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CRITICAL LARGE DEVIATIONS OF ONE-DIMENSIONAL ANNEALED BROWNIAN MOTION IN A POISSONIAN POTENTIAL

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We derive a large deviation principle for the position at large times t of a one-dimensional annealed Brownian motion in a Poissonian potential in the critical spatial scale $t^{1/3}$. Here "annealed" means that averages are taken with respect to both the path and environment measures. In contrast to the d-dimensional case for $d \geq 2$ in the critical scale $t^{d/(d+2)}$ as treated by Sznitman, the rate function which measures the large deviations exhibits three different regimes. These regimes depend on the position of the path at time t. Our large deviation principle has a natural application to the study of a one-dimensional annealed Brownian motion with a constant drift in a Poissonian potential.

0. Introduction. The main goal of the present article is to derive a large deviation principle for $t^{-1/3}Z_t$, where Z_t is the position at time t of an "annealed" one-dimensional Brownian motion moving in a soft repulsive Poissonian potential. Let $Z_\bullet$ denote a canonical Brownian motion, P_0 the Wiener measure on $C(\mathbb{R}_+, \mathbb{R})$ and $\mathbb{P}$ the law of a Poisson point process of constant intensity $\nu > 0$ on the space Ω of simple pure point measures on $\mathbb{R}$. We define the annealed weighted measure as

$$Q_t = \frac{1}{S_t} \exp\left\{-\int_0^t V(Z_s, \omega)\, ds\right\} P_0(dw)\mathbb{P}(d\omega), \tag{0.1}$$

where S_t is the normalizing constant.

For a point configuration $\omega = \sum_i \delta_{x_i} \in \Omega$ and $x \in \mathbb{R}$, the Poissonian potential is defined as $V(x, \omega) = \sum_i W(x - x_i)$. The function W modeling the soft obstacles is nonnegative, bounded, measurable, compactly supported and not a.s. equal to zero. Here the expression "soft obstacles" signifies that the term $\exp\{-\int_0^t V(Z_s, \omega)\, ds\}$ in (0.1) represents a penalty for the Brownian particle spending time in the support of some $W(\cdot - x_i)$.

Recall that the asymptotic behavior of the normalizing constant S_t, $t \to \infty$, restricted to $d = 1$ is

$$S_t = \exp\{-c(1, \nu)t^{1/3}(1 + o(1))\}, \tag{0.2}$$

where $c(1, \nu) = \inf_{c \geq 0}(\nu c + \pi^2/2c^2)$; see [3].

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Our main result is:

THEOREM 1. (i) (*Large deviations.*) *Under* Q_t, $t^{-1/3} Z_t$ *satisfies a large deviation principle at rate* $t^{1/3}$ *with rate function* $J_1(\cdot)$, *where*

$$J_1(y) = \begin{cases} 0, & |y| \in [0, c_0], \\ \nu|y| + \dfrac{\pi^2}{2y^2} - c(1,\nu), & |y| \in [c_0, c_1], \\ \nu c_1 + \dfrac{\pi^2}{2c_1^2} + \beta_0(1)(|y| - c_1) - c(1,\nu), & |y| \geq c_1, \end{cases} \tag{0.3}$$

$c_0 = (\pi^2/\nu)^{1/3}$ *is the unique point where* $\inf_{c \geq 0}(\nu c + \pi^2/2c^2)$ *is attained*, $c_1 = (\pi^2/(\nu - \beta_0(1)))^{1/3}$, *which is possibly infinite, and* $0 < c_0 < c_1$.

Here $\beta_0(x)$ *denotes the one-dimensional annealed Liapounov exponent introduced in* [12], *which is a norm in the* x *variable and* $\beta_0(1) \leq \nu$.

(ii) (*Criterion for* $c_1 < \infty$.) *Let* $W(\cdot)$ *be a nonnegative, bounded, measurable function with compact support in* $(-l, l)$, $l \in (0, \infty)$, *not a.s. equal to zero and let* $\nu \in (0, \infty)$. *If* $\|W\|_\infty$ *is sufficiently small we have* $c_1 < \infty$.

Observe that the decay rate of the normalizing constant S_t is the same as the rate of the large deviation principle we wish to derive. Therefore we will call the spatial scale $t^{1/3}$ critical. For the reader's convenience we include a picture of $J_1(y)$ for $y \geq 0$ (see Figure 1).

For a discussion of the function $J_1(y) + c(1,\nu) = I(y)$, we refer to the comment after Theorem 1.1.

A natural application of the preceding large deviation principle, and in fact one of our main motivations for this work, is the study of the long-term behavior of a one-dimensional "annealed Brownian motion with constant drift h" in the Poissonian potential V. That is, we replace in (0.1) the measure dQ_t by

$$dQ_t^h = \frac{S_t}{\tilde{S}_t^h} \exp\{hZ_t\}\, dQ_t, \tag{0.4}$$

where $\tilde{S}_t^h$ is the normalizing constant.

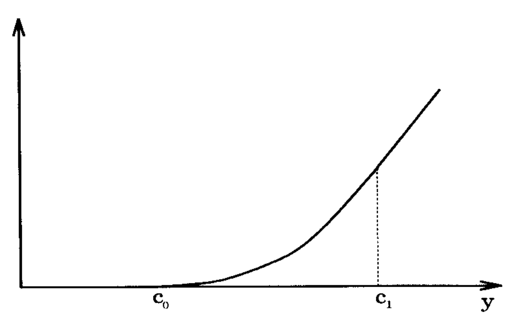

FIG. 1. *Function* $J_1(y)$.

This model appeared in the physics literature (Grassberger and Procaccia [5]) in the context of “hard obstacles” with emphasis on the transition of regime which occurs in the small $|h|$ and the large $|h|$ situations. In this context “hard obstacles” stands for the replacement of $\exp\{-\int_0^t V(Z_s,\omega)\,ds\}$ in (0.1) by $1_{\{T>t\}}$. Here T denotes the entrance time in $\bigcup_i B(x_i,a)$, the so-called trap configuration, where $B(x_i,a)$ is the closed ball with center x_i and radius a. The hard-obstacle case in fact corresponds to the singular shape function $W_{\text{h.o.}}(x)=\infty 1_{[-a,a]}(x)$. On the mathematical side it follows from Sznitman's work [12] (see also [4] for earlier results), treating both hard obstacles as well as soft obstacles, that there is a critical threshold for $|h|$ which in $d=1$ in fact equals $\beta_0(1)$. For precise statements we refer the reader to Theorem 3.1 and the subsequent discussion. Let us mention at this point that the analogue of Theorem 1 for hard obstacles ($d=1$) can be found in [8], Remark 1.4.

The result of Theorem 1 should be contrasted with the case $d\geq 2$. Indeed, it was shown in [12], part II, that:

THEOREM 2 ([12]). *For $d\geq 2$, $t^{-d/(d+2)}Z_t$ satisfies under Q_t a large deviation principle at rate $t^{d/(d+2)}$ with rate function $J_d(y)=\beta_0(y)$, $y\in\mathbb{R}^d$.*

Observe, however, that the graph of the function $J_1(y)$ for $|y|\geq c_1$ is a straight line with slope $\beta_0(1)$.

We now give an intuitive argument why the one-dimensional case is singular in the critical spatial scale $t^{d/(d+2)}$:

The proof of the upper bound in Theorem 2, the most difficult part, is crucially dependent on the so-called method of enlargement of obstacles introduced by Sznitman (for a review see [11]). Roughly speaking, this method produces a certain coarse-grained picture of the Poisson cloud configurations in terms of “clearings” of size $\sim t^{1/(d+2)}$ which are used by the process as resting places and “forests” of size $\sim t^{1/(d+2)}$. The forests represent in some sense a hostile environment for the process. To prove the upper bound in Theorems 1 and 2, one has to show with a suitable uniformity in y an asymptotic upper bound of the type

$$\begin{aligned}\limsup_{t\to\infty} t^{-d/(d+2)}\log \mathbb{E}\otimes E_0\left[Z_t\in B(yt^{d/(d+2)}), \exp\left\{-\int_0^t V(Z_s,\omega)\,ds\right\}\right]\\ \leq -(J_d(y)+c(d,\nu)),\end{aligned}\tag{0.5}$$

where $B(y)$ is the closed ball with center $y\in\mathbb{R}^d$ and radius 1 (see Theorem 1.1 of [12], part II, and Theorem 2.1 in Section 2). Here the constant $c(d,\nu)$ appears in the large t behavior of S_t, with obvious notation, where $t^{1/3}$ in (0.2) has to be replaced by $t^{d/(d+2)}$ and $c(1,\nu)$ by $c(d,\nu)$ (Donsker and Varadhan [3]).

From (0.5) it follows that the process produces a “big excursion,” that is, at time t, $Z_t\sim yt^{d/(d+2)}$. However, for $d\geq 2$ with t large enough and $y\neq 0$, $|y|t^{d/(d+2)}$ is much larger than the clearing size $t^{1/(d+2)}$. In other words, when

$d \geq 2$ and $y \neq 0$ we cannot expect that the "terminal location" $yt^{d/(d+2)}$ of Z_t lies in a clearing which contains the origin, that is, the starting point. However, we expect that on its way to $yt^{d/(d+2)}$ the process will encounter some clearings used as resting places. Indeed, one can view $c(d, \nu)$ in (0.5) as the costs attached to the time spent in clearings and $J_d(y) = \beta_0(y)$, $d \geq 2$, as the costs attached to the "big excursion" of order $\sim yt^{d/(d+2)}$. Observe, however, that $\beta_0(0) = 0$.

The fact causing the singularity for $d = 1$ is that the scale of the clearings is the same as the scale of the big excursion $yt^{1/3}$ unless, of course, $y = 0$. Therefore in $d = 1$ we can expect that for small enough $|y|$ the terminal location $yt^{1/3}$ lies in a clearing which contains the origin. In such a situation we would expect that no costs are attached to an excursion leading to $yt^{1/3}$, but costs are incurred for spending time in the clearing. However, a clearing should have an "optimal size" (which is in fact $c_0 t^{1/3}$, where c_0 was defined in Theorem 1) and therefore, as soon as $|y|t^{1/3}$ is larger than this optimal size, there should also be a cost attached to the excursion out in the forest.

It turns out that this intuition is not completely correct. Indeed, Theorem 1 shows that there is an intermediate phase, namely the situation where $c_0 t^{1/3} \leq |y|t^{1/3} \leq c_1 t^{1/3}$. In these cases there are also no costs attached to an excursion of order $|y|t^{1/3}$, but costs are incurred for spending time in a clearing of size $\sim |y|t^{1/3}$. This is, however, larger than the optimal size $c_0 t^{1/3}$. Finally, when $|y|t^{1/3} > c_1 t^{1/3}$ and provided $c_1 < \infty$, the terminal location is too far away from its starting point and Theorem 1 shows that then there is an "extra" excursion cost $\beta_0(1)(|y| - c_1) > 0$. This is analogous to the case $d \geq 2$.

We mention for the sake of completeness that the large deviations of $Z_t/\phi(t)$ under Q_t for $\phi(t) = t$, and $t^{d/(d+2)} \ll \phi(t) \ll t$ at rate $\phi(t)$ are established in Theorem 2.1 of [12], part I, for $d \geq 1$. Here $f(t) \ll g(t)$ stands for $f(t) = o(g(t))$ as $t \to \infty$. In view of the preceding discussion it should be clear that in these problems the dimension $d = 1$ is not singular. Indeed, also for $d = 1$ and for large enough t, $|y|\phi(t)$ is much larger than $t^{1/(d+2)}$, $y \neq 0$, the size of a clearing.

The paper is organized as follows. In Section 1 we prove in Theorem 1.1 the essential step for the lower bound of the large deviation principle. In Proposition 1.2 we show that at least for "small enough" shape functions W, the constant c_1 appearing in the rate function $J_1(\cdot)$ is in fact finite.

In Section 2 we establish in Theorem 2.1 the main step for the upper-bound part of the large deviation principle. The large deviation principle itself is stated in Theorem 2.2. The proof of Theorem 2.1 will first involve a "coarse graining procedure," different from the method of enlargement of obstacles, which reduces the combinatorial complexity of the problem. This "coarse graining scheme" is explained in (2.5)–(2.9).

The next step is then a partitioning of the space over which the integration is performed into an essential part E and an inessential part E^c; see (2.15). The contribution of part E^c to the total expectation, after suitably adjusting the parameters coming in the coarse graining procedure, will be negligible with respect to the lower bound derived in Section 1. This will be shown in Propositions 2.1 and 2.2.

The core of the proof of Theorem 2.1 will then be the investigation of the essential part E; see Proposition 2.3. To this end we will cover E by a family $\mathscr{G}$ of events G. This will be done in such a fashion that the cardinality of $\mathscr{G}$ is not too high; see Lemma 2.2. On the other hand, we have in the asymptotic regime simultaneously our desired upper bound for the contributions of the various G to the expectation under study; see (2.45).

We close Section 2 with an application of the large deviation result from Theorem 2.2 to the long-term behavior of $\overline{r}(t, 0, t^{1/3}y)$, where $\overline{r}$ is the averaged kernel of the Schrödinger heat semigroup $\exp\{t(\frac{1}{2}\Delta - V)\}$ in $d = 1$; see Theorem 2.3. For the long-term behavior of $\overline{r}(t, 0, t^{d/(d+2)}y)$, when $d \geq 2$, see Theorem 1.5 of [12], part II. For $\overline{r}(t, 0, \phi(t)y)$ in $d \geq 1$ for various scales $\phi(t)$, $\phi(t) \ll t^{d/(d+2)}$, or $t^{d/(d+2)} \ll \phi(t) \ll t$ or $\phi(t) = t$, see Theorem 2.4 of [12], part I.

In Section 3, Theorem 3.1, we give an application of the large deviation principle from Theorem 2.2 to the long-term behavior of a one-dimensional "annealed Brownian motion with constant drift h" in the Poissonian potential V.

1. The lower bound.

1.1. *Notation and introductory remarks.* In this section we derive an asymptotic lower bound for the expectation that the Brownian particle under the influence of the potential V is at time t near to the point $yt^{1/3}$. This asymptotic lower bound is the essential step for the lower bound of the large deviation principle given in Theorem 1. More precisely:

Let $z \in \mathbb{R}$ and $B(zt^{1/3}) = [zt^{1/3} - 1, zt^{1/3} + 1]$. We know from Proposition 1.2 of [12], part I, the existence of the so-called "annealed Liapounov exponents" $\beta_\lambda(x)$, where $\lambda \geq 0$, $x \in \mathbb{R}^d$. Among other things it is shown in Theorem 1.3 of [12], part I, that, as $|x|$ becomes large,

$$\mathbb{E} \otimes E_0\left[\exp\left\{-\int_0^{H(x)} (\lambda + V)(Z_s, \omega)\, ds\right\}\right] = \exp\{-\beta_\lambda(x)(1 + o(1))\}, \tag{1.1}$$

where $H(x)$ stands for the entrance time in $B(x)$.

The coefficients $\beta_\lambda(x)$ are norms in the x variable, and $\beta_\lambda(x) \geq |x|\kappa$, where the constant $\kappa = \kappa(d, \nu, W)$ is strictly positive. Furthermore, one has $\beta_\lambda(x) \leq K|x|$, where in dimension $d = 1$ we have that $K = \nu + \sqrt{2\lambda}$.

We introduce two constants $0 < c_0 < c_1 \leq \infty$ which play an important role in the sequel. To this end define the function $g(c) = \nu c + \pi^2/2c^2$, where $c \in [0, \infty)$. Denote by c_0 the unique $c \in [0, \infty)$ where $g(c)$ attains its global minimum, that is,

$$c_0 = \left(\frac{\pi^2}{\nu}\right)^{1/3}. \tag{1.2}$$

Furthermore, define

$$c_1 = \left(\frac{\pi^2}{\nu - \beta_0(1)}\right)^{1/3}, \tag{1.3}$$

where the constant c_1 is possibly infinite when $\nu = \beta_0(1)$. We show later in Proposition 1.1 that for "small enough potentials W" $\beta_0(1) < \nu$, so that c_1 is finite in these cases. Note that for $c_1 < \infty$ we have $g'(c_1) = \beta_0(1)$.

Let us introduce some additional notation: for $a, b \in \mathbb{R}$ we denote by $a \vee b = \max\{a, b\}$, $a \wedge b = \min\{a, b\}$ and by $(a - b)_+ = \max\{a - b, 0\}$.

1.2. *The main step in the derivation of the lower bound.* The main result which will easily imply the lower bound of the large deviation principle is the following:

THEOREM 1.1. *With the constants $c_0 < c_1$ introduced in* (1.2), *resp.* (1.3), *and $V(x, \omega)$, $x \in \mathbb{R}$, $\omega \in \Omega$, introduced in* (0.1), *the following holds:*

$$\liminf_{t \to \infty} t^{-1/3} \log \mathbb{E} \otimes E_0 \left[Z_t \in B(yt^{1/3}), \exp\left\{ -\int_0^t V(Z_s, \omega)\, ds \right\} \right] \geq -I(y), \tag{1.4}$$

where

$$I(y) = \left[\nu(c_0 \vee (c_1 \wedge |y|)) + \frac{\pi^2}{2(c_0 \vee (c_1 \wedge |y|))^2} + \beta_0(1)(|y| - c_1)_+ \right]. \tag{1.5}$$

Note that $I(y) = J_1(y) + c(1, \nu)$, where $J_1(y)$ was defined in (0.3). Let us give some comments on the structure of the function $I(y)$:

For $0 \leq |y| \leq c_0$ the function $I(y) = I(c_0) = c(1, \nu)$, where

$$c(1, \nu) = \inf_{c \geq 0} \left[\nu c + \frac{\pi^2}{2c^2} \right].$$

For $|y| \in [c_0, c_1]$ we have

$$I(y) = \inf_{c \geq |y|} \left[\nu c + \frac{\pi^2}{2c^2} \right] = \nu |y| + \frac{\pi^2}{2y^2}.$$

Finally, for $|y| > c_1$, provided c_1 is finite,

$$I(y) = \nu c_1 + \frac{\pi^2}{2c_1^2} + \beta_0(1)(|y| - c_1),$$

the tangent at the point c_1 of the function $g(c)$, with slope $\beta_0(1)$.

The geometric structure of the function $I(y)$ corresponds to the following strategy for the lower bound: in the case where $0 \leq |y| < c_1$, one possible strategy to give a lower bound on the quantity under consideration is to say that the Brownian motion remains until time t in an interval J which contains $yt^{1/3}$ and receives no Poisson point. Such an event costs approximately $\exp\{-[\nu|J| + (\pi^2/2|J|^2)t]\}$. Optimizing on the length of J, where $|J| \geq |y|t^{1/3}$, gives us the lower bound, when $0 \leq |y| < c_1$.

The interesting fact is that when $|y|$ becomes too large, it is more favorable to have an empty interval of length $c_1 t^{1/3}$, provided c_1 is finite. The particle stays most of the time in this interval and then goes from the one endpoint of this interval to $yt^{1/3}$. The probability of this is, up to correction terms, greater than

$$\exp\left\{-\beta_0(1)(|y|-c_1)t^{1/3} - \frac{\pi^2}{2c_1^2}t^{1/3} - \nu c_1 t^{1/3}\right\}.$$

PROOF OF THEOREM 1.1. We will only treat the case $y \geq 0$. The case where $y \leq 0$ is treated in the same way by using the symmetry of Brownian motion and working with the shape function $\hat{W}(x) = W(-x)$ which has the same Liapounov exponent as $W(x)$. We first look at the case where $0 \leq y < c_1$. Let I be an open interval such that $0, y \in I$, where $|I| = l$ with $l \geq y \geq 0$. Now since

$$1 - \exp\left\{-\int_0^t W(Z_s - y)\,ds\right\} \leq 1_{C_t^a}(y),$$

where C_t^a denotes the closed a-neighborhood of the support of the path up to time t, we get after performing the $\mathbb{E}$ expectation,

$$\begin{aligned}
&\mathbb{E}\otimes E_0\left[Z_t \in B(yt^{1/3}), \exp\left\{-\int_0^t V(Z_s,\omega)\,ds\right\}\right]\\
&\quad\geq E_0\big[Z_t \in B(yt^{1/3}), \exp\{-\nu|C_t^a|\}\big]\\
&\quad= \exp\{-\nu 2a\}E_0\big[Z_{t^{1/3}} \in B(y, t^{-1/3}), \exp\{-\nu t^{1/3}|C^0_{t^{1/3}}|\}\big]\\
&\quad\geq \exp\{-\nu 2a\}E_0\big[Z_{t^{1/3}} \in B(y, t^{-1/3}), T_I > t^{1/3}, \exp\{-\nu t^{1/3}|C^0_{t^{1/3}}|\}\big]\\
&\quad\geq \exp\{-\nu 2a\}\exp\{-\nu t^{1/3}|I|\}E_0\big[Z_{t^{1/3}} \in B(y, t^{-1/3}), T_I > t^{1/3}\big],
\end{aligned}\tag{1.6}$$

where T_I denotes the exit time from the interval I and where we used scaling in the third line. Using an eigenfunction expansion and denoting by $\lambda_1 > 0$ the first Dirichlet eigenvalue of $-\frac{1}{2}(d^2/dx^2)$ in I, we get for a constant $\kappa \in (0,\infty)$ independent of t, for large enough t,

$$E_0\big[Z_{t^{1/3}} \in B(y, t^{-1/3}), T_I > t^{1/3}\big] \geq \kappa t^{-1/3}\exp\{-\lambda_1 t^{1/3}\}.\tag{1.7}$$

Since $\lambda_1 = \pi^2/2|I|^2$ we get from (1.6), optimizing on the length l of I,

$$\begin{aligned}
&\liminf_{t\to\infty} t^{-1/3}\log \mathbb{E}\otimes E_0\left[Z_t \in B(yt^{1/3}), \exp\left\{-\int_0^t V(Z_s,\omega)\,ds\right\}\right]\\
&\qquad \geq -\inf_{l\geq y}\left[\nu l + \frac{\pi^2}{2l^2}\right].
\end{aligned}\tag{1.8}$$

But since

$$\inf_{l\geq y}\left[\nu l + \frac{\pi^2}{2l^2}\right] = \nu(c_0 \vee y) + \frac{\pi^2}{2(c_0\vee y)^2},$$

this shows the lower bound for $y \in [0, c_1)$.

Let us now look at the case where $c_1 \le y$, provided c_1 is finite. The strategy to give a lower bound is the same as in the proof of (2.9) in Theorem 2.1 of [12], part I:

One possibility for Z_t to be in $B(yt^{1/3})$ is to be at a time $t_1 < t$ in $B(c_1 t^{1/3})$, then to travel to $B(yt^{1/3})$ and be there between the time $t_1 + \gamma_1 t^{1/3}$ and $t_1 + \gamma_2 t^{1/3}$ where γ_1 and γ_2 are picked close enough and $t_1 + \gamma_2 t^{1/3} < t$ is close to t. The rest of the time is spent in $B(yt^{1/3})$.

To this end we introduce as in (2.11) of [12], part I, the following quantities, except that here $H(x, 1/2)$ stands for the entrance time in $B(x, 1/2)$, the closed interval with center x and radius 1/2.

For $0 < \gamma_1 < \gamma_2$ and $v \in \mathbb{R} \setminus \{0\}$, $n \ge 0$, we set

$$S_{n,v,\gamma_1} = H(nv, 1/2) \circ \vartheta_{n\gamma_1} + n\gamma_1 \tag{1.9}$$

and

$$A_{n,v,\gamma_1,\gamma_2} = \{S_{n,v,\gamma_1} \le n\gamma_2\}, \tag{1.10}$$

where $\vartheta_\bullet$ denotes the canonical shift on $C([0,\infty), \mathbb{R})$.

Thus the event $A_{n,v,\gamma_1,\gamma_2}$ means that the Brownian particle has entered $B(nv, 1/2)$ during the time interval $[n\gamma_1, n\gamma_2]$. We now want to apply Lemma 2.3 of [12], part I, where the following quantity was introduced:

$$\begin{aligned} &c(n, v, \gamma_1, \gamma_2, \lambda) \\ &\quad = -\inf_{z \in B(0,1/2)} \log \mathbb{E} \otimes E_z\biggl[A_{n,v,\gamma_1,\gamma_2}, \\ &\qquad\qquad \exp\biggl\{-\int_0^{S_{n,v,\gamma_1}} (\lambda + V)(Z_s, \omega)\, ds\biggr\}\biggr]. \end{aligned} \tag{1.11}$$

Let us recall the results from Lemma 2.3 of [12], part I, for $d = 1$. It was shown there that for $0 < \gamma_1 < \gamma_2 < \infty$, $\lambda \ge 0$, $v \in \mathbb{R} \setminus \{0\}$,

$$\lim_{n\to\infty} \frac{1}{n} c(n, v, \gamma_1, \gamma_2, \lambda) = \delta(v, \gamma_1, \gamma_2, \lambda) \in [0, \infty). \tag{1.12}$$

Moreover, if $(\gamma_1, \gamma_2) \cap [\beta'_\lambda(v)_+, \beta'_\lambda(v)_-] \ne \varnothing$, $\lambda > 0$, then

$$\delta(v, \gamma_1, \gamma_2, \lambda) \le \beta_\lambda(v), \tag{1.13}$$

provided $\beta'_\lambda(v)_+$ [resp. $\beta'_\lambda(v)_-$] denotes the right (resp. the left) derivative of the increasing concave function $\beta_\bullet(v)$ at $\lambda > 0$.

We apply this result in our situation to a sequence of positive real numbers tending to infinity. To this end we extend the definition (1.11) to $n \ge 0$ a positive real number. Note that for $r, s \in \mathbb{R}^+$ we have as in the proof of Lemma 2.3 of [12], part I, that $c(r+s) \le c(r) + c(s)$. If we can show that $\sup_{1 \le x \le 2} c(x, 1, \gamma_1, \gamma_2, \lambda) < \infty$, then (1.12) follows for real $n > 0$.

To prove $\sup_{1\le x\le 2} c(x,1,\gamma_1,\gamma_2,\lambda)<\infty$, let $z\in B(0,1/2)$ and pick $x\in[1,2]$. Observe that on the set $A_{x,1,\gamma_1,\gamma_2}$ we have $S_{x,1,\gamma_1}\le x\gamma_2\le 2\gamma_2$ and $|C^a_{S_{x,1,\gamma_1}}|\le |C^a_{x\gamma_1}|+|C^a_{H(x,1/2)}\circ\vartheta_{x\gamma_1}|$.

Using the strong Markov property, we find

$$\begin{aligned}
\mathbb{E}\otimes E_z&\left[A_{x,1,\gamma_1,\gamma_2},\exp\left\{-\int_0^{S_{x,1,\gamma_1}}(\lambda+V)(Z_s,\omega)\,ds\right\}\right]\\
&\ge \exp\{-\lambda 2\gamma_2-\nu 2(1+a)\}\\
&\quad\times E_z\left[A_{x,1,\gamma_1,\gamma_2},T_{B(0,1)}>x\gamma_1,\exp\{-\nu|C^a_{H(x,1/2)}\circ\vartheta_{x\gamma_1}|\}\right]\\
&\ge \exp\{-\lambda 2\gamma_2-\nu 2(1+a)\}\inf_{z\in B(0,1/2)}E_z\left[T_{B(0,1)}>2\gamma_1\right]\\
&\quad\times\inf_{z\in B(0,1)}E_z\left[H(2,1/2)\le(\gamma_2-\gamma_1),\exp\{-\nu|C^a_{H(2,1/2)}|\}\right],
\end{aligned}$$

where the last inequality follows from the strong Markov property and the fact that for $Z_0\in B(0,1)$, $x\in[1,2]$ we have $H(x,1/2)\le H(2,1/2)$, resp. $\gamma_2-\gamma_1\le x(\gamma_2-\gamma_1)$ P_0 a.s. Denoting by $q_I(t,x,y)$ the fundamental solution of $\partial_t u=\frac{1}{2}(\partial^2/\partial x^2)u$, $x\in I$, where I is some open interval, we see that

$$\begin{aligned}
\inf_{z\in B(0,1)}&E_z\left[H(2,1/2)\le(\gamma_2-\gamma_1),\exp\{-\nu|C^a_{H(2,1/2)}|\}\right]\\
&\ge\inf_{z\in B(0,1)}E_z\left[T_{B(0,2\,1/2)}>(\gamma_2-\gamma_1),Z_{(\gamma_2-\gamma_1)}\in B(2,1/2),\exp\{-\nu|C^a_{H(2,1/2)}|\}\right]\\
&\ge\exp\{-\nu(5+2a)\}\inf_{z\in B(0,1)}\int_{1\,1/2}^{2\,1/2}q_{B(0,2\,1/2)}(\gamma_2-\gamma_1,z,u)\,du>0.
\end{aligned}$$

Thus (1.12) holds for $n>0$ a real number tending to infinity.

Now pick for $n\ge 1$, $\lambda(n)=1/n$, $0<\gamma_1(n)<\gamma_2(n)$ such that $\gamma_1(n)<\beta'_{\lambda(n)}(1)_-<\gamma_2(n)$ and $|\gamma_2(n)-\gamma_1(n)|<1/n$. Then we have ensured that $(\gamma_1(n),\gamma_2(n))\cap[\beta'_{\lambda(n)}(1)_+,\beta'_{\lambda(n)}(1)_-]\ne\varnothing$.

Define $t_1=t-\gamma_2 t^{1/3}<t$. For large enough t such that $t_1>0$, we find

$$\begin{aligned}
\mathbb{E}\otimes E_0&\left[Z_t\in B(yt^{1/3}),\exp\left\{-\int_0^t V(Z_s,\omega)\,ds\right\}\right]\\
&=E_0\left[Z_t\in B(yt^{1/3}),\exp\left\{-\nu\int\left[1-\exp\left\{-\int_0^t W(Z_s-x)\,ds\right\}\right]dx\right\}\right]\\
&\ge E_0\Big[Z_{t_1}\in B(c_1t^{1/3}),A_{t^{1/3},y,\gamma_1,\gamma_2}\circ\vartheta_{t_1},\\
&\qquad T_{B(yt^{1/3})}\circ\vartheta_{S_{t^{1/3},y,\gamma_1}}\circ\vartheta_{t_1}>(\gamma_2-\gamma_1)t^{1/3}\\
&\qquad\times\exp\left\{-\nu\int\left[1-\exp\left\{-\int_0^t W(Z_s-x)\,ds\right\}\right]dx\right\}\Big].
\end{aligned}\tag{1.14}$$

Since for $a, b \geq 0$ we have $1 - e^{-(a+b)} \leq 1 - e^{-a} + 1 - e^{-b}$ and using the strong Markov property, we see that the last member in (1.14) is larger than

$$
\begin{aligned}
&E_0\big[Z_{t_1} \in B(c_1 t^{1/3}, 1/2), \exp\{-\nu |C^a_{t_1}|\}\big] \\
&\quad \times \inf_{z \in B(c_1 t^{1/3}, 1/2)} E_z\Bigg[A_{t^{1/3}, y, \gamma_1, \gamma_2}, \\
&\qquad \exp\bigg\{-\nu \int \bigg[1 - \exp\bigg\{-\int_0^{S_{t^{1/3}, y, \gamma_1}} W(Z_s - x)\, ds\bigg\}\bigg]\, dx\bigg\}\Bigg] \qquad (1.15)\\
&\quad \times \inf_{z \in B(y t^{1/3}, 1/2)} E_z\big[T_{B(yt^{1/3})} > (\gamma_2 - \gamma_1) t^{1/3}, \exp\{-\nu |C^a_{(\gamma_2 - \gamma_1) t^{1/3}}|\}\big] \\
&\quad = A_1 A_2 A_3.
\end{aligned}
$$

Consider A_1. Recall that $t_1 = t - \gamma_2 t^{1/3}$. Using the scaling property of Brownian motion, we get for large enough t,

$$
\begin{aligned}
&E_0\bigg[Z_{t_1} \in B\bigg(c_1 t^{1/3}, \frac{1}{2}\bigg), \exp\{-\nu |C^a_{t_1}|\}\bigg] \\
&\quad = \exp\{-\nu 2a\} \\
&\qquad \times E_0\bigg[Z_{t_1^{1/3}} \in B\bigg(c_1 \frac{1}{(1 - \gamma_2 t^{-2/3})^{1/3}}, \frac{1}{2} t_1^{-1/3}\bigg), \exp\{-\nu t_1^{1/3} |C^0_{t_1^{1/3}}|\}\bigg] \qquad (1.16)\\
&\quad \geq \exp\{-\nu 2a\} \exp\{-\nu t_1^{1/3} |I|\} E_0\bigg[Z_{t_1^{1/3}} \in B\bigg(c_1, \frac{1}{2} t_1^{-1}\bigg), T_I > t_1^{1/3}\bigg],
\end{aligned}
$$

where I is an open interval containing 0 (resp. c_1), and $|I| = l \geq c_1$.

As for $y \in [0, c_1)$ [cf. (1.7)], we get, for large t and a constant $\kappa \in (0, \infty)$,

$$
E_0\bigg[Z_{t_1^{1/3}} \in B\bigg(c_1, \frac{1}{2} t_1^{-1}\bigg), T_I > t_1^{1/3}\bigg] \geq \kappa t_1^{-1} \exp\bigg\{-\frac{\pi^2}{2|I|^2} t_1^{1/3}\bigg\}. \qquad (1.17)
$$

Inserting (1.17) into (1.16) and optimizing on the length $l \geq c_1$ of I, we find, for large t,

$$
\liminf_{t \to \infty} t^{-1/3} \log A_1 \geq -\bigg[\nu c_1 + \frac{\pi^2}{2 c_1^2}\bigg]. \qquad (1.18)
$$

As for A_3, we find by using translation invariance

$$
\begin{aligned}
A_3 &\geq \exp\{-\nu 2(a+1)\} \inf_{z \in B(0, 1/2)} E_z\big[T_{B(0)} > (\gamma_2 - \gamma_1) t^{1/3}\big] \\
&\geq \kappa \exp\{-\nu 2(a+1)\} t^{-1/3} \exp\bigg\{-\frac{\pi^2}{8} (\gamma_2 - \gamma_1) t^{1/3}\bigg\}. \qquad (1.19)
\end{aligned}
$$

For A_2, applying (1.12) in our context, we see that

$$
A_2 \geq \exp\{-t^{1/3} [\beta_{\lambda(n)}(1)(y - c_1) - \lambda(n) \gamma_1](1 + o(1))\}. \qquad (1.20)
$$

Inserting the last three inequalities into (1.14) yields

$$\begin{aligned}
&\liminf_{t\to\infty} t^{-1/3}\log \mathbb{E}\otimes E_0\biggl[Z_t\in B(yt^{1/3}), \exp\biggl\{-\int_0^t V(Z_s,\omega)\,ds\biggr\}\biggr]\\
&\quad\geq -\biggl[\nu c_1+\frac{\pi^2}{2c_1^2}+\beta_{\lambda(n)}(1)(y-c_1)\\
&\qquad\qquad +\frac{\pi^2}{8}(\gamma_2(n)-\gamma_1(n))-\lambda(n)\gamma_1(n)\biggr].
\end{aligned}\tag{1.21}$$

Letting n tend to infinity and using the continuity of $\beta_\bullet(1)$ in $\lambda\geq 0$, we finally find that

$$\begin{aligned}
&\liminf_{t\to\infty} t^{-1/3}\log \mathbb{E}\otimes E_0\biggl[Z_t\in B(yt^{1/3}), \exp\biggl\{-\int_0^t V(Z_s,\omega)\,ds\biggr\}\biggr]\\
&\quad\geq -\biggl[\nu c_1+\frac{\pi^2}{2c_1^2}+\beta_0(1)(y-c_1)\biggr].
\end{aligned}\tag{1.22}$$

This completes the proof of Theorem 1.1. □

1.3. *Criterion for* $c_1<\infty$. In this section we prove a proposition which shows that at least for sufficiently small potentials the constant c_1 introduced in (1.3) is in fact finite.

This already shows part (ii) of Theorem 1.

PROPOSITION 1.2. *Let $W(\cdot)$ be a nonnegative, bounded, measurable function with compact support in $(-l,l)$, $l>0$, not a.s. equal to zero. If $\|W\|_\infty$ is sufficiently small we have $\beta_0(1)<\nu$.*

PROOF. Let us recall some notation from [12], part I. We restrict ourselves to $d=1$.

Define, for $x\in\mathbb{R}$,

$$\begin{aligned}
f(x)&=\mathbb{E}\otimes E_0\biggl[\exp\biggl\{-\int_0^{H(x)} V(Z_s,\omega)\,ds\biggr\}\biggr]\\
&=E_0\biggl[\exp\biggl\{-\nu\int\biggl[1-\exp\biggl\{-\int_0^{H(x)} W(Z_s-y)\,ds\biggr\}\biggr]\,dy\biggr\}\biggr]
\end{aligned}\tag{1.23}$$

and

$$b(x)=-\inf_{z\in B(0)}\log f(x-z)\geq 0.\tag{1.24}$$

It was shown in Proposition 1.2 of [12], part I, that $b(x+y)\leq b(x)+b(y)$ and, furthermore,

$$\beta_0(x)=\inf_{n\geq 1}\frac{1}{n}b(nx),\tag{1.25}$$

$$\beta_0(x)\leq \nu|x|.\tag{1.26}$$

Thus we know that $\beta_0(1)\leq\min\{b(1),\nu\}$.

Now define the function

$$[0,\infty)\ni\gamma\mapsto F_W(\gamma) \tag{1.27}$$
$$= -\log E_0\left[\exp\left\{-\nu\int\left[1-\exp\{-\gamma\int_0^{H_{\{1\}}}W(Z_s-y)\,ds\}\right]dy\right\}\right],$$

where $H_{\{1\}}$ denotes the entrance time to $\{1\}$.

Let $Z_0=0$. Then since P_0 a.s. for any $z\in[-1,1]$ we have $H(1-z)\le H(2)=H_{\{1\}}$, we get with regard to (1.23) that $f(1-z)\ge f(2)$. Hence $b(1)=F_W(1)$ and since for all $x\in[-a,a]$ we have $\gamma W(x)\le\gamma\|W\|_\infty 1_{[-a,a]}(x)$ we find

$$F_W(\gamma)\le F_{1_{[-a,a]}(\cdot)}(\gamma\|W\|_\infty). \tag{1.28}$$

Hence

$$\beta_0(1)\le\min\{F_{1_{[-a,a]}(\cdot)}(\|W\|_\infty),\nu\}. \tag{1.29}$$

Since $F_\bullet(0)=0$ it suffices to show that $F_W(\gamma)$ is continuous in γ which together with (1.29) shows Proposition 1.2.

Now take $\gamma\in[0,\infty)$. On the set of full P_0 measure where $H_{\{1\}}<\infty$, we have for all $\gamma\in[0,\infty)$ that the integrand under the E_0 expectation in (1.27) is bounded above by 1. The continuity follows from dominated convergence. □

2. The upper bound.

2.1. *Statement of the main result, strategy of the proof and overview.* The main result implying the upper bound of the large deviation principle from Theorem 1 is the following theorem.

THEOREM 2.1. *Let* $0\le L_1<L_2<\infty$. *With the constants* $c_0<c_1\le\infty$ *introduced in* (1.2), *resp.* (1.3), *and the function* $I(y)$ *introduced in* (1.5) *the following holds:*

$$\limsup_{t\to\infty}\sup_{0\le L_1\le|y|\le L_2}t^{-1/3}\log\mathbb{E}\otimes E_0\left[Z_t\in B(yt^{1/3}),\exp\left\{-\int_0^t V(Z_s,\omega)\,ds\right\}\right] \tag{2.1}$$
$$\le -I(L_1).$$

Let us introduce the strategy of the proof for the preceding theorem. The basic reduction step is the fact that we can restrict ourselves to the case where the particle does not leave the interval $U_M=(-Mt^{1/3},Mt^{1/3})$ until time t, provided M is chosen large enough.

Our main goal is then to describe excursions of the particle between regions in U_M which are large and where the Poisson points are sparse and the complement of these regions. We will call large regions where the Poisson points are sparse "pseudo-holes."

We then want to show that the leading asymptotics of the object of interest comes from the term where the particle does not make too many excursions between the various pseudo-holes and spends most of its time in them. Considering "resting costs," that is, the cost that the process spends most of its time in the "pseudo holes," and "connecting costs," that is, the costs for connecting the various pseudo-holes which "lead the way to $yt^{1/3}$," will give us a variational problem leading to the function $I(y)$.

We therefore have to specify what we mean by "sparse," "large," "excursions" and so on. To this end we chop $\mathbb{R}$ into intervals of length $L(t) = t^{\delta}$, where $\delta \in (1/6, 1/3)$ [cf. (2.5)]. To explain what we mean by "sparse," we chop the previous intervals into subboxes of length $3a$, where a is the "radius of the support" of the shape function W [cf. (2.6)]. We pick a parameter $\alpha \in (0, 1/3a)$. If the number of subboxes receiving a Poisson point is smaller than $\alpha L(t)$, we declare this interval to be a "thin edge." If instead the number of occupied subboxes is bigger than $\alpha L(t)$, we declare this interval to be an edge. This is made precise in (2.7), resp. (2.8). With this construction we have a partition of $\mathbb{R}$ into edges and thin edges of size $L(t)$. We then pick a parameter $r \in (0, M)$ which will measure what we mean by large regions [cf. (2.9)]. Indeed we look at connected components L_i of thin edges such that $|L_i|$, the length of L_i, is larger than $rt^{1/3}$. By definition, at the left and right, end of such connected components we have edges, that is, intervals of length $L(t)$, a small scale compared to that of our large regions, where the Poisson points are dense enough. We finally look at the open $L(t)$ neighborhood $\mathscr{O}$ of $\bigcup_i L_i$ and call the connected components of $\mathscr{O}$ pseudo-holes. Finally, the excursions will describe the successive times of reaching $\bigcup_i L_i$ and exiting $\mathscr{O}$; see (2.10).

With the preceding construction we show that we can neglect the term where the process makes more than $L(t)$ excursions. The reason is roughly that when crossing an edge the process meets at least $\alpha L(t)$ subboxes receiving a point and owing to our assumption $\delta > 1/6$ this is very costly for the process. Furthermore, the term where the particle spends most of the time in the complement of the pseudo-holes is also negligible for our purpose, provided r is chosen small enough. This is due to the fact that the largest trap-free regions in these intervals are of order $rt^{1/3}$. Another crucial point is that we have a good control on the probability that some interval is a pseudo-hole. Indeed we are going to show that, provided α is chosen small enough, this probability is up to correction terms bounded above by the probability that the interval is empty.

We now give an overview of the several steps involved in the proof of Theorem 2.1. Section 2.2 is the first basic reduction step. In Section 2.3 we explain our coarse graining scheme already mentioned in the Introduction and then partition the space over which the integration is performed. Section 2.4 deals with the negligible part of this partitioning. In Section 2.5 we state in Proposition 2.3 our main result on the essential part. This will conclude the proof of Theorem 2.1. Section 2.6 introduces a suitable covering of the essential part by events G and in Section 2.7 we show that our desired asymptotic upper bound of Proposition 2.3 holds simultaneously for the various G of the covering. This is again proved in several steps. In Section 2.7.1 we will split up the "resting" and the "connecting" cost and in Section 2.7.2 we give an asymptotic upper bound on the probability that some interval is a “pseudo-hole.” Section 2.7.3 puts all asymptotic upper bounds together and completes the proof of Proposition 2.3 and Theorem 2.1. Finally, in Section 2.8 we state and prove our large deviation result. We are now ready to begin with the proof of Theorem 2.1.

PROOF OF THEOREM 2.1. Without loss of generality we shall from now on assume that by possibly shifting the Poisson process and $W(\cdot)$, 0 is a point of density of the function $W(\cdot)$. By this we mean that $\liminf_{x\to 0_+}(1/x) \times \int_0^x W(y)\,dy > 0$ and $\liminf_{x\to 0_+}(1/x)\int_{-x}^0 W(y)\,dy > 0$. Now pick $0 \le L_1 < L_2 < \infty$.

2.2. *The basic reduction step.* The next lemma is the first reduction step.

LEMMA 2.1. *Let $M > 0$ and set $U_M = (-Mt^{1/3}, Mt^{1/3})$. Then*

$$\limsup_{M\to\infty}\limsup_{t\to\infty} t^{-1/3}\log \mathbb{E}\otimes E_0\biggl[T_{U_M}\le t, \exp\biggl\{-\int_0^t V(Z_s,\omega)\,ds\biggr\}\biggr] = -\infty. \tag{2.2}$$

PROOF. Let $M > 0$ and denote by $H(Mt^{1/3})$ the hitting time of $Mt^{1/3}$. On $\{T_{U_M}\le t\}$ we have either $H(Mt^{1/3})\le t$ or $H(-Mt^{1/3})\le t$. Hence we find

$$\begin{aligned}\mathbb{E}\otimes E_0\biggl[T_{U_M}\le t, \exp\biggl\{-\int_0^t V(Z_s,\omega)\,ds\biggr\}\biggr] &\le \mathbb{E}\otimes E_0\biggl[\exp\biggl\{-\int_0^{H(Mt^{1/3})} V(Z_s,\omega)\,ds\biggr\}\biggr]\\ &\quad + \mathbb{E}\otimes E_0\biggl[\exp\biggl\{-\int_0^{H(-Mt^{1/3})} V(Z_s,\omega)\,ds\biggr\}\biggr].\end{aligned}$$

Since, for $x\in\mathbb{R}$

$$\limsup_{t\to\infty} t^{-1/3}\log \mathbb{E}\otimes E_0\biggl[\exp\biggl\{-\int_0^{H(xt^{1/3})} V(Z_s,\omega)\,ds\biggr\}\biggr] = -|x|\beta_0(1)$$

(see Theorem 1.3 of [12], part I), this shows Lemma 2.1. □

So for our purpose of proving Theorem 2.1 we see that writing

$$\begin{aligned}\mathbb{E}\otimes E_0\biggl[Z_t\in B(yt^{1/3}), \exp\biggl\{-\int_0^t V(Z_s,\omega)\,ds\biggr\}\biggr] &= \mathbb{E}\otimes E_0\biggl[T_{U_M}\le t, Z_t\in B(yt^{1/3}), \exp\biggl\{-\int_0^t V(Z_s,\omega)\,ds\biggr\}\biggr]\\ &\quad + \mathbb{E}\otimes E_0\biggl[T_{U_M}> t, Z_t\in B(yt^{1/3}), \exp\biggl\{-\int_0^t V(Z_s,\omega)\,ds\biggr\}\biggr]\\ &= A_0 + B_0,\end{aligned} \tag{2.3}$$

it suffices to prove (2.1) with B_0 instead of the expression under the logarithm in (2.1), provided we pick

$$M > \max\left\{\frac{I(L_1)}{\beta_0(1)}, L_2\right\}. \tag{2.4}$$

From now on we assume that we work with a fixed M satisfying (2.4).

2.3. *A coarse graining scheme and partitioning.* As already pointed out, the proof of Theorem 2.1, that is, the study of the term B_0 from (2.3), hinges on a coarse-grained description of the Poisson environment which we are now going to explain.

Define the function $L(t) = t^\delta$, where $\delta \in (1/6, 1/3)$, and chop $\mathbb{R}$ into boxes of size $L(t)$.

Let $t > 1$, $m \in \mathbb{Z}$ and define

$$B_m = \big[mL(t), (m+1)L(t)\big). \tag{2.5}$$

We now explain what we mean by saying that B_m is an edge or a thin edge.

To this end define $a = \inf\{r > 0;\ W(\cdot) = 0 \text{ on } [-r, r]^c\} \in (0, \infty)$, the radius of the support of the shape function. Chop each subbox B_m into at most $[L(t)/3a] + 1$ segments of length $3a$, except maybe the last one. Thus we define, for $1 \le i \le [L(t)/3a]$,

$$B_m^i = \big[mL(t) + (i-1)3a, mL(t) + i3a\big), \tag{2.6}$$

where we drop the a dependence in the notation.

For $A \subset \mathbb{R}$ and $\omega \in \Omega$, the space of simple pure point measures on $\mathbb{R}$, $\omega(A)$ denotes the number of points in A.

Pick $\alpha \in (0, 1/3a)$. Let $t > 1$ and $m \in \mathbb{Z}$. We define the event "the subbox B_m is an edge" by

$$B_m^e = \big\{\omega;\ |\{1 \le i \le [L(t)/3a];\ \omega(B_m^i) \ge 1\}| \ge \alpha L(t)\big\} \tag{2.7}$$

and the event "the subbox is a thin edge" by

$$B_m^{te} = \big\{\omega; |\{1 \le i \le [L(t)/3a];\ \omega(B_m^i) \ge 1\}| < \alpha L(t)\big\}, \tag{2.8}$$

where $|A|$ stands for the number of elements in the set A.

For a given point configuration $\omega \in \Omega$, we thus have a partition of $\mathbb{R}$ into subboxes B_m of size $L(t)$ consisting of edges and thin edges. Given such a partition, the next step is to single out "large regions" where the Poisson points are sparse. To this end we are going to define pseudo-holes.

Pick $r \in (0, M)$. Let $\omega \in \Omega$ and define the set $K = \{k \in \mathbb{Z};\ \omega \in B_k^{te}\}$. Consider the set of connected components L_i of $\bigcup_{m\in K} \overline{B}_m$ with length $|L_i| \ge rt^{1/3}$. Finally, let $L = \bigcup_i L_i$ and set

$$\mathscr{O} = \big\{x \in \mathbb{R};\ \operatorname{dist}(x, L) < L(t)\big\}, \tag{2.9}$$

the open $L(t)$ neighborhood of L. We write $\mathscr{O} = \bigcup_i \mathscr{O}_i$, where the $\mathscr{O}_i$ are the connected components of $\mathscr{O}$, and call $\mathscr{O}_i$ the pseudo-holes.

Observe that since the length $|\mathscr{O}_i|$ of a pseudo-hole is larger than or equal to $rt^{1/3}$, the number of pseudo-holes $\mathscr{O}_i$ intersecting U_M is bounded above by $[2M/r]+1$. Furthermore, for a pseudo-hole $\mathscr{O}_i$, the number of possible edges lying in $\mathscr{O}_i \cap U_M$ is also bounded above by $[2M/r]+1$.

We are now ready to describe the excursions of the path $Z_\bullet$. To this end we are going to introduce stopping times, namely the "returns" to L and the "departures" from $\mathscr{O}$. In analogy to [12], part II, we define [$L \subset \mathscr{O}$; L closed, $\mathscr{O}$ open from (2.9)]

$$R_1 = \inf\{t \geq 0\colon Z_t \in L\}, \tag{2.10}$$

$$D_1 = \inf\{t \geq R_1:\ Z_t \notin \mathscr{O}\} \tag{2.11}$$

and inductively ($j \geq 1$)

$$R_{j+1} = D_j + R_1 \circ \vartheta_{D_j}, \qquad D_{j+1} = D_j + D_1 \circ \vartheta_{D_j}. \tag{2.12}$$

Note that $D_j = R_j + D_1 \circ \vartheta_{R_j}$. For $t > 0$ we set, with $D_0 = 0$,

$$L_t = \frac{1}{t}\sum_{j\geq 0} R_{j+1} \wedge t - D_j \wedge t, \tag{2.13}$$

the total fraction of time until t the process spends in returning to L. Define also

$$N_t = \sum_{j \geq 1} 1_{\{R_j \leq t\}}, \tag{2.14}$$

the total number of completed returns to L up to time t.

We now start with the estimate of the quantity under the logarithm in (2.1) from Theorem 2.1. For $\eta \in (0,1)$ we have

$$\begin{aligned}
&\mathbb{E} \otimes E_0\left[Z_t \in B(yt^{1/3}), \exp\left\{-\int_0^t V(Z_s,\omega)\,ds\right\}\right]\\
&\quad = \mathbb{E} \otimes E_0\left[T_{U_M} \leq t, Z_t \in B(yt^{1/3}), \exp\left\{-\int_0^t V(Z_s,\omega)\,ds\right\}\right]\\
&\qquad + \mathbb{E} \otimes E_0\Big[N_t > [L(t)], T_{U_M} > t, Z_t \in B(yt^{1/3}),\\
&\qquad\qquad \exp\left\{-\int_0^t V(Z_s,\omega)\,ds\right\}\Big]\\
&\qquad + \mathbb{E} \otimes E_0\Big[N_t \leq [L(t)], L_t \geq \eta, T_{U_M} > t, Z_t \in B(yt^{1/3}),\\
&\qquad\qquad \exp\left\{-\int_0^t V(Z_s,\omega)\,ds\right\}\Big]\\
&\qquad + \mathbb{E} \otimes E_0\Big[N_t \leq [L(t)], L_t < \eta, T_{U_M} > t, Z_t \in B(yt^{1/3}),\\
&\qquad\qquad \exp\left\{-\int_0^t V(Z_s,\omega)\,ds\right\}\Big]\\
&\quad = A_0 + A_1 + A_2 + A_3.
\end{aligned} \tag{2.15}$$

With regard to Lemma 2.1, we already know that for our purpose of proving (2.1), A_0 is negligible owing to our choice of M for which (2.4) holds. We are in fact going to show that by suitably adjusting the parameters, the leading asymptotics of the quantity under the logarithm in (2.1) comes from the term A_3.

2.4. *The negligible terms.* In this section we proceed by proving two propositions which show that A_1 and A_2 are negligible for our purpose. We begin with A_1.

PROPOSITION 2.1. *Let $\alpha \in (0, 1/3a)$, $r \in (0, M)$. Define $\mathscr{E}_1 = \{N_t > [L(t)], T_{U_M} > t,\ Z_t \in B(yt^{1/3})\}$. We have*

$$\limsup_{t\to\infty} \sup_{|y|\in[L_1, L_2]} t^{-1/3} \log \mathbb{E} \otimes E_0\left[1_{\mathscr{E}_1} \exp\left\{-\int_0^t V(Z_s, \omega)\, ds\right\}\right] = -\infty. \tag{2.16}$$

PROOF. On the set $\{N_t > [L(t)]\}$ we have $R_{[L(t)]} \le t$. Thus

$$A_1 \le \mathbb{E} \otimes E_0\left[R_{[L(t)]} \le t, \exp\left\{-\int_0^{R_{[L(t)]}} V(Z_s, \omega)\, ds\right\}\right]. \tag{2.17}$$

For arbitrary $j \ge 0$ we have, due to (2.12) and the strong Markov property,

$$\begin{aligned} &E_0\left[R_{j+1} < \infty, \exp\left\{-\int_0^{R_{j+1}} V(Z_s, \omega)\, ds\right\}\right] \\ &\quad \le E_0\left[R_j < \infty, \exp\left\{-\int_0^{R_j} V(Z_s, \omega)\, ds\right\}\right. \\ &\qquad \times E_{Z_{R_j}}\left[D_1 < \infty, \exp\left\{-\int_0^{D_1} V(Z_s, \omega)\, ds\right\}\right. \\ &\qquad\qquad \left.\left.\times E_{Z_{D_1}}\left[\exp\left\{-\int_0^{R_1} V(Z_s, \omega)\, ds\right\}\right]\right]\right]. \end{aligned} \tag{2.18}$$

Since $Z_{D_1} \in \partial\mathscr{O}$ when $D_1 < \infty$, we shall give an upper bound on

$$\sup_{z\in\partial\mathscr{O}} E_z\left[\exp\left\{-\int_0^{R_1} V(Z_s, \omega)\, ds\right\}\right]. \tag{2.19}$$

Write $\mathscr{O} = \bigcup_i \mathscr{O}_i$. We then have $z \in \partial\mathscr{O}_i$ for some i. Let us furthermore assume that we have picked the labeling such that $\mathscr{O}_{i-1}$ is the nearest pseudo-hole to the left of $\mathscr{O}_i$ and $\mathscr{O}_{i+1}$ is the nearest to the right. If we define, for $j \in \{i-1, i, i+1\}$,

$$C_j(t) = \sup_{z\in\partial\mathscr{O}_i} E_z\left[\exp\left\{-\int_0^{R_1} V(Z_s, \omega)\, ds\right\},\ Z_{R_1} \in \mathscr{O}_j\right], \tag{2.20}$$

we find

$$\sup_{z\in\partial\mathscr{O}_i} E_z\left[\exp\left\{-\int_0^{R_1} V(Z_s, \omega)\, ds\right\}\right] \le \sum_{j=i-1}^{i+1} C_j(t). \tag{2.21}$$

We shall only give an upper bound on $C_i(t)$. The other terms can be treated in a completely analogous fashion and lead to exactly the same upper bound. In this case, denoting by z_-, resp. z_+, the leftmost, resp. the rightmost, point of $\mathscr{O}_i$, we have that $C_i(t) = \max\{C_i^-(t), C_i^+(t)\}$ where

$$C_i^-(t) = E_{z_-}\left[\exp\left\{-\int_0^{R_1} V(Z_s, \omega)\, ds\right\}, Z_{R_1} = z_- + t^\delta\right] \tag{2.22}$$

and $C_i^+(t)$ is defined by replacing in (2.22) z_- by z_+, resp. $z_- + t^\delta$ by $z_+ - t^\delta$.

We shall only give an upper bound on $C_i^-(t)$. It is clear that $C_i^+(t)$ leads to exactly the same upper bound. Observe now that $B_m = [z_-, z_- + t^\delta)$ is an edge. Therefore there exists j_l such that $1 \le j_1 < j_2 < \cdots < j_{[\alpha L(t)/2]}$ with $\mathrm{dist}(B_m^{j_l}, B_m^{j_k}) \ge 3a$, $l \ne k$, and $B_m^{j_l} \ni x_{j_l} \in \operatorname{supp} \omega$, $1 \le l \le [\alpha L(t)/2]$; cf. (2.7).

Denote for a closed subset $A \subset \mathbb{R}$ by H_A the entrance time of $Z_\bullet$ in A, that is, $H_A = \inf\{t \ge 0;\ Z_t \in A\}$. For $0 \le i \le [\alpha L(t)/2] - 1$ and t large enough, we define the following sequence of stopping times ($H_0 = 0$):

$$H_{i+1} = H_i + H_{\overline{B}_m^{j_{i+1}}} \circ \vartheta_{H_i}. \tag{2.23}$$

Since $j_1 < j_2 < \cdots < j_{[\alpha L(t)/2]}$ it follows that $Z_{H_i} = mL(t) + (j_i - 1)3a$; cf. (2.6). Using the strong Markov property, we then find for large enough t that

$$\begin{aligned} C_i^-(t) &\le E_{z_-}\left[\exp\left\{-\int_0^{H_{[\alpha L(t)/2]}} V(Z_s, \omega)\, ds\right\}\right] \\ &\le E_{z_-}\left[\prod_{i=1}^{[\alpha L(t)/2]-1} \exp\left\{-\int_{H_i}^{H_{i+1}} W(Z_s - x_{j_i})\, ds\right\}\right] \\ &= \prod_{i=1}^{[\alpha L(t)/2]-1} E_{mL(t)+(j_i-1)3a}\left[\exp\left\{-\int_0^{H_{\overline{B}_m^{j_{i+1}}}} W(Z_s - x_{j_i})\, ds\right\}\right]. \end{aligned} \tag{2.24}$$

Denote by $T_{6a} = \inf\{u \ge 0: Z_u = 6a\}$ the first hitting time of $6a$, where a is the radius of the support of the shape function W. Then in view of (2.24),

$$\begin{aligned} C_i^-(t) &\le \left(\sup_{x \in [0,3a]} E_0\left[\exp\left\{-\int_0^{T_{6a}} W(Z_s - x)\, ds\right\}\right]\right)^{[\alpha L(t)/2]-1} \\ &\le \left(E_0\left[\exp\left\{-\int_0^{T_{3a}} W(Z_s)\, ds\right\}\right]\right)^{[\alpha L(t)/2]-1} \\ &= (\kappa_a)^{[\alpha L(t)/2]-1} \end{aligned} \tag{2.25}$$

by the strong Markov property, with the obvious notation.

Observe that $\kappa_a \in (0, 1)$. We thus have shown that, for large enough t,

$$\sup_{z \in \partial\mathscr{O},\, \omega} E_z\left[\exp\left\{-\int_0^{R_1} V(Z_s, \omega)\, ds\right\}\right] \le 3\kappa_a^{[\alpha L(t)/2]-1} \tag{2.26}$$

(if $\partial\mathscr{O} = \varnothing$, the previous expectation is equal to zero).

In view of (2.18) we find, iterating the preceding inequality,

$$A_1 \le 3^{[L(t)]} \exp\left\{-[L(t)]\left(\left[\frac{\alpha L(t)}{2}\right] - 1\right) \log\left(\frac{1}{\kappa_a}\right)\right\}. \tag{2.27}$$

Since $L(t) = t^\delta$ with $\delta \in (1/6, 1/3)$, this completes the proof of Proposition 2.1. □

We now show that A_2 from (2.15) is negligible for our purpose.

PROPOSITION 2.2. *Let* $\eta \in (0,1)$ *and define* $\mathscr{E}_2 = \{T_{U_M} > t,\ Z_t \in B(yt^{1/3}),\ N_t \le [L(t)],\ L_t \ge \eta\}$. *We then have*

$$\begin{aligned} &\limsup_{r\to 0} \limsup_{\alpha\to 0} \limsup_{t\to\infty} \sup_{|y|\in[L_1, L_2]} t^{-1/3} \\ &\qquad \times \log \mathbb{E} \otimes E_0\left[1_{\mathscr{E}_2} \exp\left\{-\int_0^t V(Z_s,\omega)\,ds\right\}\right] = -\infty. \end{aligned} \tag{2.28}$$

PROOF. Let $0 \le J \le [L(t)]$. On the set $\{L_t \ge \eta\} \cap \{N_t = J\}$ we find, using (2.12)–(2.14),

$$\begin{aligned} t\eta &\le \sum_{j=0}^{J} R_{j+1} \wedge t - D_j \wedge t \\ &= \sum_{j=0}^{J-1} R_1 \circ \vartheta_{D_j} + t - D_J \wedge t, \end{aligned} \tag{2.29}$$

with $D_0 = 0$, and if $J = 0$, the sum is not present.

Now let $\lambda > 0$. We thus find, using (2.29) and Chebyshev's inequality, that

$$\begin{aligned} A_2 &\le \sum_{J=0}^{[L(t)]} \mathbb{E} \otimes E_0\left[N_t = J, L_t \ge \eta, T_{U_M} > t, \exp\left\{-\int_0^t V(Z_s,\omega)\,ds\right\}\right] \\ &\le \sum_{J=0}^{[L(t)]} \mathbb{E} \otimes E_0\Bigg[\exp\{-\lambda\eta t\} \exp\left\{\sum_{j=0}^{J-1} \lambda(R_1 \circ \vartheta_{D_j}) - \int_{D_j}^{R_{j+1}} V(Z_s,\omega)\,ds\right\} \\ &\qquad\qquad \times \exp\left\{\lambda(t - D_J \wedge t) - \int_{D_J \wedge t}^{t} V(Z_s,\omega)\,ds\right\} \\ &\qquad\qquad\qquad \times 1_{\{R_J \le t < R_{J+1} \wedge T_{U_M}\}}\Bigg], \end{aligned} \tag{2.30}$$

with the obvious notation if $J = 0$ and where we define $R_0 = 0$.

Observe now that, using integration by parts, we have, for all $h \ge 0$,

$$\begin{aligned} &1 + \int_0^h \lambda \exp\{\lambda u\} \exp\left\{-\int_0^u V(Z_s,\omega)\,ds\right\} du \\ &\qquad \ge \exp\left\{\lambda h - \int_0^h V(Z_s,\omega)\,ds\right\} \end{aligned} \tag{2.31}$$

and the left-hand side in (2.31) is increasing in h. But then we find, using (2.12) and (2.31), that for $0 \le j \le J-1$ ($J \ge 1$),

$$\begin{aligned}
&\exp\Big\{\lambda(R_1 \circ \vartheta_{D_j}) - \int_{D_j}^{R_{j+1}} V(Z_s, \omega)\, ds\Big\} 1_{\{R_J \le t < R_{J+1} \wedge T_{U_M}\}} \\
&\quad = \exp\Big\{\lambda(R_1 \wedge T_{U_M}) \circ \vartheta_{D_j \wedge T_{U_M}} \\
&\qquad\qquad - \int_0^{(R_1 \wedge T_{U_M}) \circ \vartheta_{D_j \wedge T_{U_M}}} V(Z_s \circ \vartheta_{D_j \wedge T_{U_M}}, \omega)\, ds\Big\} \\
&\qquad \times 1_{\{R_J \le t < R_{J+1} \wedge T_{U_M}\}} \\
&\quad \le \Big(1 + \lambda \int_0^{R_1 \wedge T_{U_M}} \exp\{\lambda u\} \exp\Big\{-\int_0^u V(Z_s, \omega)\, ds\Big\}\, du\Big) \circ \vartheta_{D_j \wedge T_{U_M}}.
\end{aligned} \tag{2.32}$$

For the last term in (2.30), observe that on the set $\{R_J \le t < R_{J+1} \wedge T_{U_M}\}$ we have $t - D_J \wedge t \le (R_1 \wedge T_{U_M}) \circ \vartheta_{D_J \wedge T_{U_M}}$, where we recall that $R_0 = 0$. In view of (2.31) this yields

$$\begin{aligned}
&\exp\Big\{\lambda(t - D_J \wedge t) - \int_{D_J \wedge t}^{t} V(Z_s, \omega)\, ds\Big\} 1_{\{R_J \le t < R_{J+1} \wedge T_{U_M}\}} \\
&\quad \le \Big(1 + \lambda \int_0^{(R_1 \wedge T_{U_M}) \circ \vartheta_{D_J \wedge T_{U_M}}} \exp\{\lambda u\} \\
&\qquad\qquad \times \exp\Big\{-\int_0^u V(Z_s \circ \vartheta_{D_J \wedge t}, \omega)\, ds\Big\}\, du 1_{\{D_J < t\}}\Big) \\
&\qquad \times 1_{\{R_J \le t < R_{J+1} \wedge T_{U_M}\}} \\
&\quad \le \Big(1 + \lambda \int_0^{R_1 \wedge T_{U_M}} \exp\{\lambda u\} \exp\Big\{-\int_0^u V(Z_s, \omega)\, ds\Big\}\, du\Big) \circ \vartheta_{D_J \wedge T_{U_M}},
\end{aligned} \tag{2.33}$$

where in the last inequality we have used that, on the set $\{R_J \le t < R_{J+1} \wedge T_{U_M}\} \cap \{D_J < t\}$, $D_J \wedge t = D_J \wedge T_{U_M}$. Inserting (2.32) and (2.33) into (2.30), we find:

$$\begin{aligned}
A_2 \le \sum_{J=0}^{[L(t)]} \mathbb{E} \otimes E_0\Big[&\exp\{-\lambda \eta t\} \\
&\times \prod_{j=0}^{J}\Big(1 + \lambda \int_0^{R_1 \wedge T_{U_M}} \exp\{\lambda u\} \\
&\qquad \times \exp\Big\{-\int_0^u V(Z_s, \omega)\, ds\Big\}\, du\Big) \circ \vartheta_{D_j \wedge T_{U_M}}\Big].
\end{aligned} \tag{2.34}$$

Observe that if $\partial\mathscr{O} = \varnothing$, $D_1 \wedge T_{U_M} = T_{U_M}$ and the corresponding term in the preceding product is equal to 1. We are now going to pick an appropriate λ. To this end define the open set

$$F = \mathbb{R} \setminus L, \tag{2.35}$$

where the set L was introduced in (2.9). Denote by $\lambda_\omega(F)$ the principal Dirichlet eigenvalue of $-\frac{1}{2}(d^2/dx^2) + V(\cdot, \omega)$ in F. Define

$$\lambda_\omega = \tfrac{1}{2}\lambda_\omega(F). \tag{2.36}$$

Now pick in (2.34) $\lambda = \lambda_\omega$. Using the strong Markov property in (2.34) and applying Proposition A.1 and A.2 from [13], we find with a suitable constant $\kappa_1 \in (1, \infty)$ and

$$K\big(\tfrac{1}{2}\big) = \big(\tfrac{1}{2}\big)^{-3/2}\kappa_1 \tag{2.37}$$

that

$$A_2 \le ([L(t)] + 1)K(1/2)^{[L(t)]+1}\mathbb{E}\big[\exp\big\{-\lambda_\omega \eta t\big\}\big]. \tag{2.38}$$

We now need a lower bound on λ_ω. We are going to use Lemmas 5.1 and 5.2 of [10], where lower bounds on $\lambda_\omega(U)$, $U \subset \mathbb{R}$ open, are established. To this end denote by I_k the connected components of

$$U \setminus \operatorname{supp} \omega = \bigcup_k I_k. \tag{2.39}$$

We can easily conclude from Lemmas 5.1 and 5.2 of [10], owing to our assumption that 0 is a point of density of $W(\cdot)$, that for all $\varepsilon \in (0, 1)$ there exists a constant $\gamma = \gamma(W, \varepsilon) > 0$, such that on a set of full $\mathbb{P}$ measure and for $U \subset \mathbb{R}$ open we have

$$\lambda_\omega(U) \ge \gamma \wedge \inf_k \frac{\pi^2}{2|I_k|^2}(1 - \varepsilon), \tag{2.40}$$

where $|I|$ denotes the length of I. We shall in fact only use (2.40) for $\varepsilon = \frac{1}{2}$ for the time being.

In view of (2.40) we now need an upper bound on the largest "trap free" region in the open set F from (2.35). To this end let $I \subset F$ be an interval receiving no point of ω. But then since $F = \mathbb{R} \setminus L$ and since, by the definition of L before (2.9), connected components of thin edges [cf. (2.8)] in F have a length at most $rt^{1/3}$, we find for fixed r,

$$\begin{aligned} |I| &\le rt^{1/3} + 2L(t) \\ &\le 2rt^{1/3} \end{aligned} \tag{2.41}$$

for large enough t, since $L(t) = t^\delta$ with $\delta < 1/3$.

Inserting (2.41) into (2.40), we find for all fixed r, α, for large enough t,

$$\lambda_\omega(F) \ge \frac{\kappa_2}{r^2 t^{2/3}}, \tag{2.42}$$

where $\kappa_2 = \pi^2/16$. Inserting (2.42) into (2.38), we find that

$$t^{-1/3} \log A_2 \le \frac{\log([L(t)] + 1)}{t^{1/3}} + \frac{[L(t)] + 1}{t^{1/3}} \log K\left(\frac{1}{2}\right) - \frac{1}{2}\eta\kappa_2\frac{1}{r^2}. \tag{2.43}$$

Since $L(t) = t^\delta$ with $\delta < 1/3$, this completes the proof of Proposition 2.2. $\square$

2.5. *The leading term.* In this section we finally come to the investigation of A_3 from (2.15), which is the core of the proof of Theorem 2.1. Due to our choice of M in (2.4) and in view of Proposition 2.1 and Proposition 2.2, we see that the claim of Theorem 2.1 follows once we have shown the following result.

PROPOSITION 2.3. *Define the set* $\mathscr{E}_3 = \{T_{U_M} > t,\ Z_t \in B(yt^{1/3}),\ N_t \leq [L(t)],\ L_t < \eta\}$. *We then have*

$$\limsup_{\eta\to 0}\limsup_{r\to 0}\limsup_{\alpha\to 0}\limsup_{t\to\infty}\sup_{|y|\in[L_1,L_2]} t^{-1/3} \times \log \mathbb{E}\otimes E_0\left[1_{\mathscr{E}_3}\exp\left\{-\int_0^t V(Z_s,\omega)\,ds\right\}\right] \leq -I(L_1), \tag{2.44}$$

where $I(y)$ *was introduced in* (1.5).

Before we begin the proof of Proposition 2.3, let us briefly explain the strategy. On the set $\mathscr{E}_3$ we consider $\mathscr{O}_1,\dots,\mathscr{O}_K$, the collection of pseudo-holes the process has visited up to time t. Since the particle spends at least $(1-\eta)t$ units of time in $\bigcup_i \mathscr{O}_i$ and since $T_{U_M} > t$, it follows that $1 \leq K \leq [2M/r]+1$.

We shall then split up the cost for the particle to produce an excursion leading to $yt^{1/3}$ as a cost to spend $(1-\eta)t$ units of time in $\bigcup_i \mathscr{O}_i \cap U_M$, the "resting cost," and a cost for the particle to connect the various pseudo-holes $\mathscr{O}_1,\cdots,\mathscr{O}_K$ which "lead the way to $yt^{1/3}$," the "connecting cost." Here by splitting up the cost we mean that after "summing" over all possible configurations of $\bigcup_i \mathscr{O}_i \cap U_M$, the resting cost and the connecting cost are "independent" with respect to $\mathbb{P}$; see Lemma 2.3. We are also going to show that the combinatorial complexity of the possible configurations of $\bigcup_i \mathscr{O}_i \cap U_M$ is not too large.

Let us now comment on the resting cost. As in the proof of Proposition 2.2, we are going to see, again after applying an exponential Chebyshev inequality and the uniform exponential bounds from Proposition A.1 of [13], that on $\mathscr{E}_3$ the cost for the particle, feeling the potential V, to spend at least $(1-\eta)t$ units of time in $\bigcup_i \mathscr{O}_i \cap U_M$ is for large t bounded above by $\exp\{o(t^{1/3})\}\mathbb{E}[\exp\{-\lambda_\omega(1-\eta)t\}]$. Here λ_ω is chosen close to the principal Dirichlet eigenvalue of $-\frac{1}{2}(d^2/dx^2) + V(\cdot,\omega)$ in $\bigcup_i \mathscr{O}_i \cap U_M$; cf. (2.62). Of course, since $V(\cdot,\omega) \geq 0$, a lower bound on λ_ω is given by the principal Dirichlet eigenvalue of $-\frac{1}{2}(d^2/dx^2)$ in $\bigcup_i \mathscr{O}_i \cap U_M$. An important point is then that we have a good control on the probability that, for $\mathscr{O}_i \cap U_M \neq \varnothing$, $\mathscr{O}_i$ is a pseudo-hole. Indeed, we are going to show, after picking our parameters in a good way, that this probability is up to correction terms bounded from above by $\exp\{-\nu|\mathscr{O}_i \cap U_M|\}$. This is precisely the probability that $\mathscr{O}_i \cap U_M$ receives no Poisson point.

We now explain the strategy we use to give an upper bound on the "connecting cost." For simplicity, let us assume that $y \geq 0$.

We extract from our original sequence of stopping times introduced in (2.10) a subsequence such that at two successive return times the particle is in different pseudo-holes. Moreover, this subsequence will take into account all the corresponding returns to $\mathscr{O}_i$, $1 \le i \le K$, the collection of pseudo-holes the process has visited up to time t; cf. the definition of G_5 in (2.53). If $y \ge 0$ lies at the right of all the $\mathscr{O}_i$, we will also take into account the cost for connecting the rightmost pseudo-hole with $yt^{1/3}$; cf. (2.55). Finally, if all the $\mathscr{O}_i$, $1 \le i \le K$, lie at the right of the origin, we also consider R_1, the first return to a pseudo-hole, which is then responsible for the cost of connecting the origin with the leftmost pseudo-hole; see also the remark after (2.66).

Our main tool is then Theorem 1.3 of [12], part I. Here the Liapounov exponent $\beta_0(1)$ enters. It measures how costly it is for the process to reach a certain location which is far away from its starting point when the particle can pick its own time to perform the displacement. Theorem 1.3 of [12], part I, will give us an asymptotic upper bound on the "connecting cost," where the "total connecting length" enters. A crucial point is that the "total connecting length" the particle has to travel on its way to $yt^{1/3}$ is up to correction terms bounded below by $(|y|t^{1/3} - \sum_i |\mathscr{O}_i \cap U_M|)_+$. This follows from the fact that we have considered all pseudo-holes visited until time t and $Z_t \in B(yt^{1/3})$; cf. (2.79). We finally come to the proof of Proposition 2.3.

PROOF OF PROPOSITION 2.3. As in the proof of Theorem 1.1, we only treat the case $y \ge 0$. To show (2.44), we are now going to cover $\mathscr{E}_3$ by a family $\mathscr{G}$ of events of combinatorial complexity $|\mathscr{G}| = \exp\{o(t^{1/3})\}$ as $t \to \infty$; see (2.57), resp. (2.58), in Lemma 2.2. Thus we see that once we show for such a covering that

$$\begin{aligned}(2.45)\qquad &\limsup_{\eta\to 0}\limsup_{r\to 0}\limsup_{\alpha\to 0}\limsup_{t\to\infty}\sup_{y\in[L_1,L_2]}\sup_{G\in\mathscr{G}} t^{-1/3}\\ &\quad\times \log \mathbb{E}\otimes E_0\left[1_G \exp\left\{-\int_0^t V(Z_s,\omega)\,ds\right\}\right] \le -I(L_1),\end{aligned}$$

the claim of Proposition 2.3 follows. □

The covering $\mathscr{G}$ will, in particular, keep track of the possible endpoints of $\mathscr{O}_i \cap U_M$, where the $\mathscr{O}_i$, $1 \le i \le K \le [2M/r] + 1$, are the various pseudo-holes the process returns to up to time t. Furthermore, as in the proof of Proposition 2.2, we are going to keep track of the total number of completed returns up to time t. Finally, we also take into consideration the various possible succsesive return times at which the process is in different pseudo-holes, as well as the position of the path at the corresponding departure, resp. successive return time.

2.6. *The covering lemma.* In this section we introduce a covering $\mathscr{G}$ of $\mathscr{E}_3$. We start by describing the type of events of the covering $\mathscr{G}$ of $\mathscr{E}_3$.

We begin by introducing the set $\mathscr{C} \subset t^{\delta}\mathbb{Z} \cap U_{2M}$ which consists of the "possible discretized endpoints" of $\mathscr{O}_i \cap U_M$.

For $1 \le K \le [2M/r]+1$ and large enough t such that $rt^{1/3} > t^\delta$, we define

$$\mathscr{C} = \Big\{x_i, y_i \in t^\delta\mathbb{Z} \cap U_{2M};\ x_1 < y_1 < x_2 < \cdots < x_K < y_K, \\ x_K < Mt^{1/3},\ y_1 > -Mt^{1/3},\ y_i - x_i \ge rt^{1/3}, \\ x_{i+1} - y_i \ge 2t^\delta,\ 1 \le i \le K-1\Big\} \tag{2.46}$$

and if $K = 1$, the condition $x_{i+1} - y_i \ge 2t^\delta$ is not present. Observe that since $Mt^{1/3} > rt^{1/3} > t^\delta$ we have $\mathscr{C} \neq \varnothing$.

For given $\mathscr{C}$ as in (2.46), $1 \le K \le [2M/r]+1$, we then define the event G_1 that there exists a pseudo-hole $\mathscr{O}_i$ with prescribed intersection with U_M, $1 \le i \le K$, by

$$G_1 = \bigcap_{i=1}^{K} \Big\{\mathscr{O}_i \cap U_M = \big((-Mt^{1/3}) \vee (x_i - t^\delta), (y_i + t^\delta) \wedge (Mt^{1/3})\big) \\ \text{and } \mathscr{O}_i \text{ is a pseudo hole}\Big\}. \tag{2.47}$$

Observe that, for $\mathscr{C}$ as in (2.46), $\mathscr{O}_i \cap U_M \neq \varnothing$. Observe also that in (2.47) $x_1 - t^\delta$ need not be the leftmost endpoint, resp. $y_K + t^\delta$ need not be the rightmost endpoint, of the visited pseudo-holes, since it is of course possible that these endpoints are strictly smaller than $-Mt^{1/3}$, resp. strictly larger than $Mt^{1/3}$. This is also the reason why we are working in (2.46) with $t^\delta\mathbb{Z} \cap U_{2M}$ instead of $t^\delta\mathbb{Z} \cap U_M$.

The covering $\mathscr{G}$ will now consist of two types of events, depending on whether $yt^{1/3} (\le Mt^{1/3})$ lies at the "left of the rightmost endpoint" of the pseudo-holes $\mathscr{O}_i$, $1 \le i \le K$, or not.

We are first going to describe events of the first type.

To this end we assume that we are given $\mathscr{C}$ as in (2.46) with $yt^{1/3} \le (y_K + 2t^\delta) \wedge Mt^{1/3}$. Events G of the first type will be of the form $G = \bigcap_{i=1}^5 G_i$, with G_1 from (2.47) and where the G_i, $2 \le i \le 5$, are now going to be defined; cf. (2.54).

Let $1 \le K \le [2M/r]+1$ and J: $K \le J \le [L(t)]$. In close analogy to the proof of Proposition 2.2, we define the event G_2 that before time t the process returns exactly J times to pseudo-holes and does not exit U_M before time t:

$$G_2 = \{R_J \le t < R_{J+1} \wedge T_{U_M}\} \tag{2.48}$$

and the event G_3 that Brownian motion spends at least $(1-\eta)t$ units of time in the pseudo-holes:

$$G_3 = \Big\{\sum_{j=1}^{J} D_j \wedge t - R_j \wedge t \ge t(1-\eta)\Big\}. \tag{2.49}$$

We also define the event G_4 that the position of the path at all the return times R_j, $1 \le j \le J$, lies in $\bigcup_{i=1}^K \mathscr{O}_i \cap U_M$:

$$G_4 = \Big\{Z_{R_j} \in \bigcup_{i=1}^{K} \mathscr{O}_i \cap U_M,\ 1 \le j \le J\Big\}. \tag{2.50}$$

Finally, we come to the definition of the event G_5 that at prescribed successive return times the process is in different pseudo-holes, provided $K \geq 2$. The union of the corresponding pseudo-holes equals the union of all the different pseudo-holes the process has visited up to time t.

More precisely, if $K \geq 2$ let $\widetilde{J}$ be such that $K \leq \widetilde{J} \leq J$, pick $0 = j_0 < j_1 < \cdots < j_{\widetilde{J}-1} \leq J-1$ and let $l_1 \neq l_2 \neq \cdots \neq l_{\widetilde{J}}$ be consecutively distinct with $l_i \in \{1, \cdots, K\}$, $1 \leq i \leq \widetilde{J}$, such that

$$\bigcup_{i=1}^{\widetilde{J}} \{l_i\} = \{1, \cdots, K\}. \tag{2.51}$$

We then introduce a sequence (u_{l_i}, v_{l_i}) of points, which shall keep track of the position of the path at the successive departures and returns, when it "connects" the various pseudo-holes $\mathscr{O}_i$, $1 \leq i \leq K$. For given $\mathscr{C}$ as in (2.46), let (u_{l_i}, v_{l_i}), $1 \leq i \leq \widetilde{J}-1$, be a sequence of points with

$$(u_{l_i}, v_{l_i}) \in \{(y_{l_i} + t^\delta, x_{l_{i+1}}), (x_{l_i} - t^\delta, y_{l_{i+1}})\} \tag{2.52}$$

and let $u_0 = 0$, $v_0 \in t^\delta \mathbb{Z} \cap U_M$.

Observe that, owing to (2.51), we have that $\bigcup_{i=1}^{\widetilde{J}} \mathscr{O}_{l_i} \cap U_M = \bigcup_{i=1}^{K} \mathscr{O}_i \cap U_M$. We then define the event G_5 that at time D_{j_i} the process is in u_{l_i}, one of the "endpoints of $\mathscr{O}_{l_i}$," and at the successive return time R_{j_i+1} the process is in $v_{l_i} \in \mathscr{O}_{l_{i+1}}$, $l_i \neq l_{i+1}$, by

$$G_5 = \{Z_{D_{j_i}} = u_{l_i},\ Z_{R_{j_i+1}} = v_{l_i},\ 0 \leq i \leq \widetilde{J}-1\}, \tag{2.53}$$

where we put $l_0 = 0$, $D_0 = 0$, and in the case where $K = 1$ we set $\widetilde{J} = 1$ in (2.53).

Observe, however, that because of the continuity of the path it is of course possible that $G_5 = \varnothing$. Notice also that it is possible that $G_5 \cap G_2 = \varnothing$, since it might occur that, for given $\mathscr{C}$, some u_{l_i} or some v_{l_i} does not lie in U_M.

Events of the first type are now of the form

$$G = \bigcap_{i=1}^{5} G_i. \tag{2.54}$$

We now describe events of the second type of the covering $\mathscr{G}$.

We assume that we are given $\mathscr{C}$ with $Mt^{1/3} \geq yt^{1/3} > y_K + 2t^\delta$. We then define $\widetilde{G}_i = G_i$, $1 \leq i \leq 5$.

Furthermore, we define the event $\widetilde{G}_6$ that at time D_J the process is in $y_K + t^\delta$, the rightmost endpoint of the rightmost pseudo-hole $\mathscr{O}_K$, and then enters $B(yt^{1/3})$ before time t without exiting U_M and returning to a pseudo-hole before time t:

$$\widetilde{G}_6 = \{Z_{D_J} = y_K + t^\delta,\ D_J < D_J + H(yt^{1/3}) \circ \vartheta_{D_J} \leq t < R_{J+1} \wedge T_{U_M}\}. \tag{2.55}$$

Events of the second type are then of the form

$$G = \bigcap_{i=1}^{6} \widetilde{G}_i. \tag{2.56}$$

We are now ready to state the following result.

LEMMA 2.2. *For the set $\mathscr{E}_3$ defined in Proposition 2.3, we have*

$$\mathscr{E}_3 \subset \bigcup_{G \in \mathscr{G}} G, \tag{2.57}$$

where G is either of the first type (2.54) *or of the second type* (2.56). *Furthermore, if $|\mathscr{G}|$ denotes the number of elements in $\mathscr{G}$, we have, as $t \to \infty$,*

$$|\mathscr{G}| = \exp\{o(t^{1/3})\}. \tag{2.58}$$

PROOF. The first point to observe is that on $\mathscr{E}_3$ we have $N_t \geq 1$. Indeed, if $N_t = 0$, we would have $\eta > L_t = (1/t)(R_1 \wedge t) = 1$, which is a contradiction. This also shows that on the set $\mathscr{E}_3$ there exists a pseudo-hole, possibly the whole of $\mathbb{R}$, to which the process returns strictly before time t.

Now pick $1 \leq J \leq [L(t)]$ and observe that

$$\{N_t = J\} \cap \{T_{U_M} > t\} = \{R_J \leq t < R_{J+1} \wedge T_{U_M}\}. \tag{2.59}$$

Since $t = \sum_{j \geq 1} D_j \wedge t - R_j \wedge t + tL_t$, we also have

$$\{N_t = J\} \cap \{L_t < \eta\} = \{N_t = J\} \cap \left\{\sum_{j=1}^{J} D_j \wedge t - R_j \wedge t \geq t(1-\eta)\right\}. \tag{2.60}$$

Furthermore, on the set $\{N_t = J\} \cap \{T_{U_M} > t\} \cap \{L_t < \eta\} \cap \{Z_t \in B(yt^{1/3})\}$, let $1 \leq N(\omega) \leq [2M/r] + 1$ be the number of pseudo-holes having a nonempty intersection with U_M. Now fix ω and denote by $K(w, \omega)$, $1 \leq K(w, \omega) \leq N(\omega) \leq [2M/r] + 1$, the number of different pseudo-holes the process has returned to before time t. In particular, $1 \leq K(w, \omega) \leq N_t$. Let us then pick the labeling such that $\mathscr{O}_1$ is the leftmost of these pseudo-holes, $\mathscr{O}_2$ the nearest to the right of $\mathscr{O}_1$ and so on until $\mathscr{O}_K$.

For $1 \leq i \leq K$ we now define $(x_i(\omega) - t^\delta) \in t^\delta\mathbb{Z}$ to be the nearest point to the left, resp. $(y_i(\omega) + t^\delta) \in t^\delta\mathbb{Z}$ to be the nearest point to the right, of $\mathscr{O}_i \cap U_M$. Of course, for i, $2 \leq i \leq K-1$, provided $K \geq 3$, $x_i(\omega) - t^\delta$ and $y_i(\omega) + t^\delta$ are the endpoints of $\mathscr{O}_i$; cf. the remark after (2.47). We then have $\mathscr{O}_i \cap U_M = ((-Mt^{1/3}) \vee (x_i - t^\delta),\ (y_i + t^\delta) \wedge (Mt^{1/3})) \neq \varnothing$, with $y_i - x_i \geq rt^{1/3}$, $1 \leq i \leq K$, $x_1 < y_1 < x_2 < \cdots < y_K$, and if $K \geq 2$, $x_{i+1} - y_i \geq 2t^\delta$, $1 \leq i \leq K-1$. Moreover, since $R_J \leq t < R_{J+1} \wedge T_{U_M}$ we have, as soon as $rt^{1/3} > t^\delta$, $x_i(\omega),\ y_i(\omega) \in t^\delta\mathbb{Z} \cap U_{2M}$, $1 \leq i \leq K$, $y_1 > -Mt^{1/3}$, $x_K < Mt^{1/3}$, and

$$Z_{R_i} \in \bigcup_{i=1}^{K} \mathscr{O}_i \cap U_M, \qquad 1 \leq i \leq J. \tag{2.61}$$

Let us now show (2.57).

We first assume that $y_K + 2t^\delta \geq yt^{1/3}$ has occurred. If $K(w,\omega) = 1$, there is nothing left to show, since $Z_0 = 0$ and $Z_{R_1} = v_0$ for some $v_0 \in t^\delta\mathbb{Z} \cap U_M$. Let us therefore assume that $K(w,\omega)$ and $N_t = J \geq 2$. We are now going to extract from our original sequence of stopping times $0 \leq R_1 < D_1 < \cdots < R_J \leq t < R_{J+1} \wedge T_{U_M}$ a subsequence, such that two successive return times correspond to the visit of "new" pseudo-holes, that is, $Z_{R_{j_{i+1}}} \notin$ pseudo-hole containing $Z_{R_{j_i}}$. The union of the collection of the corresponding pseudo-holes equals $\bigcup_{i=1}^K \mathscr{O}_i \cap U_M$, which is the union of all pseudo-holes the process has returned to before time t. More precisely:

Let j_1 be the smallest number in $\{1,\dots,J-1\}$ such that there exists $l_1 \neq l_2$, $l_1, l_2 \in \{1,\dots,K\}$ with $Z_{R_{j_1}} \in \mathscr{O}_{l_1}$, $Z_{R_{j_1+1}} \in \mathscr{O}_{l_2}$. In particular, we then have either $Z_{D_{j_1}} = y_{l_1} + t^\delta$, $Z_{R_{j_1+1}} = x_{l_2}$, or $Z_{D_{j_1}} = x_{l_1} - t^\delta$, $Z_{R_{j_1+1}} = y_{l_2}$.

Let j_2 be the smallest number in $\{j_1+1,\dots,J-1\}$ such that there exists $l_3 \neq l_2$, $l_3 \in \{1,\dots,K\}$ with $Z_{R_{j_2}} \in \mathscr{O}_{l_2}$, $Z_{R_{j_2+1}} \in \mathscr{O}_{l_3}$.

Continue like this until $j_{\widetilde{J}-1}$, the last number in $\{j_{\widetilde{J}-2}+1,\dots,J-1\}$, such that there exists $l_{\widetilde{J}} \neq l_{\widetilde{J}-1}$, $l_{\widetilde{J}} \in \{1,\cdots,K\}$, with $Z_{R_{j_{\widetilde{J}-1}}} \in \mathscr{O}_{l_{\widetilde{J}-1}}$, $Z_{R_{j_{\widetilde{J}-1}+1}} \in \mathscr{O}_{l_{\widetilde{J}}}$. We then have $\widetilde{J} = \widetilde{J}(w)$ and $K \leq \widetilde{J} \leq J$ with $\bigcup_{i=1}^{\widetilde{J}} \mathscr{O}_{l_i} \cap U_M = \bigcup_{i=1}^K \mathscr{O}_i \cap U_M$.

This together with (2.59), (2.60) and (2.61) already shows (2.57) in the case where $y_K + 2t^\delta \geq yt^{1/3}$ has occurred. Let us now show (2.57) in the case where $y_K + 2t^\delta < yt^{1/3}$ has occurred.

But then since $R_J \leq t < R_{J+1} \wedge T_{U_M}$, resp. $Z_t \in B(yt^{1/3})$, and because $\mathscr{O}_i$, $1 \leq i \leq K$, are all pseudo-holes the process has returned to up to time t, we know that $Z_{D_J} = y_K + t^\delta$ and $D_J < D_J + H(yt^{1/3}) \circ \vartheta_{D_J} \leq t < R_{J+1} \wedge T_{U_M}$, which shows (2.57).

Finally, it remains to show (2.58). To this end observe that

$$|\mathscr{C}| \leq ([4Mt^{1/3}/L(t)]+1)^{([2M/r]+1)} = \exp\{o(t^{1/3})\} \quad \text{as } t \to \infty.$$

Furthermore, the number of possible indices j_i is bounded above by $[L(t)]^{[L(t)]} = \exp\{o(t^{1/3})\}$, since $L(t) = t^\delta$ with $\delta < 1/3$, and the number of possible indices l_i is bounded above by $([2M/r]+1)^{[L(t)]} = \exp\{o(t^{1/3})\}$. Since $1 \leq J \leq [L(t)] = \exp\{o(t^{1/3})\}$ and $K \leq \widetilde{J} \leq J$, resp. $1 \leq K \leq [2M/r]+1$, we see that $|\mathscr{G}| = \exp\{o(t^{1/3})\}$, and the proof of Lemma 2.2 is complete. □

2.7. *The proof of* (2.45). In this section we show (2.45), which is the remaining step in the proof of Proposition 2.3.

We will restrict the estimates to the case of events G of the first type. The case of events of the second type is treated in a completely analogous fashion. Consider a nonempty G of the type (2.54). In particular, we assume from now on that $yt^{1/3} \leq Mt^{1/3} \wedge (y_K + 2t^\delta)$. Pick $\rho \in (0,1)$ and write

$$\lambda_\omega = (1-\rho)\lambda_\omega\Bigl(\bigcup_{i=1}^K \mathscr{O}_i \cap U_M\Bigr), \tag{2.62}$$

with the same notation as in the proof of Proposition 2.2; cf. (2.36). We then find, using Chebyshev's inequality, that

$$\begin{aligned}&\mathbb{E}\otimes E_0\biggl[1_G \exp\biggl\{-\int_0^t V(Z_s,\omega)\,ds\biggr\}\biggr]\\ &\quad\le \mathbb{E}\biggl[1_{G_1}\exp\{-\lambda_\omega(1-\eta)t\}\\ &\qquad\times E_0\biggl[1_{G_2\cap G_4\cap G_5}\exp\biggl\{\lambda_\omega\sum_{i=1}^J D_i\wedge t - R_i\wedge t-\int_0^t V(Z_s,\omega)\,ds\biggr\}\biggr]\biggr].\end{aligned}\tag{2.63}$$

For $\omega\in G_1$ we now find, using $D_j = R_j + D_1\circ\vartheta_{R_j}$, that the term under the E_0 expectation in (2.63) equals ($j_0=0$)

$$\begin{aligned}&1_{G_2\cap G_4\cap G_5}\exp\biggl\{\lambda_\omega\sum_{i=1}^{j_{\widetilde{J}-1}}(D_1\wedge T_{U_M})\circ\vartheta_{R_i}-\int_0^{R_{j_{\widetilde{J}-1}+1}} V(Z_s,\omega)\,ds\biggr\}\\ &\quad\times\exp\biggl\{\lambda_\omega\sum_{i=j_{\widetilde{J}-1}+1}^{J}(D_1\wedge T_{U_M})\circ\vartheta_{R_i}\wedge(t-R_i)_+\\ &\qquad\qquad -\int_{R_{j_{\widetilde{J}-1}+1}}^t V(Z_s,\omega)\,ds\biggr\},\end{aligned}\tag{2.64}$$

with the convention that if $K=1$, which, by definition, is equivalent to $\widetilde{J}=1$ [cf. (2.53)], there is no summation in the first term.

Now, as in the proof of Proposition 2.2, we get with the same constant $K(\rho)\in(1,\infty)$ from (2.37), replacing $\frac{1}{2}$ by ρ in (2.37), that the E_0 expectation of the expression in (2.63) is smaller than

$$\begin{aligned}K(\rho)^{[L(t)]}E_0\biggl[1_{H_2\cap H_4\cap G_5}\exp\biggl\{&\lambda_\omega\sum_{i=1}^{j_{\widetilde{J}-1}}(D_1\wedge T_{U_M})\circ\vartheta_{R_i}\\ &-\int_0^{R_{j_{\widetilde{J}-1}+1}} V(Z_s,\omega)\,ds\biggr\}\biggr],\end{aligned}\tag{2.65}$$

where H_2 is defined by replacing, in G_2, $R_J\le t<T_{U_M}\wedge R_{J+1}$ by $R_{j_{\widetilde{J}-1}+1}<T_{U_M}$, resp. H_4 is defined by replacing, in G_4, $1\le i\le J$ by $1\le i\le j_{\widetilde{J}-1}+1$.

The goal now is to split up the resting cost and the connecting cost. To this end we write

$$\begin{aligned}&\exp\biggl\{\lambda_\omega\sum_{i=1}^{j_{\widetilde{J}-1}}(D_1\wedge T_{U_M})\circ\vartheta_{R_i}-\int_0^{R_{j_{\widetilde{J}-1}+1}} V(Z_s,\omega)\,ds\biggr\}\\ &\quad=\prod_{l=1}^{\widetilde{J}-1}\exp\biggl\{\lambda_\omega\sum_{i=j_{l-1}+1}^{j_l}(D_1\wedge T_{U_M})\circ\vartheta_{R_i}-\int_{R_{j_{l-1}+1}}^{D_{j_l}} V(Z_s,\omega)\,ds\biggr\}\\ &\qquad\times\exp\biggl\{-\int_{D_{j_l}}^{R_{j_l+1}} V(Z_s,\omega)\,ds\biggr\}\exp\biggl\{-\int_0^{R_1} V(Z_s,\omega)\,ds\biggr\},\end{aligned}\tag{2.66}$$

where if $\widetilde{J}=1$ only the last term is present.

Here the term $\exp\{-\int_{D_{j_l}}^{R_{j_{l+1}}} V(Z_s,\omega)\,ds\}$ in (2.66) will be responsible for the connecting cost. Furthermore, in the case where all the $\mathscr{O}_i$, $1 \le i \le K$, lie at the right of the origin, that is, when $x_1 \ge t^\delta$, we are also going to take into account the term $\exp\{-\int_0^{R_1} V(Z_s,\omega)\,ds\}$, which is then responsible for the connecting cost of the origin with the leftmost pseudo-hole.

Now inserting (2.66) into (2.65) and applying the strong Markov property, we find with regard to the definition of G_5 from (2.53) that the expression in (2.65) is smaller than

$$K(\rho)^{2[L(t)]} \prod_{i=0}^{\tilde{J}-1} \left(E_{u_{l_i}} \left[\exp\left\{ -\int_0^{R_1} V(Z_s,\omega)\,ds \right\},\ Z_{R_1} = v_{l_i} \right] \right), \tag{2.67}$$

where we used once again Proposition A.1 of [13] for the terms involving λ_ω.

Observe now that since we are working with a nonempty G, the u_{l_i} with $i \ge 1$ in (2.67) are one of the endpoints of $\mathscr{O}_{l_i} \cap U_M$, resp. $v_{l_i} \in \mathscr{O}_{l_{i+1}} \cap U_M$, $l_i \ne l_{i+1}$; cf. (2.52). For the same reason we know that $\mathscr{O}_{l_i} \cap U_M$ and $\mathscr{O}_{l_{i+1}} \cap U_M$ are "nearest neighbors." Since (2.51) ensures that $\bigcup_{i=1}^{\tilde{J}} \mathscr{O}_{l_i} \cap U_M = \bigcup_{i=1}^{K} \mathscr{O}_i \cap U_M$, we can therefore find a subsequence of $(u_{l_i}, v_{l_i})_{1 \le i \le \tilde{J}-1}$, which consists of pairs of "nearest neighbor endpoints" of $\mathscr{O}_1 \cap U_M$ and $\mathscr{O}_2$, $\mathscr{O}_2$ and $\mathscr{O}_3$, and so on until $\mathscr{O}_{K-1}$ and $\mathscr{O}_K \cap U_M$, provided $K \ge 2$.

In other words, denoting this subsequence for notational convenience by $(u_i, v_i)_{1 \le i \le K-1}$ and recalling that, on $\mathscr{C}$, $y_1 > -Mt^{1/3}$, $x_K < Mt^{1/3}$, we have

$$(u_i, v_i) \in \{(y_i + t^\delta, x_{i+1}), (x_{i+1} - t^\delta, y_i)\}, \tag{2.68}$$

with $1 \le i \le K-1$, provided $K \ge 2$.

We then find that the expression in (2.67) is smaller than

$$K(\rho)^{2[L(t)]} \prod_{i=0}^{K-1} \left(E_{u_i} \left[\exp\left\{ -\int_0^{R_1} V(Z_s,\omega)\,ds \right\},\ Z_{R_1} = v_i \right] \right), \tag{2.69}$$

where we recall that we have defined $l_0 = 0$.

With regard to (2.63) and (2.65), we now get that for all G of the type (2.54)

$$\begin{aligned} &\mathbb{E} \otimes E_0 \left[1_G \exp\left\{ -\int_0^t V(Z_s,\omega)\,ds \right\} \right] \\ &\quad \le K(\rho)^{2[L(t)]} \mathbb{E} \left[1_{G_1} \exp\left\{ -\lambda_\omega (1-\eta) t \right\} \right. \\ &\qquad\qquad \left. \times \prod_{i=0}^{K-1} \left(E_{u_i} \left[\exp\left\{ -\int_0^{R_1} V(Z_s,\omega)\,ds \right\},\ Z_{R_1} = v_i \right] \right) \right] \\ &\quad = K(\rho)^{2[L(t)]} \mathbb{E}[\mathscr{G}_1], \end{aligned} \tag{2.70}$$

where $\mathscr{G}_1$ is defined as the expression given previously inside the $\mathbb{E}$ expectation.

Let us mention here that for events G of the second type of the covering defined in (2.56), we get exactly the same upper bound as in (2.70) except that there is the additional term $E_{y_K+t^\delta}[\exp\{-\int_0^{H(yt^{1/3})} V(Z_s,\omega)\,ds\},\ R_1 > H(yt^{1/3})]$ coming in the product of (2.70).

Observe now that, owing to the fact that $V(\cdot,\omega)\ge 0$, we have the following obvious lower bound on $\lambda_\omega(1-\eta)$:

$$\lambda_\omega(1-\eta) \ge \frac{\pi^2\,\gamma_1(\rho,\eta)}{2(\sum_{i=1}^K |\mathscr{O}_i\cap U_M|)^2}, \tag{2.71}$$

where $\gamma_1(\rho,\eta)=(1-\rho)(1-\eta)$. Using this in (2.70), we find

$$\begin{aligned}\mathbb{E}[\mathscr{G}_1] &\le \exp\left\{-\frac{\pi^2\,\gamma_1(\rho,\eta)}{2(\sum_{i=1}^K |\mathscr{O}_i\cap U_M|)^2}\,t\right\}\\ &\quad\times \mathbb{E}\left[1_{G_1}\prod_{i=0}^{K-1}\left(E_{u_i}\left[\exp\left\{-\int_0^{R_1} V(Z_s,\omega)\,ds\right\}, Z_{R_1}=v_i\right]\right)\right]\\ &= A\,B,\end{aligned} \tag{2.72}$$

where B is the expression involving the $\mathbb{E}$ expectation in the previous product.

2.7.1. *Splitting up the resting and the connecting cost.* Here we prove a lemma, which in view of (2.72) is the last step in splitting up the resting and the connecting cost. Indeed, we have the following result.

LEMMA 2.3. *Let $\varepsilon\in(0,1)$ and put $\gamma_2(\varepsilon)=1-\varepsilon$. With the notation from* (2.47) *and with B introduced in* (2.72)*, we have, for large enough t,*

$$\begin{aligned}B &\le \exp\left\{-\beta_0(1)\,\gamma_2(\varepsilon)\left[\left(yt^{1/3}-\sum_{i=1}^K |\mathscr{O}_i\cap U_M|\right)_+ - t^\delta\right]\right\}\\ &\quad\times\prod_{i=1}^K \mathbb{P}[\mathscr{O}_i\cap U_M \text{ contains at most } [2M/r]+5 \text{ edges}].\end{aligned} \tag{2.73}$$

PROOF. We shall only give a proof of the preceding lemma in the case where $x_1\ge t^\delta$ in (2.72). In this case we know that x_1-t^δ is the left endpoint of the leftmost pseudo-hole $\mathscr{O}_1$, resp. since we are working with a nonempty G that $v_0=x_1$. The case where $x_1<t^\delta$ is treated in a completely analogous fashion and is simpler.

To show (2.73), the idea now is to use independence with respect to $\mathbb{P}$. Observe, however, that at this point this is not possible since the terms coming in the product of B from (2.72) a priori depend on the restriction of ω to the closed $t^\delta+a$ neighborhood of the "gaps" between the $\mathscr{O}_i\cap U_M$, $1\le i\le K$, which have a nonempty intersection with $\bigcup_i \mathscr{O}_i\cap U_M$ coming in the definition of G_1. Therefore we introduce intervals I_i which will "equal $\mathscr{O}_i\cap U_M$, after having deleted the two rightmost, resp. two leftmost," subboxes from it. To this end we introduce the interval $\mathscr{O}=(-Mt^{1/3},(y_K+t^\delta)\wedge(Mt^{1/3}))$, where we recall that $(y_K+t^\delta)\wedge(Mt^{1/3})$ is the right endpoint of the truncated rightmost "pseudo-hole" $\mathscr{O}_K\cap U_M$. We then define the set F which consists of the "gaps" between the $\mathscr{O}_i \cap U_M$, $1 \le i \le K$, by

$$F = \mathscr{O} \setminus \bigcup_{i=1}^{K} (\mathscr{O}_i \cap U_M). \tag{2.74}$$

We then define the set $I \subset \bigcup_{i=1}^{K} (\mathscr{O}_i \cap U_M)$, which consists of all points belonging to $\bigcup_{i=1}^{K} \mathscr{O}_i \cap U_M$ having distance at least $2t^\delta$ from F:

$$I = \left\{ x \in \bigcup_{i=1}^{K} \mathscr{O}_i \cap U_M;\ \mathrm{dist}(x, F) \ge 2t^\delta \right\}, \tag{2.75}$$

and denote by I_1 the leftmost connected component of I and so on until I_K. Observe, however, that it is possible that $I_K = \varnothing$. Observe also that the connected components of F have a mutual distance at least $rt^{1/3}$. But then, since the terms coming in the product of B in (2.72) only depend on ω restricted to the closed $t^\delta + a$ neighborhood of some connected component of F and since [cf. the remark after (2.9)]

$$G_1 \subset \bigcap_{i=1}^{K} \{I_i \text{ contains at most } [2M/r] + 1 \text{ edges}\}, \tag{2.76}$$

we can as soon as $rt^{1/3} \ge t^\delta > a$ use independence with respect to $\mathbb{P}$ to conclude that, for large enough t,

$$\begin{aligned} B &\le \prod_{i=1}^{K} \mathbb{P}\big[I_i \text{ contains at most } [2M/r] + 1 \text{ edges}\big] \\ &\quad \times \prod_{i=0}^{K-1} \mathbb{E} \otimes E_{u_i}\left[\exp\left\{-\int_0^{R_1} V(Z_s, \omega)\, ds\right\}, Z_{R_1} = v_i\right], \end{aligned} \tag{2.77}$$

where we used that $I_i \cap I_j = \varnothing$, $1 \le i,\ \ j \le K$.

Let us mention here that in the case where $x_1 < t^\delta$, we have the same upper bound as in (2.77), except that the term corresponding to $i = 0$ in the second product from (2.77) is not present.

Since $|u_i - v_i| \ge t^\delta$, $0 \le i \le K - 1$, we can use Theorem 1.3 of [12], part I, to conclude that for all $\varepsilon \in (0, 1)$, for large enough t,

$$\begin{aligned} &\prod_{i=0}^{K-1} \mathbb{E} \otimes E_{u_i}\left[\exp\left\{-\int_0^{R_1} V(Z_s, \omega)\, ds\right\},\ Z_{R_1} = v_i\right] \\ &\qquad \le \exp\left\{-\beta_0(1)(1-\varepsilon) \sum_{i=0}^{K-1} |u_i - v_i|\right\}. \end{aligned} \tag{2.78}$$

We are now going to give a lower bound on the total "connecting length." Indeed, in the case where $v_0 = x_1 \ge t^\delta$, we have

$$\sum_{i=0}^{K-1} |u_i - v_i| \ge \left(yt^{1/3} - \sum_{i=1}^{K} |\mathscr{O}_i \cap U_M| \right)_+. \tag{2.79}$$

To show (2.79), we recall that for events G of the first type defined in (2.54) we have that $yt^{1/3} \le (Mt^{1/3}) \wedge (y_K + 2t^{\delta})$. But then, recalling that $u_0 = 0$ and $v_0 = x_1$, we find

$$\begin{aligned}\sum_{i=0}^{K-1} |u_i - v_i| + \sum_{i=1}^{K} |\mathscr{O}_i \cap U_M| - Kt^{\delta} &= (y_K + t^{\delta}) \wedge (Mt^{1/3}) \\ &\ge yt^{1/3} - t^{\delta},\end{aligned}$$

which shows (2.79).

Let us mention here that in the case where $x_1 < t^{\delta}$, instead of (2.79), we would have

$$\sum_{i=1}^{K-1} |u_i - v_i| + t^{\delta} \ge \left(yt^{1/3} - \sum_{i=1}^{K} |\mathscr{O}_i \cap U_M| \right)_+ \tag{2.80}$$

and if $K = 1$ the sum on the right-hand side of (2.80) is not present. Indeed, this follows from the fact that, for $x_1 < t^{\delta}$,

$$\sum_{i=1}^{K-1} |u_i - v_i| + \sum_{i=1}^{K} |\mathscr{O}_i \cap U_M| - (K-1)t^{\delta} \ge (y_K + t^{\delta}) \wedge (Mt^{1/3}) \ge yt^{1/3} - t^{\delta}.$$

But then, using (2.79) in (2.78) and observing that by the definition of I in (2.75), $\bigcup_{i=1}^{K} (\mathscr{O}_i \cap U_M)$ is the "open $2t^{\delta}$ neighborhood of I," we see that, for each i, $1 \le i \le K$,

$$\begin{aligned}&\{I_i \text{ contains at most } [2M/r] + 1 \text{ edges}\} \\ &\qquad \subset \{\mathscr{O}_i \cap U_M \text{ contains at most } [2M/r] + 5 \text{ edges}\},\end{aligned}$$

and in view of (2.77) the claim of Lemma 2.3 follows. □

Let us mention here that for estimates involving events of the second type of the covering $\mathscr{G}$ we have exactly the same bound as in Lemma 2.3. This follows from the fact that since $|y_K + t^{\delta} - yt^{1/3}| \ge t^{\delta}$ Theorem 1.3 of [12], part I, can also be applied to the additional term $E_{y_K + t^{\delta}}[\exp\{-\int_0^{H(yt^{1/3})} V(Z_s, \omega)\, ds\},\ R_1 > H(yt^{1/3})]$ coming in $\mathscr{G}_1$, after having used independence with respect to $\mathbb{P}$; cf. the discussion after (2.70). Furthermore, it is also easy to see that the total "connecting length" for events G of the second type of the covering is bounded below by $(yt^{1/3} - \sum_{i=1}^{K} |\mathscr{O}_i \cap U_M|)_+$.

2.7.2. *Probabilities involving pseudo-holes.* In view of Lemma 2.3, we finally have to give an asymptotic upper bound on the probability that $\mathscr{O}_i \cap U_M$ contains at most $[2M/r] + 5$ edges.

To this end observe that the number of subboxes contained in $\mathscr{O}_i \cap U_M$ is possibly strictly smaller than $[2M/r] + 5$, but of course only for $i = 1, K$. We then get for all i, $1 \le i \le K$, using the notation introduced in (2.8),

$$\begin{aligned}(2.81)\qquad &\mathbb{P}\Big[\mathscr{O}_i \cap U_M \text{ contains at most } [2M/r]+5 \text{ edges}\Big]\\ &\quad\le \mathbb{P}\Big[\mathscr{O}_i \cap U_M \text{ contains } \max\Big\{0, [|\mathscr{O}_i \cap U_M|/L(t)]\\ &\qquad\qquad - ([2M/r]+5)\Big\} \text{ thin edges}\Big]\\ &\quad\le \big[|\mathscr{O}_i \cap U_M|/L(t)\big]^{[2M/r]+5}\mathbb{P}\big[B_m^{te}\big]^{\{[|\mathscr{O}_i\cap U_M|/L(t)]-([2M/r]+5)\}_+}.\end{aligned}$$

We now need an asymptotic upper bound on the probability that some subbox B_m is a thin edge. To this end we introduce $p = 1 - e^{-\nu 3a}$ and define

$$(2.82)\qquad \Lambda^*(x) = \begin{cases} \infty, & x \notin [0,1],\\ x \log \dfrac{x}{p} + (1-x)\log \dfrac{1-x}{1-p}, & x \in [0,1].\end{cases}$$

We then have the following result.

LEMMA 2.4. *Assume $\alpha < p/6a$. We have, for large enough t,*

$$(2.83)\qquad \mathbb{P}\big[B_m^{te}\big] \le \exp\Big\{-L(t)\frac{1}{3a}\Lambda^*(\alpha 6a) + \nu 3a\Big\}.$$

PROOF. To show (2.83), we introduce i.i.d random variables in the following way; cf. (2.6):

$$(2.84)\qquad X_i = \begin{cases} 1, & N(B_m^i) \ge 1,\\ 0, & N(B_m^i) = 0.\end{cases}$$

We have $P(X_i = 1) = 1 - e^{-\nu 3a} = p$. Set $M = [L(t)/3a] \ge L(t)/3a - 1$. From Cramér's theorem we get, for large enough t,

$$\begin{aligned}(2.85)\qquad \mathbb{P}\big[B_m^{te}\big] &\le P\Big[\sum_{i=1}^M X_i < \alpha L(t)\Big]\\ &\le P\Big[\sum_{i=1}^M X_i \le M\alpha 6a\Big]\\ &\le \exp\Big\{\Lambda^*(6a\alpha) - L(t)\frac{1}{3a}\Lambda^*(6a\alpha)\Big\},\end{aligned}$$

where we used $\alpha < p/6a$ in the last inequality. Since $\Lambda^*(6a\alpha) \le \Lambda^*(0) = \nu 3a$, Lemma 2.4 now follows. □

Now applying Lemma 2.4 in (2.81), we then find for $\alpha < p/6a$, for large enough t, remembering that $|\mathscr{O}_i \cap U_M| \le 2Mt^{1/3}$,

$$\begin{aligned}(2.86)\qquad &\prod_{i=1}^{K} \mathbb{P}\big[\mathscr{O}_i \cap U_M \text{ contains at most } [2M/r]+5 \text{ edges}\big] \\ &\quad \le \exp\{o(t^{1/3})\}\exp\biggl\{-\frac{1}{3a}\Lambda^*(6a\alpha)\sum_{i=1}^{K}|\mathscr{O}_i \cap U_M|\biggr\},\end{aligned}$$

since $L(t) = t^\delta$ with $\delta < 1/3$, and $1 \le K \le [2M/r]+1$, independent of t.

2.7.3. *The final step in the proof.* In this section we put all our asymptotic upper bounds together and complete the proofs of Proposition 2.3 and Theorem 2.1.

Indeed, now using (2.86) in Lemma 2.3, we find, since $1 \le K \le [2M/r]+1$ is independent of t, in view of (2.72) and (2.70), that for all events G of the covering $\mathscr{G}$ [cf. the discussion after the proof of Lemma 2.3, resp. (2.70)], for all $\varepsilon \in (0,1)$, $\rho \in (0,1)$, for all $\alpha < p/6a$,

$$\begin{aligned}(2.87)\qquad &\limsup_{t\to\infty} \widetilde{\sup}\, t^{-1/3}\log \mathbb{E}\otimes E_0\biggl[1_G \exp\biggl\{-\int_0^t V(Z_s,\omega)\,ds\biggr\}\biggr] \\ &\quad \le -\inf_{y\in[L_1,L_2]}\inf_{l\ge 0}\biggl[\frac{1}{3a}\Lambda^*(6a\alpha)l + \frac{\pi^2\gamma_1(\rho,\eta)}{2l^2} \\ &\qquad\qquad + \beta_0(1)\gamma_2(\varepsilon)(y-l)_+\biggr],\end{aligned}$$

where $\widetilde{\sup}$ denotes the various suprema from (2.45). Note that $\gamma_1(\rho,\eta)$ was defined in (2.71), $\gamma_2(\varepsilon)$ in Lemma 2.3, and we have used that, for given $\mathscr{C}$, $\sum_{i=1}^{K}|\mathscr{O}_i \cap U_M| = L_K t^{1/3}$ for some $L_K \in (0, 2M)$.

Observe now that $\lim_{\alpha\to 0}(1/3a)\Lambda^*(6a\alpha) = \nu$ and

$$(2.88)\qquad I(y) = \inf_{l\ge 0}\biggl[\nu l + \frac{\pi^2}{2l^2} + \beta_0(1)(y-l)_+\biggr].$$

Since $\varepsilon \in (0,1)$ and $\rho \in (0,1)$ coming in (2.87) are arbitrary, performing the various remaining lim sup operations from (2.45), the claim of Proposition 2.3 now follows and and the proof of Theorem 2.1 is complete. □

2.8. *Large deviations.* We now apply Theorems 1.1 and 2.1 to obtain a large deviation result for $t^{-1/3}Z_t$ under the annealed weighted measure

$$(2.89)\qquad Q_t(dw,\,d\omega) = \frac{1}{S_t}\exp\biggl\{-\int_0^t V(Z_s,\omega)\,ds\biggr\}P_0(dw)\mathbb{P}(d\omega),$$

where S_t is the normalizing constant.

The function which is going to govern the large deviation principle is

$$(2.90)\qquad J_1(y) = I(y) - I(c_0), \qquad y \in \mathbb{R},$$

where $I(y)$ was introduced in (1.5). Observe that, for $y \in [-c_0, c_0]$, $J_1(y) = 0$. We are going to show the following result.

THEOREM 2.2. *Under Q_t, $t^{-1/3}Z_t$ obeys a large deviation principle at rate $t^{1/3}$ with rate function $J_1(\cdot)$, as $t\to\infty$, that is,*

$$\limsup_{t\to\infty} t^{-1/3}\log Q_t\big(Z_t\in t^{1/3}A\big)\le -\inf_{y\in A} J_1(y),\qquad A\subseteq\mathbb{R}\ \textit{closed}, \tag{2.91}$$

$$\liminf_{t\to\infty} t^{-1/3}\log Q_t\big(Z_t\in t^{1/3}\mathscr{O}\big)\ge -\inf_{y\in\mathscr{O}} J_1(y),\qquad \mathscr{O}\subseteq\mathbb{R}\ \textit{open}. \tag{2.92}$$

PROOF. First of all recall from [3] that $\lim_{t\to\infty} t^{-1/3}\log S_t=-I(c_0)$. The proof of the lower-bound part, now follows from Theorem 1.1.

To prove the upper-bound part, we first notice that we have exponential tightness. Indeed, for any $L>0$ we have

$$\begin{aligned}\mathbb{E}\otimes E_0&\left[|Z_t|\ge Lt^{1/3},\exp\left\{-\int_0^t V(Z_s,\omega)\,ds\right\}\right]\\ &\le \mathbb{E}\otimes E_0\left[\exp\left\{-\int_0^{H(Lt^{1/3})} V(Z_s,\omega)\,ds\right\}\right]\\ &\quad+\mathbb{E}\otimes E_0\left[\exp\left\{-\int_0^{H(-Lt^{1/3})} V(Z_s,\omega)\,ds\right\}\right].\end{aligned}\tag{2.93}$$

Using once again Theorem 1.3 of [12], part I, we get

$$\begin{aligned}&\limsup_{t\to\infty} t^{-1/3}\log\mathbb{E}\\ &\qquad\otimes E_0\left[|Z_t|\ge Lt^{1/3},\exp\left\{-\int_0^t V(Z_s,\omega)\,ds\right\}\right]\le -L\beta_0(1),\end{aligned}\tag{2.94}$$

which implies the asserted tightness.

So we only need to prove the upper-bound part for compact $A\subset\mathbb{R}$. For $\varepsilon>0$, we can cover $t^{1/3}A$ by a finite number of closed intervals of length $\varepsilon t^{1/3}$. Using Theorem 2.1 and then letting $\varepsilon\to 0$, the upper bound now follows. □

2.9. *The Schrödinger heat kernel.* We finally give an application of Theorem 2.2 which derives the asymptotic behavior of $\overline{r}(t,0,yt^{1/3})$, the averaged kernel of the Schrödinger semigroup $\exp\{t(1/2\Delta-V)\}$, where $V(x,\omega)$ was defined in (0.1), the so-called soft-obstacle case, in dimension $d=1$. The case where $\overline{r}(t,x,y)$ stands for the Dirichlet heat kernel on $\mathbb{R}\setminus\bigcup_i B(x_i,a)$ of $\exp(t1/2\Delta)$ in $d=1$, where $\omega=\sum_i\delta_{x_i}$, the hard-obstacle case, can be found in Theorem 1.3(ii) of [8]. More precisely, we define

$$\begin{aligned}\overline{r}(t,x,y)&=\mathbb{E}[r(t,x,y,\omega)]\\ &=(2\pi t)^{-1/2}\exp\left\{-\frac{(y-x)^2}{2t}\right\}\\ &\quad\times E^t_{x,y}\left[\exp\left\{-\nu\int\left[1-\exp\left\{-\int_0^t W(Z_s-y)\,ds\right\}\right]dy\right\}\right],\end{aligned}\tag{2.95}$$

where $E^t_{x,y}$ stands for the Brownian bridge measure in time t from x to y. In the hard-obstacle case the integrand in $E^t_{x,y}$ reads $\exp\{-\nu|C^a_t|\}$, where C^a_t is the a-neighborhood of the support of the path up to time t. Indeed, we have the following result.

THEOREM 2.3. *For any $y \in \mathbb{R}$,*

$$\lim_{t\to\infty} t^{-1/3} \log \overline{r}(t, 0, yt^{1/3}) = -I(y), \tag{2.96}$$

where $I(y)$ was defined in (1.5).

The proof of the previous theorem is exactly the same as the proof of Theorem 1.5 of [12], part II, where the asymptotic behavior of $\overline{r}(t, 0, yt^{d/d+2})$, with $y \in \mathbb{R}^d$, $d \geq 2$, as $t \to \infty$, was obtained in the hard- and soft-obstacle case.

Let us also mention here that the asymptotic behavior of $\overline{r}(t, 0, y\phi(t))$, $y \in \mathbb{R}^d$, $d \geq 1$, as $t \to \infty$, was obtained for scales $\phi(t) = o(t^{d/d+2})$, resp. $\phi(t) = o(t)$, and $t^{d/d+2} = o(\phi(t))$, resp. $\phi(t) = t$, in Theorem 2.4 of [12], part I, for the hard- and soft-obstacle case.

3. Brownian motion with drift in a Poisson potential.

3.1. *Statement of the results and comments.* In this section we want to apply Theorem 2.2 to the study of a one-dimensional annealed Brownian motion with a constant drift feeling the influence of the potential V.

To this end we define for $h \in \mathbb{R}$ the annealed weighted measure of Brownian motion with drift h:

$$dQ_t^h = \frac{S_t}{\widetilde{S}_t^h} \exp\{hZ_t\}\, dQ_t, \tag{3.1}$$

where dQ_t was defined in (2.89) and $\widetilde{S}_t^h$ is the normalizing constant. We are going to show the following theorem.

THEOREM 3.1. *For h with $|h| \in (0, \beta_0(1))$ the following holds:*

$$\lim_{t\to\infty} t^{-1/3} \log \mathbb{E} \otimes E_0\left[\exp\left\{hZ_t - \int_0^t V(Z_s, \omega)\, ds\right\}\right] = -c(1, \nu - |h|), \tag{3.2}$$

where

$$c(1, \nu - |h|) = \inf_{l\geq 0}\left[(\nu - |h|)l + \frac{\pi^2}{2l^2}\right] = \frac{3}{2}\big[\pi(\nu - |h|)\big]^{2/3}. \tag{3.3}$$

Furthermore, $t^{-1/3}Z_t$ satisfies under Q_t^h a large deviation principle at rate $t^{1/3}$ with rate function $H(y) = I(y) - hy - c(1, \nu - |h|)$.

Before proving the preceding theorem we give some comments.

A first point to notice is that, for $|h| < \beta_0(1) \leq \nu$, $\inf_{y\in\mathbb{R}}[I(y) - hy] = c(1, \nu - |h|)$ and that the infimum is attained at $c_0^h = (\pi^2/(\nu - |h|))^{1/3}$ if $h \geq 0$ (resp. at $-c_0^h$, if $h < 0$). This should be viewed in the context of the hard-obstacle case, where for $|h| \in (0, \nu)$ the limit law as $t \to \infty$ of $t^{-1/3}Z_{\bullet t^{2/3}}$ under

$dQ_t^{h,\text{h.o.}}$, the analogous object to (3.1) in the hard-obstacle case, was obtained in [6]. It was shown there that, for large t, $t^{-1/3}Z_{\bullet t^{2/3}}$ under $dQ_t^{h,\text{h.o.}}$ converges in law to a Brownian motion with drift, starting in 0, conditioned "never to leave the interval $(0, c_0^h)$." The case where $h = 0$, $d = 1$ can be found in [7]. For $h = 0$, $d = 2$, see [9], resp. [1], for a discrete setting. The cases $d \geq 3$ are conjectured to lead to similar results, but are still open.

Let us briefly mention why we have to restrict ourselves to certain values of h. On the one hand, the factor $\exp\{-\int_0^t V(Z_s, \omega)\,ds\}$ in the definition of dQ_t^h represents a penalty for the path visiting Poissonian points, since our shape function W is positive and $V(Z_s, \omega)$ can only pick strictly positive values when the particle comes into distance smaller than a to some Poisson point. Thus this factor favors trajectories which are in some sense "localized." On the other hand, the factor $\exp\{hZ_t\}$ rewards paths which make large excursions, thus particles which are in some sense "delocalized." So there is a competition between these two factors. It is intuitively clear that if the drift is sufficiently small, then the localized paths give the main contribution to the long-time asymptotics of our object under consideration. This is also reflected in the scale $t^{-1/3}$.

It is shown in Theorem 2.1 of [12], part II, specialized to $d = 1$, that $\beta_0(1)$, the so-called annealed Liapounov exponent, which measures how costly it is for the process to make long excursions when the particle can pick its own time to perform the displacement, is a threshold for the drift. More precisely, it is shown there that

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E} \otimes E_0\left[\exp\left\{hZ_t - \int_0^t V(Z_s, \omega)\,ds\right\}\right] \begin{cases} = 0, & |h| \leq \beta_0(1), \\ > 0, & |h| > \beta_0(1). \end{cases} \tag{3.4}$$

Hence (3.4) says that for $|h| \leq \beta_0(1)$ we have a "localizing effect," whereas for $|h| > \beta_0(1)$ one has an exponential growth. Thus the "delocalizing" factor $\exp\{hZ_t\}$ is the dominant one. We finally refer the reader to Theorem 2.2 of [12], part II, where the analogous statement to Theorem 3.1 can be found in the case $d \geq 2$, hard and soft obstacles, resp. to Theorem 4.1 from [8] for $d = 1$ and hard obstacles. Let us now begin with the proof of Theorem 3.1.

PROOF OF THEOREM 3.1. Owing to Theorem 2.2, it is enough to show (3.2) and (3.5) from below. Indeed, the large deviation result then follows from 2.1.24 of [2]. To show (3.2), it is enough to look at the case where $0 < h < \beta_0(1) (\leq \nu)$. If $h < 0$ we use the symmetry of Z_t and work with the shape function $\hat{W}(x) = W(-x)$ having the same Liapounov exponent.

We want to apply Varadhan's integral theorem (see, e.g., Theorem 2.1.10 of [2]) to the function $\phi(x) = hx$, $x \in \mathbb{R}$. To this end we have to check that

$$\lim_{L\to\infty} \limsup_{t\to\infty} t^{-1/3} \log \mathbb{E} \otimes E_0\left[\exp\left\{hZ_t - \int_0^t V(Z_s, \omega)\,ds\right\}, hZ_t \geq Lt^{1/3}\right] = -\infty. \tag{3.5}$$

To show (3.5), we find as in the proof of (2.7) from Theorem 2.2 of [12], part II, using $\{Z_t \geq y\} \subseteq \{H(y) \leq t\}$, that for large enough t, for a constant $\kappa \in (0, \infty)$, the expression under the logarithm in (3.5) is smaller than

$$\kappa \exp\left\{-(\beta_0(1) - h)\frac{L}{h} t^{1/3}\right\} \tag{3.6}$$

from which (3.5) now follows.

Theorem 2.1.10 of [2] now yields

$$\lim_{t\to\infty} t^{-1/3} \log E^{Q_t}\big[\exp\{hZ_t\}\big] = \sup_{y\in\mathbb{R}}(hy - J_1(y)), \tag{3.7}$$

where E^{Q_t} denotes expectation with respect to the annealed weighted measure Q_t introduced in (2.89).

Since $\lim_{t\to\infty} t^{-1/3} \log S_t = -I(c_0)$, we find

$$\lim_{t\to\infty} t^{-1/3} \log \mathbb{E} \otimes E_0\left[\exp\left\{hZ_t - \int_0^t V(Z_s, \omega)\, ds\right\}\right] = \sup_{y\in\mathbb{R}}(hy - I(y)). \tag{3.8}$$

Since $h > 0$, it suffices to check that

$$\inf_{y\geq 0}(I(y) - hy) = c(1, \nu - h). \tag{3.9}$$

To this end define $c_0^h = (\pi^2/(\nu - h))^{1/3}$. Observe that

$$c_0^h = \arg\min_{c\geq 0}\left(c(\nu - h) + \frac{\pi^2}{2c^2}\right), \qquad c_0^h \in (c_0, c_1).$$

By looking now separately at the cases $y \in [0, c_0]$, $y \in (c_0, c_1)$ and $y \in [c_1, \infty)$, provided c_1 is finite, it is easy to see that

$$\inf_{y\geq 0}[I(y) - hy] = \inf_{y\geq 0}\left[(\nu - h)y + \frac{\pi^2}{2y^2}\right] = c(1, \nu - h), \tag{3.10}$$

which shows (3.9) and completes the proof of Theorem 3.1. □

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