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FINITE SIZE SCALING IN THREE-DIMENSIONAL BOOTSTRAP PERCOLATION[1]

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We consider the problem of bootstrap percolation on a three-dimensional lattice and we study its finite size scaling behavior. Bootstrap percolation is an example of cellular automata defined on the d-dimensional lattice $\{1, 2, \ldots, L\}^d$ in which each site can be empty or occupied by a single particle; in the starting configuration each site is occupied with probability p, occupied sites remain occupied forever, while empty sites are occupied by a particle if at least ℓ among their $2d$ nearest neighbor sites are occupied. When d is fixed, the most interesting case is the one $\ell = d$: this is a sort of threshold, in the sense that the critical probability p_c for the dynamics on the infinite lattice $\mathbb{Z}^d$ switches from zero to one when this limit is crossed. Finite size effects in the three-dimensional case are already known in the cases $\ell \leq 2$; in this paper we discuss the case $\ell = 3$ and we show that the finite size scaling function for this problem is of the form $f(L) = \text{const}/\ln\ln L$. We prove a conjecture proposed by A. C. D. van Enter.

1. Introduction. Cellular automata are dynamical systems defined on the d-dimensional lattice $\mathbb{Z}^d$ in which each site of the lattice is occupied by one of finitely many *types* at each time t. An update rule is defined, which is *homogeneous* (all the sites follow the same rule) and *local* (transitions are determined by the configuration of types on a finite set of neighboring sites) [19, 14].

These models can be thought of as interacting particle systems and their connections with statistical mechanics models have been widely studied in past years (see, e.g., [6, 13, 18, 20]). A particular example of cellular automata, known as *bootstrap percolation*, has been introduced in [5] to model some magnetic systems. More information on the physical relevance of this model is given in [4, 7].

In bootstrap percolation only two different types are associated to each site: each site can be either occupied by a particle or empty. In the starting configuration each site is independently occupied with probability p. Occupied sites remain occupied forever, while empty sites become occupied by a particle if at least ℓ among their $2d$ nearest neighbor sites are occupied. The object of primary interest is the probability $p_{\text{full}}(\ell)$ that at the end of the dynamics, that is, in the infinite time configuration, all the sites are occupied. The basic

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question that has been addressed in physics literature is whether by changing the value of the parameter p the system exhibits a sort of phase transition, that is, whether there exists a critical value $p_c(\ell) \in [0, 1]$ such that $p_{\text{full}}(\ell) = 1$ if $p > p_c(\ell)$ and $p_{\text{full}}(\ell) < 1$ if $p < p_c(\ell)$.

For a fixed dimension d, any site occupied in the dynamics associated to ℓ is also occupied in the dynamics associated to $\ell - 1$. Hence $p_c(\ell)$ is an increasing function of ℓ. The first rigorous result on the value of $p_c(\ell)$ is due to van Enter, who used the idea of Straley's argument [7] to prove that in the case $d = 2$ and $\ell = 2$ the critical probability is equal to zero [15]. In [10, 12], Schonmann has proved that $p_c(\ell) \in \{0, 1\}$, more precisely $p_c(\ell) = 0$ if $\ell \leq d$, otherwise $p_c(\ell) = 1$; these results suggest that the most peculiar case is $\ell = d$.

Before these rigorous results the phase transition scenario in bootstrap percolation models was not clear. The technique that had been used to measure the critical probability $p_c(\ell)$ was the finite size scaling: a finite volume estimate of the critical probability was found by means of Monte Carlo simulations on a finite lattice $\Lambda_L = \{1, 2, \dots L\}^d$, for instance, the probability $p_L^{0.5}$ that one half of the samples were completely filled at the end of the dynamics, and the critical value $p_c(\ell)$ was extrapolated by means of a suitable scaling function $f(L)$. That is, the expression

$$p_L^{0.5} - p_c(\ell) \overset{L\to\infty}{\sim} f(L) \tag{1.1}$$

was supposed to be valid and Monte Carlo data were properly fitted by means of the function $f(L)$ (see [1] and references therein).

It is rather clear that the estimate of p_c strongly depends on the choice of the scaling function $f(L)$; the typical choice, when critical effects in second-order phase transition are studied, is $f(L) = \text{const} \times L^{-1/\nu}$ where ν is a suitable exponent. In fact this choice with $1/\nu = d$ is correct in the case $\ell = 1$, while estimations of $p_c(\ell)$ in the cases $\ell = 2$ and $d \geq 2$ obtained by means of Monte Carlo data analyzed through this function $f(L)$ did not fit in the rigorous scenario depicted by Schonmann's results [1]. The problem is that the power law $L^{-1/\nu}$ approaches zero too quickly and must be replaced by a slower function $f(L) = \text{const} \times (\ln L)^{-(d-1)}$ as suggested by the finite volume results of Aizenman and Lebowitz [3, 8]. Indeed the analysis of old and new data performed with the correct scaling function yields the correct estimate of the critical probability [2, 16, 17].

In [3] bootstrap percolation on finite lattices Λ_L is considered in the case $\ell = 2$ and $d \geq 2$ and it is observed that if p is kept fixed, then in the limit $L \to \infty$ the probability $p_{\text{full}}^{L,p}$ to fill $\{1, \dots, L\}^d$ tends to one whatever the value of p is. But if $p \to 0$ together with $L \to \infty$, then it is possible to find a particular regime in which the probability to fill everything tends to zero. Indeed it is proved in [3] that there exist two constants $c_+ > c_- > 0$ such that if $p \geq c_+/(\ln L)^{d-1}$ then $p_{\text{full}}^{L,p} \to 1$ when $p \to 0$ and $L \to \infty$, while if $p \leq c_-/(\ln L)^{d-1}$, then in the same limit $p_{\text{full}}^{L,p} \to 0$.

Let us focus on the case $d = 3$. The choice $\ell = 2$ is not the most delicate one; indeed, according to [12] in the case $\ell = 3$, that is, even in a situation in which it is more difficult to fill empty sites, the critical probability p_{full} is still zero. Hence one can guess that in the case $d = 3$, $\ell = 3$, the correct finite scaling function is not the Aizenman–Lebowitz one, but perhaps a function approaching zero more slowly. Our aim is to study this case and to show that results similar to those in [3], and conjectured by A. C. D. van Enter, hold with the scaling function $f(L) = \text{const}/\ln\ln L$. This problem has been proposed in [12] as Problem 3.1; we notice also that related questions have been discussed in [9, 11].

An interesting follow up would be the generalization of our results to the d-dimensional case with $\ell = d$. In this case one expects $f(L) = \text{const}/(\ln\ln\cdots\ln L)$ with the logarithm applied $d-1$ times.

In Section 2 we introduce the notations and we state the main result. Theorem 2.1 is proved in Section 3.

2. Notation and results. We first describe the model of bootstrap percolation that we are going to study. Let us consider the lattice $\mathbb{Z}^3$ and the discrete time variable $t = 0, 1, 2, \dots$. The status of the site $x \in \mathbb{Z}^3$ at time $t \in \mathbb{N}$ is described by a random variable $X_t(x)$ with values in $\{0, 1\}$. The site x is occupied at time t if $X_t(x) = 1$ and empty if $X_t(x) = 0$. We denote by $\Omega := \{0, 1\}^{\mathbb{Z}^3}$ the space of the configurations and by $X_t \in \Omega$ the configuration of the system at time t. The initial configuration X_0 is chosen by occupying independently each site of the lattice with probability p (initial density). Then the system evolves according to the following deterministic rules:

1. If $X_t(x) = 1$, then $X_{t+1}(x) = 1$ (1's are stable).
2. If $X_t(x) = 0$ and x has at least three occupied sites among its six nearest neighbors, then $X_{t+1}(x) = 1$.
3. $X_{t+1}(x) = 0$ otherwise.

We omit in our notations the dependence of the process X_t on the initial density p. When the initial density will be different from p it will be clearly stated. We denote by X the final configuration attained by the system when the dynamics stops, that is,

$$X := \lim_{t\to\infty} X_t. \tag{2.1}$$

By Schonmann's result [10, 12], we know that $X(x) = 1$ for all $x \in \mathbb{Z}^3$ whenever $p > 0$.

Let us consider a subset $\Lambda \subset \mathbb{Z}^3$. We denote by $X_{\Lambda,t}$ the process restricted to Λ with free boundary conditions; that is, the dynamics runs without taking into account sites outside Λ. Thus $X_{\Lambda,t}$, $X_{\Lambda,t}(x)$ and X_Λ will, respectively, denote the configuration at time t, the value of the random variable at time t and site x and the final configuration for the process restricted to Λ.

Given two arbitrary sets $\Lambda_1, \Lambda_2 \subset \mathbb{Z}^3$, we denote by $X^{\Lambda_2}_{\Lambda_1,t}$ the process restricted to Λ_1 and with sites in Λ_2 occupied. The sites in $\mathbb{Z}^3 \setminus (\Lambda_1 \cup \Lambda_2)$ are not taken into account to run the dynamics of $X^{\Lambda_2}_{\Lambda_1,t}$. We omit Λ_1 in the notation if $\Lambda_1 = \mathbb{Z}^3$ and we omit Λ_2 if $\Lambda_2 = \varnothing$.

DEFINITION 2.1. Following [3] we say that a set $\Lambda \subset \mathbb{Z}^3$ is *internally spanned* if it is entirely covered in the final configuration of the dynamics restricted to Λ, that is, if

$$\forall\, x \in \Lambda, \qquad X_\Lambda(x) = 1. \tag{2.2}$$

We have now introduced the bootstrap percolation model in dimension $d = 3$ and with parameter $\ell = 3$. As we have already remarked, this is the most delicate three-dimensional bootstrap percolation model. Indeed, $\ell = 3$ is the highest value of the parameter ℓ such that the critical probability in infinite volume is equal to zero: for each positive initial density p, the probability that the whole lattice is completely occupied by particles at the end of the dynamics is equal to 1.

In this particular situation we examine the question of finite size scaling, that is, following [3], we consider the process $X_{\Lambda_L,t}$ on the finite cube $\Lambda_L = \{1,\dots,L\}^3$ of size L and we perform the simultaneous limits $L \to \infty$ and $p \to 0$. We prove that there exists a cutoff between regimes in which the asymptotic probability that the cube Λ_L is internally spanned is zero or one.

THEOREM 2.1. *Let us consider the cube Λ_L of side length L and the process X_t with initial density p and let us denote by $R(L,p)$ the probability that the cube Λ_L is internally spanned. There exist two constants $c_+ > c_- > 0$ such that*:

(i) $R(L,p) \longrightarrow 1$ *if* $(L,p) \to (\infty, 0)$ *in the regime* $p > c_+/\ln\ln L$;
(ii) $R(L,p) \longrightarrow 0$ *if* $(L,p) \to (\infty, 0)$ *in the regime* $p < c_-/\ln\ln L$.

We start to prove the first part of Theorem 2.1; this is the easy part of the theorem and its proof has already been sketched in [16]. The idea of the proof relies on the notion of *critical length*. However small p is, if a fully occupied cube has size large enough, then the probability of finding on its faces two-dimensional occupied square droplets large enough to grow and cover all the faces of the cube (two-dimensional supercritical droplets) is close to 1. As p goes to 0, the size of such a cube must diverge to ∞ and this must happen quickly enough; the critical length of the occupied cubes is of order $\exp(\text{const}/p)$. In order to have a probability close to 1 of finding such cubes, the size L of the region Λ_L must be of order $\exp\exp(\text{const}/p)$.

We now make this a little more precise. From results in [12] one easily obtains that there exists a constant $c_1 > 0$ such that, given a cube Λ_l, if $p \ge c_1/\ln l$ then there exists a constant $a_1 > 0$ such that

$$P(\Lambda_l \text{ covers } \mathbb{Z}^3 \mid \Lambda_l \text{ occupied at } t=0) \ge \prod_{k=l}^{\infty} (1 - \exp(-a_1 k)). \tag{2.3}$$

We consider a large cube Λ_L and we estimate $R(L,p)$ from below:

$$\begin{aligned} &R(L,p) \ge P(\text{in } \Lambda_L\ \exists\, \Lambda_l \text{ occupied at } t=0,\ \Lambda_l \text{ covers } \mathbb{Z}^3) = \\ &P(\Lambda_l \text{ covers } \mathbb{Z}^3 \mid \text{in } \Lambda_L\ \exists\, \Lambda_l \text{ occupied at } t=0)\, P(\text{in } \Lambda_L\ \exists\, \Lambda_l \\ &\text{occupied at } t=0). \end{aligned} \tag{2.4}$$

Now, from (2.3) it follows that if $l \geq \exp(c_1/p)$ then the first factor in (2.4) tends to 1 when p goes to 0. It remains to adjust the value of L so that the second factor of (2.4) tends to 1, too. By partitioning the cube Λ_L in disjoint cubes of size l one has

$$P(\text{in } \Lambda_L \ \exists\ \Lambda_l \text{ occupied at } t = 0) \geq 1 - (1 - p^{l^3})^{(L/l)^3}. \tag{2.5}$$

By choosing $c_+ > c_1 > 0$, $l = \exp(c_1/p)$ and $L > \exp\exp(c_+/p)$, the right-hand term in the equation above tends to 1 when p goes to 0. This completes the proof of the first part of Theorem 2.1. □

3. Proof of Theorem 2.1. We prove here the second part of Theorem 2.1. Let us sketch briefly the strategy of our proof. Our goal is to find a a suitable upper bound to the probability that a cube Λ_L is internally spanned. We say that a cube is crossed if, in the final configuration of the dynamics restricted to the cube itself, there is an occupied connected set joining two opposite faces of the cube. We compute an upper bound to the probability of the simple event that a cube is crossed by building several processes which dominate the original bootstrap percolation process. The last process is simply a juxtaposition of several independent two-dimensional bootstrap percolation processes with an additional increasing step. We then rely on the known estimates [3] for the bootstrap percolation model in the case $d = 2$, $l = 2$ to bound from above the probability that a cube is crossed. If the region Λ_L is internally spanned, then it must contain cubes of every intermediate size which are crossed. The best upper bound is obtained for cubes of size of order exp(const$/p$), which is likely to be the size of a "critical droplet," yielding an upper bound of the correct order.

First, we give some additional definitions. Given a site $x \in \mathbb{Z}^3$ we denote by (x_1, x_2, x_3) its three coordinates and given a set $\Lambda \subset \mathbb{Z}^3$ we define its diameter

$$d(\Lambda) := \sup\{|x_i - y_i|: x \in \Lambda,\ y \in \Lambda,\ i \in \{1, 2, 3\}\}, \tag{3.1}$$

that is, $d(\Lambda)$ is the side length of the minimal cube surrounding the set Λ. We say that $\Lambda \subset \mathbb{Z}^3$ is a region of $\mathbb{Z}^3$ if and only if it is nearest-neighbors connected. If Λ_1 and Λ_2 are two regions of $\mathbb{Z}^3$ and $\Lambda_1 \cup \Lambda_2$ is a region as well, then $d(\Lambda_1 \cup \Lambda_2) \leq d(\Lambda_1) + d(\Lambda_2)$.

We adapt to our situation a key lemma of [3] describing what happens at a smaller scale inside an internally spanned region.

LEMMA 3.1. *Let Λ_1 be a region of $\mathbb{Z}^3$. If Λ_1 is internally spanned, then for all κ such that $1 \leq \kappa$ and $2\kappa + 1 \leq d(\Lambda_1)$ there exists some region Λ_2 included in Λ_1 which is internally spanned and such that $\kappa \leq d(\Lambda_2) < 2\kappa + 1$.*

PROOF. We build the final configuration X_{Λ_1} through the following algorithmic procedure. Let $\mathscr{C}_0$ be the collection of the sites occupied at time zero. Suppose we have built a collection of internally spanned regions $\mathscr{C}_n$; we define a rule to build the collection $\mathscr{C}_{n+1}$:

(i) If there exist two regions A, B of $\mathscr{C}_n$ such that $A \cup B$ is still a region, then we set

$$\mathscr{C}_{n+1} := \mathscr{C}_n \cup \{A \cup B\} \setminus \{A, B\}, \tag{3.2}$$

that is, $\mathscr{C}_{n+1}$ is obtained by replacing in $\mathscr{C}_n$ the two elements A and B by $A \cup B$.

(ii) If no such regions exist then we choose a site x not belonging to any set in $\mathscr{C}_n$ and having three neighbors in the set $\bigcup_{A \in \mathscr{C}_n} A$. We denote by A_i, $1 \le i \le r$, the r regions of $\mathscr{C}_n$ containing a neighbor of x, and we set

$$\mathscr{C}_{n+1} := \mathscr{C}_n \cup \left\{ \bigcup_{i=1}^{r} A_i \cup \{x\} \right\} \Big\backslash \{A_1, \dots, A_r\}. \tag{3.3}$$

(iii) If no such site x exists, the algorithm stops.

Notice that for each n, the regions of $\mathscr{C}_n$ are internally spanned. Since by hypothesis Λ_1 is internally spanned, the procedure ends for some m such that $\mathscr{C}_m = \{\Lambda_1\}$. Moreover, we have that $\max\{d(A): A \in \mathscr{C}_0\} = 1$, $\max\{d(A): A \in \mathscr{C}_m\} = d(\Lambda_1)$ and for any $n \le m-1$,

$$\max\{d(A): A \in \mathscr{C}_{n+1}\} \le 2 \max\{d(A): A \in \mathscr{C}_n\} + 1. \tag{3.4}$$

Hence there exists n such that

$$\kappa \le \max\{d(A): A \in \mathscr{C}_n\} < 2\kappa + 1, \tag{3.5}$$

which means that in $\mathscr{C}_n$ there is an internally spanned region Λ_2 such that $\kappa \le d(\Lambda_2) < 2\kappa + 1$. □

DEFINITION 3.1. Let us consider a cube Λ in $\mathbb{Z}^3$. We say that Λ is crossed, or that there is a crossing in Λ, if and only if in the final configuration X_Λ of the dynamics restricted to Λ there is an occupied region joining two opposite faces of the cube Λ.

We note that for any region A the following inclusion holds:

$$\begin{aligned} &\{A \text{ is internally spanned}\} \\ &\qquad \subset \{\text{the smallest cube surrounding } A \text{ is crossed}\}. \end{aligned} \tag{3.6}$$

Hence, by using Lemma 3.1, for any L and any κ such that $2\kappa + 1 < L$, we have

$$\begin{aligned} R(L, p) &\le P(\exists\, l,\ \kappa \le l < 2\kappa + 1,\ \exists\, \Lambda_l \subset \Lambda_L,\ \Lambda_l \text{ is crossed}) \\ &\le (\kappa + 1)\, L^3 \max_{\kappa \le l < 2\kappa+1} P(\Lambda_l \text{ is crossed}). \end{aligned} \tag{3.7}$$

Thus,

$$R(L, p) \le L^3 \min_{1 \le \kappa < (L-1)/2} (\kappa + 1) \max_{\kappa \le l < 2\kappa+1} P(\Lambda_l \text{ is crossed}). \tag{3.8}$$

We have reduced the estimate of $R(L, p)$ to the estimate of the probability that a cube Λ_l is crossed and by symmetry we can consider the case of a crossing along the first coordinate direction (denoted by e_1 in the sequel),

$$P(\Lambda_l \text{ is crossed}) \leq 3\ P(\Lambda_l \text{ is crossed along } e_1). \tag{3.9}$$

In order to estimate the right-hand term of (3.9) we reduce the problem to a two-dimensional situation by properly cutting the cube Λ_l in slices of thickness two-perpendicular to the x_1-direction. For the sake of definiteness we suppose that $\Lambda_l := \{1, 2, \dots, l\}^3$ where l is an even integer and we define the slices

$$T_k := \{x \in \Lambda_l\colon x_1 = 2k - 1 \text{ or } x_1 = 2k\}, \qquad 1 \leq k \leq \frac{l}{2}. \tag{3.10}$$

We define a map s associating to each site in a slice the only nearest neighbor along the first coordinate direction belonging to the same slice,

$$\forall\ x \in \Lambda_l, \qquad s(x) := \begin{cases} (x_1 + 1, x_2, x_3), & \text{if } x_1 \text{ is odd,} \\ (x_1 - 1, x_2, x_3), & \text{if } x_1 \text{ is even.} \end{cases} \tag{3.11}$$

The process $X_{T_k, t}^{\Lambda_l \setminus T_k}$, obtained by restricting the original process X_t to the slice T_k and by occupying all the sites in $\Lambda_l \setminus T_k$, dominates the original process in the same slice,

$$\forall\ k \in \left\{1, \dots, \frac{l}{2}\right\} \qquad \forall\ x \in T_k, \quad \forall\ t \geq 0, \quad X_t(x) \leq X_{T_k, t}^{\Lambda_l \setminus T_k}(x). \tag{3.12}$$

In each slice T_k, for k in $\{1, \dots, l/2\}$, we define a new process Y_t^k.

DEFINITION 3.2. We consider all the sites in $\Lambda_l \setminus T_k$ occupied and we define the process Y_t^k on $\{0, 1\}^{T_k}$ as follows. For any x in T_k:

(i) $Y_0^k(x) = \max(X_{T_k, 0}^{\Lambda_l \setminus T_k}(x), X_{T_k, 0}^{\Lambda_l \setminus T_k}(s(x))) = \max(X_0(x), X_0(s(x)))$.

(ii) If $Y_t^k(x) = 1$ then $Y_{t+1}^k(x) = 1$.

(iii) If $Y_t^k(x) = 0$ and x has at least three occupied sites among its six nearest neighbors in the configuration Y_t^k, then $Y_{t+1}^k(x) = 1$ and $Y_{t+1}^k(s(x)) = 1$.

(iv) $Y_{t+1}^k(x) = 0$ otherwise.

The mechanism to build $Y_{t+1}^k(x)$ is the one used for $X_{T_k, t+1}^{\Lambda_l \setminus T_k}$ followed by an additional step increasing the configuration. This mechanism ensures that for all t and any x in T_k one has $Y_t^k(x) = Y_t^k(s(x))$.

We next introduce a family of two-dimensional processes.

DEFINITION 3.3. To each slice T_k, for k in $\{1, \dots, l/2\}$, we associate a two-dimensional $l \times l$ square $Q_l^k := \{1, 2, \dots, l\}^2$. On each square Q_l^k we define a process $Z_{Q_l^k, t}$ by

$$Z_{Q_l^k, t}(x_2, x_3) := Y_t^k(2k - 1, x_2, x_3) = Y_t^k(2k, x_2, x_3) \tag{3.13}$$

for any $(x_2, x_3) \in Q_l^k$.

The processes $Z_{Q_l^k,t}$, $1 \le k \le l/2$, are independent and they are two-dimensional bootstrap percolation processes with parameter $\ell = 2$ and initial density $q = 1-(1-p)^2 = 2p - p^2$. Furthermore, these processes dominate the original process in the slices, that is, for any $k \in \{1,\dots,l/2\}$, $x = (x_1, x_2, x_3) \in T_k$ and $t \ge 0$,

$$X_{T_k,t}(x) \le Z_{Q_l^k,t}(x_2, x_3). \tag{3.14}$$

We use these two-dimensional processes to estimate the probability that a cubic region is crossed. We consider the $(l/2)\times l \times l$ parallelepiped $\mathscr{P}_l$ obtained by collecting all the $l/2$ squares Q_l^k introduced in Definition 3.3 (the kth slice along e_1 corresponds to Q_l^k) and we denote by $Z_{\mathscr{P}_l}$ the configuration on $\mathscr{P}_l$ defined as follows: $\forall\, x_1 \in \{1,\dots,l/2\}$, $\forall\, x_2, x_3 \in \{1,\dots,l\}$,

$$Z_{\mathscr{P}_l}(x_1, x_2, x_3) := Z_{Q_l^{x_1}}(x_2, x_3), \tag{3.15}$$

where $Z_{Q_l^k}$, $k \in \{1,\dots,l/2\}$, is the final configuration of the process $Z_{Q_l^k,t}$. We bound (3.9) by

$$\begin{aligned} &P(\Lambda_l \text{ is crossed along } e_1)\\ &\qquad \le P(\text{in } Z_{\mathscr{P}_l} \text{ there is a crossing along } e_1). \end{aligned} \tag{3.16}$$

We consider next the two-dimensional bootstrap percolation model with parameter $\ell = 2$ and initial density q. We define as well the concept of "being internally spanned" and we denote by $S(l)$ a square of side length l. We recall that for such a process, the final configuration is a union of separated occupied rectangular regions; that is, the distance between two occupied rectangles is strictly larger than 1. We restate a few results for this model.

LEMMA 3.2 (Aizenman–Lebowitz [3]). *For all $\kappa \ge 1$, a necessary condition for $S(l)$ to be internally spanned, where $\kappa \le l$, is that it contains at least one rectangular region whose maximal side length is in the interval $[\kappa, 2\kappa+1]$ which is also internally spanned.*

The proof of Lemma 3.2 can be found in [3].

LEMMA 3.3. *Let A be a rectangular region of side lengths l_1 and l_2, where $l_1 \le l_2$. For q small enough one has*

$$P(A \text{ is internally spanned}) \le (4 l_2 q)^{l_2/2}. \tag{3.17}$$

PROOF. If A is partitioned in $l_2/2$ disjoint slabs of width 2, a necessary condition for A to be internally spanned is that each slab contains initially an occupied site. Hence

$$
\begin{aligned}
P(A \text{ is internally spanned}) &\le (1-(1-q)^{2l_1})^{l_2/2} \\
&\le \exp\biggl(\frac{l_2}{2}\ \ln(1-\exp(2l_2\ \ln(1-q)))\biggr) \\
&\le \exp\biggl(\frac{l_2}{2}\ \ln(-2l_2\ \ln(1-q))\biggr).
\end{aligned} \tag{3.18}
$$

For q small enough, $\ln(1-q) \ge -2q$, whence

$$
P(A \text{ is internally spanned}) \le \exp\biggl(\frac{l_2}{2}\ \ln(4l_2 q)\biggr) = (4l_2 q)^{l_2/2}. \tag{3.19}
$$

□

LEMMA 3.4. *For any l in $\mathbb{N}$ and any $\kappa \le (l-1)/2$ let $\mathscr{E}$ be the event: $S(l)$ contains a rectangular region internally spanned whose maximal side length belongs to the interval $[\kappa, 2\kappa+1]$. For q small enough one has*

$$
P(\mathscr{E}) \le l^2\ (2\kappa+1)^2 \exp\biggl(-\frac{\kappa}{2}\exp(-4(2\kappa+1)q)\biggr). \tag{3.20}
$$

PROOF. We suppose q small enough to have $\ln(1-q) \ge -2q$ and we bound the probability of the event $\mathscr{E}$ as follows:

$$
\begin{aligned}
P(\mathscr{E}) &\le l^2\ (2\kappa+1)^2 \max_{\kappa \le l_2 \le 2\kappa+1} \exp\biggl(\frac{l_2}{2}\ \ln(1-\exp(2l_2\ \ln(1-q)))\biggr) \\
&\le l^2\ (2\kappa+1)^2 \exp\biggl(\frac{\kappa}{2}\ \ln(1-\exp(2(2\kappa+1)\ \ln(1-q)))\biggr) \\
&\le l^2\ (2\kappa+1)^2\ \exp\biggl(\frac{\kappa}{2}\ \ln(1-\exp(-4(2\kappa+1)q))\biggr) \\
&\le l^2\ (2\kappa+1)^2\ \exp\biggl(-\frac{\kappa}{2}\ \exp(-4(2\kappa+1)q)\biggr).
\end{aligned} \tag{3.21}
$$

□

Now we come back to the proof of the upper bound. Let α be positive. Notice that $p \le q \le 2p$. We denote by $\underline{1}$ the configuration in a square Q_l^h with all the sites occupied. We still increase the configuration $Z_{Q_l^k}$ by setting $Z_{Q_l^k} = \underline{1}$ in case $Z_{Q_l^k}$ contains at least one occupied rectangular region of maximal side length larger than α/q. Suppose that $l \ge \alpha/q$ and $\alpha/q > 3$. By applying Lemma 3.4 with $\kappa = \alpha/3q$, we get

$$
\begin{aligned}
P(Z_{Q_l^k} = \underline{1}) &\le l^2 \left(\frac{2\alpha}{3q}+1\right)^2 \exp\biggl(-\frac{\alpha}{6q}\exp\biggl(-4\biggl(\frac{2\alpha}{3q}+1\biggr)q\biggr)\biggr) \\
&\le \frac{l^2\alpha^2}{q^2}\exp\biggl(-\frac{\alpha}{6q}\exp(-4\alpha)\biggr).
\end{aligned} \tag{3.22}
$$

We suppose that α is small enough to have $\exp(-4\alpha) \ge 1/2$. Then

$$
P(Z_{Q_l^k} = \underline{1}) \le \frac{l^2\alpha^2}{q^2}\exp\biggl(-\frac{\alpha}{12q}\biggr). \tag{3.23}
$$

Let M be the (random) number of indices k such that $Z_{Q_l^k} = \underline{1}$ and let $k(1), \dots, k(M)$ be these indices arranged in increasing order. Let $\mathscr{E}_1$ be the event

$$\mathscr{E}_1 := \{\text{there is a crossing along } e_1 \text{ in } Z_{\mathscr{P}_l}\}.$$

We decompose this event as follows:

$$\begin{aligned} P(\mathscr{E}_1) &= P(\mathscr{E}_1, M = 0) \\ &\quad + \sum_{m=1}^{l/2} \sum_{i_1 < \cdots < i_m} P(\mathscr{E}_1, M = m, k(1) = i_1, \dots, k(m) = i_m). \end{aligned} \tag{3.24}$$

Let $i < j$ be two indices in $\{1, \dots, l/2\}$. By $\mathscr{E}(i, j)$ we denote the following event: there exists a sequence of H disjoint occupied rectangular regions $(R_h,\ 1 \le h \le H)$ in $Z_{\mathscr{P}_l}$ such that:

(i) R_1 is included in Q_l^i, R_H is included in Q_l^j.

(ii) The regions R_h, $2 \le h \le H-1$, are included in $\bigcup_{i<h<j} Q_l^h$.

(iii) For each h, $1 \le h \le H$, if R_h belongs to Q_l^k, then R_h is separated from the other occupied rectangular regions of $Z_{Q_l^k}$.

(iv) The maximal side length of all these regions is strictly less than α/q; we denote by r_h the maximal side length of R_h.

(v) For each h, $1 \le h \le H-1$, a site of R_h is the neighbor of a site of R_{h+1}.

(vi) All the sites of these regions are occupied in $Z_{\mathscr{P}_l}$.

We remark that H is free; however, it has to be larger than $j-i+1$. Moreover, the sequence of rectangles $(R_h,\ 2 \le h \le H-1)$ can go back and forth between the squares Q_l^{i+1} and Q_l^{j-1}. We make the convention that for any i, $\mathscr{E}(i, i) = \mathscr{E}(i, i-1)$ is the full event of probability 1. Using (3.23), we have

$$\begin{aligned} &P(\mathscr{E}_1, M = m, k(1) = i_1, \dots, k(m) = i_m) \\ &\quad \le P\bigg(M = m, k(1) = i_1, \dots, k(m) = i_m, \mathscr{E}(1, i_1 - 1), \\ &\qquad\qquad \mathscr{E}(i_1 + 1, i_2 - 1), \dots, \mathscr{E}(i_{m-1} + 1, i_m - 1),\ \mathscr{E}\bigg(i_m + 1, \frac{l}{2}\bigg)\bigg) \\ &\quad = \prod_{h=1}^{m+1} P(\mathscr{E}(i_{h-1} + 1, i_h - 1))\ P(Z_{Q_l^{i_1}} = \underline{1}) \cdots P(Z_{Q_l^{i_m}} = \underline{1}) \\ &\quad \le \bigg(\frac{l^2\alpha^2}{q^2} \exp\bigg(-\frac{\alpha}{12q}\bigg)\bigg)^m \prod_{h=1}^{m+1} P(\mathscr{E}(i_{h-1} + 1, i_h - 1)), \end{aligned} \tag{3.25}$$

where we have set $i_0 = 0$ and $i_{m+1} = l/2 + 1$.

In order to estimate $P(\mathscr{E}(i, j))$ for $i < j$, we consider a fixed sequence $r_1, \dots, r_H$ in $\{1, \dots, \alpha/q - 1\}$. The number of sequences $R_1, \dots, R_H$ of rectangles with maximal sides $r_1, \dots, r_H$ and satisfying the above requirements is smaller than

$$l^2 r_1 \times r_1^2 r_2 \times r_2^2 r_3 \times \cdots \times r_{H-1}^2 r_H \le l^2\,(r_1 r_2 \cdots r_H)^3. \tag{3.26}$$

Notice that several rectangles of the sequence $R_1, \ldots, R_H$ might belong to the same slice. However the rectangles belonging to the same slice are separated and the events that these rectangles are internally spanned depend only on the dynamics restricted to the rectangles. Hence these events are independent, so that

$$\begin{aligned} &P(R_1, \ldots, R_H \text{ are occupied in } Z_{\mathscr{P}_l}) \\ &\quad \le P(R_1, \ldots, R_H \text{ internally spanned}) \\ &\quad = P(R_1 \text{ internally spanned}) \cdots P(R_H \text{ internally spanned}) \\ &\quad \le (4r_1 q)^{r_1/2} \cdots (4r_H q)^{r_H/2}, \end{aligned} \tag{3.27}$$

where in the last inequality we have used Lemma 3.3. Thus

$$\begin{aligned} P(\mathscr{E}(i,j)) &= \sum_{H \ge j-i+1} P(\exists\, R_1, \ldots, R_H \text{ realizing } \mathscr{E}(i,j)) \\ &\le \sum_{H \ge j-i+1} \sum_{r_1, \ldots, r_H < \alpha/q} l^2\,(r_1 \cdots r_H)^3\,(4r_1 q)^{r_1/2} \cdots (4r_H q)^{r_H/2} \\ &= \sum_{H \ge j-i+1} l^2 \left(\sum_{1 \le r < \alpha/q} r^3\,(4rq)^{r/2} \right)^H. \end{aligned} \tag{3.28}$$

We estimate the sum $\sum_{1 \le r < \alpha/q} r^3\,(4rq)^{r/2}$ as follows:

$$\begin{aligned} \sum_{1 \le r < \alpha/q} r^3\,(4rq)^{r/2} &= \sum_{1 \le r \le 8} r^3\,(4rq)^{r/2} + \sum_{9 \le r < \alpha/q} r^3\,(4rq)^{r/2} \\ &\le 8^3 (32q)^{1/2} + \left(\frac{\alpha}{q}\right)^4 \max_{9 \le r < \alpha/q} (4rq)^{r/2}. \end{aligned} \tag{3.29}$$

Let $f(r) = (4rq)^{r/2}$. For α small enough, $f(r)$ is decreasing on $[9, \alpha/q[$, whence

$$\sum_{1 \le r < \alpha/q} r^3\,(4rq)^{r/2} \le 8^3 (32q)^{1/2} + \alpha^4 36^{9/2} q^{1/2} \le b_0 q^{1/2}, \tag{3.30}$$

where b_0 is a constant not depending on α. Thus,

$$P(\mathscr{E}(i,j)) \le \sum_{H \ge j-i+1} l^2 (b_0 \sqrt{q})^H. \tag{3.31}$$

Finally, for q small enough, so that $b_0\sqrt{q} < 1/2$ and $\ln b_0 \le (-1/4) \ln q$, we have

$$P(\mathscr{E}(i,j)) \le 2l^2 (b_0\sqrt{q})^{j-i+1} = 2l^2 \exp\bigl(\tfrac{1}{4}(j-i+1)\ln q\bigr). \tag{3.32}$$

Coming back to inequality (3.25),

$$
\begin{aligned}
&P(\mathscr{E}_1, M = m, k(1) = i_1, \ldots, k(m) = i_m) \\
&\qquad \le \left(\frac{l^2\alpha^2}{q^2}\exp\left(-\frac{\alpha}{12q}\right)\right)^m (2l^2)^{m+1}\exp\left[\left(\frac{l}{2}-m\right)\left(\frac{1}{4}\ln q\right)\right]
\end{aligned} \tag{3.33}
$$

and putting this estimate in (3.24),

$$
\begin{aligned}
P(\mathscr{E}_1) &= P(\mathscr{E}_1, M = 0) \\
&\quad + \sum_{m=1}^{l/2}\sum_{i_1<\cdots<i_m} P(\mathscr{E}_1,\ M = m, k(1) = i_1, \ldots, k(m) = i_m) \\
&\le \sum_{m=0}^{l/2}\left(\frac{l}{2}\right)^m\left(\frac{l^2\alpha^2}{q^2}\exp\left(-\frac{\alpha}{12q}\right)\right)^m (2l^2)^{m+1}\exp\left[\left(\frac{l}{2}-m\right)\left(\frac{1}{4}\ln q\right)\right] \\
&\le 2l^2\exp\left[\frac{l}{8}\ln q\right]\sum_{m=0}^{l/2}\left[\frac{l^5\alpha^2}{q^2}\exp\left(-\frac{\alpha}{12q}\right)\exp\left[-\frac{1}{4}\ln q\right]\right]^m.
\end{aligned} \tag{3.34}
$$

This is our estimate of the probability of a crossing along e_1 in the case $l \ge \alpha/q$. On the other hand, in the case $l < \alpha/q$ we estimate the probability of a crossing along e_1 by considering directly the event $\mathscr{E}(1, l/2)$,

$$
P(\Lambda_l \text{ is crossed along } e_1) \le P\left(\mathscr{E}\left(1, \frac{l}{2}\right)\right) \le 2l^2\exp\left[\frac{l}{8}\ln q\right]. \tag{3.35}
$$

We note that it would not have been possible to use the same strategy in the case $l \ge \alpha/q$ because the probability of having a very large internally spanned rectangle in some slice does not vanish.

Supposing that $l \le \exp(\alpha/120q)$, one has

$$
\frac{l^5\alpha^2}{q^2}\exp\left(-\frac{\alpha}{12q}\right)\exp\left[-\frac{1}{4}\ln q\right] \le \frac{\alpha^2}{q^2}\exp\left(-\frac{\alpha}{24q}-\frac{1}{4}\ln q\right). \tag{3.36}
$$

If q is sufficiently small so that the right hand term is smaller than 1, under the hypothesis $2 \le l \le \exp(\alpha/120q)$, from (3.9), (3.34), (3.35) and (3.36), one has

$$
P(\Lambda_l \text{ is crossed}) \le 6l^2\exp\left[\frac{l}{8}\ln q\right]\left(\frac{l}{2}+1\right) \le 6l^3\exp\left(\frac{l}{8}\ln q\right). \tag{3.37}
$$

Hence, there exists $\alpha > 0$ such that for p sufficiently small and $2 \le l \le \exp(\alpha/240p)$,

$$
P(\Lambda_l \text{ is crossed }) \le 6l^3\exp\left(\frac{l}{8}\ln(2p)\right). \tag{3.38}
$$

Finally, we use equations (3.8) and (3.38) to estimate the probability $R(L, p)$ and to complete the proof of Theorem 2.1. First we consider the case

$L \leq \exp(\alpha_0/p)$ with $\alpha_0 = \alpha/240$ and we write

$$\begin{aligned} R(L, p) &\leq L^3 \min_{1 \leq \kappa < (L-1)/2} (\kappa + 1) \times 6(2\kappa + 1)^3 \exp\left(\frac{\kappa}{8} \ln(2p)\right) \\ &\leq 6L^7 \exp\left(\frac{L}{24} \ln(2p)\right). \end{aligned} \tag{3.39}$$

We remark that the right-hand term goes to zero in the limit $p \to 0$ and $L \to \infty$. On the other hand, in the case $L > \exp(\alpha_0/p)$, we restrict the minimum in (3.8) to the range $2\kappa + 1 \leq \exp(\alpha_0/p)$ and we get

$$R(L, p) \leq 6L^3 \exp\left(\frac{4\alpha_0}{p}\right) \exp\left[\frac{1}{24} \exp\left(\frac{\alpha_0}{p}\right) \ln(2p)\right]. \tag{3.40}$$

From the estimate above we see that there exists a positive constant c_- such that if L is less than $\exp\exp(c_-/p)$ then $R(L, p)$ goes to 0 in the limit $L \to \infty$ and $p \to 0$. This completes the proof of Theorem 2.1. □

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