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LOCALIZATION TRANSITION FOR A POLYMER NEAR AN INTERFACE[1]

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Consider the directed process $(i, S_i)_{i \geq 0}$ where the second component is simple random walk on $\mathbb{Z}$ ($S_0 = 0$). Define a transformed path measure by weighting each n-step path with a factor $\exp[\lambda \sum_{1 \leq i \leq n}(\omega_i + h)\text{sign}(S_i)]$. Here, $(\omega_i)_{i \geq 1}$ is an i.i.d. sequence of random variables taking values ± 1 with probability 1/2 (acting as a random medium), while $\lambda \in [0, \infty)$ and $h \in [0, 1)$ are parameters. The weight factor has a tendency to pull the path towards the horizontal, because it favors the combinations $S_i > 0$, $\omega_i = +1$ and $S_i < 0$, $\omega_i = -1$. The transformed path measure describes a heteropolymer, consisting of hydrophylic and hydrophobic monomers, near an oil–water interface.

We study the free energy of this model as $n \to \infty$ and show that there is a critical curve $\lambda \to h_c(\lambda)$ where a phase transition occurs between localized and delocalized behavior (in the vertical direction). We derive several properties of this curve, in particular, its behavior for $\lambda \downarrow 0$. To obtain this behavior, we prove that as $\lambda, h \downarrow 0$ the free energy scales to its Brownian motion analogue.

0. Introduction and main results. In this paper we solve a problem that was posed by Garel, Huse, Leibler and Orland (1989) and studied by Sinai (1993). It involves a two-dimensional directed random polymer interacting with two solvents separated by an interface. Depending on the interaction, the polymer either stays near the interface (localization) or wanders away from it (delocalization). The main problem is to determine the phase transition curve.

0.1. *A random walk model.* To define the model we need two ingredients;

1. $S = (S_i)_{i \geq 0}$: a simple random walk on $\mathbb{Z}$ starting at the origin, where P, E denote its probability law and expectation.
2. $\omega = (\omega_i)_{i \geq 1}$: an i.i.d. sequence of random variables taking values ± 1 with probability 1/2, where $\mathbb{P}, \mathbb{E}$ denote its probability law and expectation.

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Fix $\lambda \in [0, \infty)$ and $h \in [0, 1)$. Given ω, define a transformed probability law $Q_n^{\lambda, h, \omega}$ on n-step paths by setting

$$\frac{dQ_n^{\lambda, h, \omega}}{dP}\big((S_i)_{i=0}^n\big) = \frac{1}{Z_n^{\lambda, h, \omega}} \exp\left[\lambda \sum_{i=1}^n (\omega_i + h)\Delta_i\right], \tag{0.1}$$

where

$$\Delta_i = \begin{cases} \mathrm{sign}(S_i), & \text{if } S_i \neq 0, \\ \mathrm{sign}(S_{i-1}), & \text{if } S_i = 0 \end{cases} \tag{0.2}$$

and $Z_n^{\lambda, h, \omega}$ is the normalizing constant or partition sum. [In (0.2) we could put $\Delta_i = 0$ if $S_i = 0$. This would be a site rather than a bond model.]

We view $Q_n^{\lambda, h, \omega}$ as modelling the following situation. Think of $(i, S_i)_{i=0}^n$ as a directed polymer on $\mathbb{Z}^2$, consisting of n monomers represented by the bonds in the path. The lower half plane is "water," the upper half plane is "oil." The monomers are of two different types, occurring in a random order indexed by ω. Namely, $\omega_i = -1$ means that monomer i "prefers water," $\omega_i = +1$ means that it "prefers oil." Since $\Delta_i = -1$ when monomer i lies in the water and $\Delta_i = +1$ when it lies in the oil, we see that the weight factor in (0.1) "encourages matches and discourages mismatches." For $h = 0$ both types of monomers interact equally strongly with the water and with the oil, being attracted by one and repelled by the other. However, for $h \in (0, 1)$ the monomers preferring oil have a stronger interaction with both the solvents than the monomers preferring water. The parameter λ is the overall interaction strength and plays the role of inverse temperature.

REMARK. In (0.1) we could put the h-dependence in the probability law of ω, for instance, by picking $\mathbb{P}(\omega_i = \pm 1) = (1 \pm h)/2$ and writing $\lambda \sum_i \omega_i \Delta_i$ in the exponent. This would describe a polymer where the two types of monomers occur with different densities but interact equally strongly with the solvents. Alternatively we could make a mix of the two types of h-dependence (or even allow for more general ω-sequences with exponential moments). For the proofs in this paper it is a slight advantage that h enters into the exponent. Nevertheless, all results carry over with only minor changes in the proofs.

The way in which the polymer behaves near the interface is the result of a competition between energy and entropy. The energy is minimal (i.e., the weight is maximal) when all the monomers are placed in their preferred solvent, but this strategy has low entropy. On the other hand, the entropy is maximal when the polymer makes large excursions away from the interface, but this strategy typically has high energy (i.e., the weight is small). What do we expect will happen under $Q_n^{\lambda, h, \omega}$ as $n \to \infty$?

1. $\lambda = 0$. The vertical motion of the polymer is free simple random walk. Since this is a null recurrent process, the polymer will not stay near the interface; that is, we have delocalization.

2. $\lambda > 0$, $h = 0$. The polymer will want to stay close to the interface, so that it can place as many monomers as possible in their preferred solvent and produce low energy. Indeed, wandering away from the interface would result in a misplacing of about half the monomers. The polymer can reduce this fraction by crossing the interface at a positive frequency. This lowers the entropy, but only by a small amount if the crossing frequency is small. The estimates in Sinai (1993) show that for this strategy the gain exceeds the loss; that is, we have localization.
3. $\lambda > 0$, $h \uparrow 1$. Now wandering away is again the winning strategy, simply because the monomers preferring water barely interact with either the water or the oil. By moving away in the upward direction the polymer can match all the monomers that prefer oil, thereby producing almost the minimal energy and almost the maximal entropy; that is, we have delocalization.

The above intuitive picture seems to suggest that there is a critical curve in the (λ, h)-plane separating the localized from the delocalized phase. It is the goal of the present paper to prove the existence of this critical curve and to derive some of its properties.

In order to give a precise definition of the two phases, we need the following preliminary result (proved in Section 1).

THEOREM 1. *For every $\lambda \in [0, \infty)$ and $h \in [0, 1)$,*

$$\lim_{n \to \infty} \frac{1}{n} \log Z_n^{\lambda, h, \omega} = \phi(\lambda, h) \tag{0.3}$$

exists $\mathbb{P}$-a.s. and is nonrandom.

The function ϕ is the specific free energy of the polymer. It is immediate from (0.1) and (0.3) that $\phi(\lambda, h)$ is continuous, nondecreasing and convex in both variables. [Note that our model makes perfect sense for $\lambda, h \in \mathbb{R}$. Obviously, in this larger parameter space, $\phi(\lambda, h)$ is everywhere finite, is symmetric and convex in both variables and hence is also continuous and unimodal in both variables.] Moreover, it is easy to show that

$$\phi(\lambda, h) \geq \lambda h. \tag{0.4}$$

Indeed, since $P(\Delta_i = +1 \text{ for } 1 \leq i \leq n) \sim C/n^{1/2} (n \to \infty)$, it follows that

$$\begin{aligned} Z_n^{\lambda, h, \omega} &= E\left(\exp\left[\lambda \sum_{i=1}^n (\omega_i + h)\Delta_i\right]\right) \\ &\geq \exp\left[\lambda \sum_{i=1}^n (\omega_i + h) + O(\log n)\right] \\ &= \exp[\lambda h n + o(n)], \quad \mathbb{P}\text{-a.s.}, \end{aligned} \tag{0.5}$$

where in the last step we use the strong law of large numbers for ω. Thus we see that the lower bound in (0.4) corresponds to the strategy where the polymer wanders away in the upward direction. This leads us to the following definition.

DEFINITION 1. We say that the polymer is:

(a) localized if $\phi(\lambda, h) > \lambda h$,
(b) delocalized if $\phi(\lambda, h) = \lambda h$.

In case (a) the polymer is able to beat on an exponential scale the trivial strategy of moving upward. It is intuitively clear that this is only possible by crossing the interface at a positive frequency, which means that the path measure localizes near the interface in a strong sense. In case (b), on the other hand, the polymer is not able to beat the trivial strategy on an exponential scale. In principle it could still do better on a smaller scale, but we do not expect this [at least not in the interior of the region described by (b)]. We shall not derive any properties of the path measure, but just stick to the above definition. (See Section 0.4 for a further discussion.)

Our first main theorem reads as follows.

THEOREM 2. *For every* $\lambda \in (0, \infty)$ *there exists* $h_c(\lambda) \in (0, 1)$ *such that the polymer is*:

$$\begin{array}{ll} \textit{localized} & \textit{if } 0 \le h < h_c(\lambda), \\ \textit{delocalized} & \textit{if } h \ge h_c(\lambda). \end{array} \tag{0.6}$$

Moreover,

$$\begin{array}{l} \lambda \to h_c(\lambda) \ \textit{is continuous and nondecreasing on } [0, \infty), \\ \lim_{\lambda \to \infty} h_c(\lambda) = 1, \ \lim_{\lambda \downarrow 0} h_c(\lambda) = 0. \end{array} \tag{0.7}$$

The proof of Theorem 2 is given in Section 2. It will also provide upper and lower bounds on $h_c(\lambda)$, namely:

$$\begin{array}{ll} \text{(i)} & \limsup_{\lambda \downarrow 0} \dfrac{1}{\lambda} h_c(\lambda) \le 1, \\ \text{(ii)} & \liminf_{\lambda \downarrow 0} \dfrac{1}{\lambda} h_c(\lambda) > 0, \\ \text{(iii)} & \lim_{\lambda \to \infty} \lambda\big(1 - h_c(\lambda)\big) \in \left[\dfrac{1}{2}\log 2, \dfrac{3}{2}\log 2\right]. \end{array} \tag{0.8}$$

0.2. *A Brownian motion model*. As $\lambda \downarrow 0$, the reward to stay close to the interface gets smaller and so the excursions of the polymer away from the interface will get longer. Therefore, intuitively we may expect to see a scaling behavior where both S and ω can be approximated by Brownian motions. To make this more precise, we first define and describe the continuous analogue of the discrete model. As we shall see in Section 0.3, the scaling happens in a way that leads to a Brownian motion model. This model retains the full complexity of the random walk model, except that the Brownian scaling property gives rise to a simpler form of the phase separation curve.

The two ingredients of the continuous model are two standard Brownian motions on $\mathbb{R}$, denoted by:

(1) $B = (B_t)_{t \geq 0}$,
(2) $\beta = (\beta_t)_{t \geq 0}$,

both starting at the origin. We write $\tilde{P}$, $\tilde{E}$, respectively, $\tilde{\mathbb{P}}, \tilde{\mathbb{E}}$, to denote their probability law and expectation. Similarly as in (0.1) and (0.2), the transformed probability law $\tilde{Q}_t^{\lambda, h, \beta}$ on paths of length t, given β, is defined by

$$\frac{d\tilde{Q}_t^{\lambda, h, \beta}}{d\tilde{P}}\big((B_s)_{0 \leq s \leq t}\big) = \frac{1}{\tilde{Z}_t^{\lambda, h, \beta}} \exp\bigg[\lambda \int_0^t \Delta_s(d\beta_s + h\,ds)\bigg]. \tag{0.9}$$

Here,

$$\Delta_s = \begin{cases} \operatorname{sign}(B_s), & \text{if } B_s \neq 0, \\ 0, & \text{if } B_s = 0, \end{cases} \tag{0.10}$$

the first integral is an Itô integral, and the parameters λ, h are both in $[0, \infty)$.

The analogue of Theorem 1 (proved in Section 3) reads as follows.

THEOREM 3. *For every* $\lambda, h \in [0, \infty)$,

$$\lim_{t \to \infty} \frac{1}{t} \log \tilde{Z}_t^{\lambda, h, \beta} = \tilde{\phi}(\lambda, h) \tag{0.11}$$

exists $\tilde{\mathbb{P}}$-*a.s. and is nonrandom.*

The function $\tilde{\phi}$ has the same qualitative properties as ϕ in (0.3), including the lower bound in (0.4). Therefore we can maintain the same distinction between phases as in Definition 1.

The Brownian scaling property tells us that

$$(B_s, \beta_s)_{s \geq 0} =_D \big(aB_{s/a^2}, a\beta_{s/a^2}\big)_{s \geq 0} \qquad \text{for all } a > 0, \tag{0.12}$$

where $=_D$ means equality in distribution. This implies that, for fixed λ, h and as a random variable in β,

$$\tilde{Z}_t^{\lambda, h, \beta} =_D \tilde{Z}_{t/a^2}^{a\lambda, ah, \beta} \qquad \text{for all } t \geq 0 \text{ and } a > 0. \tag{0.13}$$

Hence

$$\tilde{\phi}(\lambda, h) = \frac{1}{a^2} \tilde{\phi}(a\lambda, ah) \qquad \text{for all } a > 0. \tag{0.14}$$

It immediately follows from (0.14) that $\tilde{\phi}$ has the following *scaling form*:

(0.15) $\tilde{\phi}(\lambda, K\lambda) = S(K)\lambda^2$ for $K \in [0, \infty)$, with $K \to S(K)$ continuous, nondecreasing and convex, satisfying $S(K) \geq K$.

The analogue of Theorem 2 (proved in Section 3) now reads as follows.

THEOREM 4. *There exists* $K_c \in (0, 1]$ *such that*

$$S(K) = K \quad \text{if } K \ge K_c, \qquad S(K) > K \quad \text{if } 0 \le K < K_c. \tag{0.16}$$

By (0.15), Theorem 4 implies that $\tilde{\phi}(\lambda, h) = \lambda h$ for $h \ge K_c\lambda$ and $\tilde{\phi}(\lambda, h) > \lambda h$ for $h < K_c\lambda$; that is, the phase separation curve is the straight line $\lambda \to K_c\lambda$.

Although the picture here looks fairly simple, the complexity of the model is hidden in the constant K_c, which seems to be a very ungainly and complex object. We have rough bounds on K_c, but nothing like a sequence of bounds that could be expected to converge to K_c.

0.3. *Weak interaction limit*. We are now ready to formulate our main results concerning the weak interaction limit of the random walk model and its relation to the Brownian motion model.

THEOREM 5. *For every* $\lambda, h \in [0, \infty)$,

$$\lim_{a \downarrow 0} \frac{1}{a^2} \phi(a\lambda, ah) = \tilde{\phi}(\lambda, h). \tag{0.17}$$

Although (0.17) is intuitively plausible, the estimates needed for its proof are quite delicate. The reason is that our paths carry exponential weight factors, which are very sensitive to fluctuations. One should keep in mind that, at least in the localized region, the path exhibits a behavior that has an exponentially small probability under the free path measure. It is therefore clear that the result cannot be proved by a routine application of invariance principles.

We shall not prove Theorem 5 separately, as it is a consequence of the more powerful but more technical Theorem 6 below. A proof of Theorem 5 would be simpler (and more transparent) than that of Theorem 6 given in Section 4. However, the unfortunate fact is that Theorem 5 alone does not lead to a determination of the tangent at $\lambda = 0$ of the phase separation curve in the discrete model. In fact, it only yields

$$\liminf_{\lambda \downarrow 0} \frac{1}{\lambda} h_c(\lambda) \ge K_c. \tag{0.18}$$

Indeed, pick $K < K_c$. Then, by (0.15)–(0.17),

$$\lim_{a \downarrow 0} \frac{1}{a^2} \phi(a, aK) = \tilde{\phi}(1, K) > K. \tag{0.19}$$

This implies $\phi(a, aK) > Ka^2$ and hence $h_c(a) > aK$ for small enough a, which proves (0.18) after letting $a \downarrow 0$ followed by $K \uparrow K_c$. It is clear that a statement like (0.17) does not yield

$$\sup_{\lambda \downarrow 0} \frac{1}{\lambda} h_c(\lambda) \le K_c, \tag{0.20}$$

simply because $\tilde{\phi}(1, K) = K$ for $K \geq K_c$ does not imply that $\phi(a, aK) = a^2 K$ for small enough a.

In order to remedy this situation, we introduce the 'excess' free energies

$$\begin{aligned} \psi(\lambda, h) &= \phi(\lambda, h) - \lambda h, \\ \tilde{\psi}(\lambda, h) &= \tilde{\phi}(\lambda, h) - \lambda h, \end{aligned} \tag{0.21}$$

so that the delocalized region is characterized by $\psi = 0$, respectively, $\tilde{\psi} = 0$. Our main result for the weak interaction limit is the following.

THEOREM 6. *Fix* $\lambda > 0$. *Let* $h > 0$, $h' \geq 0$ *and* $\rho > 0$ *satisfy* $(1 + \rho)h' < h$. *Then*

$$\begin{aligned} \frac{1}{a^2}\psi(a\lambda, ah) &\leq (1 + \rho)\tilde{\psi}(\lambda, h'), \\ \tilde{\psi}(\lambda, h) &\leq (1 + \rho)\frac{1}{a^2}\psi(a\lambda, ah') \end{aligned} \tag{0.22}$$

for small enough a.

Theorem 6 and the continuity of ϕ and $\tilde{\phi}$ obviously imply Theorem 5. Theorem 6 is also sufficiently strong to give us the following corollary.

COROLLARY 1.

$$\lim_{\lambda \downarrow 0} \frac{1}{\lambda} h_c(\lambda) = K_c. \tag{0.23}$$

To get (0.20) from the first line in (0.22), pick $h' = K_c$, $\rho > 0$ and $\lambda = 1$, $h = (1 + 2\rho)K_c$. Since $\tilde{\psi}(1, K_c) = 0$, it follows that $\psi(a, a(1 + 2\rho)K_c) = 0$ and hence $h_c(a) \leq a(1 + 2\rho)K_c$ for small enough a. Now let $a \downarrow 0$ and $\rho \downarrow 0$.

The idea behind Theorem 6 is that by slightly varying h we can dominate the errors that arise in the approximation of the random walk by the Brownian motion.

REMARK. Theorem 6 can be shown to carry over to the version of the model where the h-dependence sits in the probability law of ω. For the Brownian motion model there is no distinction between the two versions. Apparently, the weak interaction limit is largely independent of the details of the model. This is essentially a stability result. Stability is crucial for our understanding of the localization problem, and typically hard to prove for path measures with exponential weight factors.

0.4. *Open problems.* Our distinction between the localized and the delocalized phase, as given in Definition 1, is in terms of the specific free energy rather than the path measure itself. We would like to show that in the localized phase "$(S_i)_{0 \leq i \leq n}$ truly localizes," in the sense that it stays close to the horizontal, while in the delocalized phase it does not. For instance, consider the following two questions.

1. For fixed i, does $Q_n^{\lambda, h, \omega}(S_i \in \cdot)$ converge to a nondegenerate limit law as $n \to \infty$?
2. Is there a $d = d(\lambda, h) > 0$ such that $\lim_{n\to\infty} Q_n^{\lambda, h, \omega}(|\{1 \le i \le n: S_i = 0\}|/n \in [d - \varepsilon, d + \varepsilon]) = 1$ for all $\varepsilon > 0$?

No doubt the answer is "yes" in the localized phase and "no" in the delocalized phase, but this remains to be proven. Other interesting questions are: How does the free energy behave close to the critical curve? How large are the excursions of the path away from the horizontal?

Sinai (1993) proved that if $\lambda > 0$, $h = 0$, then the path localizes in the following sense: there exist numbers $\lambda > 0$, $\delta(\lambda) > 0$ and random variables $n_0(\omega)$, $k_0(\omega)$ such that

$$\sup_{\log^\gamma n \le i \le n - \log^\gamma n} Q_n^{\lambda, 0, \omega}\big(|S_i| > k\big) \le e^{-\delta(\lambda) k} \qquad \text{for } k \ge k_0(\omega),\ n \ge n_0(\omega),\ \mathbb{P}\text{-a.s.} \tag{0.24}$$

We expect that Sinai's arguments can be extended to cover the whole localized region.

One could hope to make some progress on problems (1) and (2) above by looking at the times when the path intersects the interface. In the localized region these times admit a Gibbsian description (in the limit as $n \to \infty$). However, this leads to a Gibbs measure with a random long-range potential having both signs, which is a notoriously difficult object. Nevertheless, we expect that a limiting measure exists and that it has exponentially decaying correlations.

Even the delocalized region is not trivial. It seems intuitively clear that, at least in the interior of this region [i.e., for $h > h_c(\lambda)$], the path just behaves as simple random walk conditioned to stay positive, which is well known to have Brownian scaling with the so-called Brownian meander as limiting measure [see Bolthausen (1976)]. However, it appears to be difficult to exclude the possibility of rare returns to the interface.

Grosberg, Izrailev and Nechaev (1994) obtain localization for the case where ω is periodic instead of random.

Albeverio and Zhou (1996) prove that if $\lambda > 0$, $h = 0$, then $\log Z_n^{\lambda, 0, \omega}$ satisfies a LLN and a CLT (as a random variable in ω). However, there is no description of the mean and the variance. They further show that $\int Q_n^{\lambda, 0, \omega} \mathbb{P}(d\omega)$-a.s. both

$$\begin{aligned} &\max_{0 \le i < j \le n} \{ j - i: S_i = S_j = 0,\ S_k \ne 0 \text{ for } i < k < j \}, \\ &\max_{0 \le i \le n} |S_i| \end{aligned} \tag{0.25}$$

are of order $\log n$ as $n \to \infty$, which is typical for a localized path.

Grosberg, Izrailev and Nechaev (1994) and Sinai and Spohn (1996) study an annealed version of the model in which $Z_n^{\lambda, h, \omega}$ is averaged w.r.t. $\mathbb{P}$. The free energy and the critical curve can in this case be computed exactly.

However, the quenched version described in the present paper is qualitatively very different and considerably more complex.

1. Proof of Theorem 1. The proof consists of two parts. In Lemma 1 we prove that the claim holds when the random walk is constrained to return to the origin at time $2n$. In Lemma 2 we show how to remove this constraint.

Fix λ and h. Define

$$Z_{2n}^{\omega,*} = E\left(\exp\left[\lambda \sum_{i=1}^{2n} (\omega_i + h)\Delta_i\right] 1\{S_{2n} = 0\}\right), \tag{1.1}$$

where we recall the notation introduced in Section 0.1.

LEMMA 1. *The limit* $\lim_{n\to\infty}(1/2n)\log Z_{2n}^{\omega,*}$ *exists and is constant* $\mathbb{P}$-*a.s.*

PROOF. We need the following three properties.

I. $Z_{2n}^{\omega,*} \geq Z_{2m}^{\omega,*} Z_{2n-2m}^{T^{2m}\omega,*}$ for all $0 \leq m \leq n$, with T the left-shift $(T\omega)_i = \omega_{i+1}$.
II. $n \to (1/2n)\mathbb{E}(\log Z_{2n}^{\omega,*})$ is bounded from above.
III. $\mathbb{P}(T\omega \in \cdot) = \mathbb{P}(\omega \in \cdot)$.

Property I follows from (1.1) by inserting an extra indicator $1\{S_{2m} = 0\}$ and using the Markov property of S at time $2m$. Property II holds because

$$\begin{aligned} \mathbb{E}(\log Z_{2n}^{\omega,*}) &\leq \log \mathbb{E}(Z_{2n}^{\omega,*}) \\ &= \log E\left((\cosh\lambda)^{2n} \exp\left[\lambda h \sum_{i=1}^{2n} \Delta_i\right] 1\{S_{2n} = 0\}\right) \\ &\leq 2n(\log\cosh\lambda + \lambda h). \end{aligned} \tag{1.2}$$

Property III is trivial. Thus, $\omega \to (\log Z_{2n}^{\omega,*})_{n\geq 0}$ is a superadditive process. It therefore follows from the superadditive ergodic theorem [Kingman (1973), Theorem 1] that $\lim_{n\to\infty}(1/2n)\log Z_{2n}^{\omega,*}$ converges $\mathbb{P}$-a.s. and in mean, and is measurable w.r.t. the tail σ-field of ω. Since the latter is trivial, the limit is constant $\mathbb{P}$-a.s. □

Our original partition sum was

$$Z_{2n}^{\omega} = E\left(\exp\left[\lambda \sum_{i=1}^{2n} (\omega_i + h)\Delta_i\right]\right), \tag{1.3}$$

which is (1.1) but without the indicator. Thus, in order to prove Theorem 1 we must show that this indicator is harmless as $n \to \infty$. Since $|\log(Z_{2n}^{\omega}/Z_{2n+1}^{\omega})| \leq \lambda(1+h)$, it will suffice to consider n even.

LEMMA 2. *There exists* $C > 0$ *such that* $Z_{2n}^{\omega,*} \leq Z_{2n}^{\omega} \leq CnZ_{2n}^{\omega,*}$ *for all* n *and* ω.

PROOF. The lower bound is obvious. The upper bound is proved as follows. By conditioning on the last hitting time of 0 prior to time $2n$, we may write

$$(1.4)\qquad \begin{aligned} Z^{\omega}_{2n} &= Z^{\omega,*}_{2n} + \sum_{k=1}^{n} Z^{\omega,*}_{2n-2k} E\Bigg(\exp\Bigg[\lambda \sum_{i=2n-2k+1}^{2n} (\omega_i + h)\Delta_i\Bigg] \\ &\qquad\qquad \times 1\{A^{+}_{n,k} \cup A^{-}_{n,k}\}\Bigg| S_{2n-2k} = 0\Bigg) \\ &= Z^{\omega,*}_{2n} + \sum_{k=1}^{n} Z^{\omega,*}_{2n-2k} \frac{a_k}{b_k} E\Bigg(\exp\Bigg[\lambda \sum_{i=2n-2k+1}^{2n} (\omega_i + h)\Delta_i\Bigg] \\ &\qquad\qquad \times 1\{B^{+}_{n,k} \cup B^{-}_{n,k}\}\Bigg| S_{2n-2k} = 0\Bigg). \end{aligned}$$

Here we abbreviate the events

$$(1.5)\qquad \begin{aligned} A^{+}_{n,k} &= \{S_i > 0 \text{ for } 2n - 2k + 1 \le i \le 2n\}, \\ B^{+}_{n,k} &= \{S_i > 0 \text{ for } 2n - 2k + 1 \le i < 2n,\ S_{2n} = 0\} \end{aligned}$$

and similarly for $A^{-}_{n,k}$, $B^{-}_{n,k}$, and their probabilities

$$(1.6)\qquad \begin{aligned} a_k &= P(A^{+}_{n,k} | S_{2n-2k} = 0) = P(A^{-}_{n,k} | S_{2n-2k} = 0), \\ b_k &= P(B^{+}_{n,k} | S_{2n-2k} = 0) = P(B^{-}_{n,k} | S_{2n-2k} = 0) \end{aligned}$$

(both independent of n). The reason for the second equality in (1.4) is that $\Delta_i = +1$ for all $2n - 2k + 1 \le i \le 2n$ on the events $A^{+}_{n,k}$, $B^{+}_{n,k}$ and $\Delta_i = -1$ for all $2n - 2k + 1 \le i \le 2n$ on the events $A^{-}_{n,k}$, $B^{-}_{n,k}$ (ω is fixed).

Next, there exist $C_1, C_2 > 0$ such that $a_k \le C_1 / k^{1/2}$ and $b_k \ge C_2 / k^{3/2}$ for all $k \ge 1$. Moreover, without the factor a_k / b_k the last sum in (1.4) is precisely $Z^{\omega,*}_{2n}$. Hence

$$(1.7)\qquad Z^{\omega}_{2n} \le \left(1 + \frac{C_1}{C_2} n\right) Z^{\omega,*}_{2n}. \qquad \square$$

Lemmas 1 and 2 complete the proof of Theorem 1.

2. Proof of Theorem 2. The proof proceeds in a sequence of five steps, organized as Sections 2.1 and 2.2. Define [recall (0.21)]

$$(2.1)\qquad \psi(\lambda, h) = \phi(\lambda, h) - \lambda h.$$

Let

$$(2.2)\qquad D = \{(\lambda, h): \psi(\lambda, h) = 0\}$$

be the region of delocalization (see Definition 1).

2.1. *Existence, continuity and monotonicity of* $h_c(\lambda)$.

STEP 1. If $(\lambda, h) \in D$, then $(\lambda + \delta, h + \varepsilon) \in D$ for all $\delta, \varepsilon \ge 0$ satisfying $\varepsilon \ge \delta(1 - h)/\lambda$.

PROOF. Since $\lambda\sum_{i=1}^{n}(\omega_i + h) = \lambda hn + o(n)$ $\mathbb{P}$-a.s., we have the following equivalence [recall that $\psi \geq 0$ by (0.4)]:

$$\psi(\lambda, h) = 0$$

$$\Leftrightarrow \lim_{n\to\infty} \frac{1}{n} \log E\left(\exp\left[\lambda \sum_{i=1}^{n} (\omega_i + h)(\Delta_i - 1)\right]\right) \leq 0, \quad \mathbb{P}\text{-a.s.} \tag{2.3}$$

Thus, to prove the claim we must show that if the r.h.s. of (2.3) holds for (λ, h), then it also holds for $(\lambda + \delta, h + \varepsilon)$. To see this, write

$$\begin{aligned}(\lambda + \delta) \sum_{i=1}^{n} (\omega_i + h + \varepsilon)(\Delta_i - 1) &= \lambda \sum_{i=1}^{n} (\omega_i + h)(\Delta_i - 1) \\ &\quad + \sum_{i=1}^{n} [\delta(\omega_i + h) + \varepsilon\lambda + \delta\varepsilon](\Delta_i - 1).\end{aligned} \tag{2.4}$$

Since $\Delta_i \leq 1$ and $\omega_i \geq -1$, the last sum is less than or equal to 0 when $\delta(-1 + h) + \varepsilon\lambda \geq 0$. □

For $\lambda \in [0, \infty)$ define

$$h_c(\lambda) = \inf\{h \in [0,1]: (\lambda, h) \in D\}. \tag{2.5}$$

By continuity of ψ, we have $(\lambda, h_c(\lambda)) \in D$. It therefore follows from Step 1 that $(\lambda, h) \in D$ for all $h > h_c(\lambda)$, so that the localized and the delocalized phase are separated by a single critical curve: $\lambda \to h_c(\lambda)$.

STEP 2. (i) $\lambda \to h_c(\lambda)$ is continuous and nondecreasing on $[0, \infty)$.
(ii) $\lambda \to \lambda(1 - h_c(\lambda))$ is continuous and nondecreasing on $[0, \infty)$.

PROOF. (i) We know that $\psi(\lambda, h) \geq 0$ is convex in λ with boundary value $\psi(0, h) = 0$. Therefore, if $(\lambda, h) \notin D$ then also $(\lambda + \delta, h) \notin D$ for all $\delta > 0$. Hence $\lambda \to h_c(\lambda)$ is nondecreasing. Step 1 shows that its slope at the point λ is bounded from above by $(1 - h_c(\lambda))/\lambda$. Since this is finite for $\lambda > 0$, we get continuity on $(0, \infty)$. Continuity at $\lambda = 0$ follows from Step 3(i).
(ii) This is easily deduced from Step 1. □

2.2. *Bounds on $h_c(\lambda)$.*

STEP 3. $h_c(\lambda) \leq (1/2\lambda) \log\cosh(2\lambda)$. Consequently:

(i) $\limsup_{\lambda\downarrow 0}(1/\lambda)h_c(\lambda) \leq 1$,
(ii) $\liminf_{\lambda\to\infty} \lambda(1 - h_c(\lambda)) \geq \frac{1}{2}\log 2$.

PROOF. The claim will follow once we prove that $(\lambda, h) \in D$ for all $h \geq (1/2\lambda)\log\cosh(2\lambda)$. This will be done by checking the property in the r.h.s. of (2.3).

Estimate $\psi(\lambda, h)$ from above as follows:

$$
\begin{aligned}
\psi(\lambda, h) &= \lim_{n\to\infty} \frac{1}{n}\mathbb{E}\left(\log E\left(\exp\left[\lambda \sum_{i=1}^{n} (\omega_i + h)(\Delta_i - 1)\right]\right)\right)\\
&\le \liminf_{n\to\infty} \frac{1}{n}\log E\left(\mathbb{E}\left(\exp\left[\lambda \sum_{i=1}^{n} (\omega_i + h)(\Delta_i - 1)\right]\right)\right)\\
&= \liminf_{n\to\infty} \frac{1}{n}\log E\left(\prod_{i=1}^{n} 1_{\{\Delta_i = -1\}}\left[\frac{1}{2}e^{-2\lambda(1+h)} + \frac{1}{2}e^{-2\lambda(-1+h)}\right]\right).
\end{aligned} \tag{2.6}
$$

The first equality is a direct consequence of the superadditivity (see Section 1). The r.h.s. is less than or equal to 0 as soon as the term between square brackets is less than or equal to 1. □

STEP 4. $\liminf_{\lambda\downarrow 0}(1/\lambda)h_c(\lambda) > 0$.

PROOF. The idea is to find a strategy of the polymer for which the contribution to the free energy exceeds λh (see Definition 1). The computations below are easy but a bit lengthy, due to a necessary fine-tuning of constants. The proof comes in three parts.

(i) As was shown in Section 1,

$$
\phi(\lambda, h) = \lim_{n\to\infty} \frac{1}{n}\mathbb{E}(\log Z_n^\omega) \tag{2.7}
$$

with Z_n^ω our partition sum [see (1.3)]. We begin by rewriting Z_n^ω in terms of the excursions of S away from the origin. To that end, define

$$
\begin{aligned}
&\eta_0 = 0, \qquad \eta_{j+1} = \inf\{i > \eta_j : S_i = 0\}, \qquad j \ge 0,\\
&\tau_n = \max\{j \ge 0 : \eta_j \le n\}
\end{aligned} \tag{2.8}
$$

and

$$
\xi(x) = \log\cosh x. \tag{2.9}
$$

Let

$$
H_n(S, \omega) = \sum_{j=1}^{\tau_n} \xi\left(\lambda \sum_{i\in(\eta_{j-1}, \eta_j]} (\omega_i + h)\right) + \xi\left(\lambda \sum_{i\in(\eta_{\tau_n}, n]} (\omega_i + h)\right). \tag{2.10}
$$

Then, using the up–down symmetry of S for each excursion, we can write

$$
Z_n^\omega = E(\exp[H_n(S, \omega)]). \tag{2.11}
$$

(ii) The length of a typical free excursion has distribution f given by

$$
\sum_l z^l f(l) = 1 - \sqrt{1 - z^2}, \tag{2.12}
$$

which is the generating function for the probability of first return to the origin of simple random walk. In order to bound (2.11) from below, we shall be looking for a strategy of the path in which the excursions have distribution

$$f^{\gamma}(l) = \frac{1}{1-\gamma} f(l)\left(\sqrt{1-\gamma^2}\right)^{l}, \qquad l \geq 1. \tag{2.13}$$

This corresponds to a random walk with drift γ towards the origin (i.e., $S_{i+1} - S_i = \pm 1$ with probability $\frac{1}{2}(1 \pm \gamma[-\mathrm{sign}(S_i)])$ for $i \geq 0$). Here $0 < \gamma < 1$ is a parameter we shall optimize over.

The following lemma is an intermezzo. Abbreviate $\omega_I = \sum_{i \in I} \omega_i$.

LEMMA 3. *For all* $\lambda \in [0, \infty)$ *and* $h \in [0, 1)$

$$\begin{aligned} \phi(\lambda, h) \geq \sup_{0<\gamma<1} \frac{\gamma}{1+\gamma} \Bigg\{ & \sum_l f^{\gamma}(l) \mathbb{E}\big(\xi(\lambda \omega_{(0,l]} + \lambda h l)\big) \\ & - \frac{1+\gamma}{2\gamma} \log(1+\gamma) - \frac{1-\gamma}{2\gamma} \log(1-\gamma) \Bigg\}. \end{aligned} \tag{2.14}$$

PROOF. Let P_n^0 and P_n^{γ} denote the laws of simple random walk, respectively, random walk with drift γ, restricted to n-step paths. Then from (2.13),

$$\begin{aligned} \frac{dP_n^{\gamma}}{dP_n^0}\big((S_i)_{i=0}^n\big) &= \prod_{j=1}^{\tau_n} \frac{f^{\gamma}}{f}(\eta_j - \eta_{j-1}) \frac{\sum_{l > n - \eta_{\tau_n}} f^{\gamma}(l)}{\sum_{l > n - \eta_{\tau_n}} f(l)} \\ &\leq (1-\gamma)^{-\tau_n - 1} (1-\gamma^2)^{n/2}. \end{aligned} \tag{2.15}$$

Using Jensen's inequality, we get from (2.11) and (2.15) that

$$\begin{aligned} \log Z_n^{\omega} &= \log E_n^{\gamma}\left(\exp\left[H_n(S, \omega) - \log \frac{dP_n^{\gamma}}{dP_n^0}\right]\right) \\ &\geq E_n^{\gamma}(H_n(S, \omega)) - E_n^{\gamma}\left(\log \frac{dP_n^{\gamma}}{dP_n^0}\right) \\ &\geq E_n^{\gamma}(H_n(S, \omega)) + E_n^{\gamma}(\tau_n + 1) \log(1-\gamma) - \frac{n}{2} \log(1-\gamma^2). \end{aligned} \tag{2.16}$$

Now, $\tau_n + 1 = \min\{j \geq 0 \colon \eta_j > n\}$ is a stopping time. Moreover, a straightforward calculation yields $E_n^{\gamma}(\eta_1) = (1+\gamma)/\gamma$. Therefore, the optional sampling theorem gives us

$$\lim_{n \to \infty} \frac{1}{n} E_n^{\gamma}(\tau_n + 1) = \frac{\gamma}{1+\gamma}. \tag{2.17}$$

In order to bound $\phi(\lambda, h) = \lim_{n\to\infty}(1/n)\mathbb{E}(\log Z_n^{\omega})$, it therefore remains to consider

$$\begin{aligned} \mathbb{E}\big(E_n^{\gamma}(H_n(S, \omega))\big) &= E_n^{\gamma}\big(\mathbb{E}(H_n(S, \omega))\big) \\ &\geq E_n^{\gamma}\left(\sum_{j=1}^{\tau_n} \mathbb{E}\left(\xi\left(\lambda \sum_{i \in (\eta_{j-1}, \eta_j]} (\omega_i + h)\right)\right)\right). \end{aligned} \tag{2.18}$$

By stationarity of the ω-sequence, the summands are functions of $\eta_j - \eta_{j-1}$ only. Applying the optional sampling theorem again we get

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{n}E_n^\gamma\left(\sum_{j=1}^{\tau_n}\mathbb{E}\left(\xi\left(\lambda\sum_{i\in(\eta_{j-1},\eta_j]}(\omega_i+h)\right)\right)\right)\\ =\frac{\gamma}{1+\gamma}E_n^\gamma\left(\mathbb{E}\left(\xi\left(\lambda\sum_{j=1}^{\eta_1}(\omega_i+h)\right)\right)\right).\end{aligned}\tag{2.19}$$

(To handle the last excursion $\eta_{\tau_n+1} - \eta_{\tau_n}$, note that ξ is linearly bounded and that the excursion times have an exponential moment under P_n^γ.) Putting the estimates together we obtain the claim. □

(iii) The proof of Step 4 can now be complete as follows. Because $\xi \geq 0$ and ξ is convex, we have

$$\begin{aligned}&\mathbb{E}\big(\xi\big(\lambda\omega_{(0,l]}+\lambda hl\big)\big)\\ &\quad\geq \mathbb{P}\big(\omega_{(0,l]}\geq 0\big)\mathbb{E}\big(\xi\big(\lambda\omega_{(0,1]}+\lambda hl\big)\,|\,\omega_{(0,l]}\geq 0\big)\\ &\quad\geq \tfrac{1}{2}\xi\big(\lambda\mathbb{E}\big(\omega_{(0,l]}\,|\,\omega_{(0,l]}\geq 0\big)+\lambda hl\big).\end{aligned}\tag{2.20}$$

Next, note that there exists $A > 0$ such that $\mathbb{E}(\omega_{(0,l]}\,|\,\omega_{(0,l]} \geq 0) \geq Al^{1/2}$ for all $l \geq 1$. Now pick $h = \alpha\lambda$ and $\gamma = \beta\lambda$ in Lemma 3, insert (2.13) and (2.20), and use that $f(l) \sim [1 + (-1)^l]B/l^{3/2}(l \to \infty)$, to obtain

$$\liminf_{\lambda\downarrow 0}\frac{1}{\alpha\lambda^2}\phi(\lambda,\alpha\lambda)\geq\frac{\beta}{2\alpha}\big[BI(A,\alpha,\beta)-\beta\big],\tag{2.21}$$

where

$$I(A,\alpha,\beta)=\int_0^\infty\frac{dx}{x^{3/2}}\exp\left(-\frac{1}{2}\beta^2x\right)\xi\big(A\sqrt{x}+\alpha x\big).\tag{2.22}$$

The constants α, β can still be optimized. Pick $M > 2/BI(A, 0, 0)$ and put $\beta = M\alpha$. Then, as $\alpha \downarrow 0$, the r.h.s. of (2.21) converges to a number greater than 1. Therefore we have proved that $\phi(\lambda, \alpha\lambda) > \alpha\lambda^2$ for α, λ sufficiently small. This proves the claim in Step 4 (recall Definition 1). □

STEP 5. $\lim_{\lambda\to\infty} \lambda(1 - h_c(\lambda)) \leq \frac{3}{2}\log 2$.

PROOF. Recall Step 2(ii). The claim is proved as follows. As $\lambda \to \infty$, the path will tend to make short excursions. Therefore we bound the partition sum from below by requiring all excursions to have length 2:

$$\begin{aligned}Z_{2n}^\omega &\geq E\left(\exp\left[\lambda\sum_{i=1}^{2n}(\omega_i+h)\Delta_i\right]1\{S_{2m}=0\text{ for }0<m\leq n\}\right)\\ &=\left(\tfrac{1}{2}\right)^n\prod_{m=1}^{n}\cosh\big(\lambda[\omega_{2m-1}+\omega_{2m}+2h]\big).\end{aligned}\tag{2.23}$$

(Use the up–down symmetry of S for each excursion.) It follows that [recall (2.9)]:

$$
\begin{aligned}
\phi(\lambda, h) &= \lim_{n\to\infty} \frac{1}{2n}\mathbb{E}(\log Z_{2n}^{\omega}) \\
&\geq -\frac{1}{2}\log 2 + \frac{1}{2}\mathbb{E}\big(\xi(\lambda[\omega_1 + \omega_2 + 2h])\big) \\
&= -\frac{1}{2}\log 2 \\
&\quad + \frac{1}{2}\left\{\frac{1}{4}\xi(2\lambda(1+h)) + \frac{1}{4}\xi(2\lambda(1-h)) + \frac{1}{2}\xi(2\lambda h)\right\}.
\end{aligned}
\tag{2.24}
$$

Next, insert $\xi(x) = x - \log 2 + O(e^{-2x})$ $(x \to \infty)$. Pick $h = 1 - M/\lambda$ with $M > 0$ arbitrary. Then for $\lambda \to \infty$,

$$
\begin{aligned}
\phi\left(\lambda, 1 - \frac{M}{\lambda}\right) \geq \lambda\left(1 - \frac{M}{\lambda}\right) + \left\{\frac{1}{4}M - \frac{3}{8}\log 2 + \frac{1}{8}\xi(2M)\right\} \\
+ O(e^{-4\lambda}).
\end{aligned}
\tag{2.25}
$$

As soon as $M \geq \frac{3}{2}\log 2$, the term between braces is greater than 0, implying that $(\lambda, 1 - M/\lambda) \notin D$ for λ sufficiently large [cf. (2.2) and (2.3)]. But then $h_c(\lambda) > 1 - M/\lambda$ for λ sufficiently large [cf. (2.6)], that is, $\lambda(1 - h_c(\lambda)) < M$. $\square$

Steps 2–5 prove Theorem 2 as well as Properties (i)–(iii) in (0.8).

3. Proofs of Theorems 3 and 4. Essentially the same arguments as in the proofs of Theorems 1 and 2 carry over to the continuous case. We only indicate which points need modification.

3.1. *Proof of Theorem* 3. We cannot insert $1\{B_t = 0\}$, since $P(B_t = 0) = 0$ [compare with (1.1)]. However, this problem is easily handled through a comparison argument. Recall the notation introduced in Section 0.2.

Define

$$
\tilde{Z}_t^{\beta,*} = \inf_{|x|\leq 1} \tilde{E}\left(\exp\left[\lambda\int_0^t \Delta_s(d\beta_s + h\,ds)\right]1\{|B_t| \leq 1\}\middle| B_0 = x\right).
\tag{3.1}
$$

Then:

I. $\tilde{Z}_t^{\beta,*} \geq \tilde{Z}_u^{\beta,*}\tilde{Z}_{t-u}^{T^u\beta,*}$ for all $0 \leq u \leq t$, with T^u the left-shift $(T^u\beta)_s = \beta_{u+s} - \beta_u$.

II. $t \to (1/t)\tilde{\mathbb{E}}(\log \tilde{Z}_t^{\beta,*})$ is bounded from above.

III. $\tilde{\mathbb{P}}(T^u\beta \in \cdot) = \tilde{\mathbb{P}}(\beta \in \cdot)$ for all $u \geq 0$.

Properties I and III are obvious. Property II holds because

$$
\begin{aligned}
\tilde{\mathbb{E}}\big(\log \tilde{Z}_t^{\beta,*}\big) &\leq \log \tilde{\mathbb{E}}\big(\tilde{Z}_t^{\beta,*}\big) \\
&\leq \log \tilde{E}\left(\tilde{\mathbb{E}}\left(\exp\left[\lambda\int_0^t \Delta_s(d\beta_s + h\,ds)\right]\right)1\{|B_t| \leq 1\}\middle| B_0 = 0\right)
\end{aligned}
\tag{3.2}
$$

$$= \log \tilde{E}\left(\exp\left[\tfrac{1}{2}\lambda^2 t + \lambda h \int_0^t \Delta_s \, ds\right] 1\{|B_t| \le 1\} \middle| B_0 = 0\right)$$
$$\le t\left(\tfrac{1}{2}\lambda^2 + \lambda h\right),$$

where the equality follows from the martingale property

$$(3.3)\qquad \tilde{\mathbb{E}}\left(\exp\left[\int_0^t f(s)\, d\beta_s\right]\right) = \exp\left[\tfrac{1}{2}\int_0^t f^2(s)\, ds\right], \qquad f \in L^2([0,t)).$$

Thus, $\beta \to (\log \tilde{Z}_t^{\beta,*})_{t \ge 0}$ is a superadditive process.

In order to apply the superadditive ergodic theorem, we need an additional regularity condition that is absent in the discrete time setting, namely [see Kingman (1973), Theorem 4] the following property.

IV.

$$(3.4)\qquad \tilde{\mathbb{E}}\left(\sup_{0 \le s < t \le T} |\log \tilde{Z}_{s,t}^{\beta,*}|\right) < \infty \quad \text{for all } T < \infty,$$

where $\tilde{Z}_{s,t}^{\beta,*}$ is the partition sum over the time interval $[s, t)$; that is,

$$(3.5)\qquad \tilde{Z}_{s,t}^{\beta,*} = \inf_{|x| \le 1} \tilde{E}\left(\exp\left[\lambda \int_s^t \Delta_u (d\beta_u + h\, du)\right] 1\{|B_t| \le 1\} \middle| B_s = x\right).$$

To prove Property IV, we first note that, for all β,

$$(3.6)\qquad \inf_{0 \le s < t \le T} \tilde{Z}_{s,t}^{\beta,*} \ge \inf_{0 \le s < t \le T} \inf_{|x| \le 1} \tilde{P}(|B_t| \le 1 \mid B_s = x) > 0 \quad \text{for all } T < \infty.$$

(Use Jensen's inequality together with $\tilde{E}\Delta_u \equiv 0$.) Hence it suffices to prove (3.4) without the absolute value signs. But this we may estimate as follows:

$$\tilde{\mathbb{E}}\left(\sup_{0 \le s < t \le T} \log \tilde{Z}_{s,t}^{\beta,*}\right)$$
$$(3.7)\qquad \le \log \tilde{E}\left(\tilde{\mathbb{E}}\left(\sup_{0 \le s < t \le T} \exp\left[\lambda \int_s^t \Delta_u (d\beta_u + h\, du)\right]\right) \times 1\{|B_t| \le 1\} \middle| B_s = 0\right).$$

The exponent in (3.7) is bounded from above by $\lambda h(t-s) + \lambda \int_s^t \Delta_u \, d\beta_u$. Moreover, we note that under the law $\tilde{\mathbb{P}}$ the last integral is just Brownian motion, since $\Delta_u^2 = 1$ almost everywhere $\tilde{P}$-a.s. Thus we obtain

$$(3.8)\qquad \tilde{\mathbb{E}}\left(\sup_{0 \le s < t \le T} \log \tilde{Z}_{s,t}^{\beta,*}\right) \le \lambda h T + \log \tilde{\mathbb{E}}\left(\sup_{0 \le s < t \le T} \exp[\lambda(\beta_t - \beta_s)]\right).$$

But the last integral is finite, because $2\lambda \sup_{0 \le u \le T} |\beta_u|$ has an exponential moment. This proves Property IV.

Properties I–IV guarantee that the superadditive ergodic theorem applies:

(3.9) $\lim_{t \to \infty}(1/t)\log \tilde{Z}_t^{\beta,*}$ converges $\tilde{\mathbb{P}}$-a.s. and in mean, and is constant $\tilde{\mathbb{P}}$-a.s.

Thus we have the LLN for the quantity defined in (3.1). In order to get it for our original partition sum, it remains to remove $1\{|B_t| \le 1\}$ and $\inf_{|x|\le 1}$ from (3.1). This will be done in two pieces.

Define

$$\tilde{Z}_t^{\beta,*}(x) = \tilde{E}\left(\exp\left[\lambda\int_0^t \Delta_s(d\beta_s + h\,ds)\right]1\{|B_t| \le 1\}\middle| B_0 = x\right),$$
$$\tilde{Z}_t^{\beta}(x) = \tilde{E}\left(\exp\left[\lambda\int_0^t \Delta_s(d\beta_s + h\,ds)\right]\middle| B_0 = x\right). \tag{3.10}$$

In (3.9) we have the LLN for $\tilde{Z}_t^{\beta,*} = \inf_{|x|\le 1} \tilde{Z}_t^{\beta,*}(x)$. The key estimates are now

$$\text{(i)} \qquad \tilde{Z}_t^{\beta,*} \le \tilde{Z}_t^{\beta,*}(0) \le C(\beta)\tilde{Z}_t^{\beta,*} \quad \text{for all } t \text{ and } \beta,$$
$$\text{(ii)} \qquad \tilde{Z}_t^{\beta,*}(0) \le \tilde{Z}_t^{\beta}(0) \le Ct\tilde{Z}_t^{\beta,*}(0) \quad \text{for all } t \text{ and } \beta. \tag{3.11}$$

The lower bounds are trivial. The upper bound in (ii) is obtained from an almost literal transcription of the proof of Lemma 2. The upper bound in (i) follows from a coupling argument. Indeed, since two Brownian motions starting at 0, respectively, x, hit each other after a finite time a.s., we have $\sup_{|x|\le 1}|\tilde{Z}_t^{\beta,*}(0)/\tilde{Z}_t^{\beta,*}(x)| \le C(\beta)$ with $C(\beta) < \infty$, $\tilde{\mathbb{P}}$-a.s.

The conclusion of (3.11) is that our original partition sum $\tilde{Z}_t^{\beta} = \tilde{Z}_t^{\beta}(0)$ has the same ($\tilde{\mathbb{P}}$-a.s. constant) growth rate as $\tilde{Z}_t^{\beta,*}$ in (3.1).

3.2. *Proof of Theorem* 4. All we have to do is show that $K_c \in (0,1]$ [since the rest follows from (0.15)]. As this inclusion follows from (0.8) and (0.23), strictly speaking there is no need to give a proof here. Still, we indicate a direct proof of the lower bound for K_c because it is instructive.

Fix λ, h. In Section 3.1 we saw that

$$\tilde{\phi}(\lambda, h) = \lim_{t\to\infty} \frac{1}{t}\tilde{\mathbb{E}}\left(\log \tilde{Z}_t^{\beta}\right). \tag{3.12}$$

We begin by expressing our partition sum in terms of the excursions of B away from the origin. Let $N = \{s \ge 0\colon\ B_s = 0\}$. Then $[0,\infty)\setminus N = \bigcup_j I_j$ is a countable union of disjoint open intervals having full measure [Revuz and Yor (1991), Chapter XII]. Let

$$J_t = \left\{j\colon\ I_j \subset [0,t)\right\} \cup \{0\}, \tag{3.13}$$

where we reserve the index 0 for the interval between t and the last hitting time of the origin prior to time t. Then, using the up–down symmetry of B for each excursion, we can write

$$\tilde{Z}_t^{\beta} = \tilde{E}\left(\exp\left[\sum_{j\in J_t} \xi\left(\lambda\beta_{I_j} + \lambda h|I_j|\right)\right]\right). \tag{3.14}$$

Here, β_I denotes the increment of β over the set I, and ξ was defined in (2.9). The representation in (3.14) is the continuous analogue of (2.10) and (2.11).

Fix $\gamma > 0$. Let $\tilde{P}^\gamma$, $\tilde{E}^\gamma$ denote the probability law and expectation of Brownian motion with drift γ towards the origin. Then it follows from the Cameron–Martin formula [Chung and Williams (1990), Theorem 9.10], respectively, the Tanaka formula [Revuz and Yor (1991), Theorem VI.1.2], that

$$\begin{aligned}\frac{d\tilde{P}^\gamma}{d\tilde{P}^0}\big((B_s)_{0\le s\le t}\big) &= \exp\left[\int_0^t \{-\gamma\,\mathrm{sign}(B_s)\}\,dB_s - \frac{1}{2}\int_0^t \{-\gamma\,\mathrm{sign}(B_s)\}^2\,ds\right] \\ &= \exp\left[\gamma\big(L_t - |B_t|\big) - \frac{1}{2}\gamma^2 t\right],\end{aligned} \tag{3.15}$$

where L_t is the local time at the origin in the time interval $[0, t)$. Next, according to Tanaka's formula under $\tilde{P}^\gamma$, we have $\frac{1}{2}\tilde{E}^\gamma(L_t) = \frac{1}{2}\gamma + \tilde{E}^\gamma(B_t 1\{B_t > 0\}) = \frac{1}{2}\gamma + O(1)$. Therefore, substituting (3.15) into (3.14) and using Jensen's inequality, we obtain

$$\tilde{\phi}(\lambda, h) \ge -\frac{1}{2}\gamma^2 + \limsup_{t\to\infty} \frac{1}{t}\tilde{\mathbb{E}}\left(\tilde{E}^\gamma\left(\sum_{j\in J_t} \xi\big(\lambda\beta_{I_j} + \lambda h|I_j|\big)\right)\right). \tag{3.16}$$

It remains to compute the r.h.s. of (3.16). This is essentially parallel to (2.18)–(2.22). In order to be able to properly count excursions, one first has to cut away the excursions that have length smaller than ε and then let $\varepsilon \downarrow 0$. We leave this to the reader.

4. Proof of Theorem 6. Recall the notation introduced in Sections 0.1 and 0.2. Define for the random walk model,

$$\begin{aligned}\xi_i &= 1_{\{\Delta_i = -1\}}, \\ \psi_t(\lambda, h) &= \frac{1}{t}\mathbb{E}\left(\log E\left(\exp\left[-2\lambda \sum_{i=1}^{\lfloor t\rfloor} \xi_i(\omega_i + h)\right]\right)\right), \\ \psi(\lambda, h) &= \lim_{t\to\infty} \psi_t(\lambda, h)\end{aligned} \tag{4.1}$$

and for the Brownian motion model,

$$\begin{aligned}\xi_s &= 1_{\{\Delta_s = -1\}}, \\ \tilde{\psi}_t(\lambda, h) &= \frac{1}{t}\tilde{\mathbb{E}}\left(\log \tilde{E}\left(\exp\left[-2\lambda \int_0^t \xi_s(d\beta_s + h\,ds)\right]\right)\right), \\ \tilde{\psi}(\lambda, h) &= \lim_{t\to\infty} \tilde{\psi}_t(\lambda, h).\end{aligned} \tag{4.2}$$

By the law of large numbers for ω, respectively, β,

$$\begin{aligned}\phi(\lambda, h) &= \psi(\lambda, h) + \lambda h, \\ \tilde{\phi}(\lambda, h) &= \tilde{\psi}(\lambda, h) + \lambda h.\end{aligned} \tag{4.3}$$

It suffices to consider the case $\lambda = 1$.

4.1. *Outline of the proof of Theorem* 6. Theorem 6 is proved by a series of approximation steps. Our approximations will depend on two auxiliary parameters ε and δ, where $0 < \varepsilon < \delta$. Later on, we shall let $t \to \infty$, $a \downarrow 0$, $\varepsilon \downarrow 0$, $\delta \downarrow 0$ (in this order). There will be no danger in assuming that t/a^2, t/ε, ε/a^2, δ/ε are all integers, which we shall do in order to avoid a plethora of brackets.

Below we shall make a number of quite similar comparisons. In order to write these in a compact form, we introduce the following notation.

DEFINITION 2. Let $f_{t,\varepsilon,\delta}(a,h)$ and $g_{t,\varepsilon,\delta}(a,h)$ be real-valued functions. We write $f \prec g$ if for any $0 \le h' < h$, $\rho > 0$ satisfying $(1+\rho)h' < h$ the following is true: there exists δ_0 such that for $0 < \delta < \delta_0$ there exists $\varepsilon_0(\delta)$ such that for $0 < \varepsilon < \varepsilon_0$ there exists $a_0(\varepsilon,\delta)$ such that

$$\limsup_{t\to\infty}\left[f_{t,\varepsilon,\delta}(a,h) - (1+\rho)\, g_{t(1+\rho)^2,\,\varepsilon(1+\rho)^2,\,\delta(1+\rho)^2}\big(a(1+\rho),h'\big)\right] \le 0 \quad \text{for } 0 < a < a_0. \tag{4.4}$$

Here δ_0, ε_0, a_0 may depend on h, h', ρ. We write $f \simeq g$ if $f \prec g$ and $g \prec f$.

Note that $\prec$ is a transitive relation and therefore $\simeq$ is an equivalence relation.

The function for which we shall make such comparisons will be of the form

$$f_{t,\varepsilon,\delta}(a,h) = \frac{1}{t}\mathbb{E}\Big(\log E\big(\exp[-2aH_{t,\varepsilon,\delta}(a,h)]\big)\Big), \tag{4.5}$$

where the Hamiltonian $H_{t,\varepsilon,\delta}(a,h)$ is a random variable defined on the product space of the random walk and the random medium (having as probability measure the product of P and $\mathbb{P}$). Similar functions will be considered for the Brownian motion and medium.

Now suppose that we want to prove $f \prec f'$, where $f'_{t,\varepsilon,\delta}(a,h)$ has the Hamiltonian $H'_{t,\varepsilon,\delta}(a,h)$. We can do this in the following way:

1. Split H into two parts

$$H = H^{(I)} + H^{(II)}. \tag{4.6}$$

2. Apply Hölder, Jensen and Fubini to get, for $\rho > 0$,

$$\begin{aligned} f_{t,\varepsilon,\delta}(a,h) \le{}& \frac{1}{t(1+\rho)}\mathbb{E}\Big(\log E\big(\exp[-2a(1+\rho)H^{(I)}]\big)\Big) \\ &+ \frac{1}{t(1+\rho^{-1})}\log E\Big(\mathbb{E}\big(\exp[-2a(1+\rho^{-1})H^{(II)}]\big)\Big). \end{aligned} \tag{4.7}$$

3. The crucial point will be, for given $(1+\rho)h' < h$, to choose the splitting in such a way that

$$H^{(I)} = H^{(I)}_{t,\varepsilon,\delta}(a,h') = H'_{t(1+\rho)^2,\,\varepsilon(1+\rho)^2,\,\delta(1+\rho)^2}(a(1+\rho),h') \tag{4.8}$$

and that $H^{(II)} = H^{(II)}_{t,\varepsilon,\delta}(a,h,h')$ satisfies

$$\limsup_{t\to\infty} \frac{1}{t}\log E\Big(\mathbb{E}\Big(\exp\big[-2a(1+\rho^{-1})H^{(II)}_{t,\varepsilon,\delta}(a,h,h')\big]\Big)\Big) \le 0 \tag{4.9}$$

with δ, ε, a chosen appropriately (in the sense of Definition 2).

Clearly, (4.6)–(4.9) imply $f \prec f'$.

Before we proceed, let us agree on some conventions about constants: A, B, C are generic positive constants, not necessarily the same at different occurrences. They may depend on h, h', ρ, but not on the running parameters t, a, ε, δ.

Return to (4.1–4.3). Let

$$\begin{aligned}\Psi_{t,\varepsilon,\delta}(a,h) &= \frac{1}{a^2}\psi_{t/a^2}(a,ah),\\ \tilde{\Psi}_{t,\varepsilon,\delta}(a,h) &= \tilde{\psi}_t(1,h)\end{aligned} \tag{4.10}$$

(which in fact do not depend on δ, ε, respectively, δ, ε, a). What we finally want to prove is $\Psi \simeq \tilde{\Psi}$, since by Definition 2 this implies Theorem 6. In order to achieve this, we shall introduce three intermediate quantities $F^i_{t,\varepsilon,\delta}(a,h)$ $(i = 1,2,3)$ and prove that

$$\Psi \simeq F^1 \simeq F^2 \simeq F^3 \simeq \tilde{\Psi}. \tag{4.11}$$

The proof of (4.11) comes in four steps, organized as Sections 4.2–4.5. In order not to overburden notations, we shall often not explicitly express dependencies on a, ε, δ.

One of the crucial aspects of the proof is that the statement of Theorem 6 does not allow for error factors of the form $\exp(\kappa(a,\varepsilon,\delta)t)$ with $\kappa(a,\varepsilon,\delta)$ tending to zero as $a,\varepsilon,\delta \downarrow 0$. The reader should keep this in mind.

4.2. *Coarse graining of the RW.* We start by defining F^1. Divide time into intervals of length ε/a^2:

$$I_j = \big((j-1)\varepsilon/a^2,\, j\varepsilon/a^2\big], \qquad j \ge 1. \tag{4.12}$$

Put $\sigma_0 = 0$ and

$$\sigma_k = \inf\big\{j \ge \sigma_{k-1} + (\delta/\varepsilon)\colon S_i = 0 \text{ for some } i \in I_j\big\}, \qquad k \ge 1. \tag{4.13}$$

That is, $\sigma_1, \sigma_2, \ldots$ number the intervals in which the walk returns to the origin leaving gaps of at least $(\delta/\varepsilon) - 1$ in the numbering. Define

$$\bar{I}_k = \Bigg(\bigcup_{\sigma_{k-1} < j \le \sigma_k} I_j\Bigg) \cap \big(0, t/a^2\big], \qquad k \ge 1, \tag{4.14}$$

and put $m_{t/a^2} = \max\{k\colon \bar{I}_k \ne \emptyset\} = \min\{k\colon \sigma_k \ge t/\varepsilon\}$.

For $1 \le k < m_{t/a^2}$, we set $s_k = 1$ if the random walk is negative just prior to its first zero in I_{σ_k}, and $s_k = 0$ otherwise. For $k = m_{t/a^2}$, on the other hand, we set $s_k = 1$ if the random walk is negative at t/a^2, and $s_k = 0$ otherwise. Let

$$Z_k(\omega) = \sum_{i \in \bar{I}_k} \omega_i. \tag{4.15}$$

We can now define our first intermediate quantity:

$$\begin{aligned} F^1_{t,\varepsilon,\delta}(a,h) &= \frac{1}{t}\mathbb{E}\Big(\log E\Big(\exp\big[-2aH^1_{t,\varepsilon,\delta}(a,h)\big]\Big)\Big), \\ H^1_{t,\varepsilon,\delta}(a,h) &= \sum_{k=1}^{m_{t/a^2}} s_k\big\{Z_k(\omega) + ah|\bar{I}_k|\big\}. \end{aligned} \tag{4.16}$$

STEP 1. $\Psi \simeq F^1$.

PROOF. The proof comes in six parts.

(i) We have [recall (4.1) and (4.10)]:

$$\begin{aligned} \Psi_{t,\varepsilon,\delta}(a,h) &= \frac{1}{t}\mathbb{E}\Big(\log E\Big(\exp\big[-2aH_{t,\varepsilon,\delta}(a,h)\big]\Big)\Big), \\ H_{t,\varepsilon,\delta}(a,h) &= \sum_{i=1}^{t/a^2} \xi_i(\omega_i + ah) = \sum_{k=1}^{m_{t/a^2}} \sum_{i \in \bar{I}_k} \xi_i(\omega_i + ah). \end{aligned} \tag{4.17}$$

Remark that, by a trivial rescaling of the parameters [see (4.12)–(4.14)], we have

$$H_{t,\varepsilon,\delta}(a,\kappa h) = H_{\kappa^2 t,\kappa^2\varepsilon,\kappa^2\delta}(\kappa a, h) \quad \text{for any } \kappa \ge 0, \tag{4.18}$$

and the same for H^1. Furthermore, for any $h_1, h_2 \ge 0$,

$$\begin{aligned} &H_{t,\varepsilon,\delta}(a,h_1) - H^1_{t,\varepsilon,\delta}(a,h_2) \\ &\quad = a(h_1 - h_2)\sum_{k=1}^{m_{t/a^2}} \sum_{i \in \bar{I}_k} \xi_i + \sum_{k=1}^{m_{t/a^2}} \sum_{i \in \bar{I}_k} (ah_2 + \omega_i)(\xi_i - s_k). \end{aligned} \tag{4.19}$$

In order to prove $\Psi \prec F^1$, we split $H = H^{(I)} + H^{(II)}$ with

$$H^{(I)}_{t,\varepsilon,\delta} = H^1_{t,\varepsilon,\delta}(a,(1+\rho)h') = H^1_{t(1+\rho)^2,\varepsilon(1+\rho)^2,\delta(1+\rho)^2}(a(1+\rho),h'), \tag{4.20}$$

and take the r.h.s. of (4.19) with $h_1 = h$, $h_2 = (1+\rho)h'$ as $H^{(II)}$. On the other hand, in order to prove $F^1 \prec \Psi$, we split $H^1 = H'^{(I)} + H'^{(II)}$ with

$$H'^{(I)}_{t,\varepsilon,\delta} = H_{t,\varepsilon,\delta}(a,(1+\rho)h') = H_{t(1+\rho)^2,\varepsilon(1+\rho)^2,\delta(1+\rho)^2}(a(1+\rho),h'), \tag{4.21}$$

and take minus the r.h.s. of (4.19) with $h_1 = (1+\rho)h'$, $h_2 = h$ as $H'^{(II)}$. We shall prove that if we choose a, ε, δ small enough (in this order because of Definition 2), then also the requirement in (4.9) is met:

$$\limsup_{t\to\infty} \frac{1}{t}\log E\Big(\mathbb{E}\Big(\exp\big[-2a(1+\rho^{-1})H^{(II)}_{t,\varepsilon,\delta}(a,h,h')\big]\Big)\Big) \le 0, \tag{4.22}$$

and the same with $H'^{(II)}$ instead of $H^{(II)}$. This will prove the claim in Step 1.

(ii) To prove (4.22), we first carry out the expectation over ω:

$$\begin{aligned}
&\mathbb{E}\Big(\exp\big[-2a(1+\rho^{-1})H^{(II)}_{t,\varepsilon,\delta}(a,h,h')\big]\Big)\\
&\quad = \exp\left[-2a^2(1+\rho^{-1})(h-(1+\rho)h')\sum_{k=1}^{m_{t/a^2}}\sum_{i\in\bar{I}_k}\xi_i\right]\\
&\qquad \times \exp\left[-2a^2(1+\rho^{-1})(1+\rho)h'\sum_{k=1}^{m_{t/a^2}}\sum_{i\in\bar{I}_k}(\xi_i-s_k)\right]\\
&\qquad \times \exp\left[\sum_{k=1}^{m_{t/a^2}}\sum_{i\in\bar{I}_k}\log\cosh\big\{2a(1+\rho^{-1})(\xi_i-s_k)\big\}\right]\\
&\quad \le \exp\left[Aa^2\sum_{k=1}^{m_{t/a^2}}\sum_{i\in\bar{I}_k}|\xi_i-s_k| - Ba^2\sum_{k=1}^{m_{t/a^2}}\sum_{i\in\bar{I}_k}\xi_i\right]
\end{aligned}\tag{4.23}$$

for some constants $A, B > 0$ (which depend on h, h', ρ but not on $t, a, \varepsilon, \delta$). The crucial point is that the second summand in the exponent is able to kill the first summand for arbitrary $A, B > 0$, provided the parameters a, ε, δ are chosen appropriately. Thus, to complete the proof of $\Psi \prec F^1$, it remains to show that

$$\limsup_{t\to\infty}\frac{1}{t}\log E\left(\exp\left[Aa^2\sum_{k=1}^{m_{t/a^2}}\sum_{i\in\bar{I}_k}|\xi_i-s_k| - Ba^2\sum_{k=1}^{m_{t/a^2}}\sum_{i\in\bar{I}_k}\xi_i\right]\right) \le 0. \tag{4.24}$$

This is a problem about simple random walk and its zeroes. The only difference between $H^{(II)}$ and $H'^{(II)}$ is that the second summand on the r.h.s. comes with a minus and h_1, h_2 interchanged. However, this obviously leads to the same type of estimate as (4.23). Therefore (4.24) proves Step 1 completely.

(iii) To prove (4.24), we introduce the standard return times of the random walk:

$$\begin{aligned}
&T_0 = 0, \qquad T_l = \inf\{i > T_{l-1}: S_i = 0\}, \qquad l \ge 1,\\
&l_{t/a^2} = \min\{l: T_l \ge t/a^2\}
\end{aligned}\tag{4.25}$$

and the excursion times

$$\tau_l = T_l - T_{l-1}, \qquad 1 \le l < l_{t/a^2}, \qquad \tau_{l_{t/a^2}} = (t/a^2) - T_{l_{t/a^2-1}}. \tag{4.26}$$

We further define $\eta_l = 1$ if the sign of the lth excursion is negative, and $\eta_l = 0$ otherwise. Then, obviously, we can write the second summand in the r.h.s. of (4.24) as

$$\sum_{k=1}^{m_{t/a^2}}\sum_{i\in\bar{I}_k}\xi_i = \sum_{l=1}^{l_{t/a^2}}\tau_l\eta_l. \tag{4.27}$$

Next we estimate the first summand $\sum_{k=1}^{m_{t/a^2}}\sum_{i\in\bar{I}_k}|\xi_i - s_k|$ in terms of the same quantities. Put $t_0 = 0$, and let t_k be the first zero of the random walk in the interval I_{σ_k} $(1 \le k < m_{t/a^2})$, and $t_{m_{t/a^2}} = t/a^2$. On the time interval $(t_{k-1}, t_k]$ the random walk makes a number of excursions, and s_k just depends on the sign of the last one; that is, $s_k = 1$ if and only if this is negative. By construction, only this last excursion can have length greater than or equal to $(\delta/\varepsilon)(\varepsilon/a^2) = \delta/a^2$ [see (4.12) and (4.13)]. It follows that if i is not in an excursion of length less than δ/a^2 and i does not belong to one of the intervals I_{σ_k}, then

$$\xi_i = s_k \quad \text{for the } k \text{ with } i \in \bar{I}_k. \tag{4.28}$$

From these considerations we obtain (recall that $|I_{\sigma_k}| = \varepsilon/a^2$)

$$\sum_{k=1}^{m_{t/a^2}}\sum_{i\in\bar{I}_k}|\xi_i - s_k| \le \sum_{l=1}^{l_{t/a^2}}\tau_l 1\left\{\tau_l < \frac{\delta}{a^2}\right\} + m_{t/a^2}\frac{\varepsilon}{a^2}. \tag{4.29}$$

Combining (4.27) and (4.29) we see that, in order to prove (4.24), it now suffices to show that

$$\begin{aligned}\limsup_{t\to\infty}\frac{1}{t}\log\, E\Bigg(\exp\Bigg[&Aa^2\sum_{l=1}^{l_{t/a^2}}\tau_l 1\left\{\tau_l < \frac{\delta}{a^2}\right\}\\ &+A\varepsilon m_{t/a^2} - Ba^2\sum_{l=1}^{l_{t/a^2}}\tau_l\eta_l\Bigg]\Bigg) \le 0\end{aligned} \tag{4.30}$$

for appropriate a, ε, δ.

(iv) As the η_l's are independent of the τ_l's (0 or 1 with probability 1/2 each), we can integrate out the former and replace $-Ba^2\sum_l\tau_l\eta_l$ in the r.h.s. of (4.30) by $\sum_l\log(\frac{1}{2} + \frac{1}{2}\exp(-Ba^2\tau_l))$. We next claim that

$$A\varepsilon\left(m_{t/a^2} - 1\right) + \tfrac{1}{2}\sum_{l=1}^{l_{t/a^2}}\log\left(\tfrac{1}{2} + \tfrac{1}{2}\exp\left(-Ba^2\tau_l\right)\right) \le 0 \qquad \text{for } 0 < \varepsilon < \varepsilon_0(\delta). \tag{4.31}$$

To see why, pick any of the intervals $(t_{k-1}, t_k]$ $(1 \le k < m_{t/a^2})$. If any of the excursions on $(t_{k-1}, t_k]$ has length greater than or equal to δ/a^2, then for the l indexing this excursion we have

$$\log\left(\tfrac{1}{2} + \tfrac{1}{2}\exp\left(-Ba^2\tau_l\right)\right) \le \log\left(\tfrac{1}{2} + \tfrac{1}{2}\exp(-B\delta)\right) \tag{4.32}$$

and hence

$$A\varepsilon + \tfrac{1}{2}\log\left(\tfrac{1}{2} + \tfrac{1}{2}\exp\left(-Ba^2\tau_l\right)\right) \le 0 \quad \text{for } 0 < \varepsilon < \varepsilon_0(\delta). \tag{4.33}$$

Therefore

$$A\varepsilon + \tfrac{1}{2}\sum_{l}^{(k)}\log\left(\tfrac{1}{2} + \tfrac{1}{2}\exp\left(-Ba^2\tau_l\right)\right) \le 0 \quad \text{for } 0 < \varepsilon < \varepsilon_0(\delta), \tag{4.34}$$

where $\sum^{(k)}$ means summing over all the excursions on $(t_{k-1}, t_k]$. On the other hand, if all excursions on $(t_{k-1}, t_k]$ have length less than δ/a^2, then for all the l indexing these excursions we have

$$\log\left(\tfrac{1}{2} + \tfrac{1}{2}\exp\left(-Ba^2\tau_l\right)\right) \le -\tfrac{1}{4}Ba^2\tau_l \quad \text{for } 0 < \delta < \delta_0 \tag{4.35}$$

and so

$$\begin{aligned} A\varepsilon + \tfrac{1}{2}\sum_l^{(k)} \log\left(\tfrac{1}{2} + \tfrac{1}{2}\exp\left(-Ba^2\tau_l\right)\right) &\le A\varepsilon - \tfrac{1}{8}Ba^2\sum_l^{(k)}\tau_l \\ &\le A\varepsilon - \tfrac{1}{8}Ba^2(t_k - t_{k-1}). \end{aligned} \tag{4.36}$$

By construction, however, $t_k - t_{k-1} > [(\delta/\varepsilon) - 1](\varepsilon/a^2) = (\delta - \varepsilon)/a^2$ for $1 \le k < m_{t/a^2}$ and so the r.h.s. of (4.36) is less than or equal to 0 for $0 < \varepsilon < \varepsilon_0(\delta)$. Combining (4.34) with (4.36) and summing on k, we get (4.31). Thus, in order to prove (4.30), it now remains to show that

$$\begin{aligned} \limsup_{t\to\infty} \frac{1}{t}\log E\Bigg(\exp\Bigg[& Aa^2 \sum_{l=1}^{l_{t/a^2}} \tau_l 1\left\{\tau_l < \frac{\delta}{a^2}\right\} \\ & + \frac{1}{2}\sum_{l=1}^{l_{t/a^2}} \log\left(\frac{1}{2} + \frac{1}{2}\exp\left(-Ba^2\tau_l\right)\right)\Bigg]\Bigg) \le 0. \end{aligned} \tag{4.37}$$

(v) Observe next that

$$\begin{aligned} & Aa^2 \sum_{l=1}^{l_{t/a^2}} \tau_l 1\left\{\tau_l < \frac{\delta}{a^2}\right\} + \frac{1}{2}\sum_{l=1}^{l_{t/a^2}} \log\left(\frac{1}{2} + \frac{1}{2}\exp\left(-Ba^2\tau_l\right)\right) \\ & \quad \le Aa^2 \sum_{l=1}^{l_{t/a^2}} \tau_l' 1\left\{\tau_l' < \frac{\delta}{a^2}\right\} + \frac{1}{2}\sum_{l=1}^{l_{t/a^2}} \log\left(\frac{1}{2} + \frac{1}{2}\exp\left(-Ba^2\tau_l'\right)\right) \\ & \qquad + A\delta + \frac{1}{2}\log 2, \end{aligned} \tag{4.38}$$

where $\tau_l' = \tau_l$ $(1 \le l < l_{t/a^2})$ but $\tau'_{l_{t/a^2}} = T_{l_{t/a^2}} - T_{l_{t/a^2 - 1}}$ [compare with (4.26)]. Clearly, $A\delta + \frac{1}{2}\log 2$ is negligible after taking the $t \to \infty$ limit in (4.37). By the optional sampling theorem, it therefore suffices to prove that

$$E\left(\exp\left[Aa^2\tau_1' 1\left\{\tau_1' < \frac{\delta}{a^2}\right\} + \frac{1}{2}\log\left(\frac{1}{2} + \frac{1}{2}\exp\left(-Ba^2\tau_1'\right)\right)\right]\right) \le 1 \tag{4.39}$$

for appropriate a, δ.

(vi) For fixed $\delta > 0$, a Riemann approximation together with the asymptotic formula $P(\tau_1' = k) \sim C/k^{3/2}$ ($k \to \infty$ even) yields

$$\begin{aligned}(4.40)\quad &\lim_{a\downarrow 0}\frac{1}{a}\left\{E\left(\exp\left[Aa^2\tau_1' 1\left\{\tau_1' < \frac{\delta}{a^2}\right\}\right.\right.\right.\\ &\qquad\left.\left.\left.+\frac{1}{2}\log\left(\frac{1}{2}+\frac{1}{2}\exp\left(-Ba^2\tau_1'\right)\right)\right]\right)-1\right\}\\ &= C\int_0^\infty \frac{dx}{x^{3/2}}\left\{\exp\left[Ax1\{x<\delta\}+\frac{1}{2}\log\left(\frac{1}{2}+\frac{1}{2}\exp(-Bx)\right)\right]-1\right\}.\end{aligned}$$

Clearly, the r.h.s. of (4.40) is < 0 when $0 < \delta < \delta_0$. This proves (4.37) and completes the proof of Step 1. □

4.3. *From discrete to continuous medium*. We next replace the i.i.d. Bernoulli random variables ω_i by i.i.d. standard normal random variables $\hat{\omega}_i$. Therefore, we define our second intermediate quantity as [compare with (4.16)]

$$\begin{aligned}(4.41)\quad F^2_{t,\varepsilon,\delta}(a,h) &= \frac{1}{t}\hat{\mathbb{E}}\left(\log E\left(\exp\left[-2aH^2_{t,\varepsilon,\delta}(a,h)\right]\right)\right),\\ H^2_{t,\varepsilon,\delta}(a,h) &= \sum_{k=1}^{m_{t/a^2}} s_k\left\{Z_k(\hat{\omega})+ah|\bar{I}_k|\right\},\end{aligned}$$

where $\hat{\mathbb{E}}$ is expectation w.r.t. $\hat{\omega}$.

STEP 2. $F^1 \simeq F^2$.

PROOF. The proof comes in three parts.

(i) We couple the random variables ω_i and $\hat{\omega}_i$. Remark that these random variables enter into F^1 and F^2 only via their partial sums over intervals of length ε/a^2 [recall (4.12)–(4.15)]. We can define ω and $\hat{\omega}$ on a common probability space such that for any $j \geq 1$ [see Komlós, Major and Tusnády (1975, 1976)],

$$(4.42)\quad \mathbb{P}^*\left(\left|\sum_{i\in I_j}(\omega_i-\hat{\omega}_i)\right| \geq \left[c_1\log\frac{\varepsilon}{a^2}\right]+k\right) \leq c_2 e^{-c_3 k}, \qquad k \geq 1,$$

for some constants $c_1, c_2, c_3 > 0$. Here $\mathbb{P}^*$ denotes the coupling measure obtained by independently repeating the KMT-coupling in each ε/a^2-interval I_j. It suffices to prove $F_1 \prec F_2$. Namely, the ω_i and $\hat{\omega}_i$ enter symmetrically into (4.42), and therefore the proof of $F_2 \prec F_1$ will be exactly the same upon exchange of ω and $\hat{\omega}$.

Following our general scheme, we choose

$$\begin{aligned}(4.43)\quad H^1_{t,\varepsilon,\delta}(a,h) &= \sum_{k=1}^{m_{t/a^2}} s_k\left\{Z_k(\omega)+ah|\bar{I}_k|\right\}\\ &= H^{(I)}_{t,\varepsilon,\delta}(a,h)+H^{(II)}_{t,\varepsilon,\delta}(a,h)\end{aligned}$$

with

$$H^{(I)}_{t,\varepsilon,\delta}(a,h) = \sum_{k=1}^{m_{t/a^2}} s_k\{Z_k(\hat{\omega}) + a(1+\rho)h'|\bar{I}_k|\}$$

(4.44)
$$H^{(II)}_{t,\varepsilon,\delta}(a,h) = \sum_{k=1}^{m_{t/a^2}} s_k \sum_{i\in\bar{I}_k}(\omega_i - \hat{\omega}_i) + a(h-(1+\rho)h')\sum_{k=1}^{m_{t/a^2}} s_k|\bar{I}_k|.$$

With this choice, we have

(4.45)
$$H^{(I)}_{t,\varepsilon,\delta}(a,h) = H^2_{t(1+\rho)^2,\,\varepsilon(1+\rho)^2,\,\delta(1+\rho)^2}(a(1+\rho),h'),$$

as required by (4.8), and so we must show that (4.9) is met:

(4.46)
$$\limsup_{t\to\infty}\frac{1}{t}\log E\Big(\mathbb{E}^*\big(\exp\big[-2a(1+\rho^{-1})H^{(II)}_{t,\varepsilon,\delta}(a,h)\big]\big)\Big) \le 0.$$

(ii) To prove (4.46), we next claim that for arbitrary $A, B > 0$,

(4.47)
$$\mathbb{E}^*\left(\exp\left[Aa\sum_{k=1}^{m_{t/a^2}} s_k\left|\sum_{i\in\bar{I}_k}(\omega_i-\hat{\omega}_i)\right|\right]\right) \le \exp\left[Ba^2\sum_{k=1}^{m_{t/a^2}} s_k|\bar{I}_k|\right] \quad \text{for } 0 < a < a_0(\varepsilon).$$

To see why (4.47) is true, note that, by the independence of the coupling in disjoint ε/a^2-intervals, it suffices to prove that

(4.48)
$$\mathbb{E}^*\left(\exp\left[Aa\left|\sum_{i\in I_1}(\omega_i-\hat{\omega}_i)\right|\right]\right) \le \exp(Ba^2|I_1|).$$

But (4.42) gives

(4.49)
$$\begin{aligned}
&\mathbb{E}^*\left(\exp\left[Aa\left|\sum_{i\in I_1}(\omega_i-\hat{\omega}_i)\right|\right]\right)\\
&\quad\le \exp\left(Aa\left[c_1\log\frac{\varepsilon}{a^2}\right]\right) + \sum_{k\ge 1}\exp\left(Aa\left(\left[c_1\log\frac{\varepsilon}{a^2}\right]+k\right)\right)\\
&\qquad\times\mathbb{P}^*\left(\left|\sum_{i\in I_1}(\omega_i-\hat{\omega}_i)\right| \ge \left[c_1\log\frac{\varepsilon}{a^2}\right]+k\right)\\
&\quad\le \exp\left(Aa\left[c_1\log\frac{\varepsilon}{a^2}\right]\right)\left\{1 + c_2\sum_{k\ge 1}\exp(-k(c_3-Aa))\right\}.
\end{aligned}$$

This is clearly less than or equal to $\exp(B\varepsilon) = \exp(Ba^2|I_1|)$ when $0 < a < a_0(\varepsilon)$.

(iii) Picking $A = 2(1+\rho^{-1})$, $B \le 2(1+\rho^{-1})(h-(1+\rho)h')$ in (4.47) and recalling (4.44), we get (4.46). This completes the proof of Step 2. □

4.4. *From discrete to continuous process.* The next step consists in replacing the random walk by a Brownian motion. For the random walk we have defined in (4.13) the random times $\sigma_1, \dots, \sigma_m$ ($m = m_{t/a^2}$ for short henceforth). For convenience we put $\sigma_m = t/\varepsilon$. Write

$$a \sum_{k=1}^{m} s_k \{ Z_k(\hat{\omega}) + ah|\bar{I}_k| \} = \sum_{k=1}^{m} s_k \sum_{\sigma_{k-1} < j \le \sigma_k} \sum_{i \in I_j} \left(a\hat{\omega}_i + a^2 h \right) \tag{4.50}$$

and note that

$$\left(\sum_{\sigma_{k-1} < j \le \sigma_k} \sum_{i \in I_j} \left(a\hat{\omega}_i + a^2 h \right) \right)_{k \ge 1} =_D \left(\beta_{\bar{\sigma}_k} - \beta_{\bar{\sigma}_{k-1}} + h(\bar{\sigma}_k - \bar{\sigma}_{k-1}) \right)_{k \ge 1}, \tag{4.51}$$

where $\bar{\sigma}_k = \varepsilon \sigma_k$ are the scaled random times and $(\beta_s)_{0 \le s \le t}$ is a Brownian medium independent of the random walk.

Let Q be the distribution of

$$\Sigma = (m; s_1, \dots, s_m; \bar{\sigma}_1, \dots, \bar{\sigma}_m), \tag{4.52}$$

which of course depends on all the parameters $t, a, \varepsilon, \delta$ (Q is a probability distribution on a finite set). Then, in view of (4.41), (4.50) and (4.51), we may write (with an obvious abuse of notation):

$$\begin{aligned} F^2_{t,\varepsilon,\delta}(a, h) &= \frac{1}{t} \tilde{\mathbb{E}} \left(\log E_Q \left(\exp\left[-2aH^2_{t,\varepsilon,\delta}(a,h) \right] \right) \right), \\ H^2_{t,\varepsilon,\delta}(a, h) &= \frac{1}{a} \sum_{k=1}^{m} s_k \left\{ \beta_{\bar{\sigma}_k} - \beta_{\bar{\sigma}_{k-1}} + h(\bar{\sigma}_k - \bar{\sigma}_{k-1}) \right\}, \end{aligned} \tag{4.53}$$

where $\tilde{\mathbb{E}}$ is the expectation over β. Remark now that Σ can be interpreted as a functional on the space of continuous paths $(f(s))_{0 \le s \le t}$, defined by $f(ia^2) = aS_i$ $(0 \le i \le t/a^2)$ with linear interpolation. Replacing the law of the random walk by the law of a Brownian motion $(B_s)_{0 \le s \le t}$, we get a distribution $\tilde{Q}$ of Σ. Obviously, Q and $\tilde{Q}$ are mutually absolutely continuous. We therefore define our third intermediate quantity as

$$\begin{aligned} F^3_{t,\varepsilon,\delta}(a, h) &= \frac{1}{t} \tilde{\mathbb{E}} \left(\log E_{\tilde{Q}} \left(\exp\left[-2aH^2_{t,\varepsilon,\delta}(a,h) \right] \right) \right) \\ &= \frac{1}{t} \tilde{\mathbb{E}} \left(\log E_Q \left(\exp\left[-2aH^3_{t,\varepsilon,\delta}(a,h) \right] \right) \right), \end{aligned} \tag{4.54}$$

where

$$H^3 = H^2 - \frac{1}{2a} \log \frac{d\tilde{Q}}{dQ}. \tag{4.55}$$

STEP 3. $F^2 \simeq F^3$.

PROOF. We again use our splitting. If $(1+\rho)h' < h$, then

$$H^2_{t,\varepsilon,\delta}(a, h) = H^{(I)}_{t,\varepsilon,\delta}(a, h') + H^{(II)}_{t,\varepsilon,\delta}(a, h, h'), \tag{4.56}$$

where, as required by (4.8),

$$H^{(I)}_{t,\varepsilon,\delta}(a,h') = H^3_{t,\varepsilon,\delta}(a,(1+\rho)h')$$
$$= H^3_{t(1+\rho)^2,\,\varepsilon(1+\rho)^2,\,\delta(1+\rho)^2}(a(1+\rho),h'),$$

$$H^{(II)}_{t,\varepsilon,\delta}(a,h,h') = \frac{h-(1+\rho)h'}{a}\sum_{k=1}^{m} s_k(\bar\sigma_k - \bar\sigma_{k-1}) + \frac{1}{2a}\log\frac{d\tilde Q}{dQ}. \tag{4.57}$$

Observe that $H^{(II)}_{t,\varepsilon,\delta}(a,h,h')$ does not depend on β. According to (4.9), in order to prove $F^2 \prec F^3$ we have to show that

$$\limsup_{t\to\infty}\frac{1}{t}\log E_Q\left(\exp\left[-A\sum_{k=1}^{m} s_k(\bar\sigma_k-\bar\sigma_{k-1}) - B\log\frac{d\tilde Q}{dQ}\right]\right)\le 0 \tag{4.58}$$

with $A = 2(1+\rho^{-1})(h-(1+\rho)h')$, $B=(1+\rho^{-1})$ and for δ, ε, a appropriate. This is, however, immediate from Lemma 4 below upon putting in the lower estimate for $d\tilde Q/dQ$ and integrating out the s_k afterward. Indeed, since s_k are 0 or 1 with probability 1/2 each, the summand can be replaced by

$$\log\left(\tfrac12 + \tfrac12\exp\left(-A(\bar\sigma_k - \bar\sigma_{k-1})\right)\right) \le \log\left(\tfrac12+\tfrac12\exp(-A\delta)\right). \tag{4.59}$$

The proof of $F^3 \prec F^2$ is similar after putting in the upper estimate for $d\tilde Q/dQ$. □

LEMMA 4. *There exists* $\kappa = \kappa(a,\varepsilon,\delta) > 0$ *satisfying*

$$\lim_{\varepsilon\downarrow 0}\limsup_{a\downarrow 0}\kappa(a,\varepsilon,\delta) = 0 \quad \textit{for all } \delta>0 \tag{4.60}$$

such that

$$(1-\kappa)^m \le \frac{dQ}{d\tilde Q}(\Sigma) \le (1+\kappa)^m. \tag{4.61}$$

PROOF. The proof comes in three parts.

(i) Let k, l be positive integers such that $k+l$ is even. Define

$$q(k,l) = P(S_i \ne 0 \text{ for } k<i<k+l,\ S_{k+l}=0 \mid S_0 = 0). \tag{4.62}$$

Let further $p_k(x) = P(S_k = x | S_0 = 0)$ for $k+x$ even.

Assume that k, l are odd. (We are faced here with the usual parity problems. The case where k, l are even is handled by slight modification. We neglect such trivial points in the following discussion.) Then, via the reflection principle,

$$
\begin{aligned}
q(k,l) &= 2\sum_{x=1}^{l} p_k(x) P(S_i > 0 \text{ for } 0 < i < l,\ S_l = 0 \mid S_0 = x) \\
&= \sum_{x=1}^{l} p_k(x)\left[p_{l-1}(x-1) - p_{l-1}(x+1)\right] \qquad (4.63)\\
&= \sum_{x=1}^{l} p_k(x)\, p_l(x)\, \frac{2x}{l}.
\end{aligned}
$$

Now, $0 \le 2x/l \le 2$ $(1 \le x \le l)$, so using the Bernstein large deviation estimates for $p_k(x)$ and $p_l(x)$ we get

$$
\sum_{x=1}^{l} p_k(x)\, p_l(x)\, \frac{2x}{l} = (1+o(1)) \sum_{x=1}^{(k\wedge l)^{3/5}} p_k(x)\, p_l(x)\, \frac{2x}{l}, \tag{4.64}
$$

where $o(1)$ refers to $k, l \to \infty$ jointly. But for $k \to \infty$,

$$
p_k(x) = (1+o(1))\sqrt{\frac{2}{\pi k}}\exp\left(-\frac{x^2}{2k}\right) \quad \text{uniformly in } x \in \{1,\dots,k^{3/5}\}. \tag{4.65}
$$

Substitution into (4.63) yields a Riemann approximation (only the odd x's count),

$$
\begin{aligned}
q(k,l) &= (1+o(1))\,\frac{1}{2}\,\frac{2}{\pi}\,\frac{1}{\sqrt{kl}}\,\frac{1}{l}\int_0^\infty dx\, 2x \exp\left(-\frac{x^2}{2}\left[\frac{1}{k}+\frac{1}{l}\right]\right) \qquad (4.66)\\
&= (1+o(1))\,\frac{2}{\pi}\,\frac{\sqrt{k}}{(k+l)\sqrt{l}}.
\end{aligned}
$$

(ii) We fix now δ, ε, a (as usual with $\delta/\varepsilon, \varepsilon/a^2$ integer). For integers $j \ge 2$, $1 \le y \le \varepsilon/a^2$ we obtain from (4.66) as $a \downarrow 0$ (only half of the l's count):

$$
\begin{aligned}
&P\left(\min\left\{i > \frac{\delta}{a^2} : S_i = 0\right\} - \left(\frac{\delta}{a^2}\right) \in \left(\frac{(j-1)\varepsilon}{a^2}, \frac{j\varepsilon}{a^2}\right] \,\middle|\, S_y = 0\right)\\
&\quad = \sum_{l=(j-1)\varepsilon/a^2+1}^{j\varepsilon/a^2} q\left(\frac{\delta}{a^2} - y, l\right)\\
&\quad = (1+o(1))\,\frac{1}{2}\sum_{l=(j-1)\varepsilon/a^2+1}^{j\varepsilon/a^2} \frac{2}{\pi}\,\frac{\sqrt{(\delta/a^2) - y}}{\left[(\delta/a^2) - y + l\right]\sqrt{l}} \qquad (4.67)\\
&\quad = (1+o(1))\,\frac{2}{\pi}\left(\arctan\sqrt{\frac{j\varepsilon}{\delta - ya^2}} - \arctan\sqrt{\frac{(j-1)\varepsilon}{\delta - ya^2}}\right),
\end{aligned}
$$

where $o(1)$ refers to $a \downarrow 0$, uniformly in $0 < \varepsilon \le \delta/2$ (for a fixed $\delta > 0$), $1 \le y \le \varepsilon/a^2$ and $j \ge 2$. (The uniformity in j is of crucial importance.)

Equation (4.67) is also true for $j = 1$, although (4.66) is obviously not correct for fixed l and only $k \to \infty$. However, some rough estimate like $q(k, l) \le p_{k+l}(0) \le C/\sqrt{k+l}$ suffices to show that the small l's in (4.67) are negligible.

(iii) By weak convergence of random walk to Brownian motion, we get from (4.67) that for $0 < \varepsilon \le \delta/2$, $0 < \tilde{y} \le \varepsilon$ and $j \ge 1$:

$$\begin{aligned} &\tilde{P}\big(\inf\{u > \delta\colon B_u = 0\} - \delta \in ((j-1)\varepsilon, j\varepsilon] \mid B_{\tilde{y}} = 0\big) \\ &\qquad = \frac{2}{\pi}\left(\arctan\sqrt{\frac{j\varepsilon}{\delta - \tilde{y}}} - \arctan\sqrt{\frac{(j-1)\varepsilon}{\delta - \tilde{y}}}\right) \end{aligned} \tag{4.68}$$

(which, of course, can also be proved directly).

Now define

$$\begin{aligned} &\zeta(a, \varepsilon, \delta; y, \tilde{y}, j) \\ &\quad = \frac{P\big(\min\{i > \delta/a^2\colon S_i = 0\} - (\delta/a^2) \in ((j-1)\varepsilon/a^2, j\varepsilon/a^2] \mid S_y = 0\big)}{\tilde{P}\big(\inf\{u > \delta\colon B_u = 0\} - \delta \in ((j-1)\varepsilon, j\varepsilon] \mid B_{\tilde{y}} = 0\big)} \end{aligned} \tag{4.69}$$

and

$$\kappa(a, \varepsilon, \delta) = \sup_{1 \le y \le \varepsilon/a^2} \sup_{0 < \tilde{y} \le \varepsilon} \sup_{j \ge 1} |\zeta(a, \varepsilon, \delta; y, \tilde{y}, j) - 1|. \tag{4.70}$$

Then (4.61) follows immediately from the definition of κ, Q and $\tilde{Q}$. Combining (4.67)–(4.70), we arrive at (4.60). □

4.5. *Coarse graining of the BM*. The final step must consist in getting rid of ε, δ (we have already said goodbye to a). The quantity F^3 in (4.54) is similar to F^1 in (4.16), but all defined in terms of the Brownian motion and its zeroes in ε-intervals with gaps of size δ. The point is to remove these restrictions by letting $\varepsilon \downarrow 0$, $\delta \downarrow 0$ (in this order).

STEP 4. $F^3 \simeq \tilde{\Psi}$.

PROOF. This is quite parallel to Step 1 and we can therefore be brief. For the reader's (and our own) convenience, we stick to the proof of $F^3 \prec \Psi$ (the argument for $\psi \prec F^3$ being similar). The proof comes in six parts.

(i) Define the random function $(\phi_s)_{0 \le s \le t}$ as follows. For $1 \le k < m$, put $\phi_s = 1$ on the interval $(\bar{\sigma}_{k-1}, \bar{\sigma}_k]$ if the Brownian motion is negative just prior to its first zero in this interval, and $\phi_s = 0$ otherwise. On the last interval $(\bar{\sigma}_{m-1}, t]$ put $\phi_s = 1$ if $B_t < 0$, and $\phi_s = 0$ otherwise. Then

$$\phi_s = s_k \quad \text{for } s \in (\bar{\sigma}_{k-1}, \bar{\sigma}_k] \text{ and } 1 \le k \le m, \tag{4.71}$$

where the s_k are defined in terms of the Brownian motion.

(ii) Our quantities no longer depend on a, so we need a slight modification of our general scheme. Put

$$\begin{aligned}\overline{H}^3_{t,\varepsilon,\delta}(h) &= aH^3_{t,\varepsilon,\delta}(a,h)\\ &= \sum_{k=1}^m s_k\Big\{\big(\beta_{\overline{\sigma}_k}-\beta_{\overline{\sigma}_{k-1}}\big)+h\big(\overline{\sigma}_k-\overline{\sigma}_{k-1}\big)\Big\}\\ &= \sum_{k=1}^m \int_{\overline{\sigma}_{k-1}}^{\overline{\sigma}_k}\phi_s(d\beta_s+h\,ds),\\ \tilde{H}_t(h) &= \int_0^t \xi_s(d\beta_s+h\,ds)\\ &= \sum_{k=1}^m \int_{\overline{\sigma}_{k-1}}^{\overline{\sigma}_k}\xi_s(d\beta_s+h\,ds).\end{aligned}\tag{4.72}$$

Then

$$\begin{aligned}F^3_{t,\varepsilon,\delta}(h) &= \frac{1}{t}\tilde{\mathbb{E}}\Big(\log\,\tilde{E}\Big(\exp\big[-2\,\overline{H}^3_{t,\varepsilon,\delta}(h)\big]\Big)\Big),\\ \tilde{\Psi}_t(h) &= \frac{1}{t}\tilde{\mathbb{E}}\Big(\log\,\tilde{E}\Big(\exp\big[-2\,\tilde{H}_t(h)\big]\Big)\Big).\end{aligned}\tag{4.73}$$

Remark next that, by Brownian rescaling,

$$\begin{aligned}\overline{H}^3_{t,\varepsilon,\delta}((1+\rho)h) &=_D \frac{1}{1+\rho}\overline{H}^3_{t(1+\rho)^2,\,\varepsilon(1+\rho)^2,\,\delta(1+\rho)^2}(h),\\ \tilde{H}_t((1+\rho)h) &=_D \frac{1}{1+\rho}\tilde{H}_{t(1+\rho)^2}(h).\end{aligned}\tag{4.74}$$

Furthermore,

$$\begin{aligned}\overline{H}^3_{t,\varepsilon,\delta}(h_1)-\tilde{H}_t(h_2) = (h_1-h_2)&\sum_{k=1}^m\int_{\overline{\sigma}_{k-1}}^{\overline{\sigma}_k}\phi_s\,ds\\ &+\sum_{k=1}^m\int_{\overline{\sigma}_{k-1}}^{\overline{\sigma}_k}(\phi_s-\xi_s)(d\beta_s+h_2\,ds),\end{aligned}\tag{4.75}$$

which is completely analogous to (4.19).

(iii) It should now be clear that the argument runs parallel to Step 1, so we have to show that [compare with (4.24)]

$$\limsup_{t\to\infty}\frac{1}{t}\log\,\tilde{E}\left(\exp\left[A\int_0^t|\xi_s-\phi_s|\,ds - B\sum_{k=1}^m s_k(\overline{\sigma}_k-\overline{\sigma}_{k-1})\right]\right)\le 0\tag{4.76}$$

for ε,δ appropriate. The Brownian motion has at most a finite number of excursions of length greater than or equal to δ in the interval $(0,t]$. We denote by $J_{t,\delta}$ the complement of these excursion intervals in $(0,t]$. By the definition of ϕ_s, we have

$$\int_0^t|\xi_s-\phi_s|\,ds\le|J_{t,\delta}|+m\varepsilon.\tag{4.77}$$

[See the derivation of the corresponding estimate for the random walk in (4.29).] Substituting (4.77) into (4.76) and afterward integrating out the s_k (0 or 1 with probability 1/2 each), we see that it suffices to prove

$$\limsup_{t\to\infty}\frac{1}{t}\log\tilde{E}\Big(\exp\Big[A|J_{t,\delta}|+A\varepsilon m+\sum_{k=1}^{m}\log\Big(\frac{1}{2}+\frac{1}{2}\exp(-B(\bar{\sigma}_k-\bar{\sigma}_{k-1}))\Big)\Big]\Big)\le 0. \tag{4.78}$$

As $\bar{\sigma}_k-\bar{\sigma}_{k-1}\ge\delta$ $(1\le k<m)$, we trivially have [compare with (4.31)]

$$A\varepsilon(m-1)+\tfrac{1}{2}\sum_{k=1}^{m}\log\big(\tfrac{1}{2}+\tfrac{1}{2}\exp(-B(\bar{\sigma}_k-\bar{\sigma}_{k-1}))\big)\le 0 \qquad \text{for } 0<\varepsilon<\varepsilon_0(\delta). \tag{4.79}$$

Therefore it suffices to prove

$$\limsup_{t\to\infty}\frac{1}{t}\log\tilde{E}\Big(\exp\Big[A|J_{t,\delta}|+\frac{1}{2}\sum_{k=1}^{m}\log\Big(\frac{1}{2}+\frac{1}{2}\exp(-B(\bar{\sigma}_k-\bar{\sigma}_{k-1}))\Big)\Big]\Big)\le 0 \tag{4.80}$$

for appropriate ε, δ.

(iv) The $\bar{\sigma}_k$ are in fact stopping times for the Brownian motion. They are related to another sequence of stopping times: $\rho_0=\bar{\sigma}_0=0$ and

$$\rho_k=\inf\{t\ge\bar{\sigma}_{k-1}+\delta\colon B_t=0\},\qquad \bar{\sigma}_k=j\varepsilon\quad\text{if } \rho_k\in((j-1)\varepsilon,j\varepsilon],\qquad k\ge 1, \tag{4.81}$$

until the smallest m such that $\rho_m\ge t$. By construction, it is clear that $|(\bar{\sigma}_{k-1},\bar{\sigma}_k]\cap J_{t,\delta}|\le 2\delta$ for all k.

Next, remark that in (4.80) we may replace the last $\bar{\sigma}_m$ (which is just t) by $j\varepsilon$ if $\rho_m\in((j-1)\varepsilon,j\varepsilon]$, provided we add $\frac{1}{2}\log 2$ in the exponent (which is irrelevant in the $t\to\infty$ limit). Therefore, we prove (4.80) in this form.

(v) Clearly, $\bar{\sigma}_{k-1}$ is $F_{\rho_{k-1}}$-measurable, where $(F_s)_{s\ge 0}$ is the natural filtration of the Brownian motion. Furthermore, because $\rho_{k-1}\le\bar{\sigma}_{k-1}<\rho_{k-1}+\varepsilon$ we have

$$\bar{\sigma}_k-\bar{\sigma}_{k-1}\ge\inf\{t>\rho_{k-1}+\delta\colon B_t=0\}-\rho_{k-1}-\varepsilon. \tag{4.82}$$

Therefore, given $F_{\rho_{k-1}}$, the conditional distribution of $\bar{\sigma}_k-\bar{\sigma}_{k-1}$ dominates the conditional distribution of the r.h.s. of (4.82), which is independent of $F_{\rho_{k-1}}$ and just the distribution of $\rho_1-\varepsilon$. By the optimal sampling theorem it therefore suffices to prove [compare with (4.39)]

$$E\big(\exp\big[A\rho_1 1\{\rho_1<\delta\}+\tfrac{1}{2}\log\big(\tfrac{1}{2}+\tfrac{1}{2}\exp(-B(\rho_1-\varepsilon))\big)\big]\big)\le 1 \tag{4.83}$$

for $0<\delta<\delta_0$ and $0<\varepsilon<\varepsilon_0(\delta)$.

(vi) As ρ_1 does not depend on ε, we can first let $\varepsilon\downarrow 0$, and it therefore suffices to prove

$$E\Big(\sqrt{\tfrac{1}{2}+\tfrac{1}{2}\exp(-B\rho_1)}\Big)<\exp(-A\delta)\qquad\text{for } 0<\delta<\delta_0. \tag{4.84}$$

But ρ_1 has an explicit density [compare with (4.66)]:

$$P(\rho_1 \in \delta + ds) = \frac{1}{\pi}\frac{\sqrt{\delta}}{(\delta + s)\sqrt{s}}\, ds, \qquad s > 0. \tag{4.85}$$

Therefore

$$E\left(\sqrt{\frac{1}{2} + \frac{1}{2}\exp(-B\rho_1)}\right) = \frac{2}{\pi}\int_0^\infty \frac{dv}{1+v^2}\sqrt{\frac{1}{2} + \frac{1}{2}\exp(-B\,\delta\, v^2)}\,. \tag{4.86}$$

Since

$$\lim_{\delta\downarrow 0}\frac{1}{\delta}\left\{1 - \sqrt{\frac{1}{2} + \frac{1}{2}\exp(-B\,\delta\, v^2)}\right\} = \frac{1}{4}Bv^2 \tag{4.87}$$

and $\int_0^\infty v^2/(1+v^2)\, dv = \infty$, it follows from Fatou that

$$\lim_{\delta\downarrow 0}\frac{1}{\delta}\left\{1 - E\left(\sqrt{\frac{1}{2} + \frac{1}{2}\exp(-B\rho_1)}\right)\right\} = \infty. \tag{4.88}$$

This implies (4.84). □

Steps 1–4 combine to give (4.11), proving Theorem 6.

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