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CENTRAL LIMIT THEOREMS FOR THE WASSERSTEIN DISTANCE BETWEEN THE EMPIRICAL AND THE TRUE DISTRIBUTIONS

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If X is integrable, F is its cdf and F_n is the empirical cdf based on an i.i.d. sample from F, then the Wasserstein distance between F_n and F, which coincides with the L_1 norm $\int_{-\infty}^{\infty} |F_n(t) - F(t)|\, dt$ of the centered empirical process, tends to zero a.s. The object of this article is to obtain rates of convergence and distributional limit theorems for this law of large numbers or, equivalently, stochastic boundedness and distributional limit theorems for the L_1 norm of the empirical process. Some limit theorems for the Ornstein–Uhlenbeck process are also derived as a by-product.

1. Introduction. The Kantorovich or L_1-Wasserstein distance between two probability measures P_1 and P_2 on $\mathbb{R}$ with finite mean, defined as

$$d_1(P_1, P_2) := \inf\left\{ \int |x-y|\, d\mu(x,y):\ \mu \in \mathscr{P}(\mathbb{R}^2) \text{ with marginals } P_1, P_2 \right\}, \tag{1.1}$$

is an interesting distance in probabilistic limit theory because it metrizes weak convergence plus convergence of first absolute moments. It is well known [Shorack and Wellner (1986), page 64] that, letting F_1, F_2, respectively, be the cumulative distribution functions (cdf's) of P_1 and P_2 and Q_1, Q_2 their left continuous inverses or quantile functions, $Q_i(t) = \inf\{x\colon F_i(x) \ge t\}$, this distance can be alternatively written as

$$d_1(P_1, P_2) = \int_0^1 |Q_2(t) - Q_1(t)|\, dt = \int_{-\infty}^{\infty} |F_2(t) - F_1(t)|\, dt. \tag{1.2}$$

We will refer to any of these three expressions as the Wasserstein distance between F_1 and F_2.

Let now X, X_i, $i \in \mathbb{N}$, be i.i.d. integrable random variables with common cdf F and quantile function Q, and let

$$F_n(t) = \frac{1}{n}\sum_{i=1}^{n} I_{X_i \le t}, \qquad t \in \mathbb{R},\ n \in \mathbb{N},$$

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be their empirical measures. By Glivenko–Cantelli and the law of large numbers for $|X|$,

$$F_n(t, \omega) \to F(t) \text{ for all } t \in \mathbb{R} \quad \text{and} \quad \int |t|\, dF_n(t, \omega) \to \int |t|\, dF(t) \text{ a.s.}$$

Therefore, by dominated convergence,

$$d_1(F_n, F) \to 0 \quad \text{a.s.} \tag{1.3}$$

[If X is not integrable, then $d_1(F_n, F) = \infty$ for all n.] The object of this article is to obtain the distributional limit theorems associated to the "law of large numbers" (1.3) and, more generally, to obtain rates of convergence in probability for this law of large numbers. The relationship between $d_1(F_n, F)$ and the weighted uniform empirical process adds interest to this problem. Since $Q(\theta)$, θ uniform on $(0, 1)$, has the law of X, taking $X_i = Q(\theta_i)$ in (1.2), with θ_i i.i.d. uniform, gives, by a change of variables that, as is well known,

$$d_1(F_n, F) =_d \int_0^1 |H_n(t) - t|\, dQ(t), \qquad n \in \mathbb{N}, \tag{1.4}$$

where H_n is the nth empirical cdf corresponding to the sequence θ_i. If Q is differentiable, then the right side of (1.4) is the L_1 norm of the uniform empirical process weighted by $Q'(t)$. So, the results to be obtained in this article can be viewed in different lights, as limit theorems for a particularly important functional of the general empirical process F_n on $\mathbb{R}$, namely the L_1 norm, or as the asymptotic behavior of the Wasserstein distance between the empirical and the true distributions or as limit theorems for the L_1 norm of the weighted uniform empirical process.

In fact, the L_p norm for $p \ge 1$ of the weighted empirical process,

$$\int_0^1 [|H_n(t) - t|/q(t)]^p\, dt \tag{1.5}$$

has been studied by several authors. Csörgő, Csörgő, Horváth and Mason (1986a) (Hereafter CCHM) adapt the Kómlos–Major–Tusnády approximation of partial sums by Wiener processes to a weighted approximation of the uniform quantile process by Brownian bridges, and obtain from this a weighted approximation of the uniform empirical process by Brownian bridges [for a history and a simpler proof of the latter, see Mason (1991)], which has become an invaluable tool for the study of the empirical cdf. This approximation is then shown in CCHM (1986b) to provide an approximation in probability of integral functionals of H_n by the same functionals for Brownian bridges. M. Csörgő and Horváth (1988a, 1993) [see also M. Csörgő, Horváth and Shao (1993)] also use these approximations to obtain limit theorems for the integrals in (1.5) over different domains of integration contained in $(0, 1)$. The results of these authors on (1.5) constitute a nice complement to the celebrated Chibisov–O'Reilly theorem. For $p = 2$ and weights $q(t) = f(Q(t))$, f the density of F, these integrals are related to tests of goodness-of-fit. In the present work, the Wasserstein distance gives precise additional meaning to very general weights.

It turns out that the natural settings for distributional limit theorems for the Wasserstein distance between F_n and F are the domains of attraction of α-stable laws with $1 \le \alpha \le 2$ (assuming $\mathbb{E}|X| < \infty$ in the $\alpha = 1$ case). The technically most interesting cases are $\alpha = 1$ and $\alpha = 2$, for which there are few results in the literature. The following theorem describes the results about our problem that are contained in, or can be obtained without effort from, previous work [see Theorems 5.3.1, 5.3.2 and 5.3.3 in the Csörgő–Horváth (1993) book]. Here and elsewhere, the notation $X \in DA_\alpha(a_n)$ will indicate that the distribution of X is in the domain of attraction of an α-stable law with normalizing constants a_n. Convergence in distribution is denoted by $\to_d$ for random variables and by $\to_w$ for probability measures. Then $B(t)$, $0 \le t \le 1$, will denote a Brownian bridge process, that is, a centered Gaussian process with continuous sample paths and covariance $\mathbb{E}B(s)B(t) = s \wedge t - st$.

THEOREM 1.1. *Let X, X_i, $i \in \mathbb{N}$, be a sequence of independent random variables with a differentiable distribution function F and let Q be the associated quantile function. Assume that $X \in DA_\alpha(a_n)$ for some $1 < \alpha \le 2$ and let γ be the α-stable limit law of $\{\sum_{i=1}^n (X_i - \mathbb{E}X)/a_n\}_{n=1}^\infty$. Then*:

(a) *In the case $\alpha = 2$;*

(a1) *If, in addition, $\int_{-\infty}^\infty \sqrt{F(t)(1-F(t))}\, dt < \infty$, then*

$$\sqrt{n}\int_{-\infty}^\infty |F_n(t) - F(t)|\, dt \to_d \int_0^1 |B(t)|\, dQ(t) \tag{1.6}$$

and

(a2) *If X is symmetric, has a positive density and satisfies* $\Pr\{|X| > t\} \simeq 1/t^2$, *then there exists a constant D such that*

$$\frac{n}{\sqrt{D}a_n}\left(\int_{-\infty}^\infty |F_n(t) - F(t)|\, dt - \mathbb{E}\left(\int_{-\infty}^\infty |F_n(t) - F(t)|\, dt\right)\right) \to_d g, \tag{1.7}$$

where g is standard normal.

(b) *If $1 < \alpha < 2$ and $\gamma = \mathrm{cPois}\,\mu(c_1, c_2, \alpha)$, then*

$$\begin{aligned}\frac{n}{a_n}\int_{-\infty}^\infty |F_n(t) - F(t)|\, dt \to_d{}& \frac{1}{\alpha}\left(\frac{c_1}{\alpha}\right)^{1/\alpha}\int_0^\infty |N_1(s) - s| s^{-1-1/\alpha}\, ds\\ &+ \frac{1}{\alpha}\left(\frac{c_2}{\alpha}\right)^{1/\alpha}\int_0^\infty |N_2(s) - s| s^{-1-1/\alpha}\, ds,\end{aligned} \tag{1.8}$$

where $\{N_i(t)\colon 0 < t < \infty\}$, $i = 1, 2$, are independent Poisson processes with intensity 1.

We should recall that the centered compound Poisson measure with Lévy (or driving) measure μ satisfying $\int_{-\infty}^\infty 1 \wedge x^2\, d\mu(x) < \infty$, $\mu\{0\} = 0$ and $\int_{[-1,1]^c} |x|\, d\mu(x) < \infty$, is defined as the probability measure with characteristic function

$$(\mathrm{cPois}\,\mu)\hat{}\,(t) = \exp\left(\int_{-\infty}^{\infty} (\exp(itx) - 1 - itx)\, d\mu(x)\right); \tag{1.9}$$

we will also use below the partially centered Poisson measure

$$(\mathrm{c}_\delta\mathrm{Pois}\,\mu)\hat{}\,(t) = \exp\left(\int_{-\infty}^{\infty} (\exp(itx) - 1 - itxI_{|x|\le\delta})\, d\mu(x)\right), \tag{1.10}$$

needed when μ does not integrate $|x|$ at infinity or is not symmetric. The Lévy measure $\mu(c_1, c_2, \alpha)$, whose corresponding compound Poisson is α-stable, is defined as

$$d\mu(c_1, c_2, \alpha) = \begin{cases} c_1 x^{-1-\alpha}\, dx, & \text{if } x > 0, \\ c_2 |x|^{-1-\alpha}, & \text{if } x < 0 \end{cases} \tag{1.11}$$

[cf. Araujo and Giné (1980), Chapter 2]. It is also convenient to note that the condition $\int_{-\infty}^{\infty} \sqrt{F(t)(1-F(t))}\, dt < \infty$ implies finite second moment for X but not conversely [this condition defines the Banach space $L_{2,1}(\Omega, \Sigma, \Pr)$ [cf. Ledoux and Talagrand (1991), page 10]. In particular, the previous theorem is far from covering the basic case $\mathbb{E}X^2 < \infty$ [or, more generally, the case $X \in DA_2(b_n)$ for general b_n].

Here is a brief description of the contents of this article.

1. Parts (a1) and (b) of Theorem 1.1 are proved without assuming F differentiable and, moreover, the type of convergence is improved in the sense that it is a consequence of convergence in law of certain processes in $L_1(\mathbb{R})$; stochastic boundedness for larger classes of distributions is also considered (Section 2).
2. Two converses of (a1) are obtained [Theorem 2.1(b) and Corollary 4.5].
3. A limit theorem is proved for the partially centered Wasserstein distance when X is in the domain of attraction of a 1-stable law, with $\mathbb{E}|X| < \infty$; this generalizes Theorem 1.1(b) to this case (Section 3).
4. We show that the centered and normalized Wassertein distances, $\{nb_n^{-1}(d_1(F_n, F) - \mathbb{E}d_1(F_n, F))\}_{n=1}^{\infty}$, are stochastically bounded for all $X \in DA_2(b_n)$ (Section 4) and prove that they actually converge in law for $\mathbb{E}X^2 < \infty$ and also for $X \in DA_2(b_n)$ with $b_n \simeq \sqrt{n}(\log n)^{(\alpha+1)/2}$, $-1 < \alpha < \infty$, and $b_n \simeq \sqrt{n \log\log n}$, significantly expanding the scope of (a2), which corresponds to $\alpha = 0$ (Sections 5 and 6). Other norming sequences $\{b_n\}$ are possible, but we restrict to these for simplicity.
5. All the above theorems are shown to hold with convergence of moments as well.

These results constitute an essentially complete description of the first-order asymptotics for the Wasserstein distance $d_1(F_n, F)$ or, what is the same, for the L_1 norm of the empirical process, at least for $X \in DA_\alpha$, $\alpha \in [1, 2]$.

The results for the uncentered Wasserstein distance [corresponding to (a1) and (b)] are all obtained from the central limit theorem in the Banach space $L_1(\mathbb{R})$ for adequate processes. This CLT, obtained in the seventies, is relatively elementary (as L_1 is a well-behaved Banach space in this regard) and the proofs straightforward. We believe this method is more elementary than the one based on approximation of the weighted empirical process by the Brownian bridge, particularly if based on KMT. It has the added advantage of providing convergence of moments as well.

We devote Sections 4 to 6 to the more delicate case of the domain of attraction of the normal law, with X not in $L_{2,1}$. In Section 4 we obtain stochastic boundedness plus uniform integrability of up to the second moments by means of a new, very powerful, exponential inequality of Talagrand (1996) for empirical processes (Section 4). We should remark that this inequality, whose original proof is quite difficult, has now a simple proof based on log Sobolev inequalities (Ledoux, personal communication). Section 5 contains the convergence result for the finite variance case. The tools used are the aforementioned Talagrand's inequality and the Borell–Sudakov–Tsirel'son concentration inequality for Gaussian processes (in the simpler version of Maurey and Pisier). Section 6 is devoted to the central limit theorem in the infinite variance case. Because the norming constants are not determined by integrals over bounded intervals, but by the integrals at infinity, it does not seem that this result can be obtained using the previous inequalities in combination with approximation by integrals over bounded intervals. Here, not only strong approximation seems unavoidable, but it is perfectly suited to reduce the limit theorem for the empirical process to one for Gaussian processes. For this, we use a result of CCHM (1986a) on weighted approximation of the uniform empirical process by Brownian bridges; this result has a relatively elementary proof based on Skorohod embedding [Mason (1991), Csörgő and Horváth (1986)]. Talagrand's inequality is still helpful to obtain convergence of moments (which play an important part in the proofs). After this article had been written, Mason (1998) improved the weighted approximation from a probability statement to one on exponential moments; if we used his result in the present analysis we could avoid Talagrand's inequality. However, since Mason's result is based on KMT and there exists a simple proof of Talagrand's inequality, it is unclear whether this would amount to any real simplifications.

The derivation of (1.7) in, for example, Csörgő and Horváth (1993) uses a central limit theorem for integrals of centered functionals of stationary Markov processes with respect to Lebesgue measure [Mandl (1968)], applied to the Ornstein–Uhlenbeck process. In order to extend it, a limit theorem for integrals with respect to measures other than Lebesgue (thus, not tied to stationarity) has had to be derived, which may have some independent interest. [See Csörgő and Horváth (1988b) for limits of integral functionals of other Gaussian processes.] Our approach is quite indirect and takes most of Section 6. We view this proof and the extensive use of Banach space methods in the problem at hand as the main technical innovations in this article.

The results for the L_1 norm of the empirical process presented here have natural analogues for the L_p norm; since the changes in the proofs would only be formal, we refrain from pursuing this subject [see, e.g., Csörgő and Horváth (1993), Theorem 3.1, pages 316 and 317, for statements on the L_p norm, $p \geq 1$, corresponding to Theorem 1.1(a) above].

CCHM (1986b) obtained two very useful explicit formulas for the norming constants b_n in $DA_2(b_n)$, which we use in Sections 4 and 5. Because of their interest, we rederive them, directly from first principles of classical central limit theory, in an Appendix.

2. The uncentered Wasserstein distance. In this section we study convergence in distribution of the normalized but uncentered Wasserstein distance between the empirical and the true probabability measures. We derive results related to (a1) and (b) in Theorem 1.1, in fact stronger, by means of simple proofs based on techniques from probability in Banach spaces. These techniques are definitely more elementary than those used in Csörgő and Horváth (1993). Since part of the statements that follow are in terms of weak convergence of probability measures in $L_1 := L_1(\mathbb{R})$, we should recall that (1) if Y is a jointly mesurable process with almost all its trajectories in $L_1(\mathbb{R})$ then Y can be identified to a Borel random variable in $L_1(\mathbb{R})$ and conversely [Byczkowski (1977)], and (2) if Y_n, Y are processes with almost all their sample paths in $L_1(\mathbb{R})$, then the processes Y_n converge in law to Y in $L_1(\mathbb{R})$, $\mathscr{L}(Y_n) \to_w \mathscr{L}(Y)$ in L_1, if $\lim_{n\to\infty} \mathbb{E}f(Y_n) = \mathbb{E}f(Y)$ for all functions f: $L_1 \to \mathbb{R}$ which are bounded and continuous; in particular then, $\|Y_n\|_{L_1} \to_d \|Y\|_{L_1}$. We also denote this convergence by $Y_n \to_{\mathscr{L}} Y$ in L_1.

We also observe that, if X has distribution F, then the condition

$$\int_{-\infty}^{\infty} \sqrt{F(t)(1-F(t))}\, dt < \infty \tag{2.1}$$

is equivalent to the condition

$$\Lambda_{2,1}(X) := \int_0^{\infty} \sqrt{\Pr\{|X| > t\}}\, dt < \infty. \tag{2.1$'$}$$

The functional $\Lambda_{2,1}$ is equivalent to a norm defining the Banach space $L_{2,1}(\Omega, \Sigma, \Pr)$, dual of the weak-$L_2$ space $L_{2,\infty}$, and we have $L_{2+\delta} \subset L_{2,1} \subset L_2$ for all $\delta > 0$, as is easy to check [e.g., Ledoux and Talagrand (1991), page 10]. We mention this because this quantity appears in several other instances in probability.

THEOREM 2.1. *Let X, X_i, $i \in \mathbb{N}$, be i.i.d. random variables with common distribution F. Let*

$$Y(t) := I_{X>t} - \Pr\{X > t\}, \qquad -\infty < t < \infty, \tag{2.2}$$

and let Y_i, $i \in \mathbb{N}$, denote the processes obtained by replacing X by X_i in (2.2). *Then*:

(a) *The processes $\sum_{i=1}^{\infty} Y_i/\sqrt{n} = \sqrt{n}(F_n - F)$ converge in law in $L_1(\mathbb{R})$ to the process $B(F(t))$, $t \in \mathbb{R}$, where B is a Brownian bridge, if and only if $\Lambda_{2,1}(X) < \infty$.*

(b) *The sequence*

$$\left\| \sum_{i=1}^{\infty} \frac{Y_i}{\sqrt{n}} \right\|_{L_1} = \sqrt{n} \int_{-\infty}^{\infty} |F_n(t) - F(t)|\, dt, \qquad n \in \mathbb{N},$$

is stochastically bounded if and only if $\Lambda_{2,1}(X) < \infty$.

PROOF. It is known that a mean zero process Y with sample paths in $L_1(\mathbb{R})$ satisfies the $\sqrt{n}$ central limit theorem if and only if

$$\int_{-\infty}^{\infty}\sqrt{\mathbb{E}(Y(t))^2}\,dt<\infty. \tag{2.3}$$

[Jain (1977), Theorem 11; see also, e.g., Araujo and Giné (1980), Exercise 14, page 205; this is a direct consequence of the central limit theorem in cotype 2 spaces and can also be directly obtained from probability inequalities as, e.g., in Giné (1983) where the case of L_p with $p<1$ is considered.] For the process $Y(t)$ defined by (2.2), condition (2.3) is nothing but condition (2.1) and therefore the sequence $\{\sum_{i=1}^n Y_i/\sqrt{n}\}$ converges in law in $L_1(\mathbb{R})$ if and only if $\Lambda_{2,1}(X)<\infty$, and the limit is Gaussian. The limiting Gaussian process is $B(F(t))$ because this Gaussian process has the same covariance as $Y(t)$, hence $(Y(t_1),\dots,Y(t_k))\to_d (B(F(t_1),\dots,B(F(t_k)))$ for all $t_1,\dots,t_k$ in $\mathbb{R}$ and all $k\in\mathbb{N}$ by the central limit theorem in $\mathbb{R}^k$, and it is known that, if there is weak convergence in L_1, then the limit in law of the finite-dimensional distributions corresponding to any $t_1,\dots,t_k$ outside a set of measure zero, and all $k\in\mathbb{N}$, is the law of the corresponding finite dimensional distributions of the limit [Lawniczak (1983)]. Part (a) is proved and so is the sufficiency part of (b).

The proof of the necessity part of (b), which ultimately is an exercise on Hoffmann–Jørgensen's inequality, requires some work, as follows. By considering X^+ and X^-, we can assume $X\ge 0$. It is now convenient to write

$$Z(t):=I_{X>t},\qquad Z_i(t)=I_{X_i>t},\ i\in\mathbb{N},\ t\in\mathbb{R},$$

so that $Y(t)=Z(t)-\mathbb{E}Z(t)$ and likewise for Y_i. The stochastic boundedness hypothesis simply asserts

$$\lim_{M\to\infty}\sup_n \Pr\left\{\frac{1}{\sqrt{n}}\left\|\sum_{i=1}^n(Z_i-\mathbb{E}Z_i)\right\|_{L_1}>M\right\}=0. \tag{2.4}$$

Montgomery-Smith's (1994) Lévy type inequality for i.i.d. random vectors then implies, from (2.4), that

$$\lim_{M\to\infty}\sup_n \Pr\left\{\frac{1}{\sqrt{n}}\max_{1\le i\le n}\|Z_i-\mathbb{E}Z_i\|_{L_1}>M\right\}=0. \tag{2.5}$$

The classical inequality for independent random variables, say ξ_i,

$$\Pr\Big\{\max_i|\xi_i|>t\Big\}\ge 1-\exp\Big(-\sum\Pr\{|\xi_i|>t\}\Big)$$

then gives that there is a constant $M<\infty$ such that

$$\sup_n n\Pr\left\{\frac{1}{\sqrt{n}}\|Z-\mathbb{E}Z\|_{L_1}>M\right\}<\infty,$$

or, equivalently,

$$\Lambda_{2,\infty}(Z-\mathbb{E}Z):=\sup_{t>0}t^2\Pr\{\|Z-\mathbb{E}Z\|_{L_1}>t\}<\infty. \tag{2.6}$$

As a consequence of (2.6) we have

$$
\begin{aligned}
\mathbb{E}\max_{1\le i\le n}\frac{\|Z_i-\mathbb{E}Z_i\|_{L_1}}{\sqrt{n}} &= \frac{1}{\sqrt{n}}\int_0^\infty \Pr\Big\{\max_{1\le i\le n}\|Z_i-\mathbb{E}Z_i\|_{L_1}>t\Big\}\,dt\\
&\le 1+\sqrt{n}\int_{\sqrt{n}}^\infty \Pr\{\|Z_i-\mathbb{E}Z_i\|_{L_1}>t\}\,dt\\
&\le 1+\sqrt{n}\Lambda_{2,\infty}(Z-\mathbb{E}Z)\int_{\sqrt{n}}^\infty t^{-2}\,dt\\
&= 1+\Lambda_{2,\infty}(Z-\mathbb{E}Z)<\infty.
\end{aligned}
\tag{2.7}
$$

Now, Hoffmann–Jørgensen's inequality for general independent vectors [e.g., Ledoux and Talagrand (1991), equation (6.8), page 156] in combination with Montgomery-Smith's (1994) Lévy inequality for i.i.d. vectors gives that for every $r>0$ there exist finite positive constants c_i, $i=1,2$, depending only on r, such that

$$
\mathbb{E}\bigg\|\frac{\sum_{i=1}^n(Z_i-\mathbb{E}Z_i)}{\sqrt{n}}\bigg\|_{L_1}^r \le c_1\bigg[\mathbb{E}\max_{1\le i\le n}\frac{\|Z_i-\mathbb{E}Z_i\|_{L_1}^r}{n^{r/2}}+t_{0,n}^r\bigg],
\tag{2.8}
$$

where

$$
t_{0,n}=\inf\bigg[t\colon\ \Pr\bigg\{\frac{\|\sum_{i=1}^n(Z_i-\mathbb{E}Z_i)\|_{L_1}}{\sqrt{n}}>t\bigg\}\le c_2\bigg].
\tag{2.9}
$$

On one hand, the stochastic boundedness hypothesis (2.4) implies $\sup_n t_{0,n}<\infty$, and on the other, inequality (2.7) asserts the finiteness of the sup over n of the first summand at the right-hand side of inequality (2.8) for $r=1$. We thus conclude from inequality (2.9) for $r=1$ that

$$
\sup_n \mathbb{E}\bigg\|\frac{\sum_{i=1}^n(Z_i-\mathbb{E}Z_i)}{\sqrt{n}}\bigg\|_{L_1}<\infty.
\tag{2.10}
$$

Let now ξ be a binomial (n,p) random variable such that $4c_1/n\le p\le 1/2$, c_1 as in (2.8). Then, inequality (2.8) (Hoffmann–Jørgensen's combined with Montgomery-Smith's) gives

$$
\frac{np}{2}\le \mathbb{E}(\xi-\mathbb{E}\xi)^2\le c_1[1+(c_2^{-1}\mathbb{E}|\xi-\mathbb{E}\xi|)^2]\le\frac{np}{4}+\frac{c_1}{c_2^2}(\mathbb{E}|\xi-\mathbb{E}\xi|)^2.
$$

In other words, there exist positive finite constants C_1 and C_2 such that

$$
\mathscr{L}(\xi)=\mathrm{Bin}(n,p)\text{ with }\frac{C_1}{n}\le p\le\frac12\text{ implies }\mathbb{E}|\xi-\mathbb{E}\xi|\ge C_2\sqrt{np}.
\tag{2.11}
$$

[This also follows from symmetrization and Corollary 3.4 in Giné and Zinn (1983).] Inequality (2.11), applied to the empirical process, yields

$$
\sqrt{\Pr\{X>t\}}\le\frac{1}{C_2}\mathbb{E}\bigg|\frac{\sum_{i=1}^n(I_{X_i>t}-\Pr\{X_i>t\})}{\sqrt{n}}\bigg|
\quad\text{for }\mathrm{med}(X)<t<Q(1-C_1/n).
\tag{2.12}
$$

Then, if we integrate in inequality (2.12) and apply inequality (2.10), we obtain

$$\begin{aligned}\sup_n \int_{\mathrm{med}(X)}^{Q(1-C_1/n)} &\sqrt{\Pr\{X > t\}}\,dt\\ &\le \frac{1}{C_2}\sup_n \int_{\mathrm{med}(X)}^{Q(1-C_1/n)} \mathbb{E}\left|\frac{\sum_{i=1}^n (I_{X_i>t} - \Pr\{X_i > t\})}{\sqrt{n}}\right| dt < \infty.\end{aligned}$$

Since $Q(1 - C_1/n) \to \operatorname{ess\,sup} X$ as $n \to \infty$, this last inequality gives

$$\int_0^\infty \sqrt{\Pr\{X > t\}}\,dt < \infty,$$

that is, $\Lambda_{2,1}(X) < \infty$, proving the theorem. □

REMARK 2.1. Note, from Theorem 2.1, that condition (2.1) not only implies convergence in law of $\{\sqrt{n}\|F_n - F\|_{L_1}\}$, but convergence in law, *in the space* L_1, of processes that have these random variables as their L_1 norms, which is a stronger form of convergence; also, stochastic boundedness of the set of random variables $\{\sqrt{n}\|F_n - F\|_{L_1}\}$ implies condition (2.1), *without assuming any a priori conditions on* X. Thus, Theorem 2.1 considerably strengthens statement (a1) in Theorem 1.1.

If in Theorem 2.1(b) we consider variables not necessarily in $DA_\alpha(a_n)$, then we also obtain necessary and sufficient conditions for tightness of the normalized, uncentered Wasserstein distance. To keep things simple, we only deal with power type normalizations.

THEOREM 2.2. *Let* X, X_i, $i \in \mathbb{N}$, *be i.i.d. with common cdf* F *and let* $\alpha \in (1, 2)$. *Then, the sequence*

$$\left\{\frac{n}{n^{1/\alpha}}\int_{-\infty}^{\infty} |F_n(t) - F(t)|\,dt\right\}_{n=1}^{\infty} \tag{2.13}$$

is stochastically bounded if and only if

$$\Lambda_{\alpha,\infty}(X) := \sup_{t>0} t^\alpha \Pr\{|X| > t\} < \infty. \tag{2.14}$$

PROOF. *Necessity.* Let ξ_i be Bernoulli (p) independent random variables, and let $\xi = \sum_{i=1}^n \xi_i$. Using the classical Marcinkiewicz inequality we obtain that, if $p \in [1/n, 1/2]$ and $n \ge 2$, then

$$\begin{aligned}\mathbb{E}|\xi - \mathbb{E}\xi| &\ge \frac{1}{2\sqrt{2}}\mathbb{E}\left[\sum_{i=1}^n (\xi_i - p)^2\right]^{1/2} \ge \frac{1}{2\sqrt{2}}\mathbb{E}\max_{1\le i\le n}|\xi_i - p|\\ &\ge \frac{1}{2\sqrt{2}}(1-p)\Pr\{\xi \ge 1\} \ge \frac{1}{2\sqrt{2}}(1-p)\left[1 - \left(1 - \frac{1}{n}\right)^n\right] \ge \frac{3}{16\sqrt{2}}.\end{aligned}$$

That is, there exists $C > 0$ such that

$$\mathscr{L}(\xi) = \mathrm{Bin}(n, p) \text{ with } \frac{1}{n} \le p \le \frac{1}{2} \text{ implies } \mathbb{E}|\xi - \mathbb{E}\xi| \ge C. \tag{2.15}$$

[This also follows from Corollary 3.4 in Giné and Zinn (1983).] Consequently, since tightness of the sequence (2.13) implies boundedness of its first moments by a proof entirely analogous to the proof of (2.10), we have

$$\begin{aligned}\sup_n \frac{Q(1-1/n)-\mathrm{med}(X)}{n^{1/\alpha}} \\ \le \frac{1}{C}\sup_n \int_{\mathrm{med}(X)}^{Q(1-1/n)} \mathbb{E}\left|\frac{\sum_{i=1}^n (I_{X_i>t}-\Pr\{X_i>t\})}{n^{1/\alpha}}\right| dt < \infty.\end{aligned}$$

So, there exists $K<\infty$ such that $Q(1-1/n)\le Kn^{1/\alpha}$ for all $n\ge 2$, and therefore, $\Pr\{X>Kn^{1/\alpha}\}\le 1/n$ for all n, a condition equivalent to (2.14).

Sufficiency. Now we assume that condition (2.14) holds and that, without loss of generality, $X\ge 0$. Setting

$$Z_{n,i}(t) := I_{[0,\,X_i/n^{1/\alpha})}(t), \qquad t\ge 0, \tag{2.16}$$

a simple change of variables shows that

$$\begin{aligned}\frac{n}{n^{1/\alpha}}\int_0^\infty |F_n(t)-F(t)|\,dt &= \int_0^\infty \left|\sum_{i=1}^n \left(I_{X_i>tn^{1/\alpha}}-\Pr\{X>tn^{1/\alpha}\}\right)\right| dt \\ &= \left\|\sum_{i=1}^n (Z_{n,i}-\mathbb{E}Z_{n,i})\right\|_{L_1}.\end{aligned} \tag{2.17}$$

The following computations are classical. We will use repeatedly that $\|Z_{n,1}\|_{L_1} = \int_0^\infty I_{[0,\,X_1/n^{1/\alpha})}(t)\,dt = X_1/n^{1/\alpha}$ and similar computations. In particular, for any $M>0$,

$$\begin{aligned}n\left\|\mathbb{E}Z_{n,1}I_{\|Z_{n,1}\|_{L_1}>M}\right\|_{L_1} &= n\int_0^\infty \Pr\{X>n^{1/\alpha}(t\vee M)\}\,dt \\ &= nM\Pr\{X>Mn^{1/\alpha}\}+n\int_M^\infty \Pr\{X>tn^{1/\alpha}\}\,dt \\ &\le \Lambda_{\alpha,\infty}(X)\left[M^{1-\alpha}+\int_M^\infty t^{-\alpha}\,dt\right] = \frac{\alpha}{\alpha-1}\frac{\Lambda_{\alpha,\infty}(X)}{M^{\alpha-1}}.\end{aligned}$$

Since this last quantity is smaller than $M/2$ for all M larger than some $M_0<\infty$, we have, for such M's,

$$\begin{aligned}&\Pr\left\{\left\|\sum_{i=1}^n (Z_{n,i}-\mathbb{E}Z_{n,i})\right\|_{L_1}>M\right\} \\ &\quad\le \Pr\left\{\left\|\sum_{i=1}^n (Z_{n,i}-\mathbb{E}Z_{n,i}I_{\|Z_{n,i}\|_{L_1}\le M})\right\|_{L_1}>\frac{M}{2}\right\} \\ &\quad\le \Pr\left\{\left\|\sum_{i=1}^n (Z_{n,i}I_{\|Z_{n,i}\|_{L_1}\le M}-\mathbb{E}Z_{n,i}I_{\|Z_{n,i}\|_{L_1}\le M})\right\|_{L_1}>\frac{M}{4}\right\} \\ &\qquad + n\Pr\{X>Mn^{1/\alpha}\} \\ &\quad := (I_n)+(II_n).\end{aligned} \tag{2.18}$$

Now,

$$\sup_n(II_n) = \sup_n n\Pr\{X > Mn^{1/\alpha}\} \le \frac{\Lambda_{\alpha,\infty}(X)}{M^\alpha} \to 0 \quad \text{as } M \to \infty. \tag{2.19}$$

As for (I_n), we have

$$\begin{aligned}
(I_n) &\le \frac{4}{M}\int_0^\infty \mathbb{E}\left|\sum_{i=1}^n (Z_{n,i}I_{\|Z_{n,i}\|_{L_1}\le M} - \mathbb{E}Z_{n,i}I_{\|Z_{n,i}\|_{L_1}\le M})\right| dt \\
&\le \frac{4}{M}\int_0^\infty \left[\mathbb{E}\left(\sum_{i=1}^n (Z_{n,i}I_{\|Z_{n,i}\|_{L_1}\le M} - \mathbb{E}Z_{n,i}I_{\|Z_{n,i}\|_{L_1}\le M})\right)^2\right]^{1/2} dt \\
&\le \frac{4n^{1/2}}{M}\int_0^\infty [\mathbb{E}(Z_{n,1}I_{\|Z_{n,1}\|_{L_1}\le M})^2]^{1/2} dt \\
&\le \frac{4n^{1/2}}{M}\int_0^M [\Pr\{X > tn^{1/\alpha}\}]^{1/2} dt \\
&\le \frac{4\Lambda_{\alpha,\infty}(X)}{M}\int_0^M t^{-\alpha/2}\, dt = \frac{4\Lambda_{\alpha,\infty}(X)}{1-\alpha/2}\frac{1}{M^{\alpha/2}}.
\end{aligned}$$

Hence, it follows that

$$\lim_{M\to\infty}\sup_n(I_n) = 0. \tag{2.20}$$

Combining the estimates (2.19) and (2.20) with inequality (2.18) and the identity (2.17) gives stochastic boundedness of the sequence in (2.13). □

This theorem, together with Theorem 2.1, gives us estimates of the rate of convergence to zero of $d_1(F_n, F)$ for relatively large classes of random variables. It also tells us that, if we exclude the $L_{2,1}$ case, domains of attraction of α-stable laws, $1 < \alpha < 2$, are, in a sense, the natural classes of random variables for which to look for more precise results about the convergence rate to zero of $d_1(F_n, F)$. The cases $\alpha = 1$ with X integrable and, particularly, the large class of random variables in the domain of attraction of the normal law but not in $L_{2,1}$ are extreme. In both cases the behavior of $d_1(F_n, F)$ is radically different from the other cases, and will be treated in subsequent sections. Next we look at the cases in between these two extremes.

Concretely, we prove a stronger form of Theorem 1.1(b) for X in the domain of attraction of an α-stable law with $1 < \alpha < 2$. The limit will be given in terms of generalized Poisson measures, thus obtaining another (arguably not as nice) representation of the limit in Theorem 1.1(b). We begin with a simple argument showing that the limit law in Theorem 1.1(b), if it exists, corresponds to the sum of the integrals of two-scaled independent Poisson processes. Let X, X_i, and a_n be as in Theorem 1.1(b). By the change of variables already encountered in the previous proof,

$$
\begin{aligned}
\frac{n}{a_n}\int_{-\infty}^{\infty}\left|F_n(t)-F(t)\right|dt &= \int_{-\infty}^{0}\left|\sum_{i=1}^{n}(I_{X_i\le ta_n}-\Pr\{X_i\le ta_n\})\right|dt \\
&\quad+\int_{0}^{\infty}\left|\sum_{i=1}^{n}(I_{X_i> ta_n}-\Pr\{X_i> ta_n\})\right|dt.
\end{aligned}
\tag{2.21}
$$

Since the integrands are (absolute values of) centered binomial random variables whose probabilities of success, respectively, $\Pr\{X\le -ta_n\}$ and $\Pr\{X > ta_n\}$, satisfy

$$
\lim_{n\to\infty} n\Pr\{X>ta_n\}=\frac{c_1}{\alpha t^{\alpha}},\qquad \lim_{n\to\infty} n\Pr\{X\le -ta_n\}=\frac{c_2}{\alpha t^{\alpha}},\qquad t>0,
\tag{2.22}
$$

by the stable convergence criterion in $\mathbb{R}$ [cf. Feller (1971) or Araujo and Giné (1980)], and since the indicators in each integrand are disjoint, it follows from elementary probability (the law of rare events, multivariate version) that the finite-dimensional distributions of the processes

$$
K_n(t):=\begin{cases}\displaystyle\sum_{i=1}^{n}(I_{X_i\le ta_n}-\Pr\{X_i\le ta_n\}), & \text{for } t<0,\\ 0, & \text{for } t=0,\\ \displaystyle\sum_{i=1}^{n}(I_{X_i> ta_n}-\Pr\{X_i> ta_n\}), & \text{for } t>0\end{cases}
\tag{2.23}
$$

converge to the corresponding finite-dimensional distributions of the process

$$
K(t)-\mathbb{E}K(t):=\begin{cases} N_1\left(\dfrac{c_1}{\alpha t^{\alpha}}\right)-\dfrac{c_1}{\alpha t^{\alpha}}, & \text{for } t>0,\\ 0, & \text{for } t=0,\\ N_2\left(\dfrac{c_2}{\alpha |t|^{\alpha}}\right)-\dfrac{c_2}{\alpha |t|^{\alpha}}, & \text{for } t<0,\end{cases}
\tag{2.24}
$$

where N_1 and N_2 are two independent standard (intensity 1) Poisson processes on $[0,\infty)$. Hence, the above-mentioned result of Lawniczak (1983) shows that, if there is convergence in law in L_1 of the processes K_n, then their limit must be the process defined by (2.24), whose L_1 norm is precisely the limit process specified in Theorem 1.1(b), so that, if this is the case, Theorem 1.1(b) follows by continuous mapping. The next theorem proves that the processes K_n do indeed converge in law in L_1 to N (tightness of $\{K_n\}$ would suffice, but proving convergence is equally easy). [See CCHM (1986b), for a representation of all α-stable laws, $0<\alpha<2$, as integrals of the processes (2.24) or modifications thereof.]

THEOREM 2.3. *Let X, X_i, $i \in \mathbb{N}$, be i.i.d. and such that*

$$\mathscr{L}\left(\frac{1}{a_n}\sum_{i=1}^{n}(X_i - \mathbb{E}X_i)\right) \to_w \text{cPois}\,\mu(c_1, c_2, \alpha) \tag{2.25}$$

for constants $0 \le c_1, c_2 < \infty$, but not both equal to zero, and $1 < \alpha < 2$, where the α-stable law cPois $\mu(c_1, c_2, \alpha)$ *is given by* (1.9) *and* (1.11). *Let $K_n(t)$, $-\infty < t < \infty$, be the processes defined by* (2.23) *from these variables X_i. Let μ_i, $i = 1, 2$, be measures on $L_1(\mathbb{R})$, respectively, concentrated on the sets of functions $\{I_{[0,x)}\colon x > 0\}$ and $\{I_{[-x,0)}\colon x > 0\}$, such that*

$$\mu_1\{I_{[0,x)}\colon x > u\} = \frac{c_1}{\alpha u^{\alpha}}, \qquad \mu_2\{I_{[x,0)}\colon x < -u\} = \frac{c_2}{\alpha u^{\alpha}}, \qquad u > 0. \tag{2.26}$$

Then the measures μ_i, $i = 1, 2$, are Lévy measures on the Banach space $L_1(\mathbb{R})$ and

$$\mathscr{L}(K_n) \to_w (\text{cPois}\,\mu_1) * (\text{cPois}\,\mu_2) \quad \text{in } L_1(\mathbb{R}). \tag{2.27}$$

Conversely, if the limit (2.27) *holds with μ_i, $i = 1, 2$, as in* (2.26) *and with constants a_n regularly varying at infinity with exponent $1/\alpha$, then $X \in DA_\alpha(a_n)$ and the limit* (2.25) *holds.*

The proof of this theorem is based on the following Poisson convergence criterion in Banach spaces [a consequence of the general CLT of de Acosta, Araujo and Giné (1978), made explicit by Mandrekar and Zinn (1980); see Araujo and Giné (1980), Theorem 3.5.9 on page 129 and Exercise 5 on page 134. In the second part of this exercise the conclusion should be tightness of the laws of $\{S_n - \mathbb{E}S_{n,\delta}\}$ for all $\delta > 0$ instead of tightness of the laws of $\{S_n\}$].

GENERALIZED POISSON CONVERGENCE CRITERION IN BANACH SPACES. Let $\{Z_{n,i}\colon i = 1, \dots, n, n \in \mathbb{N}\}$ be an infinitesimal array of row-wise independent integrable B-valued random variables, where B is a Banach space (a triangular array is infinitesimal if $\max_i \Pr\{\|Z_{n,i}\| > \varepsilon\} \to 0$ for all $\varepsilon > 0$). Then, in order that there exist a Lévy measure μ such that

$$\mathscr{L}\left(\sum_{i=1}^{n}(Z_{n,i} - \mathbb{E}Z_{n,i})\right) \to_w \text{cPois}\,\mu,$$

it is necessary and sufficient that the following three conditions be met:

(i) For all $\delta \neq 0$ in a dense set D of $\mathbb{R}^+$,

$$\sum_{i=1}^{n} \mathscr{L}(Z_{n,i})\big|_{\|x\|>\delta} \to_w \mu\big|_{\|x\|>\delta}$$

for some measure μ on B.

(ii) $\lim_{\delta\to 0}\limsup_n \mathbb{E}\|\sum_{i=1}^{n}(Z_{n,i}I_{\|Z_{n,i}\|\le\delta} - \mathbb{E}Z_{n,i}I_{\|Z_{n,i}\|\le\delta})\| = 0$.

(iii) For some $\delta \in D$ (or for all $\delta \in D$), the sequence $\{\sum_{i=1}^{n}\mathbb{E}Z_{n,i}I_{\|Z_{n,i}\|>\delta}\}$ converges in B.

Moreover, conditions (i) and (ii) together are equivalent to

$$\mathscr{L}\left(\sum_{i=1}^{n}(Z_{n,i} - \mathbb{E}Z_{n,i}I_{\|Z_{n,i}\|\le\delta})\right) \to_w \mathrm{c}_\delta\mathrm{Pois}\,\mu$$

for all δ such that $\mu\{\|x\| = \delta\} = 0$.

Let $\{Z_{n,i}\}$ be an infinitesimal array of row-wise independent vectors, as above. Since, if μ_i, $i = 1, 2$, are Lévy measures then so is $\mu_1+\mu_2$ and $\mathrm{cPois}(\mu_1+\mu_2) = \mathrm{cPois}\,\mu_1 * \mathrm{cPois}\,\mu_2$, it follows trivially from this criterion that if for disjoint measurable sets $C_k \subset B$, $i = 1, 2$, we have

$$\mathscr{L}\left(\sum_{i}(Z_{n,i}I_{Z_{n,i}\in C_k} - \mathbb{E}Z_{n,i}I_{Z_{n,i}\in C_k})\right) \to_w \mathrm{cPois}\,\mu_k, \qquad k = 1, 2, \tag{2.28}$$

then

$$\mathscr{L}\left(\sum_{i}(Z_{n,i}I_{Z_{n,i}\in C_1\cup C_2} - \mathbb{E}Z_{n,i}I_{Z_{n,i}\in C_1\cup C_2})\right) \to_w (\mathrm{cPois}\,\mu_1) * (\mathrm{cPois}\,\mu_2), \tag{2.29}$$

a fact to be used below. The same comment applies to the second part of the theorem.

To prove Theorem 2.3 we will freely use the properties of random variables in domains of attraction (DA) of stable laws in $\mathbb{R}$, as described, for example, in Feller (1971) [or in Araujo and Giné (1980)].

Proof of Theorem 2.3. We begin with the direct part. By [(2.28) $\Rightarrow$ (2.29)] we may assume $X_i \ge 0$ and

$$\frac{1}{a_n}\sum_{i=1}^{n}(X_i - \mathbb{E}X_i) \to_{\mathscr{L}} \mathrm{cPois}\,\mu(c, 0, \alpha). \tag{2.25$'$}$$

Then,

$$K_n(t) = \sum_{i=1}^{n}(Z_{n,i}(t) - \mathbb{E}Z_{n,i}(t)), \qquad t > 0,$$

and $K_n(0) = 0$, with

$$Z_{n,i} := I_{[0,\, X_i/a_n)}, \qquad i = 1, \dots, n,\ n \in \mathbb{N},$$

which are $L_1[0,\infty)$-valued random variables. Since $\|Z_{n,i}\|_{L_1} = X_i/a_n$, we have $\max_{i\le n}\Pr\{\|Z_{n,i}\|_{L_1} > \varepsilon\} = \Pr\{X > \varepsilon a_n\} \to 0$ for all $\varepsilon > 0$, showing that the array $\{Z_{n,i}\}$ is infinitesimal. So, we can apply the above generalized Poisson convergence criterion for $B = L_1$. Let μ be the measure obtained from μ_1 in (2.26) replacing c_1 by c. We must prove

$$\mathscr{L}\left(\sum_{i=1}^{n}(Z_{n,i} - \mathbb{E}Z_{n,i})\right) \to_w \mathrm{cPois}\,\mu \quad \text{in } L_1[0,\infty), \tag{2.27$'$}$$

which we do by checking (i)–(iii) above.

PROOF OF (i). Since the map $T: \mathbb{R} \cup \{0\} \to L_1[0, \infty)$ given by $T(x) = I_{[0,x)}$ is an isometry and the image of T is closed in L_1, proving (i) is equivalent, by the continuous mapping theorem, to showing

$$n(\mathscr{L}(Z_{n,1})|_{\|x\|_{L_1} > \delta}) \circ T \to_w (\mu|_{\|x\|_{L_1} > \delta}) \circ T, \qquad \delta > 0,$$

that is, to showing

$$n\mathscr{L}(X/a_n)\big|_{x > \delta} \to \mu_\delta, \qquad \delta > 0,$$

where $\mu_\delta\{x > u\} = c/(\alpha(u \vee \delta)^\alpha)$. This is equivalent to proving

$$n \Pr\{X > ua_n\} \to \frac{c}{\alpha u^\alpha}$$

for all $u > 0$, which holds because of (2.25′) and the stable DA criterion in $\mathbb{R}$.

PROOF OF (ii). First we observe that, by stable convergence in $\mathbb{R}$ and the asymptotic properties of regularly varying functions [Feller (1971), Theorem 1b, page 281],

$$\frac{u(\Pr\{X > u\})^{1/2}}{\int_0^u (\Pr\{X > s\})^{1/2}\, du} \to 1 - \frac{\alpha}{2} \quad \text{as } u \to \infty. \tag{2.30}$$

Then, this gives

$$\begin{aligned}
&\mathbb{E}\left\| \sum_{i=1}^n (Z_{n,i} I_{\|Z_{n,i}\|_{L_1} \le \delta} - \mathbb{E} Z_{n,i} I_{\|Z_{n,i}\|_{L_1} \le \delta}) \right\|_{L_1} \\
&\quad = \int_0^\delta \mathbb{E}\left| \sum_{i=1}^n I_{ta_n < X_i \le \delta a_n} - \Pr\{ta_n < X_i \le \delta a_n\} \right| dt \\
&\quad \le \int_0^\delta \left[\mathbb{E}\left(\sum_{i=1}^n I_{ta_n < X_i \le \delta a_n} - \Pr\{ta_n < X_i \le \delta a_n\} \right)^2 \right]^{1/2} dt \\
&\quad \le \int_0^\delta n^{1/2} (\Pr\{X > ta_n\})^{1/2}\, dt \\
&\quad = \frac{n^{1/2}}{a_n} \int_0^{\delta a_n} (\Pr\{X > u\})^{1/2}\, du \\
&\quad \asymp \frac{\delta}{1 - \alpha/2} [n \Pr\{X > \delta a_n\}]^{1/2},
\end{aligned}$$

where $A_n \asymp B_n$ indicates that $A_n / B_n \to 1$ as $n \to \infty$. Now, by the stable DA criterion in $\mathbb{R}$, the limit of the last sequence as $n \to \infty$ is $c^{1/2}\delta^{1-\alpha/2}/(\alpha^{1/2}(1 - \alpha/2))$, which in turn tends to zero as $\delta \to 0$, proving (ii).

PROOF OF (iii). We obviously can assume X is nondegenerate. Then, if $X \in DA_\alpha$, the tail probability function $\Pr\{X > u\}$ is regularly varying at infinity with exponent $-\alpha$ and the representation of slowly varying functions in, for example, Feller (1971), page 282, gives the existence of two functions $c(y) \to c \in (0, \infty)$ and $\varepsilon(y) \to 0$ as $y \to \infty$ such that

$$\Pr\{X > u\} = u^{-\alpha} c(u) \exp\left(\int_1^u \frac{\varepsilon(y)}{y}\, dy\right).$$

As a consequence, given $\varepsilon > 0$ that we take to satisfy $\varepsilon < \alpha - 1$, there exist $u_0 < \infty$ such that

$$\Pr\{X > u\} \le 2\left(\frac{u}{v}\right)^{-\alpha+\varepsilon} \Pr\{X > v\} \tag{2.31}$$

for all $u \ge v \ge u_0$. If, more concretely, X satisfies (2.25′) then, as mentioned above,

$$\mathbb{E}\sum_{i=1}^n Z_{n,i} I_{\|Z_{n,i}\|_{L_1} > \delta} = n \Pr\{X > (t \vee \delta) a_n\} \to \frac{c}{\alpha (t \vee \delta)^\alpha} \tag{2.32}$$

as $n \to \infty$. In order to prove (iii), we must show this limit holds in L_1 for some $\delta > 0$. We show it does for $\delta = 1$: by (2.31), for n large enough independently of $t \ge 1$, we have

$$n \Pr\{X > t a_n\} \le \frac{2n \Pr\{X > a_n\}}{t^{\alpha-\varepsilon}} \le \frac{K}{t^{\alpha-\varepsilon}}$$

for some $K < \infty$, also independent of t, because $a_n \to \infty$ and the sequence $\{n \Pr\{X > a_n\}\}$ converges. Since $K/t^{\alpha-\varepsilon}$ is integrable on $[1, \infty)$, the dominated convergence theorem and (2.32) show that

$$\int_0^\infty \left| \mathbb{E}\sum_{i=1}^n Z_{n,i} I_{\|Z_{n,i}\|_{L_1} > 1} - \frac{c}{\alpha (t \vee 1)^\alpha} \right| \to 0$$

as $n \to \infty$, proving (iii) and hence the direct part of the theorem, by the generalized Poisson convergence criterion in L_1.

For the converse, we still can asume $X \ge 0$, and we just note, from the proof of (i) and the convergence criterion in L_1, that, if the limit (2.27) holds, then

$$n \Pr\{X > u a_n\} \to \frac{c}{\alpha u^\alpha}$$

for all $u > 0$, which ensures that X satisfies (2.25′) [by Lemma 3 in Feller (1971), page 277, and the stable DA criterion in $\mathbb{R}$]. □

Theorem 2.3 thus improves on the type of convergence in Theorem 1.1(b). Note that the proof we just gave consists of a relatively simple exercise on the stable convergence criterion in Banach spaces, a result known since 1978. In view of the comment that follows, it is the change of variables in (2.21), which yields the definition of the processes K_n, that makes this reduction possible.

REMARK 2.2. Next we comment on a curiosity, but skip details. The processes K_n defined by (2.20) are not the most natural ones satisfying

$$\|K_n\|_{L_1} = \frac{n}{a_n}\|F_n - F\|_{L_1},$$

the first choice for such processes being, for example, for $X \geq 0$,

$$\tilde{K}_n := \frac{n}{a_n}\sum_{i=1}^{\infty}(I_{X_i>t} - \Pr\{X > t\}).$$

It is easy to see that, under the conditions of Theorem 2.3, *the sequence of processes* $\{\tilde{K}_n\}$ *is not uniformly tight* in $L_1[0,\infty)$. Hence, these processes provide an example of a sequence of B-valued random variables $\{\tilde{K}_n\}$ which is not uniformly tight while the sequence of their norms, $\{\|\tilde{K}_n\|_{L_1}\}$, does converge in law. They also provide an example of two sequences of random vectors $\{\tilde{K}_n\}$ and $\{K_n\}$ which have the same norms, but one of then converges in law whereas the other is not even tight.

Theorems 1.1(a1) and (b) can be improved in another direction, namely, *convergence of moments*.

THEOREM 2.4. *Let X, X_i be i.i.d., with common distribution F, and let F_n denote the empirical distribution based on $X_1,\dots,X_n$, $n \in \mathbb{N}$, as usual. Then the following holds*:

(a) *If $X \in L_{2,1}$ and $\mathbb{E}|X|^p < \infty$ for some $p \geq 2$, then*

$$\lim_{n\to\infty}\mathbb{E}\left[\sqrt{n}\int_{-\infty}^{\infty}|F_n(t) - F(t)|\,dt\right]^r = \mathbb{E}\left[\int_{-\infty}^{\infty}|B(F(t))|\,dt\right]^r \tag{2.33}$$

for all $0 < r \leq p$.

(b) *If $X \in DA_\alpha(a_n)$ for some $\alpha \in (1,2)$ and K is the Poisson process defined by* (2.4), *then*

$$\lim_{n\to\infty}\mathbb{E}\left[\frac{n}{a_n}\int_{-\infty}^{\infty}|F_n(t) - F(t)|\,dt\right]^\beta = \mathbb{E}\left[\int_{-\infty}^{\infty}|K(t) - \mathbb{E}K(t)|\,dt\right]^\beta \tag{2.34}$$

for all $0 < \beta < \alpha$.

PROOF. Part (a) follows from general principles. Theorem 5.1 in de Acosta and Giné (1979) [cf. Araujo and Giné (1980), page 136], applied to the processes Y_i in the proof of Theorem 2.1 above, asserts that, if $\mathbb{E}\|Y_1\|_{L_1}^p < \infty$ for some $p \geq 2$, then the pth moment of $\|\sum_{i=1}^n Y_i\|_{L_1}$ converges to the pth moment of the limit since Y_1 satisfies the central limit theorem in L_1 by Theorem 2.1; this result applies to give part (a) because, as a simple computation shows, $\|Y_1\|_{L_1} \leq |X| + \mathbb{E}|X|$.

In order to prove part (b), by Theorem 2.3 and uniform integrability, it suffices to show that

$$\sup_n \mathbb{E}\left(\frac{1}{a_n}\int_0^\infty\left|\sum_{i=1}^n(I_{X_i>t} - \Pr\{X > t\})\right|dt\right)^\beta < \infty \tag{2.35}$$

and

$$\sup_n \mathbb{E}\left(\frac{1}{a_n}\int_0^\infty \left|\sum_{i=1}^n (I_{X_i\le -t} - \Pr\{X \le -t\})\right| dt\right)^\beta < \infty \tag{2.35$'$}$$

for all $1 \le \beta < \alpha$. Both bounds having similar proofs, we will only consider the first. We can assume without loss of generality that $X \ge 0$ and that $c \ne 0$ in (2.25′). By the generalized Minkowski inequality,

$$\begin{aligned}
&\left[\mathbb{E}\left(\frac{1}{a_n}\int_0^\infty \left|\sum_{i=1}^n (I_{X_i> t} - \Pr\{X > t\})\right| dt\right)^\beta\right]^{1/\beta}\\
&\quad\le \frac{1}{a_n}\int_0^\infty \left[\mathbb{E}\left|\sum_{i=1}^n (I_{X_i> t} - \Pr\{X > t\})\right|^\beta\right]^{1/\beta} dt\\
&\quad\le \frac{1}{a_n}\int_0^{Q(1-1/n)} \left[\mathbb{E}\left|\sum_{i=1}^n (I_{X_i> t} - \Pr\{X > t\})\right|^2\right]^{1/2} dt\\
&\qquad + \frac{1}{a_n}\int_{Q(1-1/n)}^\infty \left[\mathbb{E}\left|\sum_{i=1}^n (I_{X_i> t} - \Pr\{X > t\})\right|^\beta\right]^{1/\beta} dt\\
&\quad:= A_n + B_n.
\end{aligned} \tag{2.36}$$

Boundedness of A_n is a trivial consequence of regular variation of the tail probabilities $\Pr\{X > t\}$; by (2.30),

$$\begin{aligned}
A_n &\le \frac{1}{a_n}\int_0^{Q(1-1/n)} (n\Pr\{X > t\})^{1/2}\, dt\\
&\asymp \frac{1}{1-\alpha/2}\frac{1}{a_n} Q(1-1/n)(n\Pr\{X > Q(1-1/n)\})^{1/2},
\end{aligned}$$

which is uniformly bounded. By definition of Q, $n\Pr\{X > Q(1-1/n)\} \le 1$, and moreover, since $n\Pr\{X \ge ua_n\} \to c/(u^\alpha \alpha)$, we have that $Q(1-1/n)$ is bounded by $2a_n(\alpha/c)^{1/\alpha}$ from some n on.

To prove boundedness of B_n, we first note that if E_i are independent events with $\Pr E_i = p$, that is, if $Z = \sum_{i=1}^n I_{E_i}$ is a binomial (n, p) random variable, and if $np \le \lambda < \infty$, then there are constants $C(\lambda, r)$ such that

$$\mathbb{E}|Z - \mathbb{E}Z|^r \le C(\lambda, r)np \tag{2.37}$$

for all $r \ge 1$. To see this, we apply Hoffmann–Jørgensen's inequality in combination with Montgomery-Smith's inequality, precisely, inequality (2.8) with L_1 norms replaced by absolute values, and bound $t_{0,n}$ in (2.9) using Markov's inequality, to obtain

$$\begin{aligned}
\mathbb{E}|Z - \mathbb{E}Z|^r &\le c_1(n\mathbb{E}|I_{E_1} - \Pr E_1|^r + (\mathbb{E}|Z - \mathbb{E}Z|)^r/c_2^r)\\
&\le (c_1 2^{r-1} n(p + p^r) + c_2^{-r}(2np)^r)\\
&\le C(\lambda, r)np,
\end{aligned}$$

where the last inequality follows because $p^r \le p$ and $(np)^r \le \lambda^{r-1}np$.

Since $n\Pr\{X > t\} \le 1$ if $t \ge Q(1-1/n)$, inequality (2.37) then gives

$$B_n \le \frac{1}{a_n}\int_{Q(1-1/n)}^{\infty}[C(1,\beta)n\Pr\{X > t\}]^{1/\beta}\,dt.$$

The properties of regular variation [Feller (1971), page 281], then imply that the integral at the right-hand side of this inequality is asymptotically of the order of

$$(\alpha/\beta - 1)^{-1}Q(1-1/n)(n\Pr\{X > Q(1-1/n)\})^{1/\beta}/a_n.$$

Here $Q(1-1/n)$ cancels with $1/a_n$ (as shown above, when bounding $\{A_n\}$) and $n\Pr\{X > Q(1-1/n)\} \le 1$, proving that $\sup B_n < \infty$, and concluding the proof of (2.35). $\square$

Theorem 2.4 cannot be improved since, for $X \in \Lambda_{2,1}$ or $X \in DA_\alpha(n^{1/\alpha})$, it gives convergence for the moment of order r if and only if $\mathbb{E}(\|F_1 - F\|_{L_1})^r < \infty$.

3. Integrable random variables in domains of atraction of 1-stable laws. The Wasserstein distance $d_1(F_n, F)$ is defined if and only if $\mathbb{E}|X| < \infty$, hence, in particular, it is defined for integrable variables in domains of attaction of 1-stable laws. However, the limit law in Theorem 1.1(b) does not exist if $X \in DA_1$ since $|N(t) - t|/t^2$ is just $1/t$ for the a.s. positive amount of time that $N(t)$ is zero. It turns out that we still have convergence, with the same normalizers, *provided we choose the right centerings*. We begin with an analogue of Theorem 2.3.

THEOREM 3.1. *Let X, X_i, $i \in \mathbb{N}$, be i.i.d.* integrable *random variables such that*

$$\mathscr{L}\left(\frac{1}{a_n}\sum_{i=1}^{n}(X_i - \mathbb{E}X_i I_{|X_i|\le\delta})\right) \to_w \mathrm{c}_\delta\mathrm{Pois}\,\mu(c_1, c_2, 1) \tag{3.1}$$

for constants $0 \le c_1, c_2 < \infty$, but not both equal to zero, and $\delta > 0$, where the 1-stable law $\mathrm{c}_\delta\mathrm{Pois}\,\mu(c_1, c_2, \alpha)$ is given by (1.10). *Let*

$$Z_{n,i}(t) := \begin{cases} I_{X_i > ta_n}, & \text{for } t > 0,\\ 0, & \text{for } t = 0,\\ I_{X_i \le ta_n}, & \text{for } t < 0,\end{cases} \tag{3.2}$$

as above. Let μ_i, $i = 1, 2$, be measures on $L_1(\mathbb{R})$, respectively, concentrated on the sets of functions $\{I_{[-x,0)}: x > 0\}$ and $\{I_{[0,x)}: x > 0\}$, such that

$$\mu_1\{I_{[0,x)}: x > u\} = \frac{c_1}{u}, \qquad \mu_2\{I_{[x,0)}: x < -u\} = \frac{c_2}{u},\ \ u > 0. \tag{3.3}$$

Then the measures μ_i, $i = 1, 2$, are Lévy measures on the Banach space $L_1(\mathbb{R})$ and, for all $\delta > 0$,

$$\mathscr{L}\left(\sum_{i=1}^{n}(Z_{n,i} - \mathbb{E}Z_{n,i}I_{\|Z_{n,i}\|_{L_1}\le\delta})\right) \to_w (\mathrm{c}_\delta\mathrm{Pois}\,\mu_1) * (\mathrm{c}_\delta\mathrm{Pois}\,\mu_2) \quad \text{in } L_1(\mathbb{R}). \tag{3.4}$$

Conversely, if X is integrable and the limit (3.4) *holds with* μ_i, $i = 1, 2$, *as in* (3.3) *and with constants* a_n *regularly varying with exponent* 1 *at infinity, then* $X \in DA_1(a_n)$ *and the limit* (3.1) *holds.*

PROOF. As in the previous section, there is no loss of generality in assuming $X \geq 0$ and nondegenerate. In this case, the processes $Z_{n,i}$ are as in the proof of Theorem 2.3. Proceeding as in the proof of that theorem, we see that the triangular array $\{Z_{n,i}\}$ satisfies conditions (i) and (ii) of the Poisson convergence criterion in L_1. [However, it does not satisfy (iii); while there is pointwise convergence of $\mathbb{E}\sum_{i=1}^n Z_{n,i} I_{\|Z_{n,i}\|_{L_1}>\delta}$, convergence in L_1 fails because it involves the nonintegrable functions c/t.] Moreover, the measure $\mu = \mu_1$ gives mass zero to all the sets $\{f\colon \|f\|_{L_1} = \delta\}$. Hence, by the Poisson convergence criterion, second part, the limit (3.4) is satisfied. The converse is also as in Theorem 2.3. □

THEOREM 3.2. *Let X, X_i be be i.i.d. integrable random variables in the domain of attraction of a* 1*-stable law, with normalizing constants a_n and parameters c_1, c_2 for the limit law, as in* (3.1). *Set*

$$\beta_n := \int_{a_n}^{\infty} \Pr\{|X| > t\}\, dt. \tag{3.5}$$

Then,

(a)

$$\lim_{n\to\infty} \beta_n = 0, \qquad \lim_{n\to\infty} \frac{n}{a_n} = \infty \quad \textit{and} \quad \lim_{n\to\infty} \frac{n\beta_n}{a_n} = \infty. \tag{3.6}$$

(b) *The sequence*

$$\left\{ \frac{n}{a_n} \left[\int_{-\infty}^{\infty} \left| F_n(t) - F(t) \right| dt - \beta_n \right] \right\}_{n=1}^{\infty} \tag{3.7}$$

converges in distribution to the law of the random variable

$$\sum_{i=1}^{2} c_i \left[\int_{c_i}^{\infty} |N_i(t) - t| \frac{dt}{t^2} + \int_0^{c_i} (|N_i(t) - t| - t) \frac{dt}{t^2} \right], \tag{3.8}$$

where N_i are two independent standard Poisson processes.

PROOF. Without loss of generality, we restrict the proof to the case $X \geq 0$, which means in particular that $c_1 \neq 0$ and $c_2 = 0$, and take $c = c_1$. Then $\beta_n \to 0$ because $a_n \to \infty$ and X is integrable. Also, since $n \Pr\{X > a_n\} \to c$, if $\liminf_n n/a_n < \infty$, then $t \Pr\{X > t\} \not\to 0$, hence $\mathbb{E}X = \infty$. So, $n/a_n \to \infty$. Finally, since the function $\Pr\{X > t\}$ is regularly varying at infinity with exponent -1, we can apply Theorem 1(a) in Feller [(1971), page 281] with $\gamma = -1$ and $p = 0$ to obtain

$$\lim_{n\to\infty} \frac{a_n \Pr\{X > a_n\}}{\beta_n} = 0,$$

or, since $\{n \Pr\{X > a_n\}\}$ converges to a positive number,

$$\lim_{n\to\infty} \frac{a_n}{n\beta_n} = 0,$$

which completes the proof of (a).

To prove (b) we first observe that, in Theorem 3.1, we can replace the truncated moments $\sum_{i=1}^n \mathbb{E} Z_{n,i} I_{\|Z_{n,i}\|_{L_1}\le 1}$ at the left of (3.4) by the set of centerings C_n defined as

$$C_n(t) := n \Pr\{X > ta_n\} I_{[0,1)}(t), \qquad t \in \mathbb{R},$$

and still have weak convergence [to a shift of the limit in (3.4)], that is,

$$\left\{\sum_{i=1}^n Z_{n,i} - C_n\right\}_{n=1}^{\infty} \text{ converges in law in } L_1[0,\infty). \tag{3.4$'$}$$

This is so because the difference between these centerings and the original ones, $n \Pr\{X \ge a_n\} I_{[0,1)}(t)$, converges in L_1 [to the function $cI_{[0,1)}(t)$]. We can write

$$\begin{aligned}
&\frac{n}{a_n}\left[\int_0^\infty \left|F_n(t) - F(t)\right| dt - \beta_n\right] \\
&\quad = \int_0^1 \left|\sum_{i=1}^n (I_{X_i > ta_n} - \Pr\{X > ta_n\})\right| dt \\
&\qquad + \int_1^\infty \left(\left|\sum_{i=1}^n (I_{X_i > ta_n} - \Pr\{X > ta_n\})\right| - n \Pr\{X > ta_n\}\right) dt \\
&\quad = \int_0^\infty \Bigg[\left|\sum_{i=1}^n Z_{n,i}(t) - C_n(t)\right| I_{[0,1)}(t) \\
&\qquad\qquad + \left(\left|\sum_{i=1}^n (Z_{n,i}(t) - \Pr\{X > ta_n\})\right| - n \Pr\{X > ta_n\}\right) I_{[1,\infty)}(t)\Bigg] dt \\
&\quad := \int_0^\infty (Y_n^{(1)}(t) + Y_n^{(2)}(t))\, dt.
\end{aligned}$$

By the multidimensional version of the law of rare events, the finite-dimensional distributions of the processes $Y_n^{(1)}(t) + Y_n^{(2)}(t)$ converge in law to the corresponding ones of the process

$$Y(t) = \left|N\left(\frac{c}{t}\right) - \frac{c}{t}\right| I_{(0,1)}(t) + \left(\left|N\left(\frac{c}{t}\right) - \frac{c}{t}\right| - \frac{c}{t}\right) I_{[1,\infty)}(t), \qquad t \ge 0,$$

where N is the standard Poisson process on $[0,\infty)$. Hence, by Lawniczak's (1983) proposition, already mentioned above, in order to prove that the variables in (3.7) converge in law to the variable in (3.8), it suffices to show that the processes $Y_n^{(1)} + Y_n^{(2)}$, $n \in \mathbb{N}$, are uniformly tight in $L_1[0,\infty)$. [As, if this is the case, we have convergence in law in L_1 of $Y_n^{(1)} + Y_n^{(2)}$ to Y and the continuous mapping theorem yields the result.] The proof thus reduces to showing that each of the two sequences $\{Y_n^{(i)}\}$, $i = 1, 2$, is uniformly tight.

Since the map $\pi_A(f) := |f|I_A$ of L_1 into $L_1[0,\infty)$ is continuous for any measurable set $A \subset [0,\infty)$, it follows from (3.4′) that the sequence $Y_n^{(1)} = |\sum_{i=1}^n Z_{n,i} - C_n|I_{[0,1)}$, $n \in \mathbb{N}$, converges in law in L_1, hence is uniformly tight in L_1.

The fact that c/t is not integrable complicates the issue of tightness for the second sequence, but it can be dealt with by approximation due to the cancellation occurring in the limit; $(|N(u) - u| - u)/u^2$ is integrable at zero because $N(u) = 0$ for an a.s. positive amount of time immediately following zero. So, we set

$$Y_{M,n} := Y_n^{(2)} I_{[1,M)}, \qquad M > 1. \tag{3.9}$$

Continuity of the map $\pi_{[1,M)}$ plus the fact that

$$n \Pr\{X > ta_n\} I_{[1,M)}(t) \to c I_{[1,M)}(t)/t \quad \text{in } L_1[0,\infty)$$

as $n \to \infty$ imply, by Theorem 3.1 and the continuous mapping theorem, that

$$Y_{M,n} \to_{\mathscr{L}} Y_M \quad \text{in } L_1[0,\infty) \tag{3.10}$$

as $n \to \infty$, where

$$Y_M(t) := \left(\left|N\left(\frac{c}{t}\right) - \frac{c}{t}\right| - \frac{c}{t}\right) I_{[1,M)}(t).$$

[The limit Y_M is determined by the law of rare events via Lawniczak's proposition, whereas the convergence in law of $Y_{M,n}$ follows from convergence in law in $L_1 \times L_1$ of the random vectors

$$\left(\left(\sum_{i=1}^n Z_{n,i} - C_n\right) I_{[1,M)}, n \Pr\{X > ta_n\} I_{[1,M)}\right)$$

together with the continuous mapping theorem.] Setting

$$Y^{(2)}(t) := \left(\left|N\left(\frac{c}{t}\right) - \frac{c}{t}\right| - \frac{c}{t}\right) I_{[1,\infty)}(t), \qquad t > 0,$$

we have

$$\begin{aligned}\int_0^\infty \left|Y^{(2)}(t) - Y_M(t)\right| dt &= \int_M^\infty \left|\left|N\left(\frac{c}{t}\right) - \frac{c}{t}\right| - \frac{c}{t}\right| dt \\ &= c\int_0^{c/M} \big||N(t) - t| - t\big| \frac{dt}{t^2} \to 0 \quad \text{a.s.}\end{aligned}$$

as $M \to \infty$ because $\max_{0\le t\le c} N(t) = N(c) < \infty$ a.s. and $N(t) = 0$ on $[0, t_1)$ with $t_1 > 0$ a.s. That is, $\|Y^{(2)} - Y_M\|_{L_1} \to 0$ a.s. In particular,

$$Y_M \to_{\mathscr{L}} Y^{(2)} \quad \text{in } L_1[0,\infty). \tag{3.11}$$

Because of (3.10) and (3.11), a typical 3ε approximation argument, concretely Theorem 4.2 in Billingsley [(1968), page 25] gives

$$Y_n^{(2)} \to_{\mathscr{L}} Y^{(2)} \quad \text{in } L_1[0,\infty) \tag{3.12}$$

provided we show that

$$\lim_{M\to\infty} \limsup_{n\to\infty} \Pr\bigl\{\|Y_{M,n} - Y_n^{(2)}\|_{L_1} \ge \varepsilon\bigr\} = 0 \tag{3.13}$$

for all $\varepsilon > 0$. Now, for $M > 1$, and letting $\tilde{Y}$ be a process whose law is the limit law of $\sum_{i=1}^n (Z_{n,i} - \mathbb{E}Z_{n,i} I_{\|Z_{n,i}\|_{L_1}\le 1})$ (Theorem 3.1), we have

$$\begin{aligned}
&\int_0^\infty \bigl|Y_{M,n} - Y_n^{(2)}\bigr|\,dt \\
&\quad = \int_M^\infty \left|\left|\sum_{i=1}^n (Z_{n,i}(t) - \Pr\{X > ta_n\})\right| - n\Pr\{X > ta_n\}\right| dt \\
&\quad \le \int_M^\infty \sum_{i=1}^n Z_{n,i}(t)\,dt \\
&\quad = \int_0^\infty \left|\sum_{i=1}^n (Z_{n,i} - \mathbb{E}Z_{n,i} I_{\|Z_{n,i}\|_{L_1}\le 1})\right| I_{[M,\infty)}(t)\,dt \\
&\quad \to_{\mathscr{L}} \int_M^\infty |\tilde{Y}(t)|\,dt,
\end{aligned}$$

where in the first inequality we use that $||a-b|-b| \le a$ for $a, b \ge 0$, and the limit follows from Theorem 3.1 and continuous mapping. Hence,

$$\limsup_{n\to\infty} \Pr\bigl\{\|Y_{M,n} - Y_n^{(2)}\|_{L_1} \ge \varepsilon\bigr\} \le \Pr\left\{\int_M^\infty |\tilde{Y}(t)|\,dt \ge \varepsilon\right\}.$$

By dominated convergence, $\int_M^\infty |\tilde{Y}(t)|\,dt \to 0$ a.s. as $M \to \infty$, and this proves (3.13), hence the theorem. □

For example, if $\Pr\{X > t\} \asymp 1/[t(\log t)^{1+\delta}]$, then $n/a_n \asymp (\log n)^{1+\delta}$ and $\beta_n \asymp 1/[\delta(\log n)^\delta]$.

Weak convergence of (normalized, centered versions of) $\|F_n - F\|_{L_1}$ in the case of variables in domains of attraction of 1-stable laws does not seem to have been treated before in the literature. Theorem 3.2 completes the picture on the asymptotic behavior of the Wasserstein distance between the empirical and the true distributions from the side of weakest integrability.

As in the case of domains of attraction of stable laws with index $\alpha \in (1,2)$, we also have convergence of moments in Theorem 3.2.

THEOREM 3.3. *Let X, X_i, $i \in \mathbb{N}$, be i.i.d. integrable random variables in $DA_1(a_n)$, that is, satisfying the limit* (3.1) *for some $c_1, c_2 \ge 0$. Let β_n be as in*

Theorem 3.2. *Then*,

$$\begin{aligned}(3.14)\qquad &\lim_{n\to\infty}\mathbb{E}\left[\frac{n}{a_n}\left(\int_{-\infty}^{\infty}|F_n(t)-F(t)|\,dt-\beta_n\right)\right]^\beta\\ &\quad=\mathbb{E}\left[\sum_{i=1}^{2}c_i\left(\int_{c_i}^{\infty}|N_i(t)-t|\frac{dt}{t^2}+\int_0^{c_i}(|N_i(t)-t|-t)\frac{dt}{t^2}\right)\right]^\beta\end{aligned}$$

for all $0<\beta<1$.

PROOF. Proceeding as in the proof of Theorem 2.4 and assuming (w.l.o.g.) that $X\ge 0$, it suffices to show that

$$\sup_n\mathbb{E}\left[\frac{1}{a_n}\int_0^{a_n}\left|\sum_{i=1}^n(I_{X_i>t}-\Pr\{X>t\})\right|dt\right]^\beta<\infty \tag{3.15}$$

and

$$\sup_n\mathbb{E}\left|\frac{1}{a_n}\int_{a_n}^{\infty}\left(\left|\sum_{i=1}^n(I_{X_i>t}-\Pr\{X>t\})\right|-n\Pr\{X>t\}\right)dt\right|^\beta<\infty. \tag{3.16}$$

It is easy to check that (3.15) holds, exactly the way boundedness of $\{A_n\}$ is established in Theorem 2.4:

$$\frac{1}{a_n}\int_0^{a_n}\mathbb{E}\left|\sum_{i=1}^n(I_{X_i>t}-\Pr\{X>t\})\right|dt\le\frac{1}{a_n}\int_0^{a_n}(n\Pr\{X>t\})^{1/2}\,dt\to 2\sqrt{c}.$$

To prove (3.16) we just note that, using that $(\sum a_i)^\beta\le\sum a_i^\beta$ for $a_i\ge 0$ and $0<\beta<1$, regular variation and that $n\Pr\{X>a_n\}\to c$,

$$\begin{aligned}&\mathbb{E}\left|\frac{1}{a_n}\int_{a_n}^{\infty}\left(\left|\sum_{i=1}^n(I_{X_i>t}-\Pr\{X>t\})\right|-n\Pr\{X>t\}\right)dt\right|^\beta\\ &\quad\le\mathbb{E}\left|\frac{1}{a_n}\int_{a_n}^{\infty}\left|\left|n\Pr\{X>t\}-\sum_{i=1}^n I_{X_i>t}\right|-n\Pr\{X>t\}\right|dt\right|^\beta\\ &\quad\le\mathbb{E}\left|\frac{1}{a_n}\int_{a_n}^{\infty}\sum_{i=1}^n I_{X_i>t}\,dt\right|^\beta=\mathbb{E}\left(\frac{1}{a_n}\sum_{i=1}^n(X_i-a_n)^+\right)^\beta\\ &\quad\le\mathbb{E}\left(\frac{1}{a_n}\sum_{i=1}^n X_iI_{X_i>a_n}\right)^\beta\le\frac{n}{a_n^\beta}\mathbb{E}X^\beta I_{X>a_n}\\ &\quad=\beta n\Pr\{X>a_n\}+\frac{\beta n}{a_n^\beta}\int_{a_n}^{\infty}t^{\beta-1}\Pr\{X>t\}\,dt\\ &\quad\asymp\left(1+\frac{1}{1-\beta}\right)\beta n\Pr\{X>a_n\}\asymp\left(1+\frac{1}{1-\beta}\right)\beta c.\end{aligned}$$

□

4. The domain of attraction of the normal law I: rates. It will be shown that, for X in the domain of attraction of the normal law, with normalizing constants b_n, the Wasserstein distances between the empirical and the true distributions, centered at expected values and normalized by b_n/n are indeed stochastically bounded; actually, that their pth moments are uniformly bounded for $0 < p < 2$. Stochastic boundedness is proved by applying a recent exponential inequality of Talagrand (1996). The sizes of the centering and norming are then compared and this gives, as corollaries, both, a law of large numbers for the uncentered Wasserstein distance and a variation on Theorem 2.1(b).

Let X, X_i, $i \in \mathbb{N}$, be i.i.d., with $X \in DA_2(b_n)$. We recall from the theory of domains of attraction [Feller (1971) or Araujo and Giné (1980)] that, setting $U(t) = \mathbb{E}X^2 I_{|X|\le t}$, the function $U(t)$ is slowly varying, that, equivalently,

$$\lim_{t\to\infty} \frac{t^2 \Pr\{|X| \ge t\}}{U(t)} = 0 \tag{4.1}$$

and that, if

$$\lim_{n\to\infty} \frac{n}{b_n^2} U(b_n) = 1 \tag{4.2}$$

and $\mathbb{E}X^2 = \infty$, then $\sum_{i=1}^n (X_i - \mathbb{E}X_i)/b_n$ is asymptotically standard normal; in fact we can take the sequence in (4.2) to be constantly 1. In what follows we will take b_n satisfying (4.2) if $\mathbb{E}X^2 = \infty$ and $b_n = \sqrt{n}$ if $\mathbb{E}X^2 < \infty$. We also set

$$Q(y) = \inf[t\colon\ \Pr\{X \le t\} \ge y], \qquad y \in (0,1), \tag{4.3}$$

to be the quantile function of X, as in the introduction, and recall that, if F is the distribution function of X,

$$F(Q(y)-) \le y \le F(Q(y)), \qquad y \in (0,1) \tag{4.4}$$

and

$$Q(y) \le x \text{ if and only if } y \le F(x), \qquad y \in (0,1),\ x \in \mathbb{R}. \tag{4.5}$$

The following lemma allows for truncation of the domain of integration in $nd_1(F_n, F)/b_n$. Equation (4.6) was obtained by CCHM (1986b), (A.5), (A.20), using a particular form of b_n.

LEMMA 4.1. *If $X \in DA_2(b_n)$, then*

$$\lim_{n\to\infty} \frac{|Q(1-1/n)| \vee |Q(1/n)|}{b_n} = 0 \tag{4.6}$$

[CCHM (1986b)] and

$$\lim_{n\to\infty} \mathbb{E}\left[\frac{n}{b_n}\int_{(Q(1/n),\,Q(1-1/n))^c} |F_n(t) - F(t)|\,dt\right]^r = 0 \tag{4.7}$$

for all $0 < r < 2$.

PROOF. Since $n\Pr\{|X|\ge b_n\}\to 0$ [by (4.1) and (4.2)] whereas $n\Pr\{|X|\ge |Q(1-1/n)|\vee|Q(1/n)|\}\ge 1$ [by (4.4)], it follows that eventually $b_n>|Q(1-1/n)|\vee|Q(1/n)|$. The limit (4.6) is obvious if $\mathbb{E}X^2<\infty$: if X is bounded then the numerator is bounded and the denominator is $\sqrt{n}$, whereas if, for example, X^+ is unbounded then $y_n:=Q(1-1/n)\to\infty$ and

$$\frac{y_n^2}{n}\le y_n^2\Pr\{X\ge y_n\}\to 0$$

and likewise for $Q(1/n)$ if X^- is unbounded. If $\mathbb{E}X^2=\infty$, also, for example, assuming $y_n=Q(1-1/n)\to\infty$, the limits (4.1) and (4.2) together with the previous observation give

$$\frac{b_n^2}{y_n^2}\asymp\frac{nU(b_n)}{y_n^2}\ge\frac{U(b_n)}{y_n^2\Pr\{X\ge y_n\}}\ge\frac{U(y_n)}{y_n^2\Pr\{X\ge y_n\}}\to\infty,$$

proving (4.6).

The limit (4.7) is equivalent to

$$\begin{aligned}&\lim_{n\to\infty}\mathbb{E}\left[\frac{1}{b_n}\int_{Q(1-1/n)}^{\infty}\left|\sum_{i=1}^n(I_{X_i>t}-\Pr\{X>t\})\right|dt\right]^r\\ (4.7')\qquad &=\lim_{n\to\infty}\mathbb{E}\left[\frac{1}{b_n}\int_{-\infty}^{Q(1/n)}\left|\sum_{i=1}^n(I_{X_i\le t}-\Pr\{X\le t\})\right|dt\right]^r=0.\end{aligned}$$

Since both limits in (4.7′) have similar proofs, we only prove one of them and assume $1\le r<2$. By (4.5), if $t\ge Q(1-1/n)$ then $n\Pr\{X>t\}\le 1$ so that we can apply inequality (2.37) to $\sum_{i=1}^n(I_{X_i>t}-\Pr\{X>t\})$ for t in this range. This and the generalized Minkowswki inequality give

$$\begin{aligned}&\left(\mathbb{E}\left[\frac{1}{b_n}\int_{Q(1-1/n)}^{\infty}\left|\sum_{i=1}^n(I_{X_i>t}-\Pr\{X>t\})\right|dt\right]^r\right)^{1/r}\\ (4.8)\qquad &\le\frac{1}{b_n}\int_{Q(1-1/n)}^{\infty}\left(\mathbb{E}\left|\sum_{i=1}^n(I_{X_i>t}-\Pr\{X>t\})\right|^r\right)^{1/r}dt\\ &\le\frac{C(1,r)^{1/r}}{b_n}\int_{Q(1-1/n)}^{\infty}(n\Pr\{X>t\})^{1/r}\,dt.\end{aligned}$$

Now we decompose these last integrals into two parts, from $Q(1-1/n)$ to b_n and from b_n to ∞. For the first, since $n\Pr\{X>Q(1-1/n)\}\le 1$ and $n\Pr\{X>b_nt\}\to 0$ as $n\to\infty$ by (4.1), (4.2) and regular variation of $U(t)$, it follows by bounded convergence that

$$(4.9)\qquad \frac{1}{b_n}\int_{Q(1-1/n)}^{b_n}(n\Pr\{X>t\})^{1/r}\,dt=\int_{Q(1-1/n)/b_n}^{1}(n\Pr\{X>b_nt\})^{1/r}\,dt\to 0$$

as $n\to\infty$. For the second integral, setting $\delta_n=\sup_{t\ge b_n}t^2\Pr\{X\ge t\}/U(t)$, which tends to zero as $n\to\infty$ by (4.1), and using regular variation [Feller

(1971), Theorem 1(a) applied to $U(t)$] and the limit (4.2), we obtain

$$\begin{aligned}
\frac{1}{b_n}\int_{b_n}^{\infty}(n\Pr\{X>t\})^{1/r}\,dt &= \frac{b_n^{2/r-1}\int_{b_n}^{\infty}(\Pr\{X>t\})^{1/r}\,dt}{(b_n^2/n)^{1/r}} \\
&\asymp \frac{b_n^{2/r-1}\int_{b_n}^{\infty}(\Pr\{X>t\})^{1/r}\,dt}{(U(b_n))^{1/r}} \\
&\le \frac{\delta_n^{1/r}b_n^{2/r-1}}{(U(b_n))^{1/r}}\int_{b_n}^{\infty}\left(\frac{U(t)}{t^2}\right)^{1/r}dt \\
&\asymp \frac{1}{2/r-1}\delta_n^{1/r}\to 0.
\end{aligned} \tag{4.10}$$

Now, combining (4.8)–(4.10), we obtain that the first limit in (4.7′) is zero. □

As indicated above, the main result of this section will be proved by application of an inequality of Talagrand (1996). For ease of reference we state it.

TALAGRAND'S EXPONENTIAL INEQUALITY. Let $\mathbf{X}_i$ be independent random variables with values in a measurable space $(S,\mathscr{S})$, let $\mathscr{F}$ be a countable class of measurable functions on S and let

$$Z := \sup_{f\in\mathscr{F}}\sum_{i=1}^{n} f(\mathbf{X}_i). \tag{4.11}$$

Let

$$U := \sup_{f\in\mathscr{F}}\|f\|_\infty \quad\text{and}\quad V := \mathbb{E}\left[\sup_{f\in\mathscr{F}}\sum_{i=1}^{n} f^2(\mathbf{X}_i)\right]. \tag{4.12}$$

Then there exists a universal constant (independent of $\mathbf{X}_i$, n and S) such that

$$\Pr\{|Z-\mathbb{E}Z|\ge t\}\le K\exp\left(-\frac{1}{K}\frac{t}{U}\log\left(1+\frac{tU}{V}\right)\right). \tag{4.13}$$

THEOREM 4.2. *Let $X\in DA_2(b_n)$ and set*

$$Z_n := n\int_{-\infty}^{\infty}\left|F_n(t)-F(t)\right|dt,\qquad n\in\mathbb{N}. \tag{4.14}$$

Then,

$$\sup_n \mathbb{E}\left|\frac{Z_n-\mathbb{E}Z_n}{b_n}\right|^r<\infty \tag{4.15}$$

for all $0<r<2$. (In particular, the sequence $\{(Z_n-\mathbb{E}Z_n)/b_n\}_{n=1}^{\infty}$ is stochastically bounded.)

PROOF. We can decompose each variable Z_n into the sum of two, one for which the integral runs from 0 to ∞ and the other with limits $-\infty$ and 0. Each of the two resulting sequences can be treated in the same way, hence we will only consider the first. We may as well assume $Q(1-1/n) > 0$. Lemma 4.1 allows reduction of the domain of integration of theses variables to $(0, Q(1-1/n))$ [resp. $(Q(1/n), 0)$]. Hence, defining

$$\tilde{Z}_n := \int_0^{Q(1-1/n)} \left| \sum_{i=1}^{n} (I_{X_i > t} - \Pr\{X > t\}) \right| dt, \tag{4.14$'$}$$

where $\{X_i\}$ is as usual an i.i.d. sequence with $\mathscr{L}(X_i) = \mathscr{L}(X)$, we must show that the rth moments of the variables in the sequence

$$\left\{ \frac{\tilde{Z}_n - \mathbb{E}\tilde{Z}_n}{b_n} \right\}_{n=1}^{\infty} \tag{4.15$'$}$$

are uniformly bounded. For ease of notation we set $Q_n = Q(1-1/n)$. To apply Talagrand's inequality, we define

$$S := S_n := \{ h_x \in L_1(\mathbb{R}) \colon x > 0,\ h_x(t) := (I_{x>t} - \Pr\{X > t\}) I_{0<t<Q_n} \},$$

$\mathscr{S}$ as the restriction of the Borel sigma algebra of $L_1(\mathbb{R})$ to S, and

$$\mathbf{X}_i(t) := h_{X_i}(t) = (I_{X_i > t} - \Pr\{X > t\}) I_{0<t<Q_n}.$$

Since $L_1(\mathbb{R})$ is separable, there is a countable family $\mathscr{F}$ of elements of the unit ball of $L_\infty(\mathbb{R})$ such that

$$\|h\|_{L_1} = \sup_{f \in \mathscr{F}} \langle f, h \rangle = \sup_{f \in \mathscr{F}} |\langle f, h \rangle| = \sup_{f \in \mathscr{F}} \left| \int f(t) h(t)\, dt \right|$$

for all $h \in L_1$. We take $\mathscr{F}$ in Talagrand's theorem to be this class, more exactly, the functions that act on $h \in L_1$ as $h \to \langle f, h \rangle$, where $f \in \mathscr{F}$ (since these maps are continuous in L_1, they are Borel measurable). Note that $\operatorname{ess\,sup} |f| \le 1$ for all $f \in \mathscr{F}$. Next we estimate U and V from (4.12) in our case,

$$\begin{aligned} U &= \sup_{f \in \mathscr{F}} \|f\|_\infty = \sup_{f \in \mathscr{F}} \sup_{h_x \in S_n} |\langle f, h_x \rangle| \\ &= \sup_{f \in \mathscr{F}} \sup_{x>0} \left| \int_0^{Q_n \wedge x} f(t)\, dt - \int_0^{Q_n} f(t) \Pr\{X > t\}\, dt \right| \\ &= \sup_{f \in \mathscr{F}} \sup_{x>0} \left| \int_0^{Q_n \wedge x} f(t)(1 - \Pr\{X > t\})\, dt - \int_{Q_n \wedge x}^{Q_n} f(t) \Pr\{X > t\}\, dt \right| \\ &\le Q_n \end{aligned} \tag{4.16}$$

and

$$
\begin{aligned}
V &= \mathbb{E}\Big[\sup_{f\in\mathscr{F}}\sum_{i=1}^{n}\langle f,\mathbf{X}_i\rangle^2\Big]\\
&= \mathbb{E}\Big[\sup_{f\in\mathscr{F}}\sum_{i=1}^{n}\Big(\int_0^{(X_i\vee 0)\wedge Q_n} f(t)\,dt-\int_0^{Q_n} f(t)\Pr\{X>t\}\,dt\Big)^2\Big]\\
&\le 2\mathbb{E}\Big[\sup_{f\in\mathscr{F}}\sum_{i=1}^{n}\Big(\int_0^{(X_i\vee 0)\wedge Q_n} f(t)\,dt\Big)^2\Big] \qquad (4.17)\\
&\qquad + 2\sup_{f\in\mathscr{F}}\sum_{i=1}^{n}\Big(\int_0^{Q_n} f(t)\Pr\{X>t\}\,dt\Big)^2\\
&\le 2n\mathbb{E}[(X\vee 0)\wedge Q_n]^2+2n[\mathbb{E}((X\vee 0)\wedge Q_n)]^2\\
&\le 4n\mathbb{E}(|X|\wedge b_n)^2\le 5b_n^2
\end{aligned}
$$

for all n large enough [recall (4.1), (4.2) and (4.6)]. Taking into account that the function $-u^{-1}\log(1+u/c^2)$ is nondecreasing for $u>0$, plugging the estimates (4.16) and (4.17) into Talagrand's inequality (4.13) gives

$$
\begin{aligned}
\Pr\big\{\big|\tilde{Z}_n-\mathbb{E}\tilde{Z}_n\big|>t\big\} &\le K\exp\Big(-\frac{1}{K}\frac{t}{U}\log\Big(1+\frac{tU}{5b_n^2}\Big)\Big)\\
&\le K\exp\Big(-\frac{1}{K}\frac{t}{Q_n}\log\Big(1+\frac{tQ_n}{5b_n^2}\Big)\Big)
\end{aligned}
$$

for all n satisfying (4.17). Then, if n_0 is such that (4.17) and $0<Q_n\le b_n$ both hold for all $n\ge n_0$ [recall (4.6)], we have

$$
\sup_{n\ge n_0}\Pr\big\{\big|\tilde{Z}_n-\mathbb{E}\tilde{Z}_n\big|>tb_n\big\}\le K\exp\Big(-\frac{1}{K}t\log\Big(1+\frac{t}{5}\Big)\Big).
$$

This implies that $\sup_{n\ge n_0}\mathbb{E}|(\tilde{Z}_n-\mathbb{E}\tilde{Z}_n)/b_n|^r<\infty$ for all $r>0$. Since the essential supremum of the variable $\max_{n\le n_0}|\tilde{Z}_n-\mathbb{E}\tilde{Z}_n|$ is finite, this proves the theorem. □

REMARK 4.1 (The order of magnitude of $\mathbb{E}\tilde{Z}_n$). If Z is a binomial (n,p) variable, then Hölder's inequality gives

$$
\mathbb{E}|Z-\mathbb{E}Z|\le\sqrt{np(1-p)},
$$

which is of the right order for $np(1-p)$ large. Therefore, assuming, as in the previous proof, that $X\ge 0$, we have

$$
0\le\mathbb{E}\tilde{Z}_n\le\sqrt{n}\int_0^{Q(1-1/n)}\sqrt{\Pr\{X>t\}}\,dt. \tag{4.18}
$$

In particular, if $X\in L_{2,1}$, the centering in (4.15′) satisfies

$$
0\le\mathbb{E}\tilde{Z}_n\le\Lambda_{2,1}(X)\sqrt{n}.
$$

Hence, the centering in (4.15) is not needed in the $L_{2,1}$ case, and the above theorem recovers the sufficiency part of Theorem 2.1(b). Next we bound $\mathbb{E}\tilde{Z}_n$ from below. The case $X \in L_{2,1}$ has already been considered in Section 2. By Theorem 2.4 and Lemma 4.1, if $X \in L_{2,1}$ then

$$\lim_{n\to\infty} \frac{\mathbb{E}\tilde{Z}_n}{\sqrt{n}} = \lim_{n\to\infty} \frac{\mathbb{E}Z_n}{\sqrt{n}} = \int_{-\infty}^{\infty} \mathbb{E}|B(F(t))|\,dt = \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} \sqrt{F(t)(1-F(t))}\,dt.$$

So, we can assume $\Lambda_{2,1}(X) = \infty$, and, without loss of generality, $X \geq 0$. The binomial estimates (2.11) and (2.15) give

$$\begin{aligned}
\mathbb{E}\tilde{Z}_n &\geq \int_{\mathrm{med}(X)}^{Q(1-1/n)} \mathbb{E}\left|\sum_{i=1}^{n}(I_{X_i>t} - \Pr\{X>t\})\right| dt \\
&\geq C_2\sqrt{n} \int_{\mathrm{med}(X)}^{Q(1-C_1/n)} \sqrt{\Pr\{X>t\}}\,dt + \int_{Q(1-C_1/n)}^{Q(1-1/n)} C\,dt \\
&\geq (C_2 + C/\sqrt{C_1})\sqrt{n} \int_{\mathrm{med}(X)}^{Q(1-1/n)} \sqrt{\Pr\{X>t\}}\,dt.
\end{aligned}$$

If $\Lambda_{2,1}(X) = \infty$, $\int_{\mathrm{med}(X)}^{Q(1-1/n)} \sqrt{\Pr\{X>t\}}\,dt / \int_0^{\mathrm{med}(X)} \sqrt{\Pr\{X>t\}}\,dt \to \infty$ and therefore, for all n large enough,

$$\mathbb{E}\tilde{Z}_n \geq \tfrac{1}{2}(C_2 + C/\sqrt{C_1})\sqrt{n} \int_{\mathrm{med}(X)}^{Q(1-1/n)} \sqrt{\Pr\{X>t\}}\,dt.$$

This and (4.18) give

$$\mathbb{E}\tilde{Z}_n \simeq \sqrt{n} \int_0^{Q(1-1/n)} \sqrt{\Pr\{X>t\}}\,dt. \tag{4.19}$$

Here the exact meaning of $\simeq$ is that the quotient of the two quantities becomes bounded and bounded away from 0 as $n \to \infty$, with bounds independent from X. In connection with (4.19), we should mention that this estimate is improved to the limit (5.18) in Corollary 5.4 below, in particular showing that the centering $\mathbb{E}Z_n$ in (4.15), Theorem 4.2, can be replaced by the easier to compute

$$\gamma_n = \sqrt{\frac{2n}{\pi}} \int_{Q(1/n)}^{Q(1-1/n)} \sqrt{F(t)(1-F(t))}\,dt. \tag{4.20}$$

Theorem 4.2 determines the size in probability of the centered Wasserstein distance for $X \in DA_2 \setminus L_{2,1}$ ($L_{2,1}$ is considered in Theorem 2.1); in order to retrieve information from Theorem 4.2 about the size of the uncentered Z_n as well, we must compare the sizes of the centerings $\mathbb{E}Z_n$ and the normings b_n occurring in (4.15) for all $X \in DA_2 \setminus L_{2,1}$, and this is done in the next proposition. The norming constants b_n admit several forms besides the implicit one in (4.2), as shown in CCHM (1986b); in particular, b_n can be taken to be

$$b_n = \sqrt{n}\left(\int_{1/n}^{1-1/n} Q^2(t)\,dt\right)^{1/2} \tag{4.21}$$

if $\mathbb{E}X^2 = \infty$. [See the Appendix for an alternative proof of (4.21)]. Recall that we take $b_n = \sqrt{n}$ if $\mathbb{E}X^2 < \infty$.

PROPOSITION 4.3. *Let $X \in DA_2(b_n) \setminus L_{2,1}$ and let*

$$Z_n = n\int_{-\infty}^{\infty} |F_n(t) - F(t)|\,dt, \qquad n \in \mathbb{N},$$

as in (4.14). *Then,*

$$\lim_{n\to\infty} \frac{\mathbb{E}Z_n}{b_n} = \infty. \tag{4.22}$$

PROOF. Almost as in (4.14), we set

$$\tilde{Z}_n := n\int_{Q(1/n)}^{Q(1-1/n)} |F_n(t) - F(t)|\,dt.$$

A slight modification of the arguments in Remark 4.1 shows that

$$\mathbb{E}\tilde{Z}_n \simeq \sqrt{n}\int_{Q(1/n)}^{Q(1-1/n)} \sqrt{F(t)(1-F(t))}\,dt = \sqrt{n}\int_{1/n}^{1-1/n} \sqrt{t(1-t)}\,dQ(t). \tag{4.19$'$}$$

In particular, if $\mathbb{E}X^2 < \infty$ and $\Lambda_{2,1}(X) = \infty$, then $\mathbb{E}Z_n/\sqrt{n} \to \infty$. So, we can assume from now on that $\mathbb{E}X^2 = \infty$. Then, by (4.21) and (4.19$'$), (4.22) reduces to showing

$$\lim_{n\to\infty} \frac{\int_{1/n}^{1-1/n} \sqrt{t(1-t)}\,dQ(t)}{(\int_{1/n}^{1-1/n} Q^2(t)\,dt)^{1/2}} = \infty. \tag{4.23}$$

Integration by parts yields

$$\int_{1/n}^{1-1/n} \sqrt{t(1-t)}\,dQ(t) = \sqrt{\frac{1}{n}\left(1-\frac{1}{n}\right)}\left(Q\left(1-\frac{1}{n}\right) - Q\left(\frac{1}{n}\right)\right) - \int_{1/n}^{1-1/n} \frac{(\frac{1}{2}-t)Q(t)}{\sqrt{t(1-t)}}\,dt.$$

By (4.6) in Lemma 4.1, the first summand at the right of this identity is of a smaller order than b_n and, therefore, it suffices to prove

$$\lim_{n\to\infty} \frac{\int_{1/n}^{1-1/n} [t(1-t)]^{-1/2}|Q(t)|\,dt}{(\int_{1/n}^{1-1/n} Q^2(t)\,dt)^{1/2}} = \infty. \tag{4.24}$$

Set

$$f(x) := \int_x^{1-x} |Q(t)|[t(1-t)]^{-1/2}\,dt$$

and

$$g(x) := \left(\int_x^{1-x} Q^2(t)\,dt\right)^{1/2}, \qquad 0 < x < 1/2,$$

and note that $\lim_{x\to 0} g(x) = \infty$ (as $\mathbb{E}X^2 = \infty$). Both f and g are absolutely continuous and their a.s. derivatives are, respectively,

$$f'(x) = -\frac{|Q(1-x)| + |Q(x)|}{\sqrt{x(1-x)}} \quad \text{and} \quad g'(x) = -\frac{Q^2(1-x) + Q^2(x)}{2\sqrt{\int_x^{1-x} Q^2(s)\,ds}}.$$

We claim that

$$\lim_{x\to 0} \frac{f'(x)}{g'(x)} = \infty. \tag{4.25}$$

To see this we write

$$\begin{aligned}\frac{\sqrt{1-x}}{2}\frac{f'(x)}{g'(x)} &= \frac{\sqrt{\int_x^{1-x} Q^2(s)\,ds}}{\sqrt{x}(|Q(1-x)| + (Q^2(x)/|Q(1-x)|))} \\ &\quad + \frac{\sqrt{\int_x^{1-x} Q^2(s)\,ds}}{\sqrt{x}((Q^2(1-x)/|Q(x)|) + |Q(x)|)} \\ &:= b_1(x) + b_2(x).\end{aligned}$$

Let $\{x_k\}$ be a sequence of positive numbers converging to zero such that $|Q(1-x_k)|/|Q(x_k)| \to c \in [0,\infty]$ as $k \to \infty$. If $c < \infty$ then, for k large enough,

$$b_2(x_k) \ge \frac{1}{c+1}\frac{\sqrt{\int_{x_k}^{1-x_k} Q^2(s)\,ds}}{\sqrt{x_k}(|Q(1-x_k)| + |Q(x_k)|)} \to \infty$$

as $k \to \infty$ because of (4.1) and (4.4) [see also (4.6)]. Likewise, if $c = \infty$ then $b_1(x_k) \to \infty$, which proves (4.25).

If Q were continuous, f and g would be everywhere differentiable and the result would follow from (4.25) by l'Hôpital's rule. The general case requires a simple extra argument. By absolute continuity of f and g we have

$$\frac{f(x) - f(x_0)}{g(x) - g(x_0)} = \frac{\int_x^{x_0} f'(s)\,ds}{\int_x^{x_0} g'(s)\,ds}. \tag{4.26}$$

For $M > 0$ fixed, we can choose x_0 satisfying $f'(x)/g'(x) > M$ for all $0 < x < x_0$. Now we fix $\varepsilon > 0$ and choose $x_1 < x_0$ such that

$$\frac{f(x_0)}{f(x)} < \varepsilon \quad \text{and} \quad \frac{g(x_0)}{g(x)} < \varepsilon$$

for all $0 < x < x_1$. Then, if $x < x_1$,

$$\frac{f(x) - f(x_0)}{g(x) - g(x_0)} = \frac{f(x)}{g(x)}\frac{(1 - f(x_0)/f(x))}{(1 - g(x_0)/g(x))} = \frac{f(x)}{g(x)}\theta_x$$

for some $\theta_x \in ((1-\varepsilon)/(1+\varepsilon), (1+\varepsilon)/(1-\varepsilon))$. Therefore, by (4.26),

$$f(x)/g(x) > M(1-\varepsilon)/(1+\varepsilon),$$

proving (4.24). □

Combining Theorem 4.2 and Proposition 4.3 yields some interesting consequences. First we will focus on a law of large numbers for Z_n.

COROLLARY 4.4. *Let X, X_i, $i \in \mathbb{N}$, be i.i.d. random variables in $DA_2(b_n) \setminus L_{2,1}$ and let Z_n, $n \in \mathbb{N}$, be as defined in* (4.14). *Then*

$$\frac{Z_n}{\mathbb{E}Z_n} \to 1$$

in probability.

PROOF. By Theorem 4.2 and Proposition 4.3,

$$\frac{b_n}{\mathbb{E}Z_n}\frac{Z_n - \mathbb{E}Z_n}{b_n} \to 0$$

in probability. □

Note that, by Lemma 4.1, the size of $\mathbb{E}Z_n$ is determined up to constants by equation (4.19′), assuming $X \notin L_{2,1}$. Actually, the approximation (4.19′) is asymptotically correct (Corollary 5.4 below).

Theorem 4.2 also implies the following version of Theorem 2.1(b) (here X is restricted to be in DA_2 but we consider more general normings).

COROLLARY 4.5. *Let X, X_i, $i \in \mathbb{N}$, be i.i.d. random variables in $DA_2(b_n)$ and let Z_n, $n \in \mathbb{N}$, be as defined in* (4.14). *Then the sequence $\{Z_n/b_n\}_{n=1}^{\infty}$ is stochastically bounded if and only if $\Lambda_{2,1}(X) < \infty$.*

PROOF. Sufficiency is already contained in Theorem 2.1 and necessity is an immediate consequence of Theorem 4.2 and Proposition 4.3. □

5. The domain of attraction of the normal law II: limit theorems, finite variance case. Theorem 4.2 suggests that if $X \in DA_2(b_n)$ then the centered and normalized Wasserstein distances, $\{(Z_n - \mathbb{E}Z_n)/b_n\}_{n=1}^{\infty}$, converge in law. Such a result would constitute the central limit theorem associated to the law of large numbers in Corollary 4.4 and would basically complete the weak limit theory for the Wasserstein distance between the empirical and the true distributions. Here we prove this CLT in the finite variance case.

An important tool we will use here and in the next section is the Borell–Sudakov–Tsirel'son concentration inequality for Gaussian processes [Sudakov and Tsirel'son (1974), Borell (1975)]. Since we do not need its full power, we state and use the Maurey–Pisier version, which has an elementary proof [Pisier (1986); see also Ledoux and Talagrand (1991), page 57].

GAUSSIAN CONCENTRATION INEQUALITY. Let Z be a centered Gaussian E-valued random vector, E a separable Banach space, and let

$$\sigma^2 = \sup_{f \in D} \mathbb{E}\langle f, Z\rangle^2, \tag{5.1}$$

where D is a countable subset of the unit ball of E' such that $\|x\| = \sup_{f\in D}|\langle f, x\rangle|$ for all $x \in E$. Then, $\mathbb{E}\|Z\| < \infty$ and

$$\Pr\big\{\big|\|Z\| - \mathbb{E}\|Z\|\big| \geq t\big\} \leq 2\exp\left(-\frac{2}{\pi^2}\frac{t^2}{\sigma^2}\right). \tag{5.2}$$

Given a random variable X we let, as usual, F and Q be respectively its cumulative distribution and quantile functions, and we also set, as in (4.14),

$$Z_n = n\|F_n - F\|_{L_1},$$

where F_n, $n \in \mathbb{N}$, are the empirical cumulative distribution functions associated to a sequence of i.i.d. random variables $\{X_i\}$ with the same law as X. As usual, $B(t)$ will denote the Brownian bridge process. Theorem 2.1(a) shows that

$$\frac{Z_n}{\sqrt{n}} \to_d \int_{-\infty}^{\infty}|B(F(t))|\,dt = \int_0^1 |B(t)|\,dQ(t)$$

whenever $X \in L_{2,1}$. Moreover, by moment convergence [Theorem 2.4(a)], $\mathbb{E}Z_n/\sqrt{n} \to \int_0^1 \mathbb{E}|B(t)|\,dQ(t) = \sqrt{2/\pi}\int_0^1\sqrt{t(1-t)}\,dQ(t)$. The following theorem generalizes this to $X \in L_2$, although in this case we do not have separate convergence of $Z_n/\sqrt{n}$ and $\mathbb{E}Z_n/\sqrt{n}$ [see, e.g., Proposition 4.3].

THEOREM 5.1. *If $\mathbb{E}X^2 < \infty$ then, for any sequences $c_n \searrow 0$ and $d_n \nearrow 1$, $c_n \neq 0$, $d_n \neq 1$, the sequence*

$$\int_{c_n}^{d_n}(|B(t)| - \mathbb{E}|B(t)|)\,dQ(t), \qquad n \in \mathbb{N}, \tag{5.3}$$

is Cauchy in L_p for every p and, moreover, if we denote its limit by

$$\int_0^1(|B(t)| - \mathbb{E}|B(t)|)\,dQ(t), \tag{5.4}$$

then

$$\frac{Z_n - \mathbb{E}Z_n}{\sqrt{n}} \to_d \int_0^1(|B(t)| - \mathbb{E}|B(t)|)\,dQ(t) \tag{5.5}$$

as $n \to \infty$, with convergence of moments of order $p < 2$.

PROOF. We can and do assume X is nondegenerate [otherwise, both sides of (5.5) are zero]. First we show that the sequence (5.3) is Cauchy in L_p. To prove this we use Gaussian concentration. Let $n < m$, and let D denote an L_1 norm determining countable subset of the unit ball of $L_\infty([d_n, d_m], dQ)$. Then, the Gaussian concentration inequality applies with $E = L_1([d_n, d_m], dQ)$ and

$Z = \{B(t): d_n \le t \le d_m\}$. The parameter specified by (5.1) in this case is

$$\begin{aligned}\sigma_{n,m}^2 &= \sup_{f\in D} \mathbb{E}\left(\int_{d_n}^{d_m} f(t)B(t)\,dQ(t)\right)^2\\ &= \sup_{f\in D} \int_{d_n}^{d_m}\int_{d_n}^{d_m} f(s)f(t)\mathbb{E}(B(s)B(t))\,dQ(s)\,dQ(t)\\ &= \int_{d_n}^{d_m}\int_{d_n}^{d_m} (s\wedge t - st)\,dQ(s)\,dQ(t).\end{aligned}$$

It is well known that if K is a function of finite total variation on $(0,1)$ and if U is uniform on $(0,1)$, then

$$\mathrm{Var}[K(U)] = \int_0^1\int_0^1 (s\wedge t - st)\,dK(s)\,dK(t)$$

[e.g., Shorack and Wellner (1986), page 43]. Applying this to

$$K(u) = Q(u)I_{[d_n,d_m]}(u) + Q(d_n)I_{u<d_n} + Q(d_m)I_{u>d_m},$$

we obtain from the above that

$$\begin{aligned}\sigma_{n,m}^2 &= \mathrm{Var}[Q(d_n)\vee(X\wedge Q(d_m))]\\ &= \mathrm{Var}[0\vee(X-Q(d_n))\wedge(Q(d_m)-Q(d_n))]\\ &\le 2\,\mathrm{Var}(XI_{X>Q(d_n)}),\end{aligned}$$

where we use that $\mathscr{L}(X) = \mathscr{L}(Q(U))$ in the first identity and that $|\phi(x)-\phi(y)| \le |x-y|$ for all x, y implies $\mathrm{Var}\,\phi(Z) \le 2\,\mathrm{Var}\,Z$ in the inequality. Therefore, the concentration inequality (5.2) gives that, for all $\varepsilon > 0$,

$$\Pr\left\{\left|\int_{d_n}^{d_m}(|B|-\mathbb{E}|B|)\,dQ\right| > \varepsilon\right\} \le 2\exp\left(-\frac{2\varepsilon^2}{\pi^2\sigma_{n,m}^2}\right) \to 0$$

as $n\wedge m \to \infty$. A similar inequality holds for $|\int_{c_m}^{c_n}(|B|-\mathbb{E}|B|)\,dQ|$. Hence, the sequence (5.3) is Cauchy in L_p for all p. Actually, it is Cauchy in some exponential Orlicz norms as well.

Without loss of generality, we will assume $X \ge 0$ for the rest of the proof. Also, by Lemma 4.1, we can replace $Z_n - \mathbb{E}Z_n$ by $\tilde{Z}_n - \mathbb{E}\tilde{Z}_n$ at the left side of the limit (5.5), with $\tilde{Z}_n$ as defined in (4.14′). Next we prove the limit (5.5) with this replacement. We consider the truncated variables $X^{(r)}$, $X_i^{(r)}$, defined by truncation of the X's as follows:

$$X^{(r)} := X \wedge Q(1-1/r),$$

and likewise for $X_i^{(r)}$, and denote $F^{(r)}$, $Q^{(r)}$ and $F_n^{(r)}$, respectively, their common cdf, quantile function and empirical cdf. We also set

$$Z_n^{(r)} = n\|F_n^{(r)} - F^{(r)}\|_{L_1}.$$

As indicated above, by Theorem 2.1(a) and moment convergence [Theorem 2.4(a)], since $X^{(r)} \in L_{2,1}$,

$$\frac{Z_n^{(r)} - \mathbb{E}Z_n^{(r)}}{\sqrt{n}} \to_d \int_0^1 (|B(t)| - \mathbb{E}|B(t)|)\, dQ^{(r)}(t). \tag{5.6}$$

Also, since this last integral is just $\int_0^{1-1/r}(|B(t)|-\mathbb{E}|B(t)|)\,dQ(t)$, the first part of the proof shows that

$$Z^{(r)} := \int_0^1 (|B(t)| - \mathbb{E}|B(t)|)\, dQ^{(r)}(t) \to_d Z := \int_0^1 (|B(t)| - \mathbb{E}|B(t)|)\, dQ(t). \tag{5.7}$$

Hence, by the usual 3ε argument [e.g., Billingsley (1968), page 25], it suffices to prove that

$$\begin{aligned} &\lim_{r\to\infty} \limsup_{n\to\infty} \left| \frac{\tilde{Z}_n - \mathbb{E}\tilde{Z}_n}{\sqrt{n}} - \frac{Z_n^{(r)} - \mathbb{E}Z_n^{(r)}}{\sqrt{n}} \right| \\ &\quad = \lim_{r\to\infty} \limsup_{n\to\infty} \frac{1}{\sqrt{n}} \left| \int_{Q(1-1/r)}^{Q(1-1/n)} \left(\left| \sum_{i=1}^n (I_{X_i>t} - \Pr\{X>t\}) \right| \right.\right. \\ &\qquad \left.\left. - \mathbb{E}\left| \sum_{i=1}^n (I_{X_i>t} - \Pr\{X>t\}) \right| \right) dt \right| = 0 \quad \text{in } L_p \end{aligned} \tag{5.8}$$

for all p. This is achieved by means of Talagrand's exponential inequality, as in the proof of Theorem 4.2. Just as in that proof, (4.16) and (4.17) in the present case become

$$U \le Q(1-1/n) - Q(1-1/r) \quad \text{and} \quad V \le 2n\mathbb{E}X^2 I_{X>Q(1-1/r)}.$$

Using this and that, by Chebyshev's inequality,

$$Q^2(1-1/n) \le n\mathbb{E}X^2 I_{|X|\ge Q(1-1/n)},$$

Talagrand's inequality gives

$$\begin{aligned} &\Pr\left\{ \left| (\tilde{Z}_n - Z_n^{(r)}) - \mathbb{E}(\tilde{Z}_n - Z_n^{(r)}) \right| > u\sqrt{n\mathbb{E}X^2 I_{X\ge Q(1-1/r)}} \right\} \\ &\qquad \le K \exp\left[-\frac{1}{K} u \log\left(1 + \frac{u}{2}\right) \right] \end{aligned}$$

for all $u > 0$. Now (5.8) follows since $\mathbb{E}X^2 I_{X\ge Q(1-1/r)} \to 0$ as $r \to \infty$. $\square$

We should mention that $\mathbb{E}Z_n$, which may be difficult to compute, satisfies

$$\lim_{n\to\infty} \frac{\mathbb{E}Z_n}{\sqrt{n}\sqrt{2/\pi}\int_{1/n}^{1-1/n} \sqrt{t(1-t)}\, dQt} = 1,$$

an estimate more precise than (4.19), (4.19′). This is proved in Corollary 6.3.

6. The domain of attraction of the normal law III: limit theorems, infinite variance case. In this section we prove a central limit theorem for $\|F_n - F\|_{L_1} - \mathbb{E}\|F_n - F\|_{L_1}$ under the assumption that the tail probabilities of X are smooth and of the order $(\log t)^\alpha/t^2$, $-1 \le \alpha < \infty$ [the case $\alpha = 0$ is Theorem 1.1(a2)]. By weighted approximation of empirical processes by Brownian bridges, this will reduce to proving a central limit theorem for

$$\int_{-s}^{s} (|V(t) - \mathbb{E}|V(t)|)|t|^{\alpha/2}\,dt,$$

where V is the Ornstein–Uhlenbeck process and $s \to \infty$. [The case $\alpha = 0$ is known: Mandl (1968), page 95.] Whereas we conjecture that the central limit theorem to be proved in this section holds for any X in the domain of attraction of the normal law and with infinite second moment, our method would only give the CLT for a subset of these (smooth, regularly varying tails); for this reason and for convenience, we restrict ourselves to the simpler tail probabilities $\Pr\{|X| > t\} \asymp (\log t)^\alpha/t^2$, $-1 \le \alpha < \infty$, which, at any rate, basically cover all the tail sizes for DA_2 with infinite variance.

Whereas the proofs of the results in Sections 2 to 5 do not use approximation of the empirical process by Brownian bridges, the results in this section are based on such representations. The value of the normalizing constants is determined by the integrals at the tails and therefore we cannot approximate the L_1 norm of the empirical process by integrals with bounded range, as done in Theorem 5.1. We use Mason's (1991) and Csörgő and Horváth (1986) version of the weighted strong approximations, which has an elementary proof based on Skorokhod embedding [as opposed to the deeper and more difficult Komlós–Major–Tusnády (KMT) strong approximation]. This approximation is as follows, and was originally stated [with a proof based on KMT] in CCHM (1986a).

WEIGHTED APPROXIMATION OF THE EMPIRICAL PROCESS BY BROWNIAN BRIDGES. There exists a sequence of independent uniform (0,1) random variables U_i, $i \in \mathbb{N}$, and a sequence of Brownian bridges B_n, $n \in \mathbb{N}$, sitting on the same probability space, such that, for all $0 \le \nu < 1/4$,

$$\sup_{1/n \le s \le 1-1/n} n^\nu |\alpha_n(s) - B_n(s)|[s(1-s)]^{\nu - 1/2} = O_P(1), \tag{6.1}$$

where

$$H_n(s) := \frac{1}{n}\sum_{i=1}^{n} I_{U_i \le s} \quad \text{and} \quad \alpha_n(s) := \sqrt{n}(H_n(s) - s), \qquad n \in \mathbb{N}, \tag{6.2}$$

are respectively the uniform empirical distribution functions and processes.

We are interested in the following corollary of the weighted approximation. CCHM (1986b) state a formally weaker result [their Theorem 2.1; see also CCHM (1986a), Theorems 3.1 and 3.2], but their methods also give it. We present it with a short proof for the reader's convenience.

COROLLARY 6.1 [Essentially contained in CCHM (1986a, b)]. *With α_n and B_n as in the weighted approximation theorem, if Q is the quantile function of* *a random variable X in $DA_2(b_n)$, then*

$$\lim_{n\to\infty} \frac{\sqrt{n}}{b_n} \int_{1/n}^{1-1/n} |\alpha_n(t) - B_n(t)|\, dQ(t) = 0 \tag{6.3}$$

in probability.

PROOF. The approximation of α_n by Brownian bridges (6.1) gives that, for any $\varepsilon \in (0, 1/4)$,

$$\begin{aligned}
&\frac{\sqrt{n}}{b_n} \int_{1/n}^{1-1/n} \big|\alpha_n(t) - B_n(t)\big|\, dQ(t) \\
&\quad \le \frac{n^{1/2-\varepsilon}}{b_n} \int_{1/n}^{1-1/n} [t(1-t)]^{1/2-\varepsilon}\, dQ(t) \\
&\qquad\qquad \times \sup_{1/n \le s \le 1-1/n} n^{\varepsilon}\big|\alpha_n(s) - B_n(s)\big|[s(1-s)]^{\varepsilon-1/2} \\
&\quad = \frac{n^{1/2-\varepsilon}}{b_n} \int_{Q(1/n)}^{Q(1-1/n)} [F(t)(1-F(t))]^{1/2-\varepsilon}\, dt \times O_P(1).
\end{aligned}$$

We can decompose the integral in the last term into two parts, from $Q(1/n)$ to the median of X, and from there to $Q(1-1/n)$. Since both integrals can be treated in the same way, we only consider one. By the properties of regular variation [Feller (1971)] applied to U and by (4.1), (4.2) and (4.6), setting $M := \sup_{t>0} t^2 \Pr\{X > t\}/U(t)$, which is finite, we have

$$\begin{aligned}
&\frac{n^{1/2-\varepsilon}}{b_n} \int_{\mathrm{med}(X)}^{Q(1-1/n)} [F(t)(1-F(t))]^{1/2-\varepsilon}\, dt \\
&\quad \le \frac{n^{1/2-\varepsilon}}{b_n} \int_{\mathrm{med}(X)}^{Q(1-1/n)} [\Pr\{X > t\}]^{1/2-\varepsilon}\, dt \\
&\quad \le \frac{M^{1/2-\varepsilon} n^{1/2-\varepsilon}}{b_n} \int_{\mathrm{med}(X)}^{Q(1-1/n)} \left(\frac{U(t)}{t^2}\right)^{1/2-\varepsilon} dt \\
&\quad \asymp \frac{M^{1/2-\varepsilon} n^{1/2-\varepsilon}}{b_n} \frac{1}{2\varepsilon} [U(Q(1-1/n))]^{1/2-\varepsilon} [Q(1-1/n)]^{2\varepsilon} \\
&\quad \lesssim \frac{M^{1/2-\varepsilon}}{2\varepsilon} \left(\frac{nU(b_n)}{b_n^2}\right)^{1/2-\varepsilon} \left(\frac{Q(1-1/n)}{b_n}\right)^{2\varepsilon} \to 0
\end{aligned}$$

as $n \to \infty$. $\square$

Let X be a random variable in $DA_2(b_n)$ for some admissible sequence $\{b_n\}$ of normalizing constants. We keep the notation of previous sections, and also set

$$\gamma_n := \sqrt{n}\sqrt{\frac{2}{\pi}}\int_{Q(1/n)}^{Q(1-1/n)} \sqrt{F(t)(1-F(t))}\,dt \tag{6.4}$$
$$= \sqrt{n}\sqrt{\frac{2}{\pi}}\int_{1/n}^{1-1/n} [t(1-t)]^{1/2}\,dQ(t) = \sqrt{n}\int_{1/n}^{1-1/n} \mathbb{E}|B(t)|\,dQ(t).$$

Then, Lemma 4.1 and Corollary 6.1 show that the sequences

$$\left\{\frac{Z_n-\gamma_n}{b_n}\right\} \quad \text{and} \quad \left\{\frac{\sqrt{n}}{b_n}\int_{1/n}^{1-1/n}(|B(t)|-\mathbb{E}|B(t)|)\,dQ(t)\right\} \tag{6.5}$$

are weak convergence equivalent. If X is symmetric and has tails of the order t^{-2}, concretely, if

$$dQ(t) = \frac{c\,dt}{[t(1-t)]^{3/2}}$$

then, transforming B into the stationary Ornstein–Uhlenbeck process $V(t)$, $-\infty < t < \infty$, by the equation

$$B(t) = t^{1/2}(1-t)^{1/2}V\left(\frac{1}{2}\log\frac{t}{1-t}\right), \tag{6.6}$$

yields

$$\frac{\sqrt{n}}{b_n}\int_{1/n}^{1-1/n}(|B(t)|-\mathbb{E}|B(t)|)\,dQ(t) \tag{6.7}$$
$$\simeq \frac{c}{\sqrt{\log(n-1)}}\int_{-\log(n-1)/2}^{\log(n-1)/2}(|V(t)|-\mathbb{E}|V(t)|)\,dt$$

for a specific constant c, and convergence in distribution follows by a central limit theorem for stationary Markov processes [Mandl (1968), page 95], as observed by Csörgő and Horváth (1993). However, we have been unable to find in the literature any central limit theorems for integrals of centered functionals of stationary processes with respect to measures other than Lebesgue. On the other hand, by the tightness result of Section 4, the Csörgő–Horváth result [Theorem 1.1(a2) above], and the CLT in Section 5, it is natural to conjecture that the sequences in (6.5) converge in distribution for general $X \in DA_2(b_n)$.

As mentioned above, we will prove convergence in law of the sequences in (6.5), but only for random variables X with smooth tail probabilities of the order of $(\log t)^\alpha/t^2$, $-1 \le \alpha < \infty$. This shows, in particular, that Lebesgue measure dt can be replaced in the second sequence in (6.7) by other measures and still have a central limit theorem. However, our method of proof does not seem to extend to all of $DA_2(b_n)$.

Here is how we attack the problem. Application of the Borell–Sudakov–Tsirel'son inequality, as in the proof of Theorem 5.1, yields that the second sequence in (6.5) is tight with subsequential convergence of moments as well and that all its subsequential limits have tails of the order of $c_1\exp(-c_2t^2)$;

hence, if these limits are infinitely divisible, they are normal [Horn (1972)]. Then, convergence follows by showing that the limits are indeed infinitely divisible and that the second moments do converge. We prove that the second moments converge by direct computation and prove that the limits are infinitely divisible by showing that the sequences in (6.5) are equivalent to the sequence of row sums of an infinitesimal triangular array.

To carry out this program, we need some lemmas and propositions. We begin by applying the Gaussian concentration inequality to the second sequence in (6.5).

PROPOSITION 6.2. *Let Q be the quantile function of $X \in DA_2(b_n)$ and let B be a Brownian bridge. Then, there is $c \in (0, \infty)$ such that*

$$\Pr\left\{\frac{\sqrt{n}}{b_n}\left|\int_{1/n}^{1-1/n}(|B(t)| - \mathbb{E}|B(t)|)\,dQ(t)\right| > u\right\} \leq 2\exp(-cu^2) \tag{6.8}$$

for all $u > 0$ and $n \in \mathbb{N}$. In particular, the sequence

$$G_n := \frac{\sqrt{n}}{b_n}\int_{1/n}^{1-1/n}(|B(t)| - \mathbb{E}|B(t)|)\,dQ(t), \qquad n \in \mathbb{N}, \tag{6.9}$$

is stochastically bounded; all its subsequential limits in law G satisfy

$$\mathbb{E}\exp(\lambda G^2) < \infty \tag{6.10}$$

for all $\lambda < c$, and if $G_{n_k} \to_d G$, then also $\mathbb{E}|G_{n_k}|^r \to \mathbb{E}|G|^r$ for all $r > 0$ and $\mathbb{E}\exp(\lambda G_{n_k}^2) \to \mathbb{E}\exp(\lambda G^2)$.

PROOF. As is well known [CCHM (1986b)], the norming sequence b_n can be taken to be

$$b_n = \sqrt{n}\left(\int_{1/n}^{1-1/n}\int_{1/n}^{1-1/n}(u \wedge v - uv)\,dQ(u)\,dQ(v)\right)^{1/2} \tag{6.11}$$

(see also the Appendix for another proof). Now, we apply the Gaussian concentration inequality as in the proof of Theorem 5.1. Taking $d_n = 1/n$ and $d_m = 1 - 1/n$ in the computation of $\sigma_{n,m}$ in that proof, we get $\sigma_{n,m}^2 = b_n^2/n$, which gives (6.8) with $c = 2/\pi^2$ and, in particular, stochastic boundedness of $\{G_n\}$. The rest of the statements of this proposition follow from (6.8) by uniform integrability. □

At this point we should slightly depart from our program and compare Theorem 4.2 and Proposition 6.2. To begin, note that, because of the weak convergence equivalence between $\{G_n\}$ and $\{(Z_n - \gamma_n)/b_n\}$, Proposition 6.2 provides an alternative to Theorem 4.2 for proving stochastic boundedness of a suitably centered and normalized version of $\{Z_n\}$. Although the normings are the same (b_n), the centerings are not; they are $\mathbb{E}Z_n$ in Theorem 4.2 and γ_n here. While $\mathbb{E}Z_n$ is quite difficult to compute, γ_n is a simple function of the distribution of X; hence, it may be of practical interest to compare these centerings. The comparison done in Remark 4.1 gave $\mathbb{E}Z_n \simeq \gamma_n$ only in the sense that the lim sup and lim inf of their quotients are finite, but not necessarily equal, constants. In order for us to be able to interchange $\mathbb{E}Z_n$ by γ_n in Theorem 4.2, in Corollary 4.3, in Theorem 5.1 and in the limit theorem to be proved in this section, it is necessary that

$$\lim_{n\to\infty} \frac{\mathbb{E}Z_n - \gamma_n}{b_n} = 0, \tag{6.12}$$

which, at this point, we only know for $X \in L_{2,1}$ (in this case the numerators are bounded). This is what we observe now as a consequence of *both* Theorem 4.1 and Proposition 6.2. Note that, since $\mathbb{E}Z_n/b_n \to \infty$ and $\gamma_n/b_n \to \infty$ if $X \in DA_2(b_n) \setminus L_{2,1}$ by Proposition 4.3 [see also (4.23)], the limit (6.12) implies in particular

$$\lim_{n\to\infty} \frac{\mathbb{E}Z_n}{\gamma_n} = 1, \tag{6.13}$$

a substantial improvement on Remark 4.1 [which is, however, used in the proof of (6.12) and (6.13)].

COROLLARY 6.3. *Let $X \in DA_2(b_n)\setminus L_{2,1}$ and let Z_n and γ_n, $n \in \mathbb{N}$, be respectively defined by equations* (4.14) *and* (6.4). *Then*, *the limit* (6.12) *is satisfied* [*and therefore so is* (6.13)].

PROOF. Proposition 6.2 implies that any subsequential limit in law G of the sequence $\{G_n\}$ satisfies $\mathbb{E}G = 0$, by convergence of moments; together with the weak convergence equivalence of the two sequences in (6.5) (which follows by weighted approximation), it also implies that the sequence $\{(Z_n - \gamma_n)/b_n\}$ is stochastically bounded and has the same subsequential limits G as the sequence $\{G_n\}$, hence, also centered. By Theorem 4.2, the sequence $\{(Z_n - \mathbb{E}Z_n)/b_n\}$ is also stochastically bounded. Tightness of the two sequences implies $\sup_n |\mathbb{E}Z_n - \gamma_n|/b_n < \infty$. This and Theorem 4.2 once more then imply that

$$\sup_n \mathbb{E}\left|\frac{Z_n - \gamma_n}{b_n}\right|^r < \infty$$

for $0 < r < 2$. Hence, by uniform integrability, if $\{(Z_{n_k} - \gamma_{n_k})/b_{n_k}\}$ converges in law to G then

$$\lim_{k\to\infty} \mathbb{E}\left(\frac{Z_{n_k} - \gamma_{n_k}}{b_{n_k}}\right) = \mathbb{E}G = 0.$$

Now the corollary follows because this happens for a subsequence of every subsequence. □

Back to the central limit theorem in the infinite variance case, we show next that the second moment of G_n converges, but only in the particular case when the tail probabilities of X are regularly varying.

PROPOSITION 6.4. *Let Q be the quantile function of a random variable X in $DA_2(b_n)$. Assume X has regularly varying tails with exponent -2 and $\mathbb{E}X^2 = \infty$. Let B be a Brownian bridge and let G_n be as defined by* (6.9). *Then*

$$\lim_{n\to\infty} \mathbb{E}G_n^2 = 1 + \frac{2\log 2}{\pi} - \frac{13}{3\pi}. \tag{6.14}$$

PROOF. We will assume for simplicity that X has a symmetric distribution [equivalently, that $Q(1-x) = -Q(x)$, $x \in (0,1)$] and that Q has a continuous derivative q. Integration by parts and the type of reasoning in the proof of Proposition 4.3 extend this proof to general Q. Let (Z_1, Z_2) be a centered random vector with bivariate normal distribution such that $\mathrm{Var}(Z_1) = \mathrm{Var}(Z_2) = 1$ and $\mathrm{Cov}(Z_1, Z_2) = \rho$. Set $K(\rho) := \mathrm{Cov}(|Z_1|, |Z_2|)$. Then, an elementary but cumbersome computation that we omit yields

$$K(\rho) = \frac{2}{\pi}\rho \arcsin\rho - \frac{1}{\pi}\rho^2\sqrt{1-\rho^2} + \frac{2}{\pi}\big(\sqrt{1-\rho^2} - 1\big).$$

Therefore, if we take b_n as in (5.17), we have

$$\begin{aligned}\mathbb{E}G_n^2 = \Bigg[\int_{1/n}^{1-1/n}\int_{1/n}^{1-1/n} &\sqrt{u(1-u)v(1-v)}\\ &\times K\left(\frac{u\wedge v - uv}{\sqrt{u(1-u)v(1-v)}}\right) dQ(u)\,dQ(v)\Bigg]\\ \times \Bigg[\int_{1/n}^{1-1/n}\int_{1/n}^{1-1/n} &(u\wedge v - uv)\,dQ(u)\,dQ(v)\Bigg]^{-1}.\end{aligned}$$

Set

$$K_1(\rho) = \frac{2}{\pi}\rho\arcsin\rho,$$

$$K_2(\rho) = -\frac{1}{\pi}\rho^2\sqrt{1-\rho^2}$$

and

$$K_3(\rho) = \frac{2}{\pi}\Big(\sqrt{1-\rho^2} - 1\Big).$$

Then,

$$K_1(\rho) = \frac{2}{\pi}\sum_{n=0}^{\infty}\binom{-1/2}{n}\frac{(-1)^n}{2n+1}\rho^{2n+2}, \tag{6.15}$$

$$K_2(\rho) = -\frac{1}{\pi}\sum_{n=0}^{\infty}\binom{1/2}{n}(-1)^n\rho^{2n+2}, \tag{6.16}$$

$$K_3(\rho) = \frac{2}{\pi}\sum_{n=1}^{\infty}\binom{1/2}{n}(-1)^n\rho^{2n} \tag{6.17}$$

for all $|\rho| \le 1$. We further define

$$f_k(x) = \int_x^{1-x} \int_x^{1-x} (u \wedge v - uv) \left(\frac{u \wedge v - uv}{\sqrt{u(1-u)v(1-v)}} \right)^k dQ(u)\, dQ(v), \qquad k \ge 1$$

and

$$g(x) = \int_x^{1-x} \int_x^{1-x} (u \wedge v - uv)\, dQ(u)\, dQ(v),$$

and set

$$\alpha_k = \lim_{n \to \infty} \frac{f_k(1/n)}{g(1/n)}.$$

Then, using (6.15), we obtain

$$\begin{aligned}
&\lim_{n\to\infty} \left[\int_{1/n}^{1-1/n} \int_{1/n}^{1-1/n} \sqrt{u(1-u)v(1-v)} \right. \\
&\qquad\qquad \left. \times K_1\left(\frac{u \wedge v - uv}{\sqrt{u(1-u)v(1-v)}} \right) dQ(u)\, dQ(v) \right] \\
&\quad \times \left[\int_{1/n}^{1-1/n} \int_{1/n}^{1-1/n} (u \wedge v - uv)\, dQ(u)\, dQ(v) \right]^{-1} \\
&\quad = \frac{2}{\pi} \sum_{n=0}^{\infty} \binom{-1/2}{n} \frac{(-1)^n}{2n+1} \alpha_{2n+1}.
\end{aligned} \tag{6.18}$$

[It is straightforward to see that the left term is bounded from below by the right term, since all the terms in the series expansion are positive. The fact that (6.15) is valid for $\rho = 1$ suffices to conclude (6.18).] In order to compute α_k, recall that symmetry of X implies $Q(1-x) = -Q(x)$ and $q(1-x) = q(x)$. Thus,

$$\begin{aligned}
f_k'(x) &= -4 \frac{x^{k/2+1}}{(1-x)^{k/2}} q(x) \int_x^{1-x} \frac{u^{k/2+1}}{(1-u)^{k/2}}\, dQ(u), \\
g'(x) &= -4xq(x)Q(1-x)
\end{aligned}$$

and, consequently,

$$\lim_{x\to 0} \frac{f_k'(x)}{g'(x)} = \lim_{x \to 0} \frac{x^{k/2} \int_{1/2}^{1-x} (1-u)^{-k/2}\, dQ(u)}{Q(1-x)}.$$

Integration by parts yields

$$\frac{x^{k/2} \int_{1/2}^{1-x} (1-u)^{-k/2}\, dQ(u)}{Q(1-x)} = 1 - \frac{k}{2} \frac{x^{k/2} \int_{1/2}^{1-x} Q(u)(1-u)^{-k/2-1}\, du}{Q(1-x)}.$$

Since X has regularly varying tails with exponent -2, $Q(1-u)$ is regularly varying at 0 with exponent $-1/2$ [see, e.g., Resnick (1987)]. Now, since, if a function $L(x)$ is regularly varying at 0 with exponent σ then $L(1/y)$ is regularly varying at ∞ with exponent $-\sigma$, the usual properties of regular variation [Feller (1971)] give

$$\lim_{x\to 0} \frac{x^{k/2}\int_{1/2}^{1-x} Q(u)(1-u)^{-k/2-1}\,du}{Q(1-x)} = \frac{2}{k+1}.$$

Therefore, $\lim_{x\to 0} f_k'(x)/g'(x) = 1/(k+1)$ and, by l'Hôpital's rule,

$$\alpha_k = \lim_{x\to 0} \frac{f_k(x)}{g(x)} = \frac{1}{k+1}. \tag{6.19}$$

Plugging (6.19) into (6.18), we obtain

$$\begin{aligned}
&\lim_{n\to\infty}\biggl[\int_{1/n}^{1-1/n}\int_{1/n}^{1-1/n}\sqrt{u(1-u)v(1-v)} \\
&\qquad\qquad \times K_1\biggl(\frac{u\wedge v-uv}{\sqrt{u(1-u)v(1-v)}}\biggr)\,dQ(u)\,dQ(v)\biggr] \\
&\quad \times\biggl[\int_{1/n}^{1-1/n}\int_{1/n}^{1-1/n}(u\wedge v-uv)\,dQ(u)\,dQ(v)\biggr]^{-1} \\
&\quad = \frac{2}{\pi}\sum_{n=0}^{\infty}\binom{-1/2}{n}\frac{(-1)^n}{2n+1}\frac{1}{2n+2} = 1-\frac{2}{\pi}.
\end{aligned} \tag{6.20}$$

The last identity follows upon noticing that

$$x\sin^{-1}x+\sqrt{1-x^2}-1 = \sum_{n=0}^{\infty}\binom{-1/2}{n}\frac{(-1)^n}{2n+1}\frac{1}{2n+2}x^{2n+2}, \qquad |x|\le 1.$$

We can apply the same reasoning to K_2 and K_3 to obtain

$$\begin{aligned}
&\lim_{n\to\infty}\biggl[\int_{1/n}^{1-1/n}\int_{1/n}^{1-1/n}\sqrt{u(1-u)v(1-v)} \\
&\qquad\qquad \times K_2\biggl(\frac{u\wedge v-uv}{\sqrt{u(1-u)v(1-v)}}\biggr)\,dQ(u)\,dQ(v)\biggr] \\
&\quad \times\biggl[\int_{1/n}^{1-1/n}\int_{1/n}^{1-1/n}(u\wedge v-uv)\,dQ(u)\,dQ(v)\biggr]^{-1} \\
&\quad = -\frac{1}{\pi}\sum_{n=0}^{\infty}\binom{1/2}{n}(-1)^n\frac{1}{2n+2} = -\frac{1}{3\pi}
\end{aligned} \tag{6.21}$$

and

$$
\begin{aligned}
&\lim_{n\to\infty}\biggl[\int_{1/n}^{1-1/n}\int_{1/n}^{1-1/n}\sqrt{u(1-u)v(1-v)} \\
&\qquad\qquad \times K_3\biggl(\frac{u\wedge v-uv}{\sqrt{u(1-u)v(1-v)}}\biggr)\,dQ(u)\,dQ(v)\biggr] \\
&\qquad \times\biggl[\int_{1/n}^{1-1/n}\int_{1/n}^{1-1/n}(u\wedge v-uv)\,dQ(u)\,dQ(v)\biggr]^{-1} \\
&\qquad = \frac{2}{\pi}\sum_{n=1}^{\infty}\binom{1/2}{n}(-1)^n\frac{1}{2n}=\frac{2}{\pi}(\log 2-1).
\end{aligned}\tag{6.22}
$$

Now, (6.20), (6.21) and (6.22) complete the proof. □

By Propositions 6.2 and 6.4, all the subsequential limits in distribution G of the sequence G_n, which are centered, have the same second moment. Hence, *if they are Gaussian*, they coincide and convergence follows.

The next proposition shows that if an infinitely divisible distribution has tail probabilities of smaller order than those of a Poisson random variable, then it must be normal, a result due to Horn (1972), who gave an analytic proof and placed unnecessary restrictions on the function H in (6.23). The proof we give here, entirely probabilistic except for use of unicity in the Lévy–Khinchin formula, seems to be the simplest for the particular statement that interests us. Csörgő and Mason (1991) obtain general and quite precise integrability results, with yet different proofs, and give many references to the extensive literature on integrability of infinitely divisible distributions.

PROPOSITION 6.5 [Horn (1972)]. *If X is infinitely divisible, $c\in(0,\infty)$ and $H(t)$, $t>0$, is a positive function such that, both*

$$
\limsup_{t\to\infty}\frac{H(t)}{t\log t}=\infty \quad\textit{and}\quad \Pr\{|X|\ge t\}\le c\exp(-H(t)) \quad\textit{for all } t>0,\tag{6.23}
$$

then X is normal (or degenerate).

PROOF. We first prove the proposition assuming X is symmetric. Then, by Lévy–Khinchin for symmetric variables,

$$\mathscr{L}(X)=N(0,\sigma^2)*\mathrm{Pois}\,\mu,$$

where σ^2 is a nonnegative number and μ is a symmetric Lévy measure [see, e.g., Araujo and Giné (1980), Chapter 2, for this and for several properties of general Poisson measures to be used below. In the proof of unicity of the Lévy–Khinchin formula, page 56, Exercise 4, the order of the integral and the exponential in the hint's second line should be reversed]. If $\mu=\alpha\delta_0$, then $\mathrm{Pois}\,\mu=\delta_0$ and $\mathscr{L}(X)=N(0,\sigma^2)$. If $\mu\ne\alpha\delta_0$ there is an interval $[a,b]$, $0<a<b<\infty$, such that $\mu[a,b]=\lambda\in(0,\infty)$. Let

$$\mu_1:=\mu|_{[a,b]},\qquad \mu_2:=\mu|_{[-b,-a]},\qquad \mu_3:=\mu-\mu_1-\mu_2$$

and let X_i, $i = 1, 2, 3$, be independent random variables such that

$$\mathscr{L}(X_i) = \text{Pois}\,\mu_i,\ i = 1, 2 \quad \text{and} \quad \mathscr{L}(X_3) = N(0, \sigma^2) * \text{Pois}\,\mu_3.$$

Then X_3 is symmetric, $X_1 \geq 0$, $X_2 \leq 0$ and $\mathscr{L}(X_2) = \mathscr{L}(-X_1)$; in fact, we can take

$$X_1 = \sum_{i=0}^{N_1} \xi_i \quad \text{and} \quad X_2 = -\sum_{i=0}^{N_2} \eta_i$$

with N_i, $i = 1, 2$, independent Poisson random variables with parameter λ, ξ_i, η_i i.i.d. with law $\lambda^{-1}\mu|_{[a,b]}$ for $i \geq 1$, independent of N_1 and N_2, and $\xi_0 = \eta_0 = 0$. Therefore,

$$\begin{aligned}
\Pr\{|X| \geq t\} &= \Pr\{|X_1 + X_2 + X_3| \geq t\} \geq \Pr\{X_1 + X_2 \geq t,\ X_3 \geq 0\} \\
&\geq \frac{1}{2}\Pr\{X_1 + X_2 \geq t\} \geq \frac{1}{2}\Pr\{X_1 \geq t,\ X_2 = 0\} \\
&= \frac{1}{2}\Pr\{X_1 \geq t\}\Pr\{N_2 = 0\} \\
&= \frac{e^{-\lambda}}{2}\Pr\left\{\sum_{i=0}^{N_1} \xi_i \geq t\right\} \geq \frac{e^{-\lambda}}{2}\Pr\{aN_1 \geq t\}.
\end{aligned}$$

Now, assuming $t/a \in \mathbb{N}$, we have that for all $c > 1/a$ there is $t_0 < \infty$ such that

$$\Pr\{aN_1 \geq t\} \geq \Pr\{N_1 = t/a\} = e^{-\lambda}\frac{\lambda^{t/a}}{(t/a)!} \gtrsim \exp(-ct\log t), \qquad t \geq t_0,$$

by Stirling's formula. Therefore, there exists a constant c_1 such that

$$\Pr\{|X| \geq t\} \geq \exp(-c_1 t \log t)$$

for all t large enough, contradicting (6.23).

If X is infinitely divisible but not necessarily symmetric, the Lévy–Khinchin formula for its distribution is

$$\mathscr{L}(X) = \delta_a * N(0, \sigma^2) * c_\delta\text{Pois}\,\mu$$

for some $a, \sigma \in \mathbb{R}$ and Lévy measure μ, and if X' is an independent copy of X, then

$$\mathscr{L}(X - X') = N(0, 2\sigma^2) * \text{Pois}(\mu + \bar{\mu}),$$

where $\bar{\mu}(A) = \mu(-A)$ for all Borel sets A. In particular, the exponent of the characteristic function of $X - X'$ is $-\sigma^2 t^2 + \int_{-\infty}^{\infty}(\cos tx - 1)\,d(\mu + \bar{\mu})(x)$. If X satisfies (6.23), so does $X - X'$ because $\Pr\{|X - X'| \geq 2t\} \leq 2\Pr\{|X| \geq t\}$. Then, $X - X'$ being symmetric and infinitely divisible, the first part of the proof shows that $X - X'$ is in fact normal, say with variance τ^2. Therefore, the exponent of its characteristic function satifies

$$-\sigma^2 t^2 + \int_{-\infty}^{\infty}(\cos tx - 1)\,d(\mu + \bar{\mu})(x) = -\frac{\tau^2 t^2}{2}, \qquad t \in \mathbb{R}.$$

Since the integral is not positive for any t, it follows that $\eta^2 := \tau^2 - 2\sigma^2$ is nonnegative. Hence,

$$\operatorname{Pois}(\mu + \bar{\mu}) = N(0, \eta^2),$$

which contradicts the unicity of the Lévy–Khinchin formula unless $\mu + \bar{\mu} = 2c\delta_0$ for some $c \geq 0$; that is, unless $\mu = c\delta_0$, implying that X is normal and proving the proposition. □

We will apply this proposition with $H(t) = Kt^2$.

For our final step, we consider symmetric random variables X such that

$$\Pr\{|X| > t\} \simeq \frac{(\log t)^\alpha}{t^2}$$

for large t, and with $\alpha \geq -1$. Such variables are in the domain of attraction of the normal law, have infinite variance and their corresponding norming constants are, up to a multiplicative constant that we can ignore for the moment,

$$b_n = \sqrt{n}(\log n)^{(\alpha+1)/2} \text{ if } \alpha > -1 \quad \text{and} \quad b_n = \sqrt{n \log\log n} \text{ if } \alpha = -1,$$

as is easily checked. Actually, we modify the law of X a little bit in order to produce a less cumbersome proof (although these modifications will not alter the essence of the proof). To wit, we take a differentiable quantile function Q with $Q(1/2) = 0$ and derivative

$$Q'(t) = \frac{|\frac{1}{2}\log(t/(1-t))|^{\alpha/2}}{t^{3/2}(1-t)^{3/2}}, \qquad 0 < t < 1, \tag{6.24}$$

for $\alpha \geq -1$. It is not difficult to see that such Q is the quantile function of a symmetric distribution as specified above, with normalizing constants b_n proportional to $\sqrt{n}(\log n)^{(\alpha+1)/2}$ if $\alpha > -1$ and to $\sqrt{n \log\log n}$ if $\alpha = -1$.

LEMMA 6.6. *Let G_n, $n \in \mathbb{N}$, be the random variables defined by* (6.9) *with quantile function Q as given by* (6.24) *for some $\alpha \geq -1$, and $b_n = \sqrt{n}(\log n)^{(\alpha+1)/2}$ for $\alpha > -1$, $b_n = \sqrt{n \log\log n}$ for $\alpha = -1$, $n \in \mathbb{N}$. Then, all the subsequential limits in law of the sequence $\{G_n\}$ are infinitely divisible.*

PROOF. The proof of this lemma will take a few sublemmas. We begin by setting up the appropriate stage. Using the equivalence (6.6) between the Brownian bridge and the Ornstein–Uhlenbeck process and with the change of variables $u = \frac{1}{2}\log[t/(1-t)]$, we have

$$G_n = \frac{\sqrt{n}}{b_n}\int_{-\log(n-1)/2}^{\log(n-1)/2} (|V(u)| - \mathbb{E}|V(u)|)\frac{e^u}{1+e^{2u}}\, dQ\left(\frac{e^{2u}}{1+e^{2u}}\right) du. \tag{6.25}$$

With a further change of variables

$$d\tilde{Q}_n\left(s + \frac{1}{2}\log(n+1)\right) := \frac{e^s}{1+e^{2s}}\, dQ\left(\frac{e^{2s}}{1+e^{2s}}\right) = 2|s|^{\alpha/2}\, ds, \tag{6.26}$$

and, using the stationarity of V, (6.25) becomes

$$G_n = \frac{\sqrt{n}}{b_n}\int_0^{\log(n-1)} (|V(s)| - \mathbb{E}|V(s)|)\, d\tilde{Q}_n(s). \tag{6.25$'$}$$

In a way similar to Mandl [(1968), page 95], we fix $\delta > 0$ and define stopping times

$$\begin{aligned} \tau_0 &= \inf\{u \ge 0\colon V(u) = 0\},\\ \tau_1 &= \inf\{u \ge \tau_0\colon V(u) = \delta\}, \dots,\\ \tau_{2n} &= \inf\{u \ge \tau_{2n-1}\colon V(u) = 0\},\\ \tau_{2n+1} &= \inf\{u \ge \tau_{2n}\colon V(u) = \delta\},\ n = 1, 2, \dots . \end{aligned} \tag{6.27}$$

Since the Ornstein–Uhlenbeck process V can also be represented as $V(t) = W(e^{2t})/e^t$, $-\infty < t < \infty$, with W a Brownian motion, these are actually stopping times for Brownian motion. In particular, by the strong Markov property, τ_0, $\tau_k - \tau_{k-1}$, $k \in \mathbb{N}$, are independent, and so are the integrals

$$\begin{aligned} &\int_0^{\tau_0} f(V(t))\, d\tilde{Q}_n(t),\ \int_{\tau_0}^{\tau_2} f(V(t))\, d\tilde{Q}_n(t)\\ &\qquad = \int_{\tau_0}^{\tau_2} f\left(\frac{W(\exp(2t)) - W(\exp(2\tau_0))}{\exp(2t)}\right) d\tilde{Q}_n(t),\\ &\qquad\quad \int_{\tau_{2k}}^{\tau_{2(k+1)}} f(V(t))\, d\tilde{Q}_n(t), \qquad k \in \mathbb{N}, \end{aligned}$$

since $W(\exp(2\tau_{2k})) = \exp(\tau_{2k})V(\tau_{2k}) = 0$. Moreover, $V(t)$ being stationary, $\tau_{2(k+1)} - \tau_{2k}$, $k \in \mathbb{N}$, are i.i.d. We will show below that

$$m_1 := \mathbb{E}\tau_0 < \infty \quad \text{and} \quad m_2 := \mathbb{E}(\tau_2 - \tau_0) = \mathbb{E}(\tau_{2k} - \tau_{2(k-1)}) < \infty. \tag{6.28}$$

Then, defining

$$k_n := \max\{k \in \mathbb{N}\colon m_1 + km_2 \le \log(n-1)\}, \tag{6.29}$$

we have

$$\begin{aligned} G_n &= \frac{\sqrt{n}}{b_n}\left(\int_0^{\tau_2} + \sum_{k=1}^{k_n-1}\int_{\tau_{2k}}^{\tau_{2(k+1)}} + \int_{\tau_{2k_n}}^{\log(n-1)}\right)(|V(t)| - \mathbb{E}|V(t)|)\, d\tilde{Q}_n(t)\\ &:= (I)_n + (II)_n + (III)_n, \end{aligned} \tag{6.30}$$

where $(II)_n$ is a sum of independent not necessarily identically distributed random variables. Hence, by the converse part of the general central limit theorem in $\mathbb{R}$ [e.g., Araujo and Giné (1980), page 61], in order to prove that the subsequential limit laws of $\{G_n\}$ are infinitely divisible it is sufficient to show that

$$(I_n) \to 0 \text{ in pr.}, \qquad (III_n) \to 0 \text{ in pr.} \tag{6.31}$$

and that the triangular array of row-wise independent random variables

$$\frac{\sqrt{n}}{b_n}\int_{\tau_{2k}}^{\tau_{2(k+1)}}(|V(t)|-\mathbb{E}|V(t)|)\,d\tilde{Q}_n(t), \qquad 0<k<k_n,\ n\in\mathbb{N},$$

is infinitesimal; that is,

$$\max_{1\le k<k_n}\Pr\left\{\frac{\sqrt{n}}{b_n}\left|\int_{\tau_{2k}}^{\tau_{2(k+1)}}(|V(t)|-\mathbb{E}|V(t)|)\,d\tilde{Q}_n(t)\right|>\varepsilon\right\}\to 0 \tag{6.32}$$

as $n\to\infty$ for all $\varepsilon>0$.

CLAIM 6.6.1. The stopping times τ_k, $k=0,1,\dots,$ have finite moments of all orders.

PROOF. The distributions of τ_0 and $\tau_{2k}-\tau_{2k-1}$ are easy to obtain, using that the distribution of $\sup_{0\le s\le u} W(s)$ is that of $\sqrt{u}|g|$, g standard normal [e.g., Billingsley (1968), page 72]. Here is how to obtain the latter: we recall also that $W(u_0v)-W(u_0)$ has the same law as $\sqrt{u_0}W(v-1)$, and get

$$\begin{aligned}
&\Pr\{\tau_{2k}-\tau_{2k-1}>t\}\\
&\qquad=\Pr\{\tau_2-\tau_1>t\}\\
&\qquad=\mathbb{E}\Pr\Big\{\inf_{0<s\le t}W(\exp(2(s+\tau_1)))>0|W(\exp(2\tau_1))=\delta\exp(\tau_1)\Big\}\\
&\qquad=\Pr\Big\{\inf_{0<s\le t}W(\exp(2(s+\tau)))-W(\exp(2\tau))>-\delta\exp(\tau)\Big\}\\
&\qquad=\Pr\Big\{\inf_{0<s\le t}W(\exp(2s)-1)>-\delta\Big\}\\
&\qquad=\Pr\Big\{\sup_{0<s\le t}[-W(\exp(2s)-1)]<\delta\Big\}\\
&\qquad=\Pr\Big\{|g|<\delta/\sqrt{\exp(2t)-1}\Big\}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\Pr\{\tau_0>t\}&=2\Pr\Big\{\sup_{1<s\le e^{2t}}W(s)<0,\ W(1)<0\Big\}\\
&=2\Pr\Big\{W(1)+\sup_{1<s\le e^{2t}}(W(s)-W(1))<0,\ W(1)<0\Big\}\\
&=\frac{2}{\pi}\tan^{-1}\frac{1}{\sqrt{\exp(2t)-1}}.
\end{aligned}$$

Therefore,

$$\Pr\{\tau_0>t\}\asymp\frac{2}{\pi e^t},\qquad \Pr\{\tau_{2k}-\tau_{2k-1}>t\}\asymp\frac{2\delta}{e^t}$$

as $t \to \infty$. On the other hand, $\tau_{2k-1} - \tau_{2k-2}$, which is equal in distribution to $\tau_1 - \tau_0$, has to do with the exit time of Brownian motion from the one-sided barrier $\delta\sqrt{t}$. Concretely,

$$\Pr\{\tau_1 - \tau_0 > t\} = \mathbb{E}\Pr\Big\{\sup_{0<s\le t} V(s+\tau_0) < \delta | V(\tau_0) = 0\Big\},$$

and for τ_0 fixed, this equals

$$\begin{aligned}\Pr\bigg\{\frac{W(e^{2s}e^{2\tau})}{e^{\tau}} < \delta e^s \text{ for all } s \le t \bigg| \frac{W(e^{2\tau})}{e^{\tau}} = 0\bigg\}\\ = \Pr\{W(e^{2s}) < \delta e^s \text{ for all } s \le t | W(1) = 0\}.\end{aligned}$$

This probability is shown in Uchiyama [(1980), Theorem 1.1] to be of the order of $c/e^{\lambda t}$ for some $c < \infty$ and $0 < \lambda < 1$ as $t \to \infty$, thus concluding the proof of Claim 6.6.1. □

CLAIM 6.6.2. We have that $\lim_{n\to\infty}(I)_n = 0$ in probability.

PROOF. First we show that the centering is not relevant for the evaluation of $(I)_n$. For $a > 0$ and n large enough so that $\log(n-1) \asymp \log n$ is much larger than $2a$, in the case $\alpha > -1$ we have

$$\begin{aligned}\frac{\sqrt{n}}{b_n}\int_0^a \mathbb{E}|V(t)|\, d\tilde{Q}_n(t) &= \frac{2\sqrt{2/\pi}}{(\log n)^{(\alpha+1)/2}}\int_{(\log(n-1))/2-a}^{(\log(n-1))/2} t^{\alpha/2}\, dt\\ &= \frac{1}{1+\alpha/2}\frac{2\sqrt{2/\pi}}{(\log n)^{(\alpha+1)/2}}\\ &\quad\times\bigg[\bigg(\frac{1}{2}\log(n-1)\bigg)^{1+\alpha/2} - \bigg(\frac{1}{2}\log(n-1) - a\bigg)^{1+\alpha/2}\bigg]\\ &\le \frac{3a\sqrt{2/\pi}}{(\log n)^{(\alpha+1)/2}}\bigg(\frac{1}{2}\log(n-1)\bigg)^{\alpha/2}\\ &\le \frac{K}{(\log n)^{1/2}} \to 0,\end{aligned}$$

where K is a constant that depends on a and α but not on n. The same computation for $\alpha = -1$ gives

$$\frac{\sqrt{n}}{b_n}\int_0^a \mathbb{E}|V(t)|\, d\tilde{Q}_n(t) \le \frac{K}{((\log n)(\log\log n))^{1/2}}.$$

Now, given $\varepsilon > 0$ and $M > 0$, then, for all n large enough,

$$\begin{aligned}&\Pr\{(I)_n > \varepsilon\}\\ &\quad\le \Pr\{\tau_2 > M\} + \Pr\bigg\{\sup_{0<x\le M}\bigg|\frac{\sqrt{n}}{b_n}\int_0^x (|V(t)| - \mathbb{E}|V(t)|)\, d\tilde{Q}_n(t)\bigg| > \varepsilon\bigg\}\\ &\quad\le \Pr\{\tau_2 > M\} + \Pr\bigg\{\frac{\sqrt{n}}{b_n}\int_0^M (|V(t)| + \mathbb{E}|V(t)|)\, d\tilde{Q}_n(t) > \frac{\varepsilon}{2}\bigg\}.\end{aligned}$$

Since the first summand at the right tends to zero as $M \to \infty$, it suffices to prove that the second tends to 0 as $n \to \infty$ for all $M < \infty$ and all $\varepsilon > 0$, and this follows from the previous computation and Markov's inequality. $\square$

CLAIM 6.6.3. We have that $\lim_{n\to\infty}(III)_n = 0$ in probability.

PROOF. By Claim 6.6.1 and the definition of k_n (6.29), the central limit theorem in $\mathbb{R}$ implies that the sequence $\{(\tau_{2k_n} - \log(n-1))/\sqrt{k_n}\}_{n=2}^{\infty}$ is stochastically bounded. Therefore, it follows from

$$\Pr\{(III_n) > \varepsilon\} \le \Pr\left\{\left|\frac{\tau_{2k_n} - \log(n-1)}{\sqrt{k_n}}\right| > M\right\}$$
$$+ \Pr\left\{\sup_{-M\le x\le M} \frac{\sqrt{n}}{b_n}\left|\int_{\log(n-1)}^{x\sqrt{k_n}+\log(n-1)} (|V| - \mathbb{E}|V|)\, d\tilde{Q}_n\right| > \varepsilon\right\}$$

that it suffices only to prove that the second summand at the right tends to 0 as $n \to \infty$ for all $\varepsilon > 0$ and $M < \infty$. The previous proof still works here for $\alpha = -1$, but it doesn't for $\alpha > -1$ because $\sqrt{k_n}$ is of the order of $(\log n)^{1/2}$; thus, we resort to the theory of Gaussian processes and metric entropy bounds. We set, for $n \ge 2$ and $-M \le x \le M$,

$$\begin{aligned} L_n(x) &= \frac{\sqrt{n}}{b_n}\int_{\log(n-1)}^{x\sqrt{k_n}+\log(n-1)} (|V| - \mathbb{E}|V|)\, d\tilde{Q}_n \\ &= \frac{\sqrt{n}}{b_n}\int_{\log(n-1)/2}^{x\sqrt{k_n}+\log(n-1)/2} (|V(t)| - \mathbb{E}|V(t)|)t^{\alpha/2}\, dt. \end{aligned}$$

So, if for $-M \le x < y \le M$ we denote by $\|\cdot\|$ the L_1 norm for the measure $t^{\alpha/2}\, dt$ on the interval $[x\sqrt{k_n} + \frac{1}{2}\log(n-1), y\sqrt{k_n} + \frac{1}{2}\log(n-1)]$, we have

$$L_n(y) - L_n(x) = \frac{1}{(\log n)^{(\alpha+1)/2}}(\|V\| - \mathbb{E}\|V\|)$$

[we are only considering $\alpha > -1$, hence $b_n = n^{1/2}(\log n)^{(\alpha+1)/2}$]. We apply the Gaussian concentration inequality to estimate the size of $L_n(y) - L_n(x)$. To this effect we must compute $\sigma^2(\langle f, V\rangle)$ for $\|f\|_\infty \le 1$. For such f, $0 < a < b$ and $\alpha \ge 0$ we have

$$\begin{aligned} \mathbb{E}\left(\int_a^b V(t)f(t)t^{\alpha/2}\, dt\right)^2 &= \int_a^b\int_a^b \exp(-|s-u|)f(s)f(u)s^{\alpha/2}u^{\alpha/2}\, ds\, du \\ &\le 2\int_a^b\int_a^s \exp(-s)s^{\alpha/2}\exp(u)u^{\alpha/2}\, du\, ds \\ &\le 2b^\alpha\int_a^b \exp(-s)(\exp(s) - \exp(a))\, ds \\ &< 2b^\alpha(b-a), \end{aligned} \tag{6.33}$$

and, likewise, if $-1 \le \alpha < 0$,

$$\mathbb{E}\left(\int_a^b V(t)f(t)t^{\alpha/2}\,dt\right)^2 \le 2a^\alpha \int_a^b e^{-s}(e^s - e^a)\,ds < 2a^\alpha(b-a). \tag{6.33$'$}$$

Taking into account that $k_n \simeq (\log(n-1) - m_1)/m_2$, these two estimates give, for n large enough,

$$\begin{aligned}\sigma^2(\langle f, V\rangle) &\le \mathbb{E}\left(\int_{x\sqrt{k_n}+\log(n-1)/2}^{y\sqrt{k_n}+\log(n-1)/2} V(t)f(t)t^{\alpha/2}\,dt\right)^2\\ &\le C(y-x)(\log n)^{\alpha+1/2},\end{aligned}$$

for some $C < \infty$ independent of n, x and y. Then, the Gaussian concentration inequality gives

$$\begin{aligned}\Pr\{|L_n(y) - L_n(x)| > t\} &\le 2\exp\left(-\frac{t^2(\log n)^{\alpha+1}}{c(\log n)^{\alpha+1/2}|x-y|}\right)\\ &= 2\exp\left(-\frac{t^2(\log n)^{1/2}}{c|y-x|}\right)\end{aligned} \tag{6.34}$$

for some $c \in (0, \infty)$ independent of n and $t > 0$. This implies, by a simple computation that we omit, that there exists $D > 0$ such that

$$\mathbb{E}\exp\left(\frac{D(L_n(y) - L_n(x))^2(\log n)^{1/2}}{|x-y|}\right) \le 3,$$

with D independent of n. Then, if ψ is the Young modulus $\psi(x) = (\exp(x^2) - 1)/2$ and $\|\cdot\|_\psi$ is the Orlicz seminorm $\|\xi\|_\psi = \inf\{c > 0\colon \mathbb{E}\psi(|\xi|/c) \le 1\}$, the increments of the processes L_n satisfy

$$\|L_n(y) - L_n(x)\|_\psi \le \frac{|x-y|^{1/2}}{D^{1/2}(\log n)^{1/4}} := d_n(x, y)$$

for all $x, y \in [-M, M]$ and $n \ge 2$. This inequality allows us to apply Pisier's modification and extension of Dudley's entropy theorem [Pisier (1983), Theorem 1.1] to get

$$\mathbb{E}\sup_{-M<x\le M}|L_n(x)| \le K\int_0^{\mathscr{D}} \sqrt{\log(2N([-M, M], d_n, \varepsilon) + 1)}\,d\varepsilon,$$

where $N(t, d, \varepsilon)$ is the ε-covering number of the pseudometric space (T, d) and $\mathscr{D}$ is the d_n-diameter of $[-M, M]$. In our case,

$$N([-M, M], d_n, \varepsilon) \lesssim \frac{2M}{D(\log n)^{1/2}\varepsilon^2} \vee 1 \quad \text{and} \quad \mathscr{D} = \frac{(2M)^{1/2}}{D^{1/2}(\log n)^{1/4}},$$

which, plugged into the previous inequality gives that there exists $K < \infty$ such that

$$\mathbb{E}\sup_{-M\le x\le M}|L_n(x)| \le \frac{K}{(\log n)^{1/4}} \to 0.$$

This shows $(III_n) \to 0$ in probability. $\square$

CLAIM 6.6.4. The limit (6.32) holds.

PROOF. Since the variables $\tau_{2k} - \tau_{2k-2}$ are i.i.d. and, setting

$$m_k := m_1 + km_2,$$

the sequence $\{(\tau_{2k} - m_k)/\sqrt{k}\}_{k=1}^{\infty}$ is stochastically bounded by Claim 6.6.1 and the central limit theorem, it follows that for all $\eta > 0$ there is $M < \infty$ such that, both

$$\max_{k \le k_n} \Pr\{\tau_{2k} - \tau_{2k-2} > M\} < \eta \quad \text{and} \quad \max_{k \le k_n} \Pr\left\{\left|\frac{\tau_{2k} - m_k}{\sqrt{k}}\right| > M\right\} < \eta. \tag{6.35}$$

As a consequence, the proof of (6.32), reduces to showing that

$$\max_{1 \le k < k_n} \mathbb{E} \sup_{\substack{-M \le x \le M \\ 0 \le y \le M}} \frac{\sqrt{n}}{b_n} \left| \int_{m_k + x\sqrt{k}}^{m_k + x\sqrt{k} + y} (|V| - \mathbb{E}|V|)\, d\tilde{Q}_n \right| \to 0 \tag{6.36}$$

as $n \to \infty$. In analogy with the previous case, we can define

$$L_{n,k}(x, y) = \frac{\sqrt{n}}{b_n} \int_{m_{n,k} + x\sqrt{k}}^{m_{n,k} + x\sqrt{k} + y} (|V(t)| - \mathbb{E}|V(t)|)|t|^{\alpha/2}\, dt, \tag{6.37}$$

where $m_{n,k} := m_k - \frac{1}{2}\log(n-1)$ [hence, $|m_{n,k}| \le \frac{1}{2}\log(n-1)$].

We assume first that $\alpha \ge 0$. In this case, in analogy with the proof of the previous claim, we first apply the Gaussian concentration inequality to the increments of this process, $L_{n,k}(\mathbf{x}) - L_{n,k}(\mathbf{x}')$, with $\mathbf{x} = (x, y)$, $\mathbf{x}' = (x', y')$, and then the entropy bound. Just as in the proof of inequality (6.34), it is easy to see that, for all $k \le k_n$ and $n \ge n_0$ independent of k, $|L_{n,k}(\mathbf{x}) - L_{n,k}(\mathbf{x}')|$ has tail probabilities dominated by $2\exp[t^2(\log n)^{1/2}/(c|\mathbf{x} - \mathbf{x}'|)]$, for some $c > 0$ independent of k, n, $\mathbf{x}$ and $\mathbf{x}'$, where $|\mathbf{x} - \mathbf{x}'| := |x - x'| \vee |y - y'|$. Hence, there exists $D \in (0, \infty)$, independent of n and k, such that

$$\mathbb{E}\exp\left(\frac{D(L_{n,k}(\mathbf{x}) - L_{n,k}(\mathbf{x}'))^2(\log n)^{1/2}}{|\mathbf{x} - \mathbf{x}'|}\right) \le 3,$$

for all $k \le k_n$ and $n \ge n_0$. If we then set

$$d_n^2(\mathbf{x}, \mathbf{x}')) = \frac{|\mathbf{x} - \mathbf{x}'|}{D(\log n)^{1/2}},$$

since a rectangle of sides parallel to the axes, repectively of sizes M and $2M$, contains about $2M^2/\delta^2$ squares of side δ, the ε-covering number of $[-M, M] \times [0, M]$ for d_n is

$$N(\varepsilon) \simeq \frac{2M^2}{D^2\varepsilon^4 \log n}.$$

Then, the entropy bound used in the proof of the previous claim now gives that the expected values in (6.36) are all bounded by a fixed constant times

$(\log n)^{-1/4}$, and therefore their maximum tends to zero, proving the claim for $\alpha \geq 0$.

We now consider the case $-1 < \alpha < 0$. Let $\delta > 0$ be such that

$$\gamma_1 := \frac{\alpha+1}{\alpha+2} - \delta > 0.$$

Then, we observe that if $|m_{n,k} + x\sqrt{k}| \leq (\log n)^{\gamma_1}$, $0 \leq y \leq M$, and n is large enough so that $M \leq (\log n)^{\gamma_1}$, we have

$$\begin{aligned}\frac{\sqrt{n}}{b_n}\int_{m_{n,k}+x\sqrt{k}}^{m_{n,k}+x\sqrt{k}+y} \mathbb{E}|V(t)||t|^{\alpha/2}\,dt &\leq \sqrt{\frac{2}{\pi}}\frac{1}{(\log n)^{(\alpha+1)/2}}\int_{-(\log n)^{\gamma_1}-M}^{(\log n)^{\gamma_1}+M} |t|^{\alpha/2}\,dt\\ &\leq \frac{8}{\alpha+2}\sqrt{\frac{2}{\pi}}(\log n)^{((\alpha+2)/2)\gamma_1 - \frac{\alpha+1}{2}}\\ &= \frac{8}{\alpha+2}\sqrt{\frac{2}{\pi}}(\log n)^{-\delta(\alpha+2)/2} \to 0\end{aligned}$$

as $n \to \infty$ uniformly in x and y. Hence,

$$\begin{aligned}&\mathbb{E}\sup_{\substack{|m_{n,k}+x\sqrt{k}|\leq(\log n)^{\gamma_1}\\ 0\leq y\leq M}} \frac{\sqrt{n}}{b_n}\left|\int_{m_{n,k}+x\sqrt{k}}^{m_{n,k}+x\sqrt{k}+y}(|V(t)| - \mathbb{E}|V(t)|)|t|^{\alpha/2}\,dt\right|\\ &\quad\leq \frac{8}{\alpha+2}\sqrt{\frac{2}{\pi}}(\log n)^{-\delta(\alpha+2)/2}\\ &\qquad + \mathbb{E}\sup_{\substack{|m_{n,k}+x\sqrt{k}|\leq(\log n)^{\gamma_1}\\ 0\leq y\leq M}} \frac{\sqrt{n}}{b_n}\left|\int_{m_{n,k}+x\sqrt{k}}^{m_{n,k}+x\sqrt{k}+y}|V(t)||t|^{\alpha/2}\,dt\right|\\ &\quad\leq \frac{16}{\alpha+2}\sqrt{\frac{2}{\pi}}(\log n)^{-\delta(\alpha+2)/2} \to 0.\end{aligned}$$

Then, in order to conclude the proof of this Claim for $-1 < \alpha < 0$, it suffices to show

$$\lim_{n\to\infty} 2\mathbb{E}\sup_{\substack{\frac{1}{2}(\log n)^{\gamma_1}\leq u\leq 2\log n\\ 0\leq y\leq M}} \frac{\sqrt{n}}{b_n}\left|\int_u^{u+y}(|V(t)| - \mathbb{E}|V(t)|)|t|^{\alpha/2}\,dt\right| = 0. \tag{6.38}$$

Since

$$\lim_{n\to\infty}\sup_{u\in\mathbb{R}} \frac{\sqrt{n}}{b_n}\int_u^{u+M}\mathbb{E}|V(t)||t|^{\alpha/2}\,dt \leq \lim_{n\to\infty}\frac{\sqrt{2/\pi}}{(\log n)^{(\alpha+1)/2}}\int_{-M/2}^{M/2}|t|^{\alpha/2}\,dt = 0,$$

we may add the centering to the integral in (6.38), use positivity of $|V|$ to replace the variable y by M in the upper limit of integration and then subtract the centering, to conclude that the limit (6.38) is equivalent to

$$\lim_{n\to\infty} 2\mathbb{E}\sup_{\frac{1}{2}(\log n)^{\gamma_1}\leq u\leq 2\log n} \frac{\sqrt{n}}{b_n}\left|\int_u^{u+M}(|V(t)| - \mathbb{E}|V(t)|)|t|^{\alpha/2}\,dt\right| = 0. \tag{6.39}$$

Redefining

$$L_n(u) = \frac{\sqrt{n}}{b_n}\int_u^{u+M}(|V(t)| - \mathbb{E}|V(t)|)|t|^{\alpha/2}\,dt,$$

and bounding $|L_n(v) - L_n(u)| \le |L_n(u)| + |L_n(v)|$ from above by $|v-u| > M$, and by $(\sqrt{n}/b_n)(|\int_u^v(|V(t)|-\mathbb{E}|V(t)|)|t|^{\alpha/2}\,dt| + |\int_{u+M}^{v+M}(|V(t)|-\mathbb{E}|V(t)|)|t|^{\alpha/2}\,dt|)$ for $|v-u| \le M$, the Gaussian concentration inequality, together with the estimate in (6.33′), gives

$$\begin{aligned}\Pr\{|L_n(v) - L_n(u)| > \varepsilon\} &\le 4\exp\biggl(-\frac{\varepsilon^2(\log n)^{\alpha+1}}{c(|v-u|\wedge M)(\log n)^{\alpha\gamma_1}}\biggr)\\ &= 4\exp\biggl(-\frac{\varepsilon^2(\log n)^{\gamma_2}}{c(|v-u|\wedge M)}\biggr),\end{aligned}$$

where c is a finite positive constant independent of n, u and v, and $\gamma_2 := \alpha + 1 - \alpha\gamma_1 = (2(\alpha+1)/\alpha+2) + \alpha\delta > 0$. As an immediate consequence, there exists $D > 0$ independent of n, u and v, such that

$$\mathbb{E}\exp\biggl(\frac{D(\log n)^{\gamma_2}}{|v-u|\wedge M}\biggr) \le 6,$$

which, by Pisier's entropy bound [Pisier (1983)], gives

$$\mathbb{E}\sup_{\frac{1}{2}(\log n)^{\gamma_1}\le u\le 2\log n}|L_n(u)| \le K\int_0^{\mathscr{D}}\sqrt{\log(5N(\varepsilon)+1)}\,d\varepsilon$$

for some universal constant $K < \infty$, where $N(\varepsilon)$ is the ε-covering number of the interval $[2^{-1}(\log n)^{\gamma_1}, 2\log n]$ for the distance $d_n(u,v) = (|v-u|\wedge M)^{1/2}/[D(\log n)^{\gamma_2}]^{1/2}$, and $\mathscr{D}$ is the d_n-diameter of this interval. Since $\mathscr{D} = M^{1/2}/[D^{1/2}(\log n)^{\gamma_2/2}]$ and $N(\varepsilon)$ is dominated by a constant times $\log n/[\varepsilon^2(\log n)^{\gamma_2}]$ for $\varepsilon < \mathscr{D}$, we obtain

$$\mathbb{E}\sup_{\frac{1}{2}(\log n)^{\gamma_1}\le u\le 2\log n}|L_n(u)| \le C\biggl(\frac{\log\log n}{(\log n)^{\gamma_2}}\biggr)^{1/2}$$

for some constant C independent of n. This proves the limit (6.39), and therefore the claim for $-1 < \alpha < 0$.

The case $\alpha = 1$ can be treated in complete analogy with the case $-1 < \alpha < 2$. Now, since $b_n = \sqrt{n\log\log n}$, the threshold $(\log n)^{\gamma_1}$ can be replaced by $(\log\log n)^{\gamma}$ for some (any) $\gamma \in (0,1)$, as for such γ,

$$\begin{aligned}&\frac{\sqrt{n}}{b_n}\int_{m_{n,k}+x\sqrt{k}}^{m_{n,k}+x\sqrt{k}+y}\mathbb{E}|V(t)||t|^{-1/2}\,dt\\ &\qquad\le\sqrt{\frac{2}{\pi}}\frac{1}{(\log\log n)^{1/2}}\int_{-(\log\log n)^{\gamma}-M}^{(\log\log n)^{\gamma}+M}|t|^{-1/2}\,dt \to 0.\end{aligned}$$

Then, Gaussian concentration followed by Pisier's entropy bound now give

$$\mathbb{E} \sup_{\frac{1}{2}(\log\log n)^{\gamma} \le u \le 2\log n} |L_n(u)| \le \frac{C}{(\log\log n)^{\gamma/2}} \to 0$$

for a constant C independent of n. We skip the details as they are similar to those for $-1 < \alpha < 0$.

This completes the proof of Lemma 6.6. $\square$

REMARK 6.1. We should mention that symmetry is not essential at all in the previous proof. We only had it there in order to produce a nice change to the Ornstein–Uhlenbeck process; also, the values of Q' in (6.24) are only relevant for t close to 0 or to 1 but its values on the midrange $t \in (a, 1-a)$ for any fixed $0 < a < 1$ are not important. Verification of these facts is elementary and is omitted.

Summarizing, we give the following theorem.

THEOREM 6.7. *Let Q be a differentiable quantile function whose derivative Q' is bounded on any interval $[a, 1-a]$, $0 < a < 1$, and such that*

$$\lim_{t\to 0} \frac{t^{3/2} Q'(t)}{|\frac{1}{2}\log t|^{\alpha/2}} = \lim_{t\to 0} \frac{t^{3/2} Q'(1-t)}{|\frac{1}{2}\log t|^{\alpha/2}} = 1 \tag{6.40}$$

for some $\alpha \in [-1, \infty)$. Let F be the corresponding cumulative distribution function, let

$$\gamma_n = \sqrt{\frac{2n}{\pi}} \int_{Q(1/n)}^{Q(1-1/n)} \sqrt{F(t)(1-F(t))}\, dt \tag{5.9}$$

and let

$$\begin{aligned} b_n &= \sqrt{\frac{8\cdot 2^{-\alpha}}{\alpha+1}} \sqrt{n}(\log n)^{(\alpha+1)/2} \quad \textit{for } \alpha > -1,\\ b_n &= \sqrt{\frac{8\cdot 2^{-\alpha}}{\alpha+1}} \sqrt{n \log\log n} \quad \textit{for } \alpha = -1. \end{aligned} \tag{6.41}$$

Let X_i, $i \in \mathbb{N}$, be i.i.d. with commmon c.d.f. F, let F_n be the corresponding empirical c.d.f. and let

$$Z_n = n \int_{-\infty}^{\infty} |F_n(t) - F(t)|\, dt, \qquad n \in \mathbb{N}. \tag{4.14}$$

Then,

$$\frac{Z_n - \gamma_n}{b_n} \to_d \sqrt{1 + \frac{2\log 2}{\pi} - \frac{13}{3\pi}}\, g, \tag{6.42}$$

where g is a standard normal random variable, with convergence of the absolute moments of any order $0 < p < 2$. Moreover γ_n can be replaced by $\mathbb{E}Z_n$ in (6.42).

PROOF. By Remark 6.1, we may as well assume that Q' has the form (6.24). A simple computation using the form c_n of the normalizing constants b_n in Proposition A.1 shows that $X_i \in DA_2(b_n)$ with standard normal limit for $\sum_{i=1}^n (X_i - \mathbb{E}X_i)/b_n$, b_n as in (6.41). Now, as mentioned above, Lemma 4.1 and the weighted approximation of the uniform empirical process by Brownian bridges (Corollary 6.1) imply that the sequences in (6.5) are weak convergence equivalent; that is, the sequence at the left side of (6.42) is weak convergence equivalent to the sequence $\{G_n\}$ in (6.9). The sequence $\{G_n\}$ is stochastically bounded and the pth absolute powers of its terms are uniformly integrable for any $p > 0$ by Proposition 6.2. All of its subsequential limits in law have Gaussian-like tail probabilities by Proposition 6.2 and are infinitely divisible by Lemma 6.6; therefore, by Proposition 6.5, they are normal. But then, by Proposition 6.4, they all coincide with the variable at the right of (6.42). Convergence of moments in (6.42) follows from the uniform integrability result in Theorem 4.2 and the limit (6.12) (Corollary 6.3). It also follows from Corollary 6.3 that γ_n can be replaced by $\mathbb{E}Z_n$ in (6.42). □

Obviously, the integral $\int_{-\infty}^{\infty} |V(t)|(|t|^{\alpha/2} \wedge 1)\, dt$ exists for $\alpha < -2$. The Gaussian part of Theorems 5.1 and 6.7 give central limit theorems for the Brownian bridge, or, what is the same, for the Ornstein–Uhlenbeck process. The statements for this last process have a particularly nice form and are somewhat surprising. Thus, we single them out in the following theorem.

THEOREM 6.8. *Let $V(t)$, $t \in \mathbb{R}$, be a stationary Ornstein–Uhlenbeck process and let $\alpha \in [-2, \infty)$. Then*:

(a) *If* $\alpha > -1$,

$$\frac{1}{\sqrt{\frac{8\cdot 2^{-\alpha}}{\alpha+1}}\, s^{(\alpha+1)/2}} \int_{-s/2}^{s/2} (|V(t)| - \mathbb{E}|V(t)|)|t|^{\alpha/2}\, dt \to_d \sqrt{1 + \frac{2\log 2}{\pi} - \frac{13}{3\pi}}\, g. \tag{6.43}$$

(b) *For* $\alpha = 1$,

$$\frac{1}{\sqrt{\frac{8\cdot 2^{-\alpha}}{\alpha+1}}(\log s)^{1/2}} \int_{-s/2}^{s/2} (|V(t)| - \mathbb{E}|V(t)|)|t|^{-1/2}\, dt \to_d \sqrt{1 + \frac{2\log 2}{\pi} - \frac{13}{3\pi}}\, g. \tag{6.44}$$

(c) *The integrals*

$$\int_{-\infty}^{\infty} (|V(t)| - \mathbb{E}|V(t)|)|t|^{\alpha/2}\, dt \quad \textit{for } -2 < \alpha < -1$$

as well as

$$\int_{-\infty}^{\infty} (|V(t)| - \mathbb{E}|V(t)|)(|t|^{-1} \wedge 1)\, dt$$

exist in the sense of convergence of all moments as the limits of integration expand to $+\infty$ *and to* $-\infty$.

For $\alpha = 0$ this result follows from Theorem 9, page 94, in Mandl (1968).

APPENDIX

We have used several times in this article two explicit formulas for the norming constants in the domain of attraction of the normal law, based on quantiles, and due to CCHM (1986b). Their proof of the equivalence of these two formulas is analytic (Propositions A.1 and A.2), but then, consistent with their approach to the central limit theorem, they show that these formulas work by means of the weighted approximation of the uniform empirical process by Brownian bridges. Because these formulas are so useful, we believe it is worthwhile that they be incorporated into the classical approach to the central limit theorem and, for this reason, here we give a direct proof of the fact that these expressions do satisfy the classical defining relation for the norming constants, namely, $nb_n^{-2}\operatorname{Var}(XI_{|X|\le b_n}) \to 1$ (or $nb_n^{-2}U(b_n) \to 1$ if $\mathbb{E}X^2 = \infty$).

PROPOSITION A.1 [CCHM (1986b)]. *Let* X, X_i, $i \in \mathbb{N}$, *be i.i.d. random variables in* $DA_2(b_n)$ *and let* Q *be their quantile function. Set*

$$c_n := \sqrt{n}\left(\int_{1/n}^{1-1/n} Q^2(t)\,dt - \left(\int_{1/n}^{1-1/n} Q(t)\,dt\right)^2\right)^{1/2}$$

and

$$d_n := \left(\int_{1/n}^{1-1/n}\int_{1/n}^{1-1/n}(s\wedge t - st)\,dQ(s)\,dQ(t)\right)^{1/2}.$$

Then

$$\lim_{n\to\infty}\frac{c_n}{b_n} = \lim_{n\to\infty}\frac{d_n}{b_n} = 1 \tag{A.1}$$

or, equivalently,

$$\frac{1}{c_n}\sum_{i=1}^n (X_i - \mathbb{E}X) \to_d g \quad \textit{and} \quad \frac{1}{d_n}\sum_{i=1}^n (X_i - \mathbb{E}X) \to_d g, \tag{A.1$'$}$$

where g *is a standard normal random variable.*

Note that, since $X =_d Q(\theta)$, θ uniform on $(0,1)$, we have $\int_{1/n}^{1-1/n} Q(t)\,dt \to \mathbb{E}X$ and $\int_{1/n}^{1-1/n} Q^2(t)\,dt \to \mathbb{E}X^2$. Hence, if $\mathbb{E}X^2 = \infty$ then

$$c_n \asymp \sqrt{n}\left(\int_{1/n}^{1-1/n} Q^2(t)\,dt\right)^{1/2},$$

which is the expression in (4.21).

PROOF. It can be easily seen that

$$d_n^2 = c_n^2 + \left(Q^2\left(\frac{1}{n}\right) + Q^2\left(1-\frac{1}{n}\right)\right) - \frac{1}{n}\left(Q\left(\frac{1}{n}\right) + Q\left(1-\frac{1}{n}\right)\right)^2 + 2\left(Q\left(\frac{1}{n}\right) + Q\left(1-\frac{1}{n}\right)\right)\int_{1/n}^{1-1/n} Q(t)\,dt \tag{A.2}$$

[see, e.g., Shorack and Wellner (1986), page 43]. Now, (A.2) and (4.6) imply that

$$\lim_{n\to\infty} \frac{d_n^2 - c_n^2}{b_n^2} = 0$$

and, therefore, it suffices to prove the first limit in (A.1). The result is trivial if $\mathbb{E}X^2 < \infty$, since then $\lim_{n\to\infty} n^{-1/2}c_n = \mathrm{Var}(X)$. Hence, we can assume $\mathbb{E}X^2 = \infty$ and then replace c_n by $\sqrt{n}(\int_{1/n}^{1-1/n} Q^2(t)\,dt)^{1/2}$, which we will also denote by c_n for ease of notation. Set $y_n = |Q(1/n)| \vee |Q(1-1/n)|$ and $z_n = |Q(1/n)| \wedge |Q(1-1/n)|$. Since $\mathbb{E}X^2 = \infty$, we have $y_n \to \infty$. We will also assume $z_n \to \infty$. Otherwise, $Q(1/n)$ or $Q(1-1/n)$ is bounded and a slight modification of the reasoning below provides the same result. Set $U(t) = \mathbb{E}X^2 I_{-t<X\le t}$, $t > 0$ (note that this definition of U differs slightly from that in Section 4, but, by slow variation, both share the same asymptotic properties). We recall from the theory of domains of attraction [Feller (1971) or Araujo and Giné (1980)] that U is slowly varying,

$$\lim_{x\to\infty} \frac{x^2 \Pr\{|X| \ge x\}}{U(x)} = 0 \tag{A.3}$$

and (A.1′) [or (A.1)] holds if and only if

$$\lim_{n\to\infty} \frac{n}{{c_n}^2} U(c_n) = 1.$$

Observe that (4.5) and the facts that $X =_d Q(\theta)$, θ uniform on $(0,1)$, imply $U(t) = \int_{F(-t)}^{F(t)} Q^2(y)dy$. Now

$$\begin{aligned}\frac{n}{{c_n}^2}U(c_n) &= \frac{\int_{F(-c_n)}^{F(c_n)} Q^2(y)\,dy}{\int_{1/n}^{1-1/n} Q^2(y)\,dy}\\ &= 1 + \frac{\int_{F(-c_n)}^{1/n} Q^2(y)\,dy}{\int_{1/n}^{1-1/n} Q^2(y)\,dy} + \frac{\int_{1-1/n}^{F(c_n)} Q^2(y)\,dy}{\int_{1/n}^{1-1/n} Q^2(y)\,dy}\\ &:= 1 + \varepsilon_{1,n} + \varepsilon_{2,n}.\end{aligned} \tag{A.4}$$

Therefore, it suffices to prove that $\lim_{n\to\infty} \varepsilon_{i,n} = 0$, $i = 1, 2$. We prove first that

$$\frac{\mathbb{E}X^2 I_{Q(1/n)<X<Q(1-1/n)}}{U(y_n)} \to 1 \tag{A.5}$$

as $n \to \infty$. To see this, suppose $|Q(1/n)| \geq |Q(1-1/n)|$. Then using (4.4), (4.5) and the fact that $X =_d Q$, we obtain

$$
\begin{aligned}
\left|\frac{\mathbb{E}X^2 I_{Q(1/n)<X<Q(1-1/n)}}{U(y_n)} - 1\right| &= \frac{\int Q^2(y) I_{Q(1-1/n)\leq Q(y)\leq -Q(1/n)}\,dy}{U(y_n)} \\
&= \frac{\int Q^2(y) I_{Q(1-1/n)<Q(y)\leq -Q(1/n)}\,dy}{U(y_n)} \\
&\quad + \frac{Q^2(1-1/n)\Pr\{X = Q(1-1/n)\}}{U(y_n)} \\
&\leq \frac{y_n^2}{nU(y_n)} + \frac{z_n^2 \Pr\{|X| \geq z_n\}}{U(y_n)} \\
&\leq \frac{y_n^2 \Pr\{|X| \geq y_n\}}{U(y_n)} + \frac{z_n^2 \Pr\{|X| \geq z_n\}}{U(y_n)}
\end{aligned}
\tag{A.6}
$$

and the same inequality holds if $|Q(1/n)| < |Q(1-1/n)|$. Now (A.3) and (A.6) prove (A.5). As a consequence,

$$
\begin{aligned}
\frac{c_n^2}{y_n^2} &= \frac{n\int_{1/n}^{1-1/n} Q^2(y)\,dy}{y_n^2} \geq \frac{n\int_{Q(1/n)<Q<Q(1-1/n)} Q^2(y)\,dy}{y_n^2} \\
&\geq \frac{\int_{Q(1/n)<Q<Q(1-1/n)} Q^2(y)\,dy}{y_n^2 \Pr\{|X| \geq y_n\}} = \frac{\mathbb{E}X^2 I_{Q(1/n)<X<Q(1-1/n)}}{y_n^2 \Pr\{|X| \geq y_n\}} \to \infty
\end{aligned}
\tag{A.7}
$$

by (A.3) and (A.5). Therefore, $F(c_n) \geq 1-1/n$ for n large enough, which implies $\varepsilon_{2,n} \geq 0$. Then

$$
\begin{aligned}
0 \leq \varepsilon_{2,n} &= \frac{n\int_{1-1/n}^{F(c_n)} Q^2(y)\,dy}{c_n^2} \leq \frac{n}{c_n}\int_{1-1/n}^{F(c_n)} Q(y)\,dy \\
&\leq \frac{n}{c_n}\int Q(y) I_{Q(1-1/n)<Q\leq c_n}\,dy + \frac{n}{c_n}\int_{1/n}^{1} Q(y) I_{Q(y)=Q(1-1/n)}\,dy \\
&= \frac{n}{c_n}\int Q(y) I_{Q(1-1/n)<Q\leq c_n}\,dy + \frac{Q(1-1/n)}{c_n}.
\end{aligned}
\tag{A.8}
$$

Note that the last inequality in the first line already shows that $\varepsilon_{2,n} \leq 1$ and, likewise $\varepsilon_{1,n} \leq 1$. This proves

$$
\frac{n}{c_n^2} U(c_n) \leq 3 \tag{A.9}
$$

for n large enough, a fact that will be used later. In order to estimate the bound given by (A.8) observe that

$$
\int_0^a \Pr\{X > t\}\,dt = \int_{[0,a]} y\,dF(y) + a\Pr\{X > a\}
$$

and therefore

$$
\begin{aligned}
&\frac{n}{c_n}\int Q(y) I_{Q(1-1/n)<Q\le c_n}\,dy \\
&\qquad = \frac{n}{c_n}\int_{Q(1-1/n)}^{c_n} \Pr\{X>t\}\,dt - n\Pr\{X>c_n\} \\
&\qquad\quad + \frac{n}{c_n} Q(1-1/n)\Pr\{X>Q(1-1/n)\}.
\end{aligned} \tag{A.10}
$$

Then, (A.3), (A.9) and slow variation of U show that for all $x>0$,

$$0=\lim_{n\to\infty}\frac{nx^2\Pr\{X>c_nx\}}{(n/c_n^2)U(c_nx)}\ge\frac{1}{3}\limsup_{n\to\infty} nx^2\Pr\{X>c_nx\};$$

that is,

$$n\Pr\{X>c_nx\}\to 0 \tag{A.11}$$

for all $x>0$. Moreover, since $n\Pr\{X>Q(1-1/n)\}\le 1$, dominated convergence gives

$$\frac{n}{c_n}\int_{Q(1-1/n)}^{c_n}\Pr\{X>t\}\,dt=\int_{Q(1-1/n)/c_n}^{1} n\Pr\{X>xc_n\}\,dx\to 0 \tag{A.12}$$

as $n\to\infty$. Now (A.7), (A.10), (A.11) and (A.12) prove that the upper bound in (A.8) tends to 0; that is, $\varepsilon_{2,n}\to 0$. Similarly, $\varepsilon_{1,n}\to 0$, which completes the proof. □

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REFERENCES

ARAUJO, A. and GINÉ, E. (1980). *The Central Limit Theorem for Real and Banach Valued Random Variables*. Wiley, New York.

BILLINGSLEY, P. (1968). *Convergence of Probability Measures.* Wiley, New York.

BORELL, C. (1975). The Brünn–Minkowski inequality in Gauss space. *Invent. Math.* **30** 207–216.

BYCZKOWSKI, T. (1977). Gaussian measures on L_p spaces $0 \leq p < \infty$. *Studia Math.* **59** 249–261.

CSÖRGŐ, M. and HORVÁTH, L. (1986). Approximations of weighted empirical and quantile processes. *Statist. Probab. Lett.* **4** 275–280.

CSÖRGŐ, M. and HORVÁTH, L. (1988a). On the distributions of L_p norms of weighted uniform empirical and quantile processes. *Ann. Probab.* **16** 142–161.

CSÖRGŐ, M. and HORVÁTH, L. (1988b). Central limit theorems for L_p-norms of density estimators. *Probab. Theory Related Fields* **80** 269–291.

CSÖRGŐ, M. and HORVÁTH, L. (1993). *Weighted Approximations in Probability and Statistics*. Wiley, New York.

CSÖRGŐ, M., CSÖRGŐ, S., HORVÁTH, L. and MASON, D. M. (1986a). Weighted empirical and quantile processes. *Ann. Probab.* **14** 31–85.

CSÖRGŐ, M., CSÖRGŐ, S., HORVÁTH, L. and MASON, D. M. (1986b). Normal and stable convergence of integral functions of the empirical distribution function. *Ann. Probab.* **14** 86–118.

CSÖRGŐ, M., HORVÁTH, L. and SHAO, Q. M. (1993). Convergence of integrals of uniform empirical and quantile processes. *Stochastic Process. Appl.* **45** 283–294.

CSÖRGŐ, S. and MASON, D. (1991). A probabilistic approach to the tails of infinitely divisible laws. In *Sums, Trimmed Sums and Extremes* 317–336. Birkhäuser, Boston.

DE ACOSTA, A., ARAUJO, A. and GINÉ, E. (1978). Poisson measures, Gaussian measures and the central limit theorem in Banach spaces. In *Advances in Probability* (J. Kuelbs, ed.) **4** 1–68. Dekker, New York.

DE ACOSTA, A. and GINÉ, E. (1979). Convergence of moments and related functionals in the central limit theorem in Banach spaces. *Z. Wahrsch. Verw. Gebiete* **48** 213–231.

FELLER, W. (1971). *An Introduction to Probability Theory and Its Applications* **2**. Wiley, New York.

GINÉ, E. (1983). The Lévy-Lindeberg central limit theorem in L_p, $0 < p < 1$. *Proc. Amer. Math. Soc.* **88** 147–153.

GINÉ, E. and ZINN, J. (1983). Central limit theorems and weak laws of large numbers in certain Banach spaces. *Z. Wahrsch. Verw. Gebiete* **62** 323–354.

HORN, R. A. (1972). On necessary and sufficient conditions for an infinitely divisible distribution to be normal or degenerate. *Z. Wahrsch. Verw. Gebiete* **21** 179–187.

JAIN, N. C. (1977). Central limit theorems and related questions in Banach space. In *Proceedings of Symposium in Pure and Applied Mathematics* **31** 55–65. Amer. Math. Soc. Providence, RI.

LAWNICZAK, A. (1983). The Lévy–Lindeberg central limit theorem in Orlicz spaces. *Proc. Amer. Math. Soc.* **89** 673–679.

LEDOUX, M. and TALAGRAND, M. (1991). *Probability in Banach Spaces*. Springer, Berlin.

MANDL, P. (1968). *Analytical Treatment of One-Dimensional Markov Processes*. Springer, New York.

MANDREKAR, V. and ZINN, J. (1980). Central limit problem for symmetric case: convergence to non-Gaussian laws. *Studia Math.* **67** 279–296.

MASON, D. M. (1991). A note on weighted approximations to the uniform empirical and quantile processes. In *Sums, Trimmed Sums and Extremes* 269–283. Birkhäuser, Boston.

MASON, D. M. (1998). An exponential inequality for a weighted approximation to the uniform empirical process with applications. Preprint.

MONTGOMERY-SMITH, S. J. (1994). Comparison of sums of independent identically distributed random variables. *Probab. Math. Statist.* **14** 281–285.

PISIER, G. (1983). Some applications of the metric entropy condition to harmonic analysis. *Banach Spaces, Harmonic Analysis and Probability. Lecture Notes in Math.* **995** 123–154. Springer, Berlin.

PISIER, G. (1986). Probabilistic methods in the geometry of Banach spaces. *Probability and Analysis. Lecture Notes in Math.* **1026** 167–241. Springer, Berlin.

RESNICK, S. I. (1987). *Extreme Values, Regular Variation and Point Processes.* Springer, New York.

SHORACK, G. R. and WELLNER, J. A. (1986). *Empirical Processes with Applications to Statistics.* Wiley, New York.

SUDAKOV, V. N. and TSIREL'SON, B. S. (1974). Extremal properties of half-spaces for spherically invariant measures. *Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)* **41** 14–24.

TALAGRAND, M. (1996). New concentration inequalities on product spaces. *Invent. Math.* **126** 505–563.

UCHIYAMA, K. (1980). Brownian first exit from and sojourn over one-sided moving boundary and application. *Z. Wahrsch. Verw. Gebiete* **54** 75–116.

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