quad \text{and} \quad \dot{g}_X(0) = X.$$

We define the Riemannian exponential map Exp: $T_p(M) \to M$ by

$$\mathrm{Exp}(X)(p) = g_X(1)$$

and note that

$$\begin{aligned} \pi(\exp(F(x))(r)) &= \mathrm{Exp}(d\pi_r(F(x)))(p) \\ &= \mathrm{Exp}(r(x))(p) \end{aligned} \tag{2.3}$$

for each $x \in \mathbf{R}^d$ and each $r \in O(M)$ with $\pi(r) = p$.

In the sequel, we will often abuse notation to the extent of writing

$$\mathrm{Exp}(X) \circ \mathrm{Exp}(Y)(p) = \mathrm{Exp}(X)(q)$$

where $q = \mathrm{Exp}(Y)(p)$, $X \in T_q(M)$ and $Y \in T_p(M)$.

We now define a process $\gamma = (\gamma(t), \ t \ge 0)$ in M by the procedure

$$\gamma(t) = \pi(R(t))$$

for each $t \ge 0$, so that in particular, $\gamma(0) = p$ (a.s.). We call γ a *compound Poisson process in M* starting at p. Clearly it is cadlag.

By (2.3), we find that for each $t \ge 0$,

$$\gamma(t) = \mathrm{Exp}(r^{N(t)-1}(X(N(t))) \circ \cdots \mathrm{Exp}(r^1)(X(2))) \circ \mathrm{Exp}(r(X(1)))(p).$$

The remainder of the section will be devoted to a discussion of the Markovianity of the process γ. The following lemma will be a vital tool in this regard.

LEMMA 2.2. *Let μ be isotropic; then for any $f \in C_b(M)$, $m \in \mathbf{N}$, $r \in O(M)$ and $q^{m-1} \in M$ such that $\pi(r^{m-1}) = q^{m-1}$, we have*

$$\mathbf{E}(f(\mathrm{Exp}(r^{m-1}(X(m)))(q^{m-1}))) = \mathbf{E}(f(\mathrm{Exp}(r_{m-1}(X(m)))(q^{m-1}))),$$

where r_{m-1} is an arbitrary frame for which $\pi(r_{m-1}) = q^{m-1}$.

PROOF. Since $X(m)$ is independent of $\sigma\{X(1), \ldots, X(m-1)\}$ we obtain

$$\mathbf{E}(f(\mathrm{Exp}(r^{m-1}(X(m)))(q^{m-1})))$$
$$= \int_{(\mathbf{R}^d)^{m-1}} \int_{\mathbf{R}^d} f(\mathrm{Exp}(r(z)(x)(q(z)))\nu(dx)\nu^{\otimes^{m-1}}(dz),$$

where $q(z) \in M$ is such that $\pi(r(z)) = q(z)$ for all $z \in (\mathbf{R}^d)^{m-1}$. Hence since ν is isotropic we have

$$\mathbf{E}(f(\mathrm{Exp}(r^{m-1}(X(m)))(q^{m-1})))$$
$$= \int_{(\mathbf{R}^d)^{m-1}} \int_{T_{q(z)}(M)} f(\mathrm{Exp}(y)(q(z)))\nu_{q(z)}(dy)\nu^{\otimes^{m-1}}(dz)$$

from which the required result follows. $\square$

We now let $S = (S(t);\ t \geq 0)$ denote the family of linear contractions on $C_b(M)$ defined by

$$S(t)(f)(p) = \mathbf{E}(f(\gamma(t))|\gamma(0) = p)$$

so that

$$S(t)(f)(p) = T(t)(f \circ \pi)(r)$$

whenever $\pi(r) = p$. We also introduce the bounded linear operator $\mathscr{A}$ on $C_b(M)$ by

$$\mathscr{A}(f)(p) = \int_{T_p(M)-\{0\}} (f(\mathrm{Exp}(y)(p)) - f(p))\nu_p(dy).$$

Note that when μ is isotropic it is easy to verify that

$$\mathscr{A}(f)(p) = \mathscr{L}(f \circ \pi)(r)$$

whenever $f \in C_b(M)$ and $\pi(r) = p$.

THEOREM 2.1. *If μ is isotropic then S is a Feller semigroup with infinitesimal generator $\mathscr{A}$.*

PROOF. Write $f = g \circ \pi$ in (2.1) and use (2.3) to obtain

$$g(\gamma(t)) - g(p) = \int_0^{t+} \int_{\mathbf{R}^d - \{0\}} [g(\mathrm{Exp}(R(u-)(x))(\gamma(u-))) - g(\gamma(u-))] Y(dx, du).$$

Now take expectations and apply Lemma 2.2 to find that

$$S(t)(g)(p) - g(p)$$
$$= \int_0^t \int_{\mathbf{R}^d - \{0\}} \mathbf{E}([g(\mathrm{Exp}(R(u)(x))(\gamma(u))) - g(\gamma(u))]|\gamma(0) = p)\nu(dx)\,du$$
$$= \int_0^t \int_{\mathbf{R}^d - \{0\}} \mathbf{E}([g(\mathrm{Exp}(r(\gamma(u))(x))(\gamma(u))) - g(\gamma(u))]|\gamma(0) = p)\nu(dx)\,du,$$

where $r(\gamma(u))$ is an arbitrary frame for which $\pi(r(\gamma(u))) = \gamma(u)$ for each $u \geq 0$. Now using Fubini's theorem, we obtain

$$\begin{aligned} &S(t)(g)(p) - g(p) \\ &\quad = \int_0^t \mathbf{E}\left[\int_{\mathbf{R}^d - \{0\}} [g(\mathrm{Exp}(r(\gamma(u))(x))(\gamma(u))) - g(\gamma(u))]\nu(dx)|\gamma(0) = p\right] du \\ &\quad = \int_0^t \mathbf{E}\left[\int_{T_{\gamma(u)}(M) - \{0\}} [g(\mathrm{Exp}(y)(\gamma(u))) - g(\gamma(u))]\nu_{\gamma(u)}(dy)|\gamma(0) = p\right] du \\ &\quad = \int_0^t \mathscr{A} S_u(f)(p)\, du \end{aligned}$$

and the required result follows. □

COROLLARY 2.1. *The process γ is a Feller process with transition semigroup* $(S(t),\ t \geq 0)$.

PROOF. We show that γ enjoys the Markov property. Let ν be an initial probability measure on M and $\tilde{\nu}$ be any probability measure on $O(M)$ for which $\tilde{\nu} \circ \pi = \nu$. We choose $0 = t_0 < \cdots < t_k < \infty$ and take $f_1, \dots, f_k$ to be bounded Borel functions on M; then since R is Markov we have

$$\begin{aligned} &\mathbf{E}\left[\prod_{i=0}^k f_i(\gamma(t_i))\right] \\ &\qquad = \mathbf{E}\left[\prod_{i=0}^k (f_i \circ \pi)(R(t_i))\right] \\ &\qquad = \mathbf{E}_{\tilde{\nu}}((f_0 \circ \pi)T(t_1)((f_1 \circ \pi)T(t_2 - t_1) \\ &\qquad\qquad \times ((f_2 \circ \pi) \cdots T(t_k - t_{k-1})(f_k \circ \pi) \cdots)(R(0))) \\ &\qquad = \mathbf{E}_\nu(f_0 S(t_1)((f_1 S(t_2 - t_1)(f_2 \cdots S(t_k - t_{k-1}) f_k) \cdots)(\gamma(0))) \end{aligned}$$

from which the Markov property of R follows (see, e.g., [22], page 76). The rest follows immediately from Theorem 2.1. □

3. Lévy processes on manifolds. In this section, to avoid complicating the argument with explosions, we will assume that M is compact and so it is automatically complete. We note that the frame bundle $O(M)$ is also compact in this case. In the sequel we will denote as F_j, the basic vector field $F(e_j)$ where $\{e_j,\ 1 \leq j \leq d\}$ is the natural basis in $\mathbf{R}^d$.

3.1. *Horizontal Lévy processes.* Let $Y = (Y(t),\ t \geq 0)$ be a d-dimensional Lévy process in $\mathbf{R}^d$ so that Y is a stochastically continuous process with stationary and independent increments for which $Y(0) = 0$ a.s. The Lévy–Itô decomposition [15] yields the following for $1 \leq i \leq d,\ t \geq 0$,

$$Y^i(t) = b^i t + B^i(t) + \int_{0<||x||<1} x^i \tilde{N}(t, dx) + \int_{||x||\geq 1} x^i N(t, dx), \tag{3.1}$$

where:

1. $b \in \mathbf{R}^d$;
2. $B = (B(t),\ t \geq 0)$ is a d-dimensional Brownian motion with covariance $\mathrm{Cov}(B^i(t), B^j(t)) = a^{ij}t$ for all $t \geq 0,\ 1 \leq i,\ j \leq d$ where $a = (a^{ij})$ is a $d \times d$ real-valued nonnegative definite symmetric matrix;
3. N is a Poisson random measure on $\mathbf{R}^+ \times (\mathbf{R}^d - \{0\})$ which is independent of B and whose intensity measure ν is a Lévy measure on $\mathbf{R}^d - \{0\}$, that is, for each $A \in \mathscr{B}(\mathbf{R}^d - \{0\}),\ t \geq 0$,

$$\mathbf{E}(N(t, A)) = t\nu(A),$$

where $\int_{\mathbf{R}^d - \{0\}} (||x^2|| \wedge 1)\nu(dx) < \infty$. $\tilde{N}$ denotes the compensator defined by

$$\tilde{N}(t, A) = N(t, A) - t\nu(A).$$

We recall the definition of a horizontal Lévy process on the orthonormal frame bundle $O(M)$ from [1].

Let $R = (R(t),\ 0 \leq t < \eta)$ be a cadlag semimartingale in $O(M)$ where η is an explosion time. We say that R is a *horizontal Lévy process* starting at $r \in O(M)$ if R solves the following stochastic differential equation in $O(M)$:

$$\begin{aligned} g(R(t)) = g(r) &+ \int_0^t F_i g(R(u-))\, dY^i(u) + \tfrac{1}{2} a^{ij} \int_0^t F_i F_j g(R(u))\, du \\ &+ \sum_{u \leq t} [g(\exp(F(\Delta Y(u)))(R(u-))) - g(R(u-)) - F_i g(R(u-))\Delta Y^i(u)] \end{aligned} \tag{3.2}$$

for all $g \in C^\infty(O(M)), t \geq 0$.

By (3.1) we can write this as

$$\begin{aligned} g(R(t)) = g(r) &+ \int_0^t F_i g(R(u))\, dB^i(u) + \int_0^t \mathscr{L}(g)(R(u))\, du \\ &+ \int_0^{t+} \int_{0<||x||<1} [g(\exp(F(x))(R(u-))) - g(R(u-))]\tilde{N}(dx, du) \\ &+ \int_0^{t+} \int_{||x||\geq 1} [g(\exp(F(x))(R(u-))) - g(R(u-))]N(dx, du), \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} \mathscr{L}(g)(r) = F(b)g(r) &+ \tfrac{1}{2} a^{ij} F_i F_j g(r) \\ &+ \int_{\mathbf{R}^d - \{0\}} [g(\exp(F(x))(r) - g(r) - \mathbf{1}_{||x||<1} F(x)g(r)]\nu(dx). \end{aligned} \tag{3.4}$$

It is shown in [1] that (3.3) has a unique cadlag solution $R = (R(t),\ 0 \leq t < \zeta)$ in $O(M)$. We will show below that $\zeta = \infty$ (a.s.). For now, it will do us no harm to assume this. We will also show that R is a Markov process with infinitesimal generator $\mathscr{L}$. We prove now that an infinitesimal generator of the form (3.4) is characteristic of a horizontal Lévy process.

PROPOSITION 3.1. *If R is a Markov process in $O(M)$ with infinitesimal generator given by* (3.4) *then R is a horizontal Lévy process.*

PROOF. We follow [9] to define the vectorial (anti-)development Y of R (see also [7] for the Brownian case) by

$$Y_{.} = \int_0^{\cdot} \theta \circ dR,$$

where θ is the canonical form of $O(M)$ with values in $\mathbf{R}^d$, that is,

$$\forall\ r \in O(M) \qquad \forall\ X \in T_r O(M), \qquad \theta_r(X) = r^{-1}(d\pi_r(X)),$$

and where the following sense is given to the Stratonovich integral of a 1-form along a cadlag semimartingale: take the usual definition for the continuous part, and at a jump time u of R consider that the "integral" presents the additive jump $\langle \theta(R(u-)), \exp_{R(u-)}^{-1}(R(u))\rangle$.

Then, for all g in $C^\infty(O(M))$, using the Stratonovich integral of the 1-form dg along R we can, by Proposition 2.4 of [9], write

$$\begin{aligned} g(R(t)) = g(r(0)) &+ \int_0^t dg \circ dR \\ &+ \sum_{u \le t} [g(R(u)) - g(R(u-)) - < dg(R(u-)), \exp_{R(u-)}^{-1}(R(u)) >]. \end{aligned}$$

Splitting the 1-form dg in horizontal and vertical parts as in Proposition 4.3 of [9], we see that the process R is a solution of (3.2).

Therefore the only thing to prove is that Y is a Lévy process in $\mathbf{R}^d$. We will do this by introducing the triplet of local characteristics (A, C, μ) of Y as in [17], Chapter II. Those characteristics are defined as follows:

1. A is the d-dimensional predictable process with finite variation appearing in the Doob decomposition of the special semimartingale $Y - \sum_{u \le .} \mathbf{1}_{|\Delta Y(u)| > 1} \Delta Y(u)$.
2. C is the continuous finite variation process with values in the set of non-negative symmetric matrices given by $C^{ij} = \langle Y^{ic}, Y^{jc}\rangle$.
3. μ is the predictable compensator of the random measure j on $\mathbf{R}^+ \times \mathbf{R}^d - \{0\}$ associated to the jumps of Y: $j(dt, dx) = \sum_{u \le t} \delta_{(u, \Delta Y(u))}(dt, dx) \mathbf{1}_{|\Delta Y(u)| > 0}$.

We then have by Theorem 2.42 of [17] that for all $g \in C^\infty(O(M))$,

$$\begin{aligned} g(R(t)) - g(R(0)) - \int_0^t \Bigg(& F_i g(R(u-)) dA_u^i + \frac{1}{2} F_i F_j g(R(u)) dC_u^{ij} \\ &+ \int_{\mathbf{R}^d - \{0\}} [g(\exp(F(y))(R(u-)) - g(R(u-)) \\ &\qquad - \mathbf{1}_{|y| < 1} F(y) g(R(u-))] \mu(dy, du) \Bigg) \end{aligned}$$

is a local martingale. Identification with the infinitesimal generator of R gives

$$A_t^i = b^i t; \qquad C_t^{ij} = a^{ij} t; \qquad \mu(dy, dt) = \nu(dy)\, dt.$$

Now by Theorem 4.19 of [17] we see that such a triplet characterizes Lévy processes and so Y is a Lévy process as required. □

3.2. *Isotropic Lévy process on M.* Let R be a horizontal Lévy process and $X = (X(t),\ 0 \le t < \zeta)$ be defined by $X(t) = \pi(R(t))$ so that X is cadlag with values in M. The process X is a candidate to be a "Lévy process on M"; however, as is shown in [1], it is not, in general, a homogeneous Markov process.

As was shown in [1], the problem is that the law of X may depend on the choice of the initial frame $R(0) = r$ at $X(0) = p$ (see [15], page 283 where a similar discussion is developed for Brownian motion in M). In order to emphasize the dependence, we will in this paragraph write $R(t) = R(r, Y, t)$ for the solution of (3.2) and $X(t) = X(r, Y, t)$ for its projection on M. Let r' be another orthonormal frame at p ($\pi(r') = p$) and let $O \in \mathscr{O}(d)$ be the orthogonal transformation such that $r' = rO$ [recall that the structure group $\mathscr{O}(d)$ acts on $O(M)$ on the right]. It follows from (3.2) and from the relation

$$d\pi_{rO}(F(y)) = d\pi_r(F(Oy)) \qquad \forall\ y \in \mathbf{R}^d$$

that

$$X(r', Y, t) = X(r, OY, t) \qquad \forall\ t \ge 0.$$

Hence we see that the processes $X(r, Y, \cdot)$ and $X(r', Y, \cdot)$ will agree in law if the processes Y and OY agree in law for all $O \in \mathscr{O}(d)$. Now if we write $\mathbf{R}^d = M(d) \setminus \mathscr{O}(d)$ where $M(d)$ is the group of all isometries of $\mathbf{R}^d$ then we see that we require Y to be a spherically symmetric Lévy process on $\mathbf{R}^d$ in the sense of [11], [12] and [2]. It follows that Y is characterized by the Lévy–Khintchine formula

$$\mathbf{E}(\exp(iu.Y(t))) = \exp\left\{t\left[-\tfrac{1}{2}a\|u\|^2 + \int_{\mathbf{R}^d-\{0\}}(e^{iu.x} - 1)\nu(dx)\right]\right\}$$

for all $t \ge 0,\ u \in \mathbf{R}^d$ where $a \ge 0$ and the Lévy measure ν is isotropic in that

$$\nu(OA) = \nu(A)$$

for all $A \in \mathscr{B}(\mathbf{R}^d - \{0\}),\ O \in \mathscr{O}(d)$.

In the case where Y is spherically symmetric as above, we say that the horizontal Lévy process R is *isotropic* and that its projection X to M is an *isotropic Lévy process on M*. We will see below that such isotropic processes in M are always homogeneous Markov as is required.

We note first of all that the infinitesimal generator for an isotropic horizontal Lévy process takes the form

$$\mathscr{L}(g)(r) = \tfrac{1}{2}a\Delta_H g(r) + \int_{\mathbf{R}^d-\{0\}}[g(\exp(F(x))(r)) - g(r)]\nu(dx) \tag{3.5}$$

for all $g \in C^\infty(O(M))$, $r \in O(M)$ where Δ_H is the horizontal Laplacian,

$$\Delta_H = \sum_{i=1}^{d} F_i F_i.$$

3.3. *Infinitesimal generator of an isotropic Lévy process on M.* Let $(B_a(t), t \geq 0)$ be a Brownian motion process in $\mathbf{R}^d$ with covariance matrix aI where $a > 0$. By the Eels–Elworthy construction we know that the solution R of the SDE in $O(M)$ given by

$$g(R(t)) = g(r) + \int_0^t F_i g(R(u)) \circ dB_a^i(u)$$

for all $t \geq 0$, $g \in C^\infty(\mathbf{R}^d)$, when projected on M yields a diffusion process with infinitesimal generator $\frac{1}{2}a\Delta$ (called *Brownian motion on M* when $a = 1$) where Δ is the Laplace–Beltrami operator defined by

$$\Delta f(p) = \Delta_H(f \circ \pi)(r)$$

for all $f \in C^\infty(M)$, $r \in O(M)$ with $\pi(r) = p$.

Conversely, it is proved (see [7], page 310 or [15], page 288) that every diffusion with infinitesimal generator $\frac{1}{2}a\Delta$ is obtained by this way. We now prove an analogous result for isotropic Lévy processes on M.

THEOREM 3.1. *Let X be a cadlag semimartingale in M. The process X is an isotropic Lévy process on M if and only if X is a Feller process with infinitesimal generator $\mathscr{A}$ given by*

$$\mathscr{A}f(p) = \frac{1}{2}a\Delta(f)(p) + \int_{T_p(M)-\{0\}} [f(\mathrm{Exp}(y)(p)) - f(p)]\nu_p(dy) \tag{3.6}$$

for each $f \in C^\infty(M)$, $p \in M$ where $a \geq 0$ and $(\nu_p)_{p\in M}$ is a field of measures on $T_p(M)$ deduced from an isotropic Lévy measure ν on $\mathbf{R}^d$ by $\nu_p(A) = \nu(r^{-1}(A))$ whenever $\pi(r) = p$.

PROOF OF FIRST IMPLICATION OF THEOREM 3.1. Let X be the projection $\pi(R)$ of an isotropic horizontal Lévy process R on $O(M)$. Denoting by $\mathscr{L}$ the generator of R and observing that

$$\mathscr{A}(f)(p) = \mathscr{L}(f \circ \pi)(r)$$

whenever $\pi(r) = p$, then the fact that X is a Feller process follows by a similar argument to that of Theorem 2.1 and Corollary 2.1 where we note that the continuity of $R(t)$ as a function of its initial point (which follows from the arguments of [10]) ensures that the semigroup associated to (3.5) preserves $C(O(M))$ and hence its projection preserves $C(M)$. □

PROOF OF SECOND IMPLICATION OF THEOREM 3.1. Let X be a Feller process starting at $p_0 \in M$ with infinitesimal generator given by (3.6) and denote by

R its horizontal lift in $O(M)$ (see [9]) starting from $r_0 \in O(M)$ such that $\pi(r_0) = p_0$.

In view of Proposition 3.1, the process R will be an isotropic horizontal Lévy process if R is shown to admit the generator $\mathscr{L}$ defined in (3.5).

That the differential part of the operator $\mathscr{L}$ is $\frac{1}{2}a\Delta_H$ is clear since it would be the unique term for continuous X and R (i.e., for X a diffusion on M with generator $\frac{1}{2}a\Delta$).

Therefore let us concentrate on the jumps of R. They will induce in the operator $\mathscr{L}$ the integral term with respect to measure ν if we can prove that, for all predictable $h\colon\ \Omega \times \mathbf{R}^+ \times O(M) \times O(M) \to \mathbf{R}$ such that $E(\sum_{u\le t} |h(u, R(u-), R(u))| < \infty$ for all $t \ge 0$,

$$\sum_{u\le \cdot} h(u, R(u-), R(u)) - \int_0^{\cdot} \int_{\mathbf{R}^d - \{0\}} h(u, R(u-), \exp\ F(x).R(u-))\nu(dx)\,du$$

is a local martingale.

By [9] we know that the process R jumps when and only when X does, and that for u a jump time, $R(u)$ is given by the parallel transport $\tau_{X(u-),X(u)}R(u-)$ of $R(u-)$ along the geodesic in M from $X(u-)$ to $X(u)$. More precisely let u be a fixed jump time and denote $p = X(u-)$ and $r = R(u-)$. Take $y \in T_p(M)$ such that $X(u) = \mathrm{Exp}(y)(p)$. There exists some $x \in \mathbf{R}^d$ such that $y = r(x)$ and then $R(u) = \exp\ F(x)(r)$. Hence

$$\begin{aligned}
&\sum_{u\le t} h(u, R(u-),\ R(u))\\
&\qquad = \sum_{u\le t} h(u, R(u-),\ \tau_{X(u-),\ X(u)}R(u-))\\
&\qquad = \text{a local martingale}\\
&\qquad\qquad + \int_0^t \int_{T_{X(u-)}(M)-\{0\}} h(u, R(u-), \tau_{X(u-),\ \mathrm{Exp}\, y.X(u-)}R(u-))\nu_{X(u-)}(dy)\,du\\
&\qquad = \text{a local martingale } + \int_0^t \int_{\mathbf{R}^d-\{0\}} h(u, R(u-), \exp F(x).R(u-))\nu(dx)\,du
\end{aligned}$$

and the theorem is proved. □

Note. Let $\mathscr{D}$ be a second-order differential operator on M which is expressed in a chart (U, ϕ) as

$$\mathscr{D}(f \circ \phi)(p) = \frac{1}{2}a^{ij}(y)\frac{\partial^2 f}{\partial y^i \partial y^j}(y) + b^j(y)\frac{\partial f}{\partial y^j}(y)$$

for all $f \in C^\infty(\mathbf{R}^d)$ where $p \in U$ with $\phi(p) = y$ and each $(a^{ij}(y))_{1\le i,\ j\le d}$ is a strictly positive definite matrix in any choice of local coordinates. We can choose a metric and a compatible connection on M (see [15], page 288) such that, for all $r \in O(M)$ with $\pi(r) = p$,

$$\tfrac{1}{2}\tilde{F}_i\tilde{F}_j(f \circ \pi)(r) = \mathscr{D}f(p),$$

where $(\tilde{F}_1, \dots, \tilde{F}_d)$ is the system of basic horizontal vector fields on $O(M)$ corresponding to the choosen connection.

Therefore, for each isotropic Lévy measure ν on $\mathbf{R}^d$, we can construct an isotropic Lévy process on M with prescribed infinitesimal generator

$$\mathscr{A} f(p) = \mathscr{D} f(p) + \int_{T_p(M)-\{0\}} [f(\mathrm{Exp}(y)(p)) - f(p)]\nu_p(dy).$$

4. The interlacing construction. We will now give an alternative construction of the processes R and X, which utilizes the interlacing technique of [2].

4.1. *Horizontal interlacing.* We begin by studying the horizontal processes. Let $(R(t),\ 0 \le t < \zeta)$ be a horizontal Lévy process on $O(M)$ starting at $r_0 \in O(M)$. Note that at this stage we are not assuming that R is isotropic nor that $\zeta = \infty$ (a.s.).

We will find it convenient to rewrite equation (3.3) in a different way. First we fix $0 < \varepsilon < 1$; we then have

$$\begin{aligned} g(R(t)) &= g(r_0) + \int_0^t F_i g(R(u-))\, dB^i(u) + \int_0^t \mathscr{L}'_\varepsilon g(R(u-))\, du \\ &\quad + \int_0^{t+} \int_{0<\|x\|<\varepsilon} \big[g\big(\exp(F(x))(R(u-))\big) - g(R(u-))\big]\tilde{N}(dx, du) \qquad (4.1) \\ &\quad + \int_0^{t+} \int_{\|x\|\ge\varepsilon} \big[g\big(\exp(F(x))(R(u-))\big) - g(R(u-))\big]N(dx, du), \end{aligned}$$

where

$$\begin{aligned} \mathscr{L}'_\varepsilon g(r) &= F(b'(\varepsilon))g(r) + \tfrac{1}{2}a^{ij} F_i F_j g(r) \\ &\quad + \int_{0<\|x\|<\varepsilon} \big[g\big(\exp(F(x))(r)\big) - g(r) - F(x)g(r)\big]\nu(dx) \end{aligned}$$

and

$$b'(\varepsilon) = b - \int_{\varepsilon<\|x\|<1} x\nu(dx).$$

Now we begin to construct R by interlacing. To this end we let $(c_n,\ n \in \mathbf{N})$ be a decreasing sequence of positive numbers with $\sup_{n\in\mathbf{N}} c_n \le \varepsilon$ and $\lim_{n\to\infty} c_n = 0$. Let $B_r(0)$ denote the open ball in $\mathbf{R}^d$ which is centered on the origin. We define a sequence of Borel sets $(V_n,\ n \in \mathbf{N})$ in $\mathbf{R}^d$ by $V_n = B_\varepsilon(0) - B_{c_n}(0)$ so that $V_n \uparrow B_\varepsilon(0)$ as $n \to \infty$. Although it is not essential for this part of the construction, we will find it useful later to also introduce a sequence $(b_n,\ n \in \mathbf{N})$ in $\mathbf{R}^d$ for which $\lim_{n\to\infty} b_n = b$ and for each $n \in \mathbf{N}$, we define $b'_n(\varepsilon)$ as above for $b'(\varepsilon)$ except that b is replaced on the right-hand side by b_n.

For each $n \in \mathbf{N}$, we consider the Poisson process

$$N^{(n)} = (N(V_n \times [0, t]),\ t \ge 0)$$

with intensity $\nu(V_n)$ and we denote as $(\tau_m^{(n)},\ m \in \mathbf{N})$, the sequence of interarrival times for $N^{(n)}$. Let also $(X_m^{(n)}, m \in \mathbf{N})$ be the sequence of i.i.d. $\mathbf{R}^d$-valued random variables given by

$$X_m^{(n)} = \int_{V_n} x N^{(n)}(dx, \{\tau_m^{(n)}\}). \tag{4.2}$$

Each $X_m^{(n)}$ takes values in V_n and has law $q^{(n)}$ where

$$q^{(n)}(A) = \frac{\nu(A)}{\nu(V_n)}$$

whenever $A \in \mathscr{B}(V_n)$. Moreover the $X_m^{(n)}$'s are all independent of the $\tau_m^{(n)}$'s.

Now define a sequence of horizontal Brownian flows with drift on $O(M)$, $(\beta^{(n)},\ n \in \mathbf{N})$, as follows:

$$d\beta^{(n)}(t) = F(b^{(n)})(\beta^{(n)}(t))\,dt + F^i(\beta^{(n)}(t)) \circ dB_i(t), \tag{4.3}$$

where $\circ$ denotes the Stratonovich differential, each $\beta^{(n)}(0)$ is the identity map on M (a.s.) and

$$b^{(n)} = b_n'(\varepsilon) - \int_{V_n} x\nu(dx).$$

Since $O(M)$ is compact, it follows that each $\beta^{(n)}$ is a stochastic flow of diffeomorphisms of $O(M)$ (see, e.g., [19], page 192).

We now define a sequence $(\eta^{(n)},\ n \in \mathbf{N})$ of stochastic flows of diffeomorphisms of $O(M)$ by interlacing as follows:

$$\begin{aligned}
\eta^{(n)}(t) &= \beta^{(n)}(t), \qquad 0 \le t < \tau_1^{(n)},\\
\eta^{(n)}(\tau_1^{(n)}) &= \exp(F(X_1^n)) \circ \beta^{(n)}(\tau_1^{(n)}),\\
\eta^{(n)}(t) &= \beta^{(n)}(t) \circ \beta^{(n)}(\tau_1^{(n)})^{-1} \circ \eta^{(n)}(\tau_1^{(n)}), \qquad \tau_1^{(n)} < t < \tau_2^{(n)},\\
\eta^{(n)}(\tau_2^{(n)}) &= \exp(F(X_2^{(n)})) \circ \eta^{(n)}(\tau_2^{(n)}-)
\end{aligned} \tag{4.4}$$

and we continue by induction.

For each $r \in O(M)$, we then find that $\eta^{(n)}(t)(r)$ satisfies the stochastic differential equation

$$\begin{aligned}
g(\eta^n(t)(r)) = g(r) &+ \int_0^t F_i g(\eta^n(u-)(r))\,dB^i(s) + \int_0^t \mathscr{L}^n g(\eta^n(u-)(r))\,du\\
&+ \int_0^{t+} \int_{V_n} [g(\exp(F(x))(\eta^n(u-)(r))) - g(\eta^n(u-)(r))]\tilde{N}(dx, du)
\end{aligned}$$

for all $g \in C^\infty(O(M))$, where

$$\begin{aligned}
\mathscr{L}^n(g)(r) = {} & F(b_n'(\varepsilon))g(r) + a^{ij} F_i F_j g(r)\\
&+ \int_{V_n} [g(\exp(F(x))(r)) - g(r) - F(x)g(r)]\nu(dx).
\end{aligned}$$

For $r \in O(M)$, we now introduce the horizontal Lévy process $(\eta_r(s,t),\ 0 \leq s \leq t < \alpha)$ (where α is an explosion time) which is the solution of the following stochastic differential equation:

$$g(\eta_r(s,t)) = g(r) + \int_s^t F_i g(\eta_r(s,u-))\, dB^i(u) + \int_s^t \mathscr{L}'_\varepsilon g(\eta_r(s,u-))\, du$$
$$+ \int_s^{t+}\int_{0\leq\|x\|<\varepsilon} [g(\exp(F(x))(\eta_r(s,u-))) - g(\eta_r(s,u-))]\tilde{N}(dx,du). \tag{4.5}$$

In the sequel we will write $\eta_r(t)$ for $\eta_r(0,t)$.

THEOREM 4.1.

$$\lim_{n\to\infty} \eta^n(t)(r) = \eta_r(t) \quad (a.s.)$$

for all $r \in O(M)$ and the convergence is uniform on compacts in $[0,\alpha)$. Furthermore, $\alpha = \infty$ (a.s.).

PROOF. By Whitney's embedding theorem there exists a smooth embedding ι of $O(M)$ into a closed submanifold of $\mathbf{R}^b$ where $b > \frac{1}{2}d(d+1)$. Following the procedure of [1], we express each of the processes $(\eta^n(t)(r),\ t \geq 0)$ and $(\eta_r(t),\ t \geq 0)$ in local coordinates in a neighborhood of r and then extend these to processes satisfying appropriate SDEs in $\mathbf{R}^b$ which we write as $\hat{\eta}^n(t)(r)$ and $\hat{\eta}_r(t)$, respectively. Following [1] again, we see that $\hat{\eta}^n(t)(r) \in \iota(O(M))$ for all $t \geq 0$ and $\hat{\eta}_r(t) \in \iota(O(M))$ for all $0 \leq t \leq \alpha$ (see also [5]). Hence we can write each $\hat{\eta}^n(t)(r) = \iota(\eta^n(t)(r))$ and $\hat{\eta}_r(t) = \iota(\eta_r(t))$. We can now use the argument of the Appendix in [2] to conclude that

$$\lim_{n\to\infty} \iota(\eta^n(t)(r)) = \iota(\eta_r(t)) \quad (a.s.)$$

and the required convergence follows from the homeomorphism property of ι. To see that $\alpha = \infty$ (a.s.), note that $(\iota(\eta^n(t)(r)),\ n \in \mathbf{N})$ is a Cauchy sequence in $\mathbf{R}^b$ for all $t \geq 0$ (a.s.) and hence $(\eta^n(t)(r)),\ n \in \mathbf{N})$ is Cauchy in $O(M)$ for all $t \geq 0$ (a.s.). However, by the Hopf–Rinow theorem, $O(M)$ is complete and so $(\eta^n(t)(r)),\ n \in \mathbf{N})$ converges in $O(M)$ for all $t \geq 0$ and the required result follows. □

Note. We have not proved that the maps $r \mapsto \eta_r(t)$ are diffeomorphisms (a.s.). In fact, we do not need this result in the sequel but we remark that it follows from general considerations (see [20]).

The process η_r is a horizontal Lévy process starting at time $s = 0$ from $r \in O(M)$ with generator $\mathscr{L}'_\varepsilon$ and whose jumps are all "bounded by ε." To recover the original process R from the process η_r, we must insert the "large jumps" (compare the SDEs (4.1) and (4.5) solved by R and η_r). We proceed in a similar way to the above.

So let $(\sigma(m),\ m \in \mathbf{N})$ be the interarrival times of the Poisson process $(N(B_\varepsilon(0)^c \times [0,t]),\ t \geq 0)$ with intensity $\nu(B_\varepsilon(0)^c)$ and let $(Z_m,\ m \in \mathbf{N})$ be the sequence of random variables

$$Z_m = \int_{B_\varepsilon(0)^c} xN(dx, \{\sigma(m)\}). \tag{4.6}$$

They are i.i.d. random variables taking values in $B_\varepsilon(0)^c$ each with law $\tilde{q}$ where $\tilde{q}(C) = (\nu(C)/\nu(B_\varepsilon(0)^c))$ for $B \in \mathscr{B}(B_\varepsilon(0)^c)$ and independent of all the $\sigma(m)$'s.

We now construct a process $\tilde{R}$ as follows:

$$\begin{aligned} \tilde{R}(t) &= \eta_r(0,t), \qquad 0 \le t < \sigma(1), \\ \tilde{R}(\sigma(1)) &= \exp(F(Z_1)) \circ \tilde{R}(\sigma(1)-), \\ \tilde{R}(t) &= \eta_{\tilde{R}(\sigma(1))}(\sigma(1),t), \qquad \sigma(1) < t < \sigma(2), \\ \tilde{R}(\sigma(2)) &= \exp(F(Z_2)) \circ \tilde{R}(\sigma(2)-) \end{aligned} \tag{4.7}$$

and so on inductively.

Note that since there are only a finite number of "large" jumps in each finite time interval, it follows from the above construction that $\zeta = \infty$ (a.s.).

The next proposition follows immediately.

PROPOSITION 4.1. *The process $\tilde{R}$ defined by* (4.7) *solves the SDE* (4.1) *and hence is equal to R a.s.*

4.2. *Interlacing in M.* In this section, $\hat{O}$ will denote the natural action of $\mathscr{O}(d)$ on $C^\infty(O(M))$ given by $\hat{O}g(r) = g(rO)$ for each $O \in \mathscr{O}(d)$ and $g \in C^\infty(O(M))$. Let R be an isotropic horizontal Lévy process in $O(M)$, then by a similar argument to Proposition 4.3 of [1], we deduce that $\hat{O}\mathscr{L}^n = \mathscr{L}^n$ for each $n \in \mathbf{N}$ and each $O \in \mathscr{O}(d)$ from which we can easily deduce that each $b^n = 0$, that $(a^{ij}) = aI$ with $a > 0$ and that ν is isotropic in the above interlacing argument.

We can now easily construct X as an interlaced process by projection of the above construction.

First we need some more notation and conceptual structure. For each $n \in \mathbf{N}$, define $\gamma^{(n)}(t) = \pi(\beta^{(n)}(t)$ for all $t \ge 0$ where $\beta^{(n)}$ is defined by (4.3) with $b^{(n)} = 0$ and $(a^{ij}) = aI$. Then each $\gamma^{(n)}$ is a Brownian motion starting at p with generator $\frac{1}{2}a\Delta$.

Let r be a frame at p; then since r is a linear isometry it follows that $r(B_\varepsilon(0)) = U_\varepsilon^r(p)$ is an open neighborhood of 0 in $T_p(M)$ and $r(V_n) = V_n^r(p)$ gives rise to a sequence of Borel sets in $T_p(M)$ such that $V_n^r(p) \uparrow U_\varepsilon^r(p)$ as $n \to \infty$.

Since ν is isotropic, we obtain a field of Lévy measures $\{\nu_p,\ p \in M\}$ defined on each $T_p(M)$. For each $n \in \mathbf{N}$, consider the sequence of random variables $(X_m^{(n)},\ m \in \mathbf{N})$ defined above in (4.2), then the prescription $X_m^{(n)}(p) = r(X_m^{(n)})$ defines an i.i.d. sequence of random variables taking values in $V_n^r(p)$ whose laws are independent of the choice of frame r satisfying $\pi(r) = p$. We can similarly associate to the sequence $(Z_m,\ m \in \mathbf{N})$ introduced in (4.6), a sequence of i.i.d. random variables $(Z_m(p),\ m \in \mathbf{N})$ taking values in $U_\varepsilon^r(p)^c$.

We construct X as follows. First for each $n \in \mathbf{N}$, we interlace the process $\gamma^{(n)}$ with jumps from the field of sequences $(X_m^{(n)}(p),\ m \in \mathbf{N},\ p \in M)$ [by projection of the scheme in (4.4)] so that the jump at the time $\tau_m^{(n)}$ is along the geodesic $\mathrm{Exp}(X_m^{(n)})(\gamma^{(n)}(\tau_m^{(n)}))$. By Theorem 4.1 and continuity of π, we deduce that this construction gives rise to a sequence which converges uniformly on bounded intervals (a.s.) to a Lévy process $\tilde{X}$ on M with generator $\tilde{\mathscr{A}}$ where for $f \in C^\infty(M),\ p \in M$,

$$\tilde{\mathscr{A}} f(p) = \tfrac{1}{2} a \Delta f(p) + \int_{U_\varepsilon^r(p)} (f(\mathrm{Exp}(y)(p)) - f(p)) \nu_p(dy)$$

We can now construct X by interlacing $\tilde{X}$ with jumps from the field $(Z_m(p),\ m \in \mathbf{N},\ p \in M)$ [by projection of the scheme in (4.7)] so that the jump at the time $\sigma(m)$ is along the geodesic $\mathrm{Exp}(Z_m)(\tilde{X}(\sigma(m)-))$.

4.3. *Example*: *symmetric spaces.* Let (G, K) be a Riemannian symmetric pair (see [13], page 209) so that G is a connected Lie group, K is a closed subgroup, the group $Ad_G(K)$ is compact and there exists an involutive analytic diffeomorphism σ of G such that K lies between the set of fixed points of σ and its identity component; then $M = G \setminus K$ is a Riemannian globally symmetric space. Furthermore (see [18]) if we assume that G acts effectively on M then G is bundle isomorphic to $O(M)$ and hence the structure group K is isomorphic to $O(d)$. In this case, as is discussed in [1], the horizontal Lévy process in $O(M)$ is (up to isomorphism) a Lévy process in G in the sense of [3]. Now let ρ be an isotropic Lévy process in M as described above and let α_t be the law of each $\rho(t)$ for $t \geq 0$. We say that the process ρ is *spherical* if

$$\alpha_t(\tau(k)A) = \alpha_t(A)$$

for all $t > 0,\ k \in K$ and $A \in \mathscr{B}(M)$, where τ is the natural action of G on M, that is,

$$\tau(g)hK = ghK$$

for all $g, h \in G$. It was shown in [2] (see also [11], [12]) that ρ is spherical if and only if its generator is of the form

$$\mathscr{A} f(p) = \tfrac{1}{2} a \Delta f(p) + \int_{T_p(M)-\{0\}} \big(f(\mathrm{Exp}(y)(p)) - f(p)\big) \nu_p(dy)$$

where $a \geq 0$ and each Lévy measure ν_p is spherically symmetric. Now it is easy to see that the requirement that ν_p is spherical is precisely that ν is isotropic. Hence we deduce that the most general isotropic Lévy process is spherical. Note that we do not have to make a compactness assumption on M in this case since non-explosion in a finite time interval (a.s.) follows in this case from the corresponding non-explosion of the process in G (see [3]).

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