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LARGE DEVIATION PRINCIPLE FOR RANDOM WALK IN A QUENCHED RANDOM ENVIRONMENT IN THE LOW SPEED REGIME

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We consider a one-dimensional random walk $(X_n)_{n \times \mathbb{N}}$ in a random environment of zero or strictly positive drifts. We establish a full large deviation principle for X_n/n of the correct order $n/(\log n)^2$ in the low speed regime, valid for almost every environment. This completes the large deviation picture obtained earlier by Greven and den Hollander and Gantert and Zeitouni in the case of zero and positive drifts. The proof uses coarse graining along with concentration of measure techniques.

1. Introduction.

The model and the main result. In this paper, we continue the study, initiated by Greven and den Hollander [6], of the large deviation behavior of a nearest neighbor random walk on $\mathbb{Z}$ with quenched (frozen-in) site-dependent transition probabilities, which are themselves generated randomly. In [6] an almost sure (with respect to the environment) large deviation principle has been derived governing the rescaled position X_n/n of the random walk. The corresponding rate function I: $[-1, 1] \to \mathbb{R} \cup \{\infty\}$ (which is independent of the environment) has been identified as the solution of a variational problem involving specific relative entropy with respect to a certain Markov process. For certain distributions of the environment, however, the rate function has been found to *vanish* over parts of the interval $[-1, 1]$, indicating thereby a slower decay of the corresponding tail probabilities. In a subsequent work [4], Gantert and Zeitouni identified the correct order of decay for the most important types of environment distributions (exhibiting the anomalous large deviation behavior mentioned above): the case of "positive and negative drifts" and the case of "positive and zero drifts" (for a recent review of these and related results we refer to [5].) It is this latter case which we will focus on in the present paper and establish a full large deviation principle for X_n/n of the correct order $n/(\log n)^2$ completing thereby the large deviations analysis for this type of environment.

We start with the sample space $\Omega = [1/2, 1)^{\mathbb{Z}} = \{\omega = (\omega_x)_{x \in \mathbb{Z}} \mid 1/2 \leq \omega_x < 1\}$ which serves as a random environment. For a given probability

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distribution α on $[1/2, 1)$, we denote by $\mathbf{P}^\alpha$ the product measure $\otimes^{\mathbb{Z}} \alpha$ on Ω. If the distribution α puts nonzero, but not all, weight on the singleton $\{1/2\}$ we speak about an environment with *positive and zero drifts*. In order to describe a random walk in the random environment, we first set $W = \{(w_n)_{n \in \mathbb{N}} \mid w_n \in \mathbb{Z}\}$ and $X_n(w) = w_n$. For every fixed ω, we consider the Markov chain $(X_n)_{n \geq 0}$ on $\mathbb{Z}$ starting at x with transition probabilities

$$\mathbf{P}_x^\omega[X_{n+1} = y \mid X_n = z] = \begin{cases} \omega_z, & \text{if } y = z+1, \\ 1 - \omega_z, & \text{if } y = z-1, \\ 0, & \text{otherwise}, \end{cases} \tag{1}$$

where $\mathbf{P}_x^\omega[\cdot]$ denotes the measure on the path space W for the given environment ω. We usually write $\mathbf{P}^\omega$ instead of $\mathbf{P}_0^\omega$. The *annealed* measure is defined as

$$\mathbb{P}_x[\cdot] = \int \mathbf{P}_x^\omega[\cdot] \mathbf{P}^\alpha(d\omega)$$

and we omit the subscript x when $x = 0$. Note that $\mathbf{P}_x^\omega$ can be viewed as the conditional measure (of $\mathbb{P}_x$) on the path space given the environment ω and is usually referred to as the quenched measure. For a summary of results concerning the behavior of a random walk in random environment (RWRE) governed by quenched or annealed measures, we refer to the introduction in [3] and to [5].)

Abbreviate $\rho_x = \rho_x(\omega) = (1 - \omega_x)/\omega_x$ and set $\langle \rho \rangle = \int \rho_0(\omega) \mathbf{P}^\alpha(d\omega) = \mathbf{E}^\alpha[\rho_0]$, where here and throughout $\mathbf{E}^\alpha[\ ]$ denotes expectation with respect to the measure $\mathbf{P}^\alpha$. From now on we will assume $0 < \alpha(\{1/2\}) < 1$ throughout, that is, we will focus on environments with positive and zero drifts. Note that in this case $\langle \rho \rangle < 1$. Solomon's result [9] guarantees that the RWRE is transient, and for $\mathbf{P}^\alpha$-a.e. ω,

$$\lim_{n \to \infty} n^{-1} X_n = (1 - \langle \rho \rangle)/(1 + \langle \rho \rangle) =: v_\alpha.$$

As mentioned above, Greven and den Hollander [6] established a large deviation principle for X_n/n of order n. In the case of positive and zero drifts, the corresponding rate function $J(v)$, $v \in [-1, 1]$ turns out to be strictly positive on $[-1, 0) \cup (v_\alpha, 1]$ and vanishes on $(0, v_\alpha)$. Indeed, quite recently Gantert and Zeitouni [4] showed that on $(0, v_\alpha)$ the correct order of decay is $n/(\log n)^2$. Our main result is the following large deviation principle.

THEOREM 1 (Positive and zero drifts). *Suppose that* $0 < \alpha(\{1/2\}) < 1$. *We have for* $\mathbf{P}^\alpha$ *a.e.* ω, *for any open and nonempty* $G \subset (0, v_\alpha)$ *with* $\overline{G} \subset [0, v_\alpha)$,

$$\begin{aligned} -\inf_{v \in G} I(v) &\leq \liminf_{n \to \infty} \frac{(\log n)^2}{n} \log \mathbf{P}\big(n^{-1} X_n \in G\big) \\ &\leq \limsup_{n \to \infty} \frac{(\log n)^2}{n} \log \mathbf{P}\big(n^{-1} X_n \in \overline{G}\big) \leq -\inf_{v \in \overline{G}} I(v), \end{aligned} \tag{2}$$

where $\overline{G}$ denotes the closure of G and $I(v)$ is given by

$$(3)\qquad I(v)=\frac{\big(\pi\log\alpha(\{1/2\})\big)^2}{8}(1-v/v_\alpha).$$

We remark that the lower bound in (2) was in fact already shown in Theorem 3 of [4].

Explanation of the rate function. To give some idea about the particular form of the rate function, let us briefly sketch how the lower bound can be derived. The basic idea is that whenever the walk is slower than v_α, it must have lost time in relatively long regions made of "fair sites" (a site x will be called fair if $\omega_x = 1/2$). By a famous result of Erdős and Rényi on longest head runs, we know that for a.e. environment and all large n we can find in $[0, vn]$ a fair region of length $\sim \log n/|\log\alpha(\{1/2\})|$. Since by Solomon's result typically $X_{nv/v_\alpha} \sim nv$, we can expect that when $X_n/n \le v$, the walk has to stay $\sim (1 - v/v_\alpha)n$ steps in such a fair region. It turns out that on this level of accuracy all excursions to the left of this region can be discarded. This simply corresponds to putting a reflection on the leftmost site of the fair region. Using classical asymptotic bounds on the tail of the exit time of a simple random walk from an interval, we can derive a lower bound $\sim \exp\{-(n/8(\log n)^2)\pi^2|\log\alpha(\{1/2\})|^2(1-v/v_\alpha)\}$ on the probability that the reflected walk stays $\sim (1-v/v_\alpha)n$ steps in our fair stretch. This basically yields the lower bound in (2).

We will now give an outline of the strategy of the proof of the upper bound. One of the main ingredients is coarse graining. Here the time will be split up in larger units corresponding to visits in three different types of (partly overlapping) regions. To describe them, we first divide $\mathbb{Z}$ into blocks of intermediate scale $(\log n)^{1-\delta}$, where $\delta > 0$ is a small constant. We will distinguish between biased and fair blocks corresponding to the proportion of fair sites (a site x is called fair if ω_x is close to $1/2$) in the block. Fair [or type (1)] regions are "connected components" of fair blocks to which we attach the neighboring biased block on both sides. The role of the biased block attached to the left of such a region is to ignore all the small but numerous left excursions. The biased block on the right simply guarantees that after leaving the fair region to the right, the walk automatically has a barrier to the left which keeps it away from reentering the fair stretch too often. In particular, once we have a control on the number of left crossings of biased blocks, we will be able to bound the number of visits in type (1) regions. Next we introduce regions where the walk basically has a constant speed v_α. These regions are called type (2) regions and are made of pairs of biased blocks satisfying some additional requirements; see (9) for a precise definition. The remaining pairs of biased blocks, whose number turns out to be too small to be of any importance, will be called type (3) regions.

Let $S_n^{(\mathrm{i})}$ denote the total time until n the walk spends in type (i) regions; compare (12). In view of the arguments leading to the lower bound we split up the event $\{X_n/n \le v\}$ as follows:

$$\begin{aligned} \{X_n/n \le v\} &= \{X_n/n \le v, S_n^{(1)} \ge n(1 - v/v_\alpha - 2\eta)\} \\ &\cup \{X_n/n \le v, S_n^{(2)} \ge n(v/v_\alpha + \eta)\} \\ &\cup \{X_n/n \le v, S_n^{(3)} \ge n\eta\} \\ &=: A_1 \cup A_2 \cup A_3, \end{aligned} \tag{4}$$

where $\eta > 0$ is a small parameter which eventually will tend to zero.

In order to estimate $\mathbf{P}^\omega[A_1]$ we note that for a.e. environment, when n is large enough, the length of the longest fair region intersecting $[-n, n]$ is of the order $\log n/|\log \alpha(\{1/2\})|$. We will show that the number of fair regions is a.s. bounded by $n/(\log n)^3 = o(n/(\log n)^2)$. This, together with a classical result on the tail of the exit time of a simple random walk from an interval, will lead to the correct upper bound in (2). This also explains why we call $\mathbf{P}^\omega[A_1]$ the leading term.

To control the negligible term $\mathbf{P}^\omega[A_3]$ we will use that the number of type (3) regions is small. This can be shown by using the martingale method, which gives a strong enough bound on the probability that a double block is of type (3). Finally, we will show that $\mathbf{P}^\omega[A_2]$ is also negligible. This will be done by observing that $S_n^{(2)}$ is the sum of $\sim nv/(\log n)^{1-\delta}$ "independent" but not identically distributed random variables, namely, the exit times from double blocks. However, the expected value as well as the tail of these random variables can be controlled uniformly. In fact, the control of the expected value is an immediate consequence of the definition of type (2) regions. The control of the tail will be established by using the martingale method once more.

Related results. Let us briefly comment on previous work closely related to this topic. The type of coarse graining we use has been introduced in a Brownian motion context in [8] and was adapted in [7] to analyze the large deviation behavior of RWRE in the annealed setting. Perhaps the closest problem to ours has been studied by Sznitman in [10]. In that article, large deviations of the position of Brownian motion in a quenched random (positive) potential have been obtained on the critical scale $t/(\log t)^{2/d}$. Besides intuitive similarities between both models, there are some striking analogies between the results in $d = 1$.

The paper is organized as follows. In Sections 2.1 and 2.2 we define our basic objects and prove some preliminary lemmas. In Section 2.3 we state Propositions 1 and 2 which provide the necessary estimates on $\mathbf{P}^\omega[A_i]$, $i \in \{1, 2, 3\}$. We also show how they imply the upper bound part of Theorem 1. In Section 2.4 we give the proof of Proposition 2, which deals with the leading term. Finally, in Section 2.5 we give the proof of Proposition 1, which takes care of the negligible term.

2. The upper bound. Our first observation is that in order to prove the upper bound in Theorem 1 it is enough to show the following, slightly weaker statement. For each fixed $v \in (0, v_\alpha)$, for $\mathbf{P}^\alpha$-a.e. ω,

$$\limsup_{n\to\infty} \frac{(\log n)^2}{n} \log \mathbf{P}\big(n^{-1}X_n \le v\big) \le -I(v). \tag{5}$$

Indeed, setting $\Omega_q = \{\omega; \limsup_{n\to\infty}((\log n)^2/n)\log \mathbf{P}(n^{-1}X_n \le q) \le -I(q)\}$, the upper bound in (2) follows for each $\omega \in \bigcap_{q\in\mathbb{Q}^+, q<v_\alpha} \Omega_q$ (which has $\mathbf{P}^\alpha$ measure 1) by approximating $v \in (0, v_\alpha)$ by $q_k \in \mathbb{Q}^+$ from above and using the continuity of $I(\cdot)$. We suppose that from now on, we are working with an α for which the assumptions of Theorem 1 are valid. In particular, we have $0 < \alpha(\{1/2\}) < 1$.

2.1. *Basic definitions.*

Intervals and regions. In this section we introduce three types of regions according to the environment ω, namely, fair regions, regular double biased blocks and irregular double biased blocks. To this end, we start by introducing fair and biased blocks in the same spirit as in [7]. Let $\delta \in (0, 1/3)$ be fixed and let $n > 10$. We divide $\mathbb{Z}$ into blocks B_j of length $\lfloor(\log n)^{1-\delta}\rfloor$: $(j \in \mathbb{Z})$,

$$B_j = \Big[j\big\lfloor(\log n)^{1-\delta}\big\rfloor, (j+1)\big\lfloor(\log n)^{1-\delta}\big\rfloor\Big) \cap \mathbb{Z}. \tag{6}$$

The closed block $\overline{B}_j$ is defined as $[j\lfloor(\log n)^{1-\delta}\rfloor, (j+1)\lfloor(\log n)^{1-\delta}\rfloor] \cap \mathbb{Z}$. Next we pick a $\xi \in (0, 1/2)$, such that

$$p(\xi) := \alpha([1/2 + \xi, 1]) > 0 \tag{7}$$

and $\varepsilon \in (0, p(\xi))$. We say that the *site* x is *biased* if $\omega_x \ge 1/2 + \xi$. Note that $p(\xi)$ is the probability that x is biased. We call the block B_j *biased* if the proportion of biased sites in B_j exceeds ε. Otherwise the block is called *fair*. We next define when a biased block is *strongly biased*, which we abbreviate by s-biased. To this end we pick $\gamma \in (0, (1-\delta)/2)$, and chop B_j into intervals of length $\lfloor(\log n)^\gamma\rfloor$. If the last one is shorter than $\lfloor(\log n)^\gamma\rfloor$, then we attach it to the previous interval. In this way we obtain what we call *subblocks*. Note that the length of a subblock cannot exceed $2\lfloor(\log n)^\gamma\rfloor$. We say that B_j is *s-biased*, if each of its subblocks itself is biased. If this is not the case, we call B_j *w-biased* (weakly biased).

Type (1) *regions*. We are now ready to define fair regions. Let $(F_i^{(1)})_{i\in I(1)}$ be the collection of all stretches of maximal length consisting of consecutive fair blocks. For every $i \in I(1)$, we attach to $F_i^{(1)}$ (when present) the neighboring *closed* biased blocks on the left and on the right. The resulting interval is denoted by $\mathscr{F}_i^{(1)}$ and is called a *fair* [or *type* (1)] *region*. Note that two different fair regions might have a biased block in common.

Type (2) *regions*. Next we define *regular double biased blocks* [or type (2) regions]. For $i \in \mathbb{Z}$ such that B_{i-1} and B_i are biased, we set $\mathscr{B}_i = \overline{B}_{i-1} \cup \overline{B}_i$.

We introduce the stopping time

$$V_i^{(n)} = \inf\{k \geq 0; X_k = (i+1)\lfloor(\log n)^{1-\delta}\rfloor\}. \tag{8}$$

For given ω let us denote by $E_i^{\langle\rho\rangle,\omega}[\cdot]$ the quenched expectation starting from $i\lfloor(\log n)^{1-\delta}\rfloor$, where we have replaced the environment to the left of $\mathscr{B}_i$ by the constant $1/(1+\langle\rho\rangle) > 1/2$, that is, $\omega_x = 1/(1+\langle\rho\rangle)$ for all $x < (i-1)\lfloor(\log n)^{1-\delta}\rfloor$, and the rest of the environment is left untouched.

Pick $\zeta > 0$. We say that $\mathscr{B}_i$ is a *regular double biased block* [or *type* (2) *region*], if B_{i-1} and B_i are s-biased and

$$E_i^{\langle\rho\rangle,\omega}[V_i^{(n)}] \leq (1/v_\alpha + \zeta)(\log n)^{1-\delta}. \tag{9}$$

Type (3) *regions*. Finally, *irregular double biased blocks* [*type* (3) *regions*] are those double biased blocks which are not regular.

Stopping times. We define a sequence of stopping times which allows us to identify the time the walk spends in various regions up to time n. To this end we define the *boundary* of the region (interval) $\mathscr{R}$ to be the set containing the leftmost and rightmost point of $\mathscr{R}$. The *interior* of a region consists of all points of $\mathbb{Z}$ which lie strictly between the boundary of this region. Let us denote the set of *division points* $\lfloor(\log n)^{1-\delta}\rfloor\mathbb{Z}$ by $\mathscr{Z}$. Observe that for $z \in \mathscr{Z}$, there exists a unique region such that z is contained in its interior. We denote this region by $\mathscr{F}(z)$ and set ($i \in \{1,2,3\}$),

$$\chi(z) = i \quad \text{if } \mathscr{F}(z) \text{ is of type (i)}. \tag{10}$$

We now define *epochs* of time associated with *visits* of regions. First we set $T_0 = 0$ and for $k \geq 1$,

$$T_k = \inf\{t > T_{k-1}; X_t \in \partial\mathscr{F}(X_{T_{k-1}})\}. \tag{11}$$

T_k indicates the end of the kth epoch as well as the beginning of the $(k+1)$th epoch. Note that for $\mathbf{P}^\alpha$ a.e. ω, all these variables are finite (and therefore also well defined).

Observe that $T_{k+1} = T_k + T_1 \circ \vartheta_{T_k}$, where ϑ denotes the canonical shift. For $i \in \{1,2,3\}$ we define

$$S_n^{(\mathrm{i})} = \sum_j (T_j \wedge n - T_{j-1}), \tag{12}$$

where the sum runs over all $j \geq 1$: $\chi(X_{T_{j-1}}) = i$, $T_{j-1} < n$. In words, $S_n^{(\mathrm{i})}$ represents the total amount of time until n the walk spends visiting regions of type (i). In particular, $n = \sum_{i=1}^3 S_n^{(\mathrm{i})}$.

Crossings. We close this section by introducing the notion of left crossings of biased blocks up to time n. As in [7] we define for $x \in \mathbb{Z}$, $k \geq 1$,

$$\tau_x^1 = \inf\{t \geq 0 \mid X_t = x\} \text{ and for } k \geq 2,\ \tau_x^k = \inf\{t > \tau_x^{k-1} \mid X_t = x\}, \tag{13}$$

the successive times of hitting the site x. During the time intervals $(\tau_x^k \wedge n, \tau_x^{k+1} \wedge n]$, $(k \geq 1)$, excursions take place (starting at x) either to the left or to the right of x (unless the time interval is empty.) The *height* of such an excursion is defined by

$$H_x^k = \begin{cases} \max\{X_t - x \mid t \in (\tau_x^k \wedge n, \tau_x^{k+1} \wedge n]\}; \\ \qquad X_{\tau_x^k+1} > x \text{ (right excursion)}, \\ \min\{X_t - x \mid t \in (\tau_x^k \wedge n, \tau_x^{k+1} \wedge n]\}; \\ \qquad X_{\tau_x^k+1} < x \text{ (left excursion)}, \\ 0; \quad (\tau_x^k \wedge n, \tau_x^{k+1} \wedge n] = \varnothing. \end{cases} \tag{14}$$

The number of left crossings of the block B_j until time n is given by

$$N_{j,n} = \left|\left\{k \mid H^k_{(j+1)\lfloor(\log n)^{1-\delta}\rfloor} \le -\lfloor(\log n)^{1-\delta}\rfloor\right\}\right| \tag{15}$$

and the total number of left crossings of biased blocks up to time n can be written as

$$N_n^{\leftarrow} = \sum_{j \in J^b} N_{j,n}, \tag{16}$$

where $J^b = \{j \in \mathbb{Z}; B_j \text{ is biased}\}$, the set of indices of biased blocks.

Parameter ranges. We finally give a list of all our parameters with their allowed range. It will be necessary to introduce one further small parameter η whose use will become apparent only later; compare (35). Nevertheless, in order to have a complete reference, we include it in the following list:

$$\begin{array}{lll} v: & 0 < v < v_\alpha & \text{fixed} \\ \delta: & 0 < \delta < 1/3 & \text{fixed} \\ \gamma: & 0 < 2\gamma < 1 - \delta & \text{fixed} \\ \xi: & 0 < \xi \text{ and } p(\xi) > 0 & (\xi \sim 0) \\ \eta: & 0 < \eta < 1 - v/v_\alpha & (\eta \sim 0) \\ \varepsilon: & 0 < \varepsilon < p(\xi) \wedge \dfrac{\eta}{4(1/v_\alpha + 1)} & (\varepsilon \sim 0) \\ \zeta: & 0 < \zeta < \eta/4 & (\zeta \sim 0), \end{array} \tag{17}$$

where $p(\xi) = \alpha[1/2 + \xi, 1] = \mathbf{P}^\alpha[0 \text{ is biased}]$. For future use we notice that by this choice of parameters the following inequality holds:

$$\eta > 2\zeta v + 2\varepsilon(1/v_\alpha + 2\zeta), \tag{18}$$

which plays a role in the proof of Proposition 4 and (87).

We will adopt the convention that dependence on v, δ and γ (as well as on the measure α) will always be suppressed in contrast to the other parameters, which sometimes will be used explicitly in the notation.

2.2. *Preliminary lemmas.* In this section we provide some preliminary lemmas which will be constantly used in the sequel.

LEMMA 1. *Let* $\varepsilon > 0$ *and define the event*

$$(19)\qquad \begin{aligned} G_1(\varepsilon, n) &= \{\exists \text{ left crossing of length } \geq \varepsilon n \text{ up to time } n\} \\ &= \bigcup_{0 \leq t < s \leq n} \{X_s \leq X_t - \varepsilon n\}. \end{aligned}$$

For $\mathbf{P}^{\alpha}$ *a.e.* ω,

$$(20)\qquad \limsup_{n \to \infty} \frac{1}{n} \log \mathbf{P}^{\omega}[G_1(\varepsilon, n)] \leq -\frac{\varepsilon}{2} |\log \langle \rho \rangle|.$$

PROOF. Using Chebyshev's inequality and (19) from [7] we find for large enough n,

$$\mathbf{P}^{\alpha}\Big[\big\{\omega; \mathbf{P}^{\omega}[G_1(\varepsilon, n)] \geq \langle \rho \rangle^{(\varepsilon/2)n}\big\}\Big] \leq \langle \rho \rangle^{-(\varepsilon/2)n} 2n(n+1)^2 \frac{\langle \rho \rangle^{\varepsilon n}}{1 - \langle \rho \rangle}.$$

Since $\langle \rho \rangle < 1$, the claim follows from Borel–Cantelli. □

Note that G_1 is negligible for the purpose of proving the upper bound in Theorem 1, since $\limsup_{n \to \infty} (\log n)^2/n \log \mathbf{P}^{\omega}[\{X_n/n \leq vn\} \cap G_1] = -\infty$. The next lemma provides a $\mathbf{P}^{\alpha}$ a.s. upper bound on the number of fair blocks.

LEMMA 2. *Let* $k \geq 1$ *and pick* ε, ξ *according to* (17). *For* $\mathbf{P}^{\alpha}$ *a.e.* ω, $\exists n_0 = n_0(\omega, k, \varepsilon, \xi)$ *such that for all* $n \geq n_0$,

$$(21)\qquad \text{number of fair blocks intersecting } [-n, n] \leq \frac{n}{(\log n)^k}.$$

PROOF. By Cramér's theorem,

$$(22)\qquad q := \mathbf{P}^{\alpha}[B_1 \text{ is fair}] \leq \exp\Big\{-\big\lfloor (\log n)^{1-\delta} \big\rfloor \Lambda^*_{p(\xi)}(\varepsilon)\Big\},$$

where $\Lambda^*_{p(\xi)}(\cdot)$ denotes the logarithmic moment generating function of a Bernoulli variable with parameter $p(\xi) = \mathbf{P}^{\alpha}[\omega_0 \geq 1/2 + \xi] \in (0,1)$. Note that $\Lambda^*_{p(\xi)}(\varepsilon) > 0$ since $\varepsilon < p(\xi)$; compare (17). Next we define Y_i to be the indicator function in the event that B_i is fair. Let I_n be the set of indices of blocks intersecting $[-n, n]$ and $N = |I_n|$. Set $c = n/(\log n)^k$ and $\lambda = \log(c/Nq)$. One can easily check that $c/(Nq) \to \infty$ as $n \to \infty$. Since the $(Y_i)_i$ are i.i.d. 0–1 variables, we find for large enough n using Chebyshev's inequality,

$$(23)\qquad \begin{aligned} \mathbf{P}^{\alpha}\left[\sum_{i \in I_n} Y_i > c\right] &\leq \exp(-\lambda c + N \log \mathbf{E}^{\alpha}[\exp(\lambda Y_1)]) \leq \exp(-\lambda c + Nqe^{\lambda}) \\ &\leq \exp\left(-\frac{n}{(\log n)^k}\right). \end{aligned}$$

Our claim now follows from Borel–Cantelli. □

At this point we make the following remark, which will be useful later.

REMARK 1. The proof of Lemma 2 shows the following. Assume that the fact that B_i has a certain property $(*)$ depends only on the restriction of ω to B_i. Then, as soon as we know that for $a, b > 0$, $\mathbf{P}^\alpha[B_i$ has property $(*)] \le \exp\{-a(\log n)^b\}$ for every n large enough, we can conclude that $\forall\, k \ge 1$, $\mathbf{P}^\alpha$ a.s. $\exists\, n_0(\omega)$ such that $\forall\, n \ge n_0$,

$$\left|\{i \mid B_i \text{ has property } (*), B_i \cap [-n, n] \neq \varnothing\}\right| \le \frac{n}{(\log n)^k}.$$

The next lemma provides a $\mathbf{P}^\alpha$ a.s. upper bound on the maximal length of a fair region in $[-n, n]$ when n is large. This Erdős–Rényi type upper bound will be crucial for the derivation of the correct rate function.

LEMMA 3. *Let ε, ξ be as in* (17). *Then $\mathbf{P}^\alpha$ a.s. $\exists\, n_1(\omega, \varepsilon, \xi)$ such that $\forall\, n \ge n_1$,*

$$\max_i \left|F_i^{(1)} \cap [-n, n]\right| \le \frac{1+\varepsilon}{\Lambda^*_{p(\xi)}(\varepsilon)} \log(n). \tag{24}$$

PROOF. Not surprisingly, we have only to modify the proof of the well-known theorem of Erdős–Rényi on longest head runs. We first introduce $N^+(n)$ ($N^-(n)$) to be the largest (smallest) index such that $B_{N^{+(-)}(n)} \cap [-n, n] \neq \varnothing$. By $l^{+(-)}(n)$ we denote the number of the longest run (consecutive sequence) of fair blocks to the left (right) of $B_{N^+}(B_{N^-})$ beginning at those blocks. Set $c(n) = (1+\varepsilon)(\log(n))^\delta / \Lambda^*_{p(\xi)}(\varepsilon)$ and observe that

$$\begin{aligned} &\mathbf{P}^\alpha\left[\max\{l^+(n), l^-(n)\} > c(n)\right] \\ &\quad \le 2\mathbf{P}^\alpha\left[l^+(n) > c(n)\right] < 2\mathbf{P}^\alpha[B_0 \text{ is fair}]^{c(n)} \\ &\quad \le 2\exp\left(-(1+\varepsilon)\left\lfloor(\log n)^{1-\delta}\right\rfloor(\log(n))^\delta\right), \end{aligned} \tag{25}$$

where we have used (22). It follows now from Borel–Cantelli that for $\mathbf{P}^\alpha$ a.e. ω, there exists $n_0 = n_0(\omega)$ such that for every $n \ge n_0$,

$$\max\{l^+(n), l^-(n)\} \le c(n). \tag{26}$$

For any such ω, we define $n_1(\omega)$ to be the smallest number ($\ge n_0$) with $c(n)\lfloor(\log n)^{1-\delta}\rfloor \ge 2n_0$. Let $n \ge n_1$. In order to show (24), pick one of the fair regions $F_i^{(1)}$ such that $F := F_i^{(1)} \cap [-n, n] \neq \varnothing$. Let n^* and n_* denote the rightmost and leftmost point of F. Either $n^* \ge |n_*|$ or $n^* < |n_*|$. We will focus on the first case; the second can be treated analogously. So assume $n^* \ge |n_*|$. If $n^* \ge n_0 \ge 0$, we know from (26) that $l^+(n^*) \le c(n^*) \le c(n)$, which implies

$$|F| \le l^+(n^*)\left\lfloor(\log n)^{1-\delta}\right\rfloor \le c(n)\left\lfloor(\log n)^{1-\delta}\right\rfloor \le \frac{1+\varepsilon}{\Lambda^*_{p(\xi)}(\varepsilon)}\log(n). \tag{27}$$

If $n^* < n_0$ we have $|F| \le n^* - n_* + 1 \le 2n^* + 1 \le 2n_0 \le (1+\varepsilon) \cdot\log(n_1)/\Lambda^*_{p(\xi)}(\varepsilon)$, and the proof of Lemma 3 is complete. □

We close this section with a lemma which gives control over $N_n^{\leftarrow}$, the total number of left crossings of biased blocks up to time n; compare (16). This is in the same spirit as Lemma 2 from [7].

LEMMA 4. *We have for all* $\omega \in \Omega$,

$$\limsup_{n\to\infty} \frac{(\log n)^2}{n} \log \mathbf{P}^{\omega}\left[N_n^{\leftarrow} \geq \frac{n}{(\log n)^{3-2\delta}} \right] = -\infty. \tag{28}$$

PROOF. We pick ω such that there is at least one biased block in $[-n, n]$; otherwise there is nothing to show. We denote by $J^b = \{j \in \mathbb{Z}; B_j \text{ is biased}\}$ and introduce the set $\mathscr{R} = \bigcup_{j\in J^b}\{(j+1)\lfloor(\log n)^{1-\delta}\rfloor\}$ of right endpoints of biased blocks. Next we introduce the stopping times,

$$\begin{aligned} &D = \inf\left\{t > 0 \,\middle|\, |X_t - X_0| = \left\lfloor (\log n)^{1-\delta} \right\rfloor \right\},\\ &\tau_1 = \inf\{t \geq 0 \mid X_t \in \mathscr{R}\}, \qquad D_1 = D \circ \vartheta_{\tau_1} + \tau_1 \end{aligned} \tag{29}$$

and for $k \geq 2$ we continue inductively by

$$\tau_k = \inf\{t \geq D_{k-1}; X_t \in \mathscr{R}\}, \qquad D_k = D \circ \vartheta_{\tau_k} + \tau_k, \tag{30}$$

where we set $D \circ \tau_k = \infty$ when $\tau_k = \infty$. Note that $0 \leq \tau_1 < D_1 \leq \tau_2 < D_2 \cdots$. For $k \geq 1$ we set

$$Y_k = 1_{\{D_k < \infty\}} 1_{\{X_{\tau_k} > X_{D_k}\}}, \tag{31}$$

the indicator of the kth left crossing of a biased block. Observe that since each crossing takes at least $\lfloor(\log n)^{1-\delta}\rfloor$ steps, the maximal number of back-crossings up to time n is bounded above by $m = m(n) = n/\lfloor(\log n)^{1-\delta}\rfloor$. Moreover, $N_n^{\leftarrow} \leq \sum_{k=1}^{m(n)} Y_k$. Let us denote $X_{\tau_k} = x_k$ and $y_k = x_k - \lfloor(\log n)^{1-\delta}\rfloor$. From [2], page 73, we find for all $k \geq 1$, for large n,

$$\begin{aligned} 1_{\{\tau_k < \infty\}} \mathbf{P}^{\omega}_{X_{\tau_k}}[X_D < X_0, D < \infty] &= \mathbf{P}^{\omega}_{x_k}[X_D = y_k]\\ &\leq (\log n)^{1-\delta} \prod_{x=x_k}^{y_k} \rho_x(\omega)\\ &\leq (\log n)^{1-\delta} \rho_*^{\varepsilon(\log n)^{1-\delta}}\\ &\leq \exp\left\{-\varepsilon/2(\log n)^{1-\delta} |\log \rho_*|\right\} =: p(n), \end{aligned} \tag{32}$$

where $\rho_*(\xi) = (1/2 - \xi)/(1/2 + \xi) \in (0,1)$ and where we have used that $[y_k, x_k)$ is biased. Denote by $c = c(n) = n/(\log n)^{3-2\delta}$ and introduce $\lambda = \lambda(n) = \log(c/(mp))$. It is easily seen that for large enough n, $\lambda \geq$

$\varepsilon(\log n)^{1-\delta}|\log\rho_*|/4$. Using the strong Markov property and (32), we find

$$
\begin{aligned}
&\mathbf{P}^\omega[N_n^{\leftarrow} \ge c(n)]\\
&\quad\le \mathbf{P}^\omega\left[\sum_{k=1}^{m(n)} Y_k \ge c(n)\right] \le \exp\{-\lambda c(n)\}E^\omega\left[\exp\left\{\sum_{k=1}^{m(n)} \lambda Y_k\right\}\right]\\
&\quad\le \exp\{-\lambda c(n)\}\left(E^\omega\left[\exp\left\{\sum_{k=1}^{m(n)-1} \lambda Y_k\right\}\right]\right.\\
&\qquad\qquad\left.+e^\lambda E^\omega\left[\exp\left\{\sum_{k=1}^{m(n)-1} \lambda Y_k\right\}, Y_{m(n)} = 1\right]\right)\\
&\quad= \exp\{-\lambda c(n)\}\left(E^\omega\left[\exp\left\{\sum_{k=1}^{m(n)-1} \lambda Y_k\right\}\right]\right.\\
&\qquad\qquad\left.+e^\lambda E^\omega\left[\exp\left\{\sum_{k=1}^{m(n)-1} \lambda Y_k\right\}1_{\{\tau_m<\infty\}}E^\omega_{X_{\tau_m}}[X_D < X_0, D<\infty]\right]\right) \qquad (33)\\
&\quad\le \exp\{-\lambda c(n)\}E^\omega\left[\exp\left\{\sum_{k=1}^{m(n)-1} \lambda Y_k\right\}\right](1+p(n)e^\lambda)\\
&\quad\le \exp\{-\lambda c(n)\}(1+p(n)e^\lambda)^{m(n)}\\
&\quad\le \exp\left\{-\frac{n}{(\log n)^{3-2\delta}}\left(\frac{\varepsilon}{4}(\log n)^{1-\delta}|\log\rho_*| - 1\right)\right\}\\
&\quad\le \exp\left\{-\frac{\varepsilon}{8}\frac{n}{(\log(n))^2}(\log(n))^\delta|\log\rho_*|\right\}
\end{aligned}
$$

and (28) follows □

2.3. *Proof of the upper bound.* Now we are ready to describe the first step in the proof. For fixed parameters as in (17), define

$$
\begin{aligned}
A_1(\varepsilon,\xi,n) = \{\nexists\ &\text{left crossing of length} \ge \varepsilon n\\
&\text{up to time } n, N_n^{\leftarrow} \le c(n)\},
\end{aligned}
\qquad (34)
$$

where $c(n) = n/(\log n)^{3-2\delta}$. We set

$$
\begin{aligned}
A_2(\varepsilon,\xi,\eta,n) &= A_1 \cap \{X_n/n \le v, S_n^{(1)} \le n(1-v/v_\alpha - 2\eta)\},\\
A_3(\varepsilon,\xi,\eta,n) &= A_1 \cap \{X_n/n \le v, S_n^{(1)} > n(1-v/v_\alpha - 2\eta)\},
\end{aligned}
\qquad (35)
$$

where $S_n^{(1)}$ was defined in (12) and write

$$
\{X_n/n \le v\} = (A_1^c \cap \{X_n/n \le v\}) \cup A_2 \cup A_3. \qquad (36)
$$

In order to prove statement (5), we start by looking at the set

$$(37)\quad \Omega_1(\varepsilon,\xi) = \left\{\omega \,\middle|\, \limsup_{n\to\infty} \frac{(\log(n))^2}{n} \log \mathbf{P}^{\omega}[A_1^c(\varepsilon,\xi,n)] = -\infty\right\}.$$

By Lemmas 1 and 4 we know that $\mathbf{P}^{\alpha}[\Omega_1] = 1$. Next we state two propositions from which the statement (5) will follow.

PROPOSITION 1. *Let $v \in (0, v_\alpha)$ be fixed and set*

$$(38)\quad \begin{aligned}\Omega_2(\varepsilon,\xi,\eta) = \Bigg\{\omega\in\Omega \,\Bigg|\, \limsup_{n\to\infty} \frac{(\log(n))^2}{n} \\ \times \log \mathbf{P}^{\omega}[A_2(\varepsilon,\xi,\eta)] = -\infty\Bigg\}.\end{aligned}$$

We then have $\mathbf{P}^{\alpha}[\Omega_2] = 1$.

PROPOSITION 2. *Let $v \in (0, v_\alpha)$ be fixed and set*

$$(39)\quad \begin{aligned}\Omega_3(\varepsilon,\xi,\eta) = \Bigg\{\omega\in\Omega \,\Bigg|\, \limsup_{n\to\infty} \frac{(\log(n))^2}{n} \log \mathbf{P}^{\omega}[A_3(\varepsilon,\xi,\eta)] \\ \le -I(v,\varepsilon,\xi,\eta)\Bigg\},\end{aligned}$$

where

$$(40)\quad I(v,\varepsilon,\xi,\eta) = \frac{\pi^2(1 - v/v_\alpha - 2\eta)\Lambda^*_{p(\xi)}(\varepsilon)^2}{8(1+\varepsilon)^4}.$$

We then have $\mathbf{P}^{\alpha}[\Omega_3] = 1$.

Statement (5) (and Theorem 1) follow now easily. Indeed, pick $v \in (0, v_\alpha)$. Set

$$(41)\quad \Omega_1^* = \bigcap_{\substack{\xi\in\mathbb{Q}^+\\ p(\xi)>0}} \ \bigcap_{\substack{\varepsilon\in\mathbb{Q}^+\\ \varepsilon<p(\xi)\wedge \frac{\eta}{4(1/v_\alpha+1)}}} \Omega_1(\varepsilon,\xi)$$

and for $i = 2, 3$,

$$(42)\quad \Omega_i^* = \bigcap_{\substack{\eta\in\mathbb{Q}^+\\ \eta<1-v/v_\alpha}} \ \bigcap_{\substack{\xi\in\mathbb{Q}^+\\ p(\xi)>0}} \ \bigcap_{\substack{\varepsilon\in\mathbb{Q}^+\\ \varepsilon<p(\xi)\wedge \frac{\eta}{4(1/v_\alpha+1)}}} \Omega_i(\varepsilon,\xi,\eta).$$

Finally, we set $\Omega^* = \Omega_1^* \cap \Omega_2^* \cap \Omega_3^*$. From Propositions 1, 2 and Lemmas 1 and 4 we know that $\mathbf{P}^{\alpha}[\Omega^*] = 1$. On the other hand, for any $\omega \in \Omega^*$ and for arbitrary $\varepsilon, \xi, \eta > 0$ with $0 < \eta < 1 - v/v_\alpha$, $p(\xi) > 0$, $\varepsilon < p(\xi) \wedge$

$\eta/[4(1/v_\alpha + 1)]$, we have

$$\limsup_{n\to\infty} \frac{(\log(n))^2}{n} \log \mathbf{P}^{\omega}[X_n/n \le v] \le -I(v, \varepsilon, \xi, \eta). \tag{43}$$

Since

$$\lim_{\eta\to 0}\lim_{\xi\to 0}\lim_{\varepsilon\to 0} I(v, \varepsilon, \xi, \eta) = (1 - v/v_\alpha)\pi^2 |\log \alpha(\{1/2\})|^2/8 = I(v),$$

we have that for each $\omega \in \Omega^*$, $\limsup_{n\to\infty}(\log n)^2/n \log \mathbf{P}^{\omega}[X_n/n \le v] \le -I(v)$ and (5) follows.

2.4. *The leading term.* In this section we give the proof of Proposition 2. We start with introducing the set [with $k = 3$ in (21)]

$$\Omega_3'(\varepsilon, \xi) = \{\omega \in \Omega \mid (21) \text{ and } (24) \text{ hold for all large enough } n\}. \tag{44}$$

From Lemmas 2 and 3 we know that $\mathbf{P}^{\alpha}[\Omega_3'] = 1$. We claim that $\Omega_3' \subseteq \Omega_3$, from which Proposition 2 clearly follows. Pick $\omega \in \Omega_3'$ and define $\mathscr{N}_1$ the indices of the epochs corresponding to visits in fair regions. More precisely,

$$\mathscr{N}_1 = \left\{ j \ge 1 \mid \chi\left(X_{T_{j-1}}\right) = 1, T_{j-1} \le n \right\}$$

and set $N_1 = |\mathscr{N}_1|$. The first observation is that on A_1, for n large enough (cf. (35) in [7]),

$$0 \le N_1 \le \frac{n}{(\log n)^3} + 2\frac{n}{(\log n)^{3-2\delta}} \le 3\frac{n}{(\log n)^{3-2\delta}}. \tag{45}$$

Indeed, when the walk starts it will possibly cross all fair regions from the left, making thereby only right crossings of the attached biased blocks. After one of these biased blocks has been (right-) crossed, the next crossing of the same block is necessarily a left crossing. Since on A_1, $N_n^{\leftarrow} \le n/(\log n)^{3-2\delta}$ and since for $\omega \in \Omega_3'$, the number of fair regions is bounded above by $n/(\log n)^3$ for all large n, (45) follows. For future use we make the following remark.

REMARK 2. We will show in the proof of Proposition 1 that the number of type (3) regions in $[-n, n]$ is $\mathbf{P}^{\alpha}$ a.s. bounded above by $n/(\log n)^3$, for n large enough. Using exactly the same arguments given above, one can see that for each ω for which this bound holds, on A_1 the number of visits in type (3) regions is bounded above by $3n/(\log n)^{3-2\delta}$, provided n is large enough (depending on ω). Using (45) one easily sees that the claim of Proposition 2 follows once we have shown that, for $\omega \in \Omega_3'$,

$$\limsup_{n\to\infty} \sup_{0 \le K \le 3n/(\log n)^{3-2\delta}} \frac{(\log n)^2}{n} \times \log \mathbf{P}^{\omega}\left[A_3(\varepsilon, \xi, \eta) \cap \{N_1 = K\}\right] \le -I(v, \varepsilon, \xi, \eta). \tag{46}$$

To show (46), pick K with $1 \le K \le 3n/(\log n)^{3-2\delta}$ (for $K = 0$ there is nothing to show). On the set $\{N_1 = K\}$ denote the set of (random) elements of $\mathscr{N}_1$ by $1 \le j_1 < j_2 \cdots < j_K < \infty$. Then, on $\{N_1 = K\}$ we have

$$S_n^{(1)} = \sum_{k=1}^{K} T_{j_k} \wedge n - T_{j_k - 1}. \tag{47}$$

Recall that $T_j = T_{j-1} + T_1 \circ \vartheta_{T_{j-1}}$. It is easily seen that

$$T_{j_k} \wedge n - T_{j_k-1} \le \left(T_1 \wedge T_{[-n,n]}\right) \circ \vartheta_{T_{j_k-1}}.$$

Pick $\lambda > 0$ and use Chebyshev's inequality to find

$$\begin{aligned} \mathbf{P}^{\omega}[A_3, N_1 = K] &\le \exp\{-\lambda n(1 - v/v_\alpha - 2\eta)\}\mathbf{E}^{\omega}\left[\exp\left\{\lambda S_n^{(1)}\right\}; N_1 = K\right] \\ &\le \exp\{-\lambda n(1 - v/v_\alpha - 2\eta)\} \\ &\quad \times \mathbf{E}^{\omega}\left[\exp\left\{\lambda \sum_{k=1}^{K} \left(T_1 \wedge T_{[-n,n]}\right) \circ \vartheta_{T_{j_k-1}}\right\}; N_1 = K\right] \end{aligned} \tag{48}$$

We would like to use the strong Markov property in (48). In order to do this (and to simplify notation) we rename the beginning of the kth epoch spent in a fair region (note that these are stopping times) as follows. We set

$$\tau_1 = \min\left\{T_{j-1} \mid j \ge 1, 0 \le T_{j-1} < \infty, \chi\left(X_{T_{j-1}}\right) = 1\right\} \tag{49}$$

and for $k \ge 2$,

$$\tau_k = \min\left\{T_{j-1} \mid j \ge 1, \tau_{k-1} < T_{j-1} < \infty, \chi\left(X_{T_{j-1}}\right) = 1\right\}, \tag{50}$$

where $\min \varnothing = \infty$. Observe now that on $\{N_1 = K\}$, we have $\tau_k = T_{j_k - 1}$. Coming back to (48) and using the strong Markov property, we find

$$\begin{aligned} \mathbf{P}^{\omega}[A_3, N_1 = K] &\le \exp\{-\lambda n(1 - v/v_\alpha - 2\eta)\} \\ &\quad \times \mathbf{E}^{\omega}\left[\exp\left\{\lambda \sum_{k=1}^{K} \left(T_1 \wedge T_{[-n,n]}\right) \circ \vartheta_{\tau_k}\right\}; \tau_k < \infty\right] \\ &\le \exp\{-\lambda n(1 - v/v_\alpha - 2\eta)\} \\ &\quad \times \mathbf{E}^{\omega}\left[\exp\left\{\lambda \sum_{k=1}^{K-1} \left(T_1 \wedge T_{[-n,n]}\right) \circ \vartheta_{\tau_k}\right\}\right. \\ &\quad \left. \times E^{\omega}_{X_{\tau_K}}\left[\exp\left\{\lambda\left(T_1 \wedge T_{[-n,n]}\right)\right\}\right]; \tau_k < \infty\right]. \end{aligned} \tag{51}$$

By choosing an appropriate λ, we now give a uniform upper bound (uniform in ω as well as in the starting point) of the inner expectation appearing in (51). Indeed, on $\{\tau_K < \infty\}$, $\chi(X_{\tau_K}) = 1$, hence $T_1 \wedge T_{[-n,n]}$ is the exit time from a fair region $\mathscr{F}_i^{(1)} \cap [-n, n]$ for some i. We will now use a well-known estimate for the exponential moments of these exit times. We pick $\rho \in (0,1)$

and choose

$$\lambda = \frac{\pi^2(1-\rho)}{8\left(\max_i \left|\mathscr{F}_i^{(1)} \cap [-n,n]\right|^2\right)}. \tag{52}$$

Using now Lemma 3 and (48) from [7], we see [see also (47) from [7]] that on $\{\tau_K < \infty\}$,

$$\mathbf{E}^{\omega}_{X_{\tau_K}}\left[\exp\left\{\lambda\left(T_1 \wedge T_{[-n,n]}\right)\right\}\right] \le \kappa(\rho) \in (1,\infty). \tag{53}$$

Iterating this in (51) (with $\{\tau_K < \infty\}$ replaced by $\{\tau_{K-1} < \infty\}$, and so on, we find

$$\mathbf{P}^{\omega}[A_3, N_1 = K] \le \kappa(\rho)^K \exp\{-\lambda n(1 - v/v_\alpha - 2\eta)\}. \tag{54}$$

Since $\omega \in \Omega'_3$, we find for n large enough [cf. (24)],

$$\begin{aligned} \max_i \left|\mathscr{F}_i^{(1)} \cap [-n,n]\right| &\le \frac{1+\varepsilon}{\Lambda^*_{p(\xi)}(\varepsilon)} \log(n) + 2(\log n)^{1-\delta} \\ &\le \frac{(1+\varepsilon)^2}{\Lambda^*_{p(\xi)}(\varepsilon)} \log(n). \end{aligned} \tag{55}$$

Coming back to (54) we find that for $\omega \in \Omega'_3(\varepsilon,\xi)$,

$$\begin{aligned} &\limsup_{n\to\infty} \max_{0 \le K \le 3n/(\log n)^{3-2\delta}} \frac{(\log n)^2}{n} \mathbf{P}^{\omega}[A_3, N_1 = K] \\ &\qquad \le -\frac{\pi^2(1-\rho)(1 - v/v_\alpha - 2\eta)}{8(1+\varepsilon)^4} \Lambda^*_{p(\xi)}(\varepsilon)^2. \end{aligned} \tag{56}$$

Since $\rho \in (0,1)$ was arbitrary, (46) follows and the proof of Proposition 2 is complete. □

2.5. *The negligible term.* In this section we give the proof of Proposition 1. The proof will be carried out by splitting A_2 into two events corresponding to times the walk spends in regions of type (2) and (3), respectively. Recall that $n = S_n^{(1)} + S_n^{(2)} + S_n^{(3)}$, where $S_n^{(2)}$ and $S_n^{(3)}$ is the total amount of time until n the walk spends in regular and irregular biased blocks, respectively. Then we can write

$$\begin{aligned} \mathbf{P}^{\omega}[A_2(\xi,\varepsilon,\eta)] &= \mathbf{P}^{\omega}\left[A_1, X_n/n \le v, S_n^{(2)} + S_n^{(3)} \ge n(v/v_\alpha + 2\eta)\right] \\ &\le \mathbf{P}^{\omega}\left[A_1, X_n/n \le v, S_n^{(3)} \ge \eta n\right] \\ &\qquad + \mathbf{P}^{\omega}\left[A_1, X_n/n \le v, S_n^{(2)} \ge n(v/v_\alpha + \eta)\right] \\ &=: \mathbf{P}_4^{\omega}(\xi,\varepsilon,\eta,\zeta) + \mathbf{P}_5^{\omega}(\xi,\varepsilon,\eta,\zeta) \end{aligned} \tag{57}$$

and A_1 was defined in (34).

PROPOSITION 3. *For any choice of parameters satisfying* (17) *we have for* $\mathbf{P}^\alpha$ *a.e.* ω,

$$\limsup_{n\to\infty} \frac{(\log n)^2}{n} \log \mathbf{P}_4^\omega(\xi, \varepsilon, \eta, \zeta) = -\infty. \tag{58}$$

The main ingredient of the proof is the next lemma.

LEMMA 5. *For* $\mathbf{P}^\alpha$ *a.e.* ω,

$$\#\ \textit{of irregular double biased blocks intersecting}\ [-n, n] \le \frac{n}{(\log n)^3}. \tag{59}$$

Let us postpone the proof of (59) and show first how (58) follows. In fact, we will use the same arguments which had been used to show Proposition 1. Recall that that proof was based on two facts:

1. The number of visits in type (1) regions is bounded by $3n/(\log n)^{3-2\delta}$. Note that by Lemma 5 and Remark 2 we have the same bound for the number of visits in irregular double biased blocks.
2. The length of a type (1) region can not exceed $(1+\varepsilon)^2 \log(n)/\Lambda^*_{p(\xi)}(\varepsilon)$. Clearly, a double biased block is much smaller (its length is $2\lfloor(\log n)^{1-\delta}\rfloor$.)

Note that the exponential moment (arising in the Chebyshev estimate) of the exit time from any region $\mathscr{R}_i$ of finite length can be controlled, uniformly in the environment, by (53) [where λ is given by (52) where $\bar{\mathscr{T}}_i^{(1)}$ has to be replaced by $\mathscr{R}_i$.] Choosing now $\lambda = \pi^2/(16(2\lfloor(\log n)^{1-\delta}\rfloor)^2)$ [which corresponds to the choice of $\rho = 1/2$ in (52)] we find as in the proof of Proposition 1 that

$$\begin{aligned}
\mathbf{P}^\omega\big[A_1, X_n/n \le v, S_n^{(3)} \ge \eta n\big] &\le \mathbf{P}^\omega\big[A_1, S_n^{(3)} \ge \eta n\big] \\
&\le \frac{3n}{(\log n)^{3-2\delta}} \kappa(1/2)^{3n/(\log n)^{3-2\delta}} \\
&\quad \times \exp\left(-\pi^2\eta \frac{n}{64(\log n)^{2-2\delta}}\right)
\end{aligned} \tag{60}$$

and the claim of Proposition 3 follows once we have proved Lemma 5.

PROOF OF LEMMA 5. Once we have shown

$$\mathbf{P}^\alpha[\mathscr{B}_0 \text{ is an irregular double biased block}] \le \exp\big(-a(\log n)^b\big), \tag{61}$$

where $a, b > 0$ are independent of n, Lemma 5 will follow from (61) by using Remark 1 with $k = 4$ applied to the double blocks with even (odd) indices having nonempty intersections with $[-n, n]$. Indeed, Remark 1 yields the upper bound on the number of type (3) regions: $2n/(\log n)^4 \le n/(\log n)^3$, for

n large enough. To show (61) we first set $S = \{\omega \mid \mathscr{B}_0$ is s-biased$\}$. Then

$$\begin{aligned} &\mathbf{P}^{\alpha}[\mathscr{B}_0 \text{ is of type } (3)] \\ &\quad \le \mathbf{P}^{\alpha}[S^c] \\ &\qquad + \mathbf{P}^{\alpha}\Big[S \cap \Big\{\omega \mid \mathbf{E}_0^{\langle \rho \rangle, \omega}\big[V_0^{(n)}\big] > (1/v_\alpha + \zeta)(\log n)^{1-\delta}\Big\}\Big]. \end{aligned} \tag{62}$$

The first summand in (62) is easily estimated. Indeed, on S^c there exists at least one fair subblock. Using (22) and the fact that a subblock has a length at least $\lfloor (\log n)^{1-\delta} \rfloor$, we find for large enough n

$$\mathbf{P}^{\alpha}[S^c] \le \exp\bigg(-\frac{1}{2}(\log n)^{\gamma} \Lambda^*_{p(\xi)}(\varepsilon)\bigg). \tag{63}$$

It remains to give an upper bound on the second summand in (62). To this end we will use Azuma's inequality [1] in the following form: Let $(\Omega, \mathscr{A}, P)$ be a probability space with a filtration $(\mathscr{A}_k)_{k=0,\dots,n}$ and $S \in \mathscr{A}$. Let $M_0, M_1, \dots, M_n$ be martingales with the property that for $k = 1, \dots, n$,

$$\|(M_k - M_{k-1}) 1_S\|_\infty \le c_k.$$

Then for every $t \ge 0$,

$$P\big[\{M_n - M_0 \ge t\} \cap S\big] \le \exp\bigg(-\frac{1}{2}\frac{t^2}{\sum_{k=1}^n c_k^2}\bigg). \tag{64}$$

In order to use this inequality, we first set for $x \in \mathbb{Z}$, $U_x(\omega) = \omega_x$ and $N = \lfloor (\log n)^{1-\delta} \rfloor$. For $\omega \in \Omega$ and $0 \le k \le 2N$, we define the martingale on Ω,

$$M_k(\omega) = \mathbb{E}_0\big[V_0^{(n)} \mid \sigma(U_{N-k}, \dots, U_N)\big](\omega) \tag{65}$$

where $\mathbb{E}_0$ is the annealed measure with start at the origin. [$V_0^{(n)} \in L^1(\mathbb{P}_0)$ since $\mathbb{E}_0(V_0^{(n)}) = N/v_\alpha$, which will be clear from (68) and (73).] For $\omega \in \Omega$ and $j \in \mathbb{Z}$, we define $\mathbf{E}_0^{\langle \rho \rangle, j, \omega}$ to be the quenched expectation starting from 0, where we have replaced the environment strictly to the left of j by the homogeneous environment given by $\omega_x = (1 + \langle \rho \rangle)^{-1}$, and the rest of the environment has been left untouched. In particular, using the notation introduced in (9), we have that $E_0^{\langle \rho \rangle, -N, \omega} = \mathbf{E}_0^{\langle \rho \rangle, \omega}$. We claim that for $\omega \in \Omega$, $0 \le k \le 2N$,

$$M_k(\omega) = \mathbf{E}_0^{\langle \rho \rangle, N-k, \omega}\big[V_0^{(n)}\big]. \tag{66}$$

To check (66), we have to show that for $0 \le k \le 2N$ and $A \in \sigma(U_{N-k}(\omega), \dots, U_N(\omega))$,

$$\mathbb{E}_0\big[V_0^{(n)} 1_A\big] = \mathbf{E}^{\alpha}\big[\mathbf{E}_0^{\langle \rho \rangle, N-k, \omega}\big[V_0^{(n)}\big] 1_A\big]. \tag{67}$$

We will use the representation for $V_0^{(n)}$ given by

$$V_0^{(n)} = \sum_{i=0}^{N-1} \tau_{i+1}^i, \tag{68}$$

where $\tau_{i+1}^i = \tau_1 \circ \vartheta_{\tau_i}$ with $\tau_i = \inf\{t \geq 0 \mid X_t = X_0 + i\}$, $\tau_0 = 0$. In words, τ_{i+1}^i is the time it takes to reach $(i+1)$ starting from i. Inserting (68) into (67), we see that as soon as we know for $i = 0, \ldots, N-1$,

$$\mathbb{E}_0\big[\tau_{i+1}^i 1_A\big] = \mathbf{E}^\alpha\big[\mathbf{E}_0^{\langle\rho\rangle, N-k, \omega}\big[\tau_{i+1}^i\big]1_A\big], \tag{69}$$

(67) follows. The next observation is that for fixed $k \in [0, 2N] \cap \mathbb{Z}$, we can look at the measure $\mathbf{P}^\alpha$ as the product of the measures $\mathbf{P}_1^\alpha$ and $\mathbf{P}_2^\alpha$, where $\mathbf{P}_1^\alpha$ is the product measure $\otimes_i \alpha$ on $\Omega_1 := [1/2, 1)^{\mathbb{Z}\cap(-\infty, N-k)}$ and $\mathbf{P}_2^\alpha = \otimes_i \alpha$ on $\Omega_2 := [1/2, 1)^{\mathbb{Z}\cap[N-k,\infty)}$. Pick $i \in [0, N-1]$. Then

$$\mathbb{E}_0\big[\tau_{i+1}^i 1_A\big] = \mathbf{E}^\alpha\big[1_A \mathbf{E}_0^\omega\big[\tau_{i+1}^i\big]\big] = \mathbf{E}_2^\alpha\big[1_A \mathbf{E}_1^\alpha\big[\mathbf{E}_0^\omega\big[\tau_{i+1}^i\big]\big]\big]. \tag{70}$$

Since $\mathbf{E}^\alpha[\mathbf{E}_0^{\langle\rho\rangle, N-k, \omega}[\tau_{i+1}^i]1_A] = \mathbf{E}_2^\alpha[\mathbf{E}_0^{\langle\rho\rangle, N-k, \omega}[\tau_{i+1}^i]1_A]$, we see that (69) follows once we have shown

$$\mathbf{E}_0^{\langle\rho\rangle, N-k, \omega}\big[\tau_{i+1}^i\big] = \mathbf{E}_1^\alpha\big[\mathbf{E}_0^\omega\big[\tau_{i+1}^i\big]\big]. \tag{71}$$

To check (71) we will use the following formula (cf. (12), (15) in [3]):

$$\begin{aligned}\mathbf{E}_i^\omega[\tau_1] = 1 + 2(&\rho_i(\omega) + \rho_i(\omega)\rho_{i-1}(\omega)\\ &+\rho_i(\omega)\rho_{i-1}(\omega)\rho_{i-2}(\omega) + \cdots),\end{aligned} \tag{72}$$

where we recall that $\rho_i = (1-\omega_i)/\omega_i$. If we now write $\omega = (\omega_1, \omega_2)$ with $\omega_i \in \Omega_i$ for $i = 1, 2$, by using the strong Markov property we find

$$\begin{aligned}&\mathbf{E}_1^\alpha\big[\mathbf{E}_0^\omega\big[\tau_{i+1}^i\big]\big]\\ &\quad = 1 + 2\mathbf{E}_1^\alpha[\rho_i + \rho_i\rho_{i-1} + \cdots]\\ &\quad = \begin{cases} 1 + 2(\langle\rho\rangle + \langle\rho\rangle^2 + \cdots), & i < N-k,\\ 1 + 2(\rho_i(\omega_2) + \cdots + \rho_i(\omega_2)\cdots\rho_{N-k}(\omega_2) & \\ \quad \big(1 + \langle\rho\rangle + \langle\rho\rangle^2 + \cdots\big)\big), & i \geq N-k.\end{cases}\end{aligned} \tag{73}$$

Since $\mathbf{E}_0^{\langle\rho\rangle, N-k, \omega}$ denotes the quenched (path) measure with $\omega_i = (1+\langle\rho\rangle)^{-1}$ for $i < N-k$, the right-hand side of (73) is easily seen to be equal to $\mathbf{E}_0^{\langle\rho\rangle, N-k, \omega}[\tau_{i+1}^i]$. This shows (71) and the proof of (66) is complete. To apply Azuma's inequality, note first that $M_{2N}(\omega) = \mathbf{E}_0^{\langle\rho\rangle, \omega}[V_0^{(n)}]$, and $M_0(\omega) = \mathbf{E}_0^{\langle\rho\rangle, N}[V_0^{(n)}] = N(1 + 2\sum_{k\geq 1}\langle\rho\rangle^k) = N/v_\alpha$; compare (2). Hence, by (64),

$$\begin{aligned}&\mathbf{P}^\alpha\left[S \cap \left\{\omega \,\middle|\, \mathbf{E}_0^{\langle\rho\rangle, \omega}\big[V_0^{(n)}\big] > \left(\frac{1}{v_\alpha} + \zeta\right)\big\lfloor(\log n)^{1-\delta}\big\rfloor\right\}\right]\\ &\qquad\qquad \leq \exp\left(-\frac{1}{2}\frac{(\zeta N)^2}{\sum_{k=1,\ldots,2N} c_k^2}\right).\end{aligned} \tag{74}$$

It remains to give an upper bound on c_k. To this end we pick $\omega \in S$ and by using (66) and (68) we find for $1 \le k \le 2N$,

$$\begin{aligned}
|M_k(\omega) - M_{k-1}(\omega)| &\le \sum_{i=1}^{N-1} |\mathbf{E}_i^{\langle \rho\rangle, N-k, \omega}(\tau_1) - \mathbf{E}_i^{\langle \rho\rangle, N-k+1, \omega}(\tau_1)| \\
&= \sum_{i=N-k}^{N-1} |\mathbf{E}_i^{\langle \rho\rangle, N-k, \omega}(\tau_1) - \mathbf{E}_i^{\langle \rho\rangle, N-k+1, \omega}(\tau_1)| \\
&= 2\sum_{i=N-k}^{N-1} \left| \rho_i \cdots \rho_{N-k} \frac{1}{1-\langle \rho\rangle} - \rho_i \cdots \rho_{N-k+1} \frac{\langle \rho\rangle}{1-\langle \rho\rangle}\right| \qquad (75) \\
&= \frac{2}{1-\langle \rho\rangle} \sum_{i=N-k}^{N-1} \rho_i \cdots \rho_{N-k+1} |\rho_{N-k} - \langle \rho\rangle| \\
&\le \frac{2}{1-\langle \rho\rangle} \sum_{i=N-k}^{N-1} \rho_i \cdots \rho_{N-k+1},
\end{aligned}$$

where we have adopted the convention that the product $\rho_i \cdots \rho_j$ is equal to 1 if $i < j$. For $i \ge N - k + 1$, we set

$$L_i = \left\lfloor \frac{i - (N-k+1)}{2\lfloor (\log n)^{\gamma}\rfloor} \right\rfloor$$

which is an upper bound on the number of subblocks contained in $[i, N - k + 1]$ (recall that the last subblock might have a length $2\lfloor(\log n)^{\gamma}\rfloor$). Since $\omega \in S$ implies that each subblock of $[-N, N]$ has at least $\varepsilon\lfloor(\log n)^{\gamma}\rfloor$ biased sites in it. Thus, for $1 \le k \le 2N$ and n large enough,

$$\begin{aligned}
|M_k(\omega) - M_{k-1}(\omega)| &\le \frac{2}{1-\langle \rho\rangle} \sum_{i=N-k}^{\infty} \rho_*^{\varepsilon\lfloor(\log n)^{\gamma}\rfloor L_i} \\
&\le \frac{2}{1-\langle \rho\rangle} \sum_{L\ge 0} \sum_{\substack{i \ge N-k+1 \\ L_i = L}} \rho_*^L \qquad (76) \\
&\le \frac{2}{1-\langle \rho\rangle} \sum_{L\ge 0} 2\lfloor (\log n)^{\gamma}\rfloor \rho_*^L \\
&= \frac{4\lfloor (\log n)^{\gamma}\rfloor}{(1-\langle \rho\rangle)(1-\rho_*)},
\end{aligned}$$

where $\rho_* = (1/2 - \xi)/(1/2 + \xi) \in (0,1)$. Coming back to (74), we find

$$\begin{aligned}
&\mathbf{P}^{\alpha}\left[S \cap \left\{\omega \,\middle|\, \mathbf{E}_0^{\langle \rho\rangle, \omega}\left[V_0^{(n)}\right] > \left(\frac{1}{v_\alpha} + \zeta\right)\left\lfloor (\log n)^{1-\delta}\right\rfloor\right\}\right] \qquad (77) \\
&\quad \le \exp\left(-\frac{1}{128}\zeta^2(1-\langle \rho\rangle)^2(1-\rho_*)^2(\log n)^{1-\delta-2\gamma}\right)
\end{aligned}$$

This together with (63) implies (61) and the proof of Proposition 3 is complete. □

The last step in the proof of the upper bound of Theorem 1 is to establish the following proposition.

PROPOSITION 4. *For any choice of parameters satisfying* (17) *we have, for each* $\omega \in \Omega$,

$$\limsup_{n\to\infty} \frac{(\log n)^2}{n} \log \mathbf{P}_5^{\omega}(\xi, \varepsilon, \eta, \zeta) = -\infty, \tag{78}$$

where $\mathbf{P}_5^{\omega}(\xi, \varepsilon, \eta, \zeta)$ *has been defined in* (57).

PROOF. We first introduce the random variables ($i \in \mathbb{Z}$),

$$Z_i = V_i^{(n)}/(\log n)^{1-\delta}, \tag{79}$$

where the $V_i^{(n)}$ were defined in (8). The following lemma, which provides tail estimates on Z_i in the case where the corresponding double block is regular [i.e., type (2)], is the key to the proof of Proposition 4.

LEMMA 6. *Set* $t_0 = 2(2 + v/v_\alpha)$. *There exists a constant* $\kappa(\xi) \in (0,\infty)$ *and* $n_0(\xi) \ge 10$ *such that for each* $\omega \in \Omega$ *with the property that* $\mathscr{B}_0$ *is of type* (2), *we have for* $n \ge n_0$ *and* $t \ge t_0$,

$$\mathbf{P}_0^{\langle\rho\rangle,\omega}[Z_0 > t] \le \exp(-\kappa(\xi)t). \tag{80}$$

Note the the choice of the index $i = 0$ plays no role in the statement.

Before we give the proof of Lemma 6, which involves the martingale method once more, let us first show how the claim of Proposition 4 follows. As before, we define $\mathscr{N}_2 = \{j \ge 0 \mid \chi(X_{T_{j-1}}) = 2, T_{j-1} \le n\}$ and set $N_2 = |\mathscr{N}_2|$. As in the proof of Proposition 1, it follows that on the set $A_1 \cap \{X_n/n \le v\}$, we have for large enough n,

$$\begin{aligned} 0 \le N_2 &\le \frac{n}{\lfloor (\log n)^{1-\delta} \rfloor}(v+\varepsilon) + 2\frac{n}{(\log n)^{3-2\delta}} \\ &\le \frac{n}{(\log n)^{1-\delta}}(v + 2\varepsilon). \end{aligned} \tag{81}$$

The claim of Proposition 4 will follow, once we have shown that for $\omega \in \Omega$,

$$\begin{aligned} \limsup_{n\to\infty} \max_{0\le K\le n(v+2\varepsilon)/(\log n)^{1-\delta}} &\frac{(\log n)^2}{n} \\ &\times \log \mathbf{P}^{\omega}\big[N_2 = K, S_n^{(2)} \ge n(v/v_\alpha + \eta)\big] = -\infty. \end{aligned} \tag{82}$$

In order to estimate the probability on the left-hand side, we follow closely the arguments of the proof of (46). Denote by τ_k the beginning of the kth visit of a type (2) region, where τ_k is defined as in (50) except that we are dealing with type (2) instead of type (1) regions. Adapting the arguments leading to (51), we obtain for any $\lambda > 0$, $\omega \in \Omega$,

$$\begin{aligned} \mathbf{P}^{\omega}\big[N_2 = K, S_n^{(2)} \geq n(v/v_\alpha + \eta)\big] \\ &\leq \exp(-\lambda n(v/v_\alpha + \eta)) \\ &\quad \times \mathbf{E}^{\omega}\Bigg[\exp\Bigg(\lambda \sum_{k=1}^{K-1} (T_1 \wedge T_{[-n,n]}) \circ \vartheta_{\tau_k}\Bigg) \\ &\quad \times \mathbf{E}^{\omega}_{X_{\tau_K}}\big[\exp\big(\lambda(T_1 \wedge T_{[-n,n]})\big); \tau_K < \infty\big]\Bigg] \end{aligned} \tag{83}$$

with T_1 defined in (11). We observe that on $\{\tau_K < \infty\}$,

$$\begin{aligned} \mathbf{E}^{\omega}_{X_{\tau_K}}\big[\exp\big(\lambda(T_1 \wedge T_{[-n,n]})\big)\big] &\leq \sup_{i:\, \mathscr{B}_i \text{ is of type (2)}} \mathbf{E}_i^{\langle\rho\rangle,\omega}\big[\exp(\lambda T_1)\big] \\ &\leq \sup_{i:\, \mathscr{B}_i \text{ is of type (2)}} \mathbf{E}_i^{\langle\rho\rangle,\omega}\big[\exp\big(\lambda V_i^{(n)}\big)\big]. \end{aligned} \tag{84}$$

Set $\bar\lambda = \lambda(\log n)^{1-\delta}$. Iterating in (83) we arrive at $(0 \leq K \leq n(v + 2\varepsilon)/(\log n)^{1-\delta})$,

$$\begin{aligned} \mathbf{P}^{\omega}\big[N_2 = K, S_n^{(2)} \geq n(v/v_\alpha + \eta)\big] \\ &\leq \exp\Bigg(-\bar\lambda \frac{n}{(\log n)^{1-\delta}}(v/v_\alpha + \eta) \\ &\qquad + K\bar\lambda \sup_{i:\, \mathscr{B}_i \text{ is of type (2)}} \frac{1}{\bar\lambda}\log \mathbf{E}_i^{\langle\rho\rangle,\omega}\big[\exp\big(\bar\lambda Z_i^{(n)}\big)\big]\Bigg) \\ &\leq \exp\Bigg(-\bar\lambda \frac{n}{(\log n)^{1-\delta}}\Bigg[(v/v_\alpha + \eta) - (v + 2\varepsilon) \\ &\qquad \times \sup_{i:\, \mathscr{B}_i \text{ is of type (2)}} \frac{1}{\bar\lambda}\log \mathbf{E}_i^{\langle\rho\rangle,\omega}\big[\exp\big(\bar\lambda Z_i^{(n)}\big)\big]\Bigg]\Bigg), \end{aligned} \tag{85}$$

where $Z_i^{(n)}$ was defined in (79).

Let ω be such that $\mathscr{B}_0$ is regular. Pick $n \geq n_0(\xi)$ (≥ 10), $t > t_0$ and $\bar\lambda = \kappa(\xi)/\log n$ (see Lemma 6 for definitions). Then,

$$\begin{aligned} \frac{1}{\bar\lambda}\log \mathbf{E}_0^{\langle\rho\rangle,\omega}\big[\exp\big(\bar\lambda Z_0^{(n)}\big)\big] &= \frac{1}{\bar\lambda}\log\Bigg(1 + \bar\lambda \int_0^\infty e^{\bar\lambda u}\mathbf{P}_0^{\langle\rho\rangle,\omega}\big[Z_0^{(n)} > u\big]\,du\Bigg) \\ &\leq e^{\bar\lambda t}\int_0^t \mathbf{P}_0^{\langle\rho\rangle,\omega}\big[Z_0^{(n)} > u\big]\,du \\ &\quad + \int_t^\infty e^{\bar\lambda u}\mathbf{P}_0^{\langle\rho\rangle,\omega}\big[Z_0^{(n)} > u\big]\,du. \end{aligned} \tag{86}$$

By Lemma 6, we estimate the second term by $2/\kappa \exp(-\kappa t/2)$ and the first one by $\exp(\kappa(\xi)t/\log n)(1/v_\alpha + \zeta)$, where we have used that $\mathbf{E}_0^{\langle\rho\rangle,\omega}[V_0^{(n)}] \le (\log n)^{1-\delta}(1/v_\alpha + \zeta)$. Choosing now $t = t(n) = (\log n)^{1/2}$ we find for large enough n,

$$\begin{aligned}
&\mathbf{P}^\omega\big[N_2 = K, S_n^{(2)} \ge n(v/v_\alpha + \eta)\big]\\
&\quad\le \exp\Bigg(-\frac{\kappa(\xi)n}{(\log n)^{2-\delta}}\Bigg\{(v/v_\alpha + \eta) - (v + 2\varepsilon)\\
&\qquad\qquad \times\Bigg(\exp\bigg(\frac{\kappa(\xi)}{(\log n)^{1/2}}\bigg)(1/v_\alpha + \zeta)\\
&\qquad\qquad\qquad + \frac{2}{\kappa(\xi)}\exp\bigg(-\frac{\kappa(\xi)}{2}(\log n)^{1/2}\bigg)\Bigg)\Bigg\}\Bigg)\\
&\quad\le \exp\bigg(-\frac{\kappa(\xi)n}{(\log n)^{2-\delta}}\big((v/v_\alpha + \eta) - (v + 2\varepsilon)(1/v_\alpha + 2\zeta)\big)\bigg).
\end{aligned}\tag{87}$$

Since by our choice of parameters we have that $\eta > 2\zeta v + 2\varepsilon(1/v_\alpha + 2\zeta)$ [cf. (18)], (82) follows and it remains only to prove Lemma 6.

PROOF OF LEMMA 6. For convenience we set $N = \lfloor(\log n)^{1-\delta}\rfloor$. Recall that $V_0^{(n)}$ is the hitting time of the point N. It will be useful to introduce the path measure $\mathbf{P}_0^{\leftarrow}$ which agrees with $\mathbf{P}_0^{\langle\rho\rangle,\omega}$ except that at N we now have a reflection (to the left), that is, $\omega_N = 0$. Otherwise the transition probabilities are unchanged (and, of course, depend on ω). Let $H_r(w)$ be the number of visits in N up to time r, and for given t, set $s = s(t,n) = \lfloor t(\log n)^{1-\delta}\rfloor$. We have

$$\begin{aligned}
\mathbf{P}_0^{\langle\rho\rangle,\omega}[Z_0 > t] &\le \mathbf{P}_0^{\langle\rho\rangle,\omega}\big[V_0^{(n)} > s\big]\\
&= \mathbf{P}_0^{\langle\rho\rangle,\omega}[H_s = 0] = \mathbf{P}_0^{\leftarrow}[H_s = 0].
\end{aligned}\tag{88}$$

Since we have a basically homogeneous drift in $\mathscr{B}_0$, during time n we expect $\sim \text{const}\cdot n$ hits at N under the measure $\mathbf{P}_0^{\leftarrow}$. Thus the event $\{H_s = 0\}$ is very unlikely. We will use the martingale method to estimate the corresponding probability. For $k \ge 0$ we first define the martingale,

$$M_k(w) = \mathbf{E}_0^{\leftarrow}[H_s \mid \sigma(X_0,\dots,X_k)](w). \tag{89}$$

Note that $M_0 = \mathbf{E}_0^{\leftarrow}[H_s]$ and $M_s = H_s$. Since $\mathbf{P}_0^{\leftarrow}[H_s = 0] = \mathbf{P}_0^{\leftarrow}[H_s - \mathbf{E}_0^{\leftarrow}[H_s] \le -u]$, where $u = \mathbf{E}_0^{\leftarrow}[H_s]$, Azuma's inequality now yields [cf. (64) applied to $-M_k$]

$$\begin{aligned}
\mathbf{P}_0^{\langle\rho\rangle,\omega}[Z_0 > t] &\le \mathbf{P}_0^{\leftarrow}[M_s - M_0 \le -u]\\
&\le \exp\bigg\{-\mathbf{E}_0^{\leftarrow}[H_s]^2\Big/\Big(2\sum_{k=1,\dots,s} c_k^2\Big)\bigg\}.
\end{aligned}\tag{90}$$

We first claim that since $\mathscr{B}_0$ is strongly biased we have for $1 \le k \le s$,

$$c_k \le k_1(\xi)(\log n)^{\gamma}, \tag{91}$$

where $k_1(\xi)$ is a constant depending only on ξ. To show (91), first note that for $0 \le k \le s$; $\mathbf{E}_0^{\leftarrow}[H_s \mid \sigma(X_0,\dots,X_k)] = H_k + \mathbf{E}_{X_k}^{\leftarrow}[H_{s-k}]$ by the Markov property. For $1 \le k \le s$, we now have

$$\begin{aligned}|M_k(w) - M_{k-1}(w)| &= \big|\mathbf{E}_0^{\leftarrow}\left[H_s \mid \sigma(X_0,\dots,X_k)\right] \\ &\qquad - \mathbf{E}_0^{\leftarrow}\left[H_s \mid \sigma(X_0,\dots,X_{k-1})\right]\big| \\ &\le 1 + \left|\mathbf{E}_{X_k}^{\leftarrow}[H_{s-k}] - \mathbf{E}_{X_{k-1}}^{\leftarrow}[H_{s-k+1}]\right| \\ &\le 1 + \left|\mathbf{E}_{X_k}^{\leftarrow}[H_{s-k}] - \mathbf{E}_{X_{k-1}}^{\leftarrow}[H_{s-k}]\right| \\ &\qquad + \left|\mathbf{E}_{X_{k-1}}^{\leftarrow}[H_{s-k}] - \mathbf{E}_{X_{k-1}}^{\leftarrow}[H_{s-k+1}]\right| \\ &\le 2 + \sup_{x \le N}\left|\mathbf{E}_x^{\leftarrow}[H_{s-k}] - \mathbf{E}_{x-1}^{\leftarrow}[H_{s-k}]\right|.\end{aligned} \tag{92}$$

We claim that for each $n \ge 1$,

$$\sup_{x \le N}\left|\mathbf{E}_x^{\leftarrow}[H_n] - \mathbf{E}_{x-1}^{\leftarrow}[H_n]\right| \le \frac{1}{2}\sup_{x \le N}\mathbf{E}_{x-1}^{\leftarrow}[\tau_x], \tag{93}$$

where we recall that τ_x is the hitting time of x. To show (93), we first observe that $\mathbf{E}_x^{\leftarrow}[H_n] \ge \mathbf{E}_{x-1}^{\leftarrow}[H_n]$. On the other hand,

$$\mathbf{E}_{x-1}^{\leftarrow}[H_n] = \sum_{j=1}^{\infty}\mathbf{E}_{x-1}^{\leftarrow}[H_n \mid \tau_x = j]\mathbf{P}_{x-1}^{\leftarrow}[\tau_x = j]. \tag{94}$$

Now

$$\mathbf{E}_{x-1}^{\leftarrow}[H_n \mid \tau_x = j] = 1_{j \le n}\mathbf{E}_x^{\leftarrow}\left[H_{n-j}\right] \ge \mathbf{E}_x^{\leftarrow}[H_n] - j/2,$$

where we used the fact that for $j \le n$, $H_{n-j} + j/2 \ge H_n$ and that in any case $H_n \le n/2$. Coming back to (94), we see that $\mathbf{E}_{x-1}^{\leftarrow}[H_n] \ge \mathbf{E}_x^{\leftarrow}[H_n] - 1/2\mathbf{E}_{x-1}^{\leftarrow}[\tau_x]$, which implies (93). Using (92), we have for $1 \le k \le s$,

$$c_k \le 2 + 1/2\sup_{x \le N}\mathbf{E}_{x-1}^{\leftarrow}[\tau_x] = 2 + 1/2\sup_{x \le N}\mathbf{E}_{x-1}^{\langle\rho\rangle,\omega}[\tau_x]. \tag{95}$$

Note that for $x \le N$, $\mathbf{E}_{x-1}^{\langle\rho\rangle,\omega}[\tau_x]$ is equal to the r.h.s. of (73) (with $i = x - 1$ and $k = 2N$). Exactly as in (76) we obtain for $n \ge n_0(\xi)$.

$$c_k \le 2 + 1/2\left(1 + \frac{2}{(1 - \langle\rho\rangle)(1 - \rho_*)}2(\log n)^{\gamma}\right) \le k_1(\xi)(\log n)^{\gamma} \tag{96}$$

for a certain constant $k_1(\xi) > 0$. It remains to give a lower bound on $\mathbf{E}_0^{\leftarrow}[H_s]$. Recall that $s = \lfloor t(\log n)^{1-\delta}\rfloor$. Set $s_0 := 2(1/v_\alpha + \zeta)(\log n)^{1-\delta}$ and observe that $t \ge t_0$ and $n \ge n_0 \ge 10$ implies $s > s_0$. We will use the simple fact that for any nonnegative random variable X we have $P[X \le 2E[X]] \ge 1/2$. For a later purpose we set

$$K = \left\lfloor (s - s_0)/\left(8(\log n)^{\gamma}k_1(\xi)\right)\right\rfloor. \tag{97}$$

Observe that $K \geq 1$, provided $t \geq t_0$ and $n \geq n_0(\xi)$. Now

$$\mathbf{E}_0^{\leftarrow}[H_s] \geq \frac{1}{2}\sum_{k=1}^{K} \mathbf{P}_0^{\leftarrow}\left[H_s \geq k \mid V_0^{(n)} \leq 2\mathbf{E}_0^{\leftarrow}\left[V_0^{(n)}\right]\right]. \tag{98}$$

Note that $2\mathbf{E}_0^{\leftarrow}[V_0^{(n)}] = 2\mathbf{E}_0^{\langle\rho\rangle,\omega}[V_0^{(n)}] \leq 2(1/v_\alpha + \zeta)(\log n)^{1-\delta} = s_0$, since $\mathscr{B}_0$ is regular. Hence

$$\mathbf{E}_0^{\leftarrow}[H_s] \geq \frac{1}{2}\sum_{k=1}^{K} \mathbf{P}_{N-1}^{\leftarrow}\left[H_{s-s_0} \geq k-1\right]. \tag{99}$$

For $i \geq 1$ let us denote by D_i the duration of the ith passage from $(N-1)$ to N. Note that all of these variables are $\mathbf{P}_0^{\leftarrow}$-a.s. finite (hence well defined). Thus

$$\begin{aligned}\mathbf{E}_0^{\leftarrow}[H_s] &\geq \frac{1}{2}\sum_{k=1}^{K} \mathbf{P}_{N-1}^{\leftarrow}\left[\sum_{i=1}^{k} D_i \leq s - s_0\right]\\ &= \frac{1}{2}\sum_{k=1}^{K}\left(1 - \mathbf{P}_{N-1}^{\leftarrow}\left[\sum_{i=1}^{k} D_i > s - s_0\right]\right)\\ &\geq \frac{1}{2}\sum_{k=1}^{K}\left(1 - \frac{k}{s-s_0}\mathbf{E}_{N-1}^{\leftarrow}[\tau_N]\right),\end{aligned} \tag{100}$$

where we used Chebyshev's inequality in the last line together with the fact that for each $i \geq 1$: $\mathbf{E}_{N-1}^{\leftarrow}[D_i] = \mathbf{E}_{N-1}^{\leftarrow}[\tau_N]$, where τ_N is the hitting time of the point N.

Recall that $\mathbf{E}_{N-1}^{\leftarrow}[\tau_N] = \mathbf{E}_{N-1}^{\langle\rho\rangle,\omega}[\tau_N] \leq 2k_1(\xi)(\log n)^{\gamma}$ [cf. (95), (96)]. Thanks to our choice of K [cf. (97)], we have for $0 \leq k \leq K$ that $(1 - 2k_1(\xi) \cdot(\log n)^{\gamma}/(s-s_0)) \geq 1/2$ and we arrive at

$$\mathbf{E}_0^{\leftarrow}[H_s] \geq \tfrac{1}{4}K. \tag{101}$$

Coming back to (90) we find, together with our bound on c_k from (96) ($s = \lfloor t(\log n)^{1-\delta}\rfloor$),

$$\begin{aligned}\mathbf{P}_0[Z_0 > t] &\leq \exp\left\{-\frac{K^2}{32k_1^2(\xi)s(\log n)^{2\gamma}}\right\}\\ &\leq \exp\left\{-\frac{1}{32k_1^2(\xi)}\left[\frac{t(\log n)^{1-\delta} - 1 - s_0}{8(\log n)^{\gamma}k_1(\xi)} - 1\right]^2 \frac{1}{t(\log n)^{1-\delta+2\gamma}}\right\}\end{aligned} \tag{102}$$

Observing that this estimate is monotone decaying in n, the claim of Lemma 6 follows. □

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