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SPECTRAL GAP FOR ZERO-RANGE DYNAMICS

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We give a lower bound on the spectral gap for symmetric zero-range processes. Under some conditions on the rate function, we show that the gap shrinks as n^{-2}, independent of the density, for the dynamics localized on a cube of size n^d. We follow the method outlined by Lu and Yau, where a similar spectral gap is proved for Kawasaki dynamics.

1. Introduction and results. One of the basic estimates required for the hydrodynamical limit of nongradient systems is a sharp lower bound on the spectral gap of the finite coordinate process [13]. What is needed is that the gap, for the process confined to cubes of linear size n, shrinks at a rate n^{-2}. Up to constants, this is heuristically the best possible lower bound, in view of a comparison with random walk which prevents larger bounds.

Another application of such a bound is to establish an invariance principle for conservative particle dynamics [11]. In that article, the bound is used to estimate the rate of convergence of the canonical to the grand canonical ensemble. Among the models discussed there are the zero-range processes. Our intention in this paper is to prove the required lower bound for this type of dynamics.

The symmetric zero-range processes consist of infinitely many particles moving on the lattice Z^d according to a Markovian law. The evolution of the particles may be informally described as follows. Denote by N the set of nonnegative integers, fix a nonnegative function $c\colon N \to R_+$ such that $c(0) = 0 < c(i)$ for $i \geq 1$ and fix a symmetric transition measure $p(\cdot)$ on Z^d. If there are k particles at a site x of Z^d, one of them jumps to y at rate $c(k)p(y-x)$. This happens independently at each site. To fix ideas, we shall consider in this paper only nearest-neighbor interactions: $p(x) = 1/2$ if $|x| = 1$ and 0 otherwise.

At this point some notation is required. The sites of Z^d are denoted by x, y and z, the state space N^{Z^d} by the symbol Σ and the configurations by the Greek letters η and ξ. In this way η_x stands for the total number of particles at site x for the configuration η.

These so-called zero-range processes are Markov processes with infinitesimal generator L defined by its action on cylinder functions ϕ:

$$(L\phi)(\eta) = \tfrac{1}{2} \sum_{|y-x|=1} c(\eta_x)(\phi(\eta^{x,y}) - \phi(\eta)),$$

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where

$$(\eta^{x,y})_z = \begin{cases} \eta_x - 1, & \text{if } z = x, \\ \eta_y + 1, & \text{if } z = y, \\ \eta_z, & \text{if } z \neq x, y \end{cases}$$

provided $\eta_x \geq 1$ and $x \neq y$; otherwise, $\eta^{x,y} \equiv \eta$.

To ensure that the process is well defined on the infinite lattice N^{Z^d} we shall assume throughout this article a Lipschitz condition on the rate:

(LG) $\sup_k |c(k+1) - c(k)| \leq a_1 < \infty$.

We refer to [1] for the details on the existence of this process.

As a conservative system where particles are neither created nor destroyed, it is expected that the process possesses a family of invariant measures supported on configurations of fixed density. In order to describe these measures, define the partition function $Z(\cdot)$ on R_+ by

$$Z(\alpha) = \sum_{k \geq 0} \frac{\alpha^k}{c(1) \cdots c(k)}.$$

It is clear that $Z(\cdot)$ is an increasing function. Let α^* denote the radius of convergence of Z:

$$\alpha^* = \sup\{\alpha;\ Z(\alpha) < \infty\}.$$

In order to avoid degeneracy we assume that the partition function Z diverges at the boundary of its domain of definition:

$$\lim_{\alpha \to \alpha^*} Z(\alpha) = \infty. \tag{1.1}$$

For $0 \leq \alpha < \alpha^*$, let $\bar{P}_\alpha$ be the translation invariant product measure on N^{Z^d} with marginals μ_α given by

$$\mu_\alpha\{\eta_x = k\} = \frac{1}{Z(\alpha)} \frac{\alpha^k}{c(1) \cdots c(k)} \quad \text{for } k \geq 0,\ x \in Z^d. \tag{1.2}$$

The family $\{\bar{P}_\alpha,\ 0 \leq \alpha < \alpha^*\}$ is a one parameter set of reversible measures for $\{\eta_t,\ t \geq 0\}$.

A more intuitive parameterization is through the particle density. Let $\rho(\alpha)$ be the density of particles for the measure $\bar{P}_\alpha$,

$$\rho(\alpha) = \bar{E}_\alpha[\eta_0],$$

where $\bar{E}_\alpha$ refers to expectation with respect to $\bar{P}_\alpha$.

From assumption (1.1) it follows that ρ: $[0, \alpha^*) \to R_+$ is a smooth strictly increasing bijection. Since $\rho(\alpha)$ has a physical meaning as the density of particles, instead of parameterizing the above family of measures by α, we parameterize it in terms of the density ρ and write for $\rho \geq 0$,

$$P_\rho = \bar{P}_{\alpha(\rho)}.$$

Under this convention it follows then that for $\rho \geq 0$,

$$\alpha(\rho) = E_\rho[c(\eta_0)],$$

where E_ρ refers to expectation with respect to P_ρ. Moreover, a simple computation shows that

$$\alpha'(\rho) = \frac{\alpha(\rho)}{\sigma(\alpha(\rho))^2}, \tag{1.3}$$

where $\sigma(\alpha(\rho))^2$ stands for the variance of η_0 under $\bar{P}_\alpha$: $\sigma(\alpha(\rho))^2 = \bar{E}_\alpha[(\eta_0 - \rho(\alpha))^2]$.

The associated Dirichlet form $D_\rho(\phi) = -E_\rho[\phi(\eta)(L\phi)(\eta)]$ is defined by its action on the test function ϕ:

$$D_\rho(\phi) = \tfrac{1}{4} \sum_{|x-y|=1} E_\rho\big[c(\eta_x)(\phi(\eta^{x,y}) - \phi(\eta))^2\big].$$

We now describe more precisely what we mean by the term "spectral gap." Consider the finite volume, finite particle zero-range process. This model governs the behavior of K particles jumping about in a finite cube, say to fix ideas, $\Lambda_n = \{1, 2, \dots, n\}^d$. The state space is then given as $\Sigma_{n,K} = \{\eta \in N^{\Lambda_n}: \sum_{x \in \Lambda_n} \eta_x = K\}$. For configurations $\eta \in \Sigma_{n,K}$ and test functions ϕ, the generator of this finite process takes the form

$$(L_n\phi)(\eta) = \tfrac{1}{2} \sum_{\substack{|x-y|=1 \\ x,\, y \in \Lambda_n}} c(\eta_x)[\phi(\eta^{x,y}) - \phi(\eta)].$$

The process defined by the generator L_n on the state space $\Sigma_{n,K}$ is an ergodic, reversible finite state Markov chain possessing discrete real-valued spectrum. The operator L_n in $L^2(P_{n,K})$ is negative definite and its largest eigenvalue is 0. The absolute value of the next largest eigenvalue is the so-called spectral gap of the process.

It is easily calculated that the ergodic measures $P_{n,K}$ are equal to the conditioned measure of the infinite volume invariant state on the restricted hyperplane $\Sigma_{n,K}$:

$$P_{n,K}(\cdot) = P_\rho\bigg(\cdot \,\bigg|\, \sum_{x \in \Lambda_n} \eta_x = K\bigg).$$

Note that these definitions are independent of ρ.

The measures $P_{n,K}$, as alluded to earlier, are reversible and a simple calculation yields

$$c(r+1)P_{n,K}\{\eta;\ \eta_{e_1} = r+1,\ \eta_{2e_1} = l\} = c(l+1)P_{n,K}\{\eta;\ \eta_{e_1} = r, \eta_{2e_1} = l+1\}$$

or, what will be used later,

$$E_{n,K}[c(\eta_x)|\eta_y = r] = c(r+1)\frac{P_{n,K}\{\eta_{e_1} = r+1\}}{P_{n,K}\{\eta_{e_1} = r\}} \tag{1.4}$$

for every $x \neq y$ in Λ_n. Here $E_{n,K}$ denotes expectation with respect to the measure $P_{n,K}$ and $\{e_i,\ 1 \leq i \leq d\}$ stands for the canonical basis of R^d. The density of the measure $P_{n,K}$ is denoted by $\rho = K/n^d$.

The Dirichlet form for the finite process is defined as

$$\begin{aligned} D_{n,K}(\phi) &= -E_{n,K}[\phi(L_n\phi)] \\ &= \frac{1}{4}\sum_{\substack{|x-y|=1\\ x,y\in\Lambda_n}} E_{n,K}\big[c(\eta_x)(\phi(\eta^{x,y}) - \phi(\eta))^2\big]. \end{aligned}$$

Consider now the quantity $W(n, K)$ appearing in the Poincaré inequality

$$E_{n,K}\big[(f - E_{n,K}[f])^2\big] \leq W(n,K)D_{n,K}(f) \tag{1.5}$$

for every $f\colon \Sigma_{n,K} \to (-\infty,\infty)$ in $L^2(P_{n,K})$, where $W(n,K)$ is a constant depending only on n and K.

This inequality gives another way to evaluate the spectral gap of the finite volume process. In fact, as the inequality deals with mean-zero functions, the smallest possible value of $W(n,K)$ is the reciprocal of the gap.

We are interested in establishing the bound $W(n,K) < W_0 n^2$ for a constant W_0 independent of n and the total number of particles K. This would then imply that the spectral gap is of $O(n^{-2})$.

Similar uniform $O(n^2)$ spectral gap estimates have been found for the simple exclusion process by Quastel [10], and for more general exclusion processes by Lu and Yau [8]. The aim of this article is to determine for a class of zero-range processes the spectral gap bound $W(n,K) < W_0 n^2$, where W_0 is a uniform constant. Such a demand rules out many zero-range processes, as shown by the following example.

EXAMPLE 1.1. Let $c(r) = I(r \geq 1)$. By a clever transform in dimension $d = 1$ [4], the corresponding zero-range process is mapped onto the simple exclusion process where the gap is well known [10]. Inverting back, we determine that the zero-range process possesses a spectral gap of order $(n+K)^{-2}$ which is clearly dependent on the density ρ.

Another illuminating example is the zero-range process where particles evolve independently.

EXAMPLE 1.2. Consider $c(r) = r$, the identity function. With this rate, the corresponding zero-range dynamics are nothing more than a collection of mutually independent symmetric random walks on $\{1,\dots,n\}^d$. For this process, the n^{-2} spectral gap estimate, independent of ρ, is well known.

To establish a spectral gap bound with constant W_0 independent of n and K we will impose a second assumption that rules out Example 1.1.

(M) There exists $k_0 \in N$ and $a_2 > 0$ such that $c(k) - c(j) \geq a_2$ for all $k \geq j + k_0$.

Notice that, under assumptions (LG) and (M), α^* is actually infinite. We are now in a position to state the main theorem of this article.

THEOREM 1.1. *Given the conditions* (LG) *and* (M), *there exists a constant* W_0 *independent of* n *and* K *such that* (1.5) *holds with* $W(n, K) = W_0 n^2$ *for the corresponding nearest-neighbor zero-range processes. This implies a spectral gap of at least* $\{W_0 n^2\}^{-1}$ *on a cube of volume* n^d.

We conclude this section with a few comments on the method of proof. The approach we present here to prove a spectral gap independent of the total number of particles is easily adapted to generalized exclusion processes introduced in [6]. For attractive systems, a simpler proof can be given based on coupling and positivity (cf. [3]). On the other hand, for the rate described in Example 1.1 and for similar rates, we do not have yet methods to determine a sharp spectral gap estimate. The difficulty arises in that the spectral gap depends on the total number of particles and our approach is not adapted to this situation.

To prove Theorem 1.1, we adapt Lu and Yau's method [8] to the context of zero-range processes where, by a clever induction argument, the Poincaré inequality on f, ostensibly a function of n^d coordinates, is reduced to an estimate involving a function of only one coordinate. At this point we prove a spectral gap for functions that depend only on one site. This one site spectral gap reduces the problem to the estimation of two terms. The first will yield the desired Dirichlet form, while the second will give a smaller variance to be absorbed into the variance on the left-hand side of (1.5).

Section 2 develops Lu and Yau's basic ideas. Section 3 details the estimates of the first and second terms and specifies the iteration. In Section 4 we prove a spectral gap for functions depending only on one site. As a consequence Theorem 1.1 is established for the initial case with two sites only. In Section 5 we prove some estimates on the one site marginal of the grand canonical zero-range distribution important in the induction step of the proof and necessary to obtain local central limit theorems uniform in the density. This last question is the main theme of Section 6.

2. Proof of the main estimate.

For simplicity we will prove Theorem 1.1 for dimension $d = 1$; the extension to higher dimensions is not difficult (cf. [5]). We will employ an iteration aided by an induction. We will assume that we have a spectral gap independent of the total number of particles for intervals of length $m \leq n - 1$: $W(m) = \sup_K W(m, K)$ is finite. Our efforts now will be to use this hypothesis to set up a recursive equation for $W(n)$. For n large, say $n > n_0$, we will give such a relation for $W(n)$. However, for $n \le n_0$ slightly different arguments are used to establish a recursion for $W(n)$. The results of these two relations are then combined, giving that $W(n) \le W_0 n^2$ is valid for all n. The initial induction case $n = 2$ is a consequence of the one site spectral gap and is further discussed in Section 4.

STEP 1. Let us write the identity

$$f - E_{n,K}[f] = (f - E_{n,K}[f|\eta_1]) + (E_{n,K}[f|\eta_1] - E_{n,K}[f]).$$

Through this decomposition we may express the variance given in (1.5) as

$$\begin{aligned} &E_{n,K}[(f - E_{n,K}[f])^2] \\ &\quad = E_{n,K}\big[(f - E_{n,K}[f|\eta_1])^2\big] + E_{n,K}\big[(E_{n,K}[f|\eta_1] - E_{n,K}[f])^2\big]. \end{aligned} \tag{2.1}$$

The first term on the right-hand side is easily analyzed through the induction assumption and a simple computation on the Dirichlet form. We write

$$\begin{aligned} &E_{n,K}[(f - E_{n,K}[f|\eta_1])^2] \\ &\quad = E_{n,K}[E_{n,K}[(f - E_{n,K}[f|\eta_1])^2 \mid \eta_1]] \\ &\quad = E_{n,K}[E_{n-1,K-\eta_1}[(f(\eta_1, \cdot) - E_{n-1,K-\eta_1}[f(\eta_1, \cdot)])^2]]. \end{aligned}$$

By the induction assumption this last expectation is bounded above by

$$\sum_{r=0}^{K} P_{n,K}\{\eta_1 = r\} W(n-1) D_{n-1,K-r}(f(r, \cdot)) \le W(n-1) D_{n,K}(f).$$

STEP 2. The second term in (2.1) is nothing more than the variance of $E_{n,K}[f|\eta_1]$, a function of one variable. For each fixed K and n, let $P^1_{n,K}$ be the one site marginal of the canonical measure $P_{n,K}$:

$$P^1_{n,K}(r) = P_{n,K}(\eta_1 = r) = P_\rho\left(\eta_1 = r \,\middle|\, \sum_{x=1}^{n} \eta_x = K\right).$$

Expectation with respect to $P^1_{n,K}$ is written $E^1_{n,K}$.

On $\{0, \dots, K\}$ consider the birth and death process which jumps from r to $r \pm 1$ with rates $p(r, r \pm 1)$ given by

$$p(r, r-1) = c(r) \quad \text{and} \quad p(r, r+1) = E^1_{n-1,K-r}[c(\eta_0)].$$

Denote the generator of this process by $\mathscr{L}_{n,K}$. It is elementary to check that $P^1_{n,K}$ is reversible for $\mathscr{L}_{n,K}$. Moreover, if $H : \Sigma_{n,K} \to R$ is a function of only one site [say $H(\eta) = H(\eta_1)$], then

$$D_{n,K}(H) = -(1/2) E^1_{n,K}[H \mathscr{L}_{n,K} H] =: D^1_{n,K}(H).$$

In order to estimate a spectral gap for zero-range processes, our method requires that the associated birth and death processes with generator $\mathscr{L}_{n,K}$ exhibit a spectral gap with magnitude independent of n and K. We shall prove in Section 4 the following one site spectral gap lemma.

LEMMA 2.1. *There exists a constant $B_0 = B_0(a_1, a_2, k_0)$ such that*

$$E^1_{n,K}\big[(H - E^1_{n,K}[H])^2\big] \le B_0 D^1_{n,K}(H)$$

for all $n \ge 1$, $K \ge 1$ and H in $L^2(P^1_{n,K})$.

This lemma applied to the function $E_{n,K}[f|\eta_1]$ shows that the second term of (2.1) is bounded above by $B_0 D^1_{n,K}(E_{n,K}[f|\eta_1])$. A few calculations simplify the one-coordinate Dirichlet form:

LEMMA 2.2. *For every $H = H(\eta_1)$ in $L^2(P_{n,K})$ we have*

$$D^1_{n,K}(H) = (1/2)\sum_{r=0}^{K-1} P_{n,K}\{\eta_1 = r+1\} c(r+1)(H(r+1) - H(r))^2.$$

PROOF. Recall $P^1_{n,K}(r) = P_{n,K}\{\eta_1 = r\}$. A simple computation shows that the Dirichlet form $4D^1_{n,K}(H(\eta_1))$ is equal to

$$E_{n,K}\big[c(\eta_1)(H(\eta_1-1) - H(\eta_1))^2\big] + E_{n,K}\big[E_{n,K}[c(\eta_2)|\eta_1](H(\eta_1+1) - H(\eta_1))^2\big].$$

In addition, taking advantage of the reversibility relation (1.4), we have that

$$E_{n,K}[c(\eta_2)|\eta_1 = r] = \frac{c(r+1)P^1_{n,K}(r+1)}{P^1_{n,K}(r)}. \tag{2.2}$$

Now, substituting this last identity into the previous formula, we obtain that

$$D^1_{n,K}(H(\eta_1)) = (1/2)\sum_{r=0}^{K-1} P^1_{n,K}(r+1)c(r+1)(H(r+1) - H(r))^2.$$

The lemma is thus proved. □

STEP 3. Recall that our intention is to apply Lemmas 2.1 and 2.2 to the one variable function $H(\eta_1) = E_{n,K}[f|\eta_1]$. In this respect, we derive a simpler expression for the difference

$$E_{n,K}[f|\eta_1 = r+1] - E_{n,K}[f|\eta_1 = r],$$

taking advantage of the reversibility of $P_{n,K}$.

LEMMA 2.3. *Let $M(\eta)$ be the function defined by*

$$M(\eta) = \frac{P^1_{n,K}(\eta_1)}{P^1_{n,K}(\eta_1+1)c(\eta_1+1)}\frac{1}{n-1}\sum_{x=2}^{n} c(\eta_x).$$

Then, for every $0 \le r \le K-1$, *the difference*

$$E_{n,K}[f|\eta_1 = r+1] - E_{n,K}[f|\eta_1 = r]$$

is equal to

$$\begin{aligned}(2.3)\qquad &\frac{1}{P^1_{n,K}(r+1)c(r+1)}\frac{1}{n-1}\sum_{x=2}^{n} E_{n,K}[c(\eta_x)(f(\eta^{x,1}) - f(\eta))I(\eta_1 = r)] \\ &\quad + E_{n,K}[M(\eta);\ f(\eta)|\eta_1 = r],\end{aligned}$$

where $E_{n,K}[g;\ h|\eta_1 = r]$ *is the covariance of* g *and* h *with respect to* $E_{n,K}[\cdot|\eta_1 = r]$.

PROOF. Notice first that due to the reversibility criterion (2.2),

$$E_{n,K}[M(\eta)|\eta_1 = r] = 1 \tag{2.4}$$

for every $0 \le r \le K-1$. On the other hand, exploiting again the reversibility (1.4), we obtain that

$$E_{n,K}[f|\eta_1 = r+1] = \frac{1}{P^1_{n,K}(r+1)c(r+1)} E_{n,K}[f(\eta^{x,1})c(\eta_x)I(\eta_1 = r)]$$

for every $2 \le x \le n$. Adding and subtracting an appropriate term and then averaging over x, this expression may be rewritten as

$$\begin{aligned}&\frac{1}{P^1_{n,K}(r+1)c(r+1)}\frac{1}{n-1}\sum_{x=2}^{n} E_{n,K}[(f(\eta^{x,1}) - f(\eta))c(\eta_x)I(\eta_1 = r)] \\ &\quad + E_{n,K}[f(\eta)M(\eta)|\eta_1 = r].\end{aligned}$$

The observation (2.4) concludes the proof of the lemma. □

STEP 4. From Lemmas 2.1 and 2.2, the second term on the right-hand side of (2.1) is bounded above by

$$\tfrac{1}{2}B_0 \sum_{r=0}^{K-1} P^1_{n,K}(r+1)c(r+1)\{E_{n,K}[f|\eta_1 = r+1] - E_{n,K}[f|\eta_1 = r]\}^2.$$

From Lemma 2.3 and the Schwarz inequality, this last sum is bounded above by

$$\begin{aligned}&B_0 \sum_{r=0}^{K-1} \frac{1}{P^1_{n,K}(r+1)c(r+1)} \\ &\qquad \times \left\{E_{n,K}\left[\frac{1}{n-1}\sum_{x=2}^{n} c(\eta_x)(f(\eta^{x,1}) - f(\eta))I(\eta_1 = r)\right]\right\}^2 \\ &\qquad + B_0 \sum_{r=0}^{K-1} P^1_{n,K}(r+1)c(r+1)(E_{n,K}[M;\ f|\eta_1 = r])^2.\end{aligned}$$

We denote the first line by $B_0A_1(n, K, f)$ and the second by $B_0A_2(n, K, f)$.

In Lemma 3.1 of Section 3 we establish that

$$A_1(n, K, f) \le (n/2)D_{n, K}(f).$$

In Lemma 3.2 we prove under assumptions (LG) and (M) that

$$A_2(n, K, f) \le a_1^2 B_0 W(n-1) D_{n, K}(f).$$

This inequality shall be used to perform the iteration for small values of n. On the other hand, using extensively the uniform Edgeworth expansion presented in Section 6, in Proposition 3.1, under assumptions (LG) and (M), we prove that for all $\varepsilon > 0$, there exist finite $n_0(\varepsilon)$ and $C(\varepsilon)$ such that

$$A_2(n, K, f) \le C(\varepsilon) D_{n, K}(f) + \varepsilon n^{-1} E_{n, K}[(f - E_{n, K}(f))^2]$$

for $n \ge n_0(\varepsilon)$.

In conclusion, from Step 1 and these estimates, the variance of f satisfies the following relation:

$$E_{n, K}[(f - E_{n, K}(f))^2] \le \{[1 + a_1^2 B_0^2] W(n-1) + (nB_0/2)\} D_{n, K}(f)$$

for $n \ge 2$ and

$$\begin{aligned} E_{n, K}[(f - E_{n, K}(f))^2] \le {} & \left\{ W(n-1) + \left(\frac{nB_0}{2}\right) + B_0 C(\varepsilon) \right\} D_{n, K}(f) \\ & + \frac{\varepsilon B_0}{n} E_{n, K}[(f - E_{n, K}(f))^2] \end{aligned}$$

for $n \ge n_0(\varepsilon)$. In other words, we have the following recurrence relations for the sequence $W(n)$:

$$\begin{aligned} & W(n) \le [1 + a_1^2 B_0^2] W(n-1) + \frac{nB_0}{2} \quad \text{for } n \ge 2, \\ & W(n) \le \left(1 - \frac{\varepsilon B_0}{n}\right)^{-1} \left\{ W(n-1) + \left(\frac{nB_0}{2}\right) + B_0 C(\varepsilon) \right\} \quad \text{for } n \ge n_0(\varepsilon). \end{aligned}$$

This yields immediately that $W(n) \le W_0 n^2$ for some universal constant W_0. This concludes the proof of Theorem 1.1. □

3. Technical bounds.

The aim of this section is to estimate the two terms A_1 and A_2 and thereby establish the iteration stated in Section 2. The first term will be bounded by the full Dirichlet form multiplied by a factor of n. This is accomplished in Lemma 3.1. The second term will be estimated by the full variance multiplied by εn^{-1}, where ε is an arbitrarily small number, and the Dirichlet form. This takes place through several lemmas below. With this bound, this "small" variance may be absorbed into the original variance. We first estimate $A_1(n, K, f)$.

LEMMA 3.1. *For every $n \ge 2$ and positive integer K,*

$$A_1(n, K, f) \le (n/2)D_{n, K}(f).$$

PROOF. Recall the definition of $A_1(n, K, f)$. It is equal to

$$\sum_{r=0}^{K-1} \frac{1}{P^1_{n,K}(r+1)c(r+1)} \left\{ E_{n,K}\left[\frac{1}{n-1} \sum_{x=2}^{n} c(\eta_x)(f(\eta^{x,1}) - f(\eta)) I(\eta_1 = r) \right] \right\}^2.$$

By the Schwarz inequality this sum is bounded above by

$$\begin{aligned} &\sum_{r=0}^{K-1} E_{n,K}\left[\frac{1}{n-1} \sum_{x=2}^{n} c(\eta_x)(f(\eta^{x,1}) - f(\eta))^2 I(\eta_1 = r) \right] \\ &\quad = E_{n,K}\left[\frac{1}{n-1} \sum_{x=2}^{n} c(\eta_x)(f(\eta^{x,1}) - f(\eta))^2 \right] \end{aligned}$$

since by reversibility we have that

$$E_{n,K}[c(\eta_x) I(\eta_1 = r)] = P^1_{n,K}(r+1)c(r+1)$$

for every $2 \le x \le n$, $0 \le r \le K-1$. Straightforward computations show that the last sum is bounded above by $n/2$ times the Dirichlet form of f. This completes the proof of the lemma. □

We now turn to the proof of an upper bound for $A_2(n, K, f)$. The naive way to approach this estimate is immediately to apply the Schwarz inequality to $A_2(n, K, f)$ and then try to bound the object [recall the definition of $M = M(\eta)$]

$$\frac{P^1_{n,K}(r+1)c(r+1)}{P^1_{n,K}(r)} E_{n,K}[M;\ M | \eta_1 = r]$$

uniformly on r. The Edgeworth expansion gives a bound on this quantity of type $C(a_1)n^{-1}$ and consequently $A_2(n, K, f)$ is bounded above by $C(a_1)n^{-1}E_{n,K}[f; f]$. This is not enough to ensure that the iteration in the previous section produces the estimate $W(n) < Cn^2$. In order for the iteration to succeed, an inequality such as $A_2(n, K, f) < C_2 n^{-1} E_{n,K}[f; f]$, where C_2 is a small universal constant, is required. The naive approach is not successful as there is no a priori smallness condition on the constant $C(a_1)$ arising from the Edgeworth expansion. A more subtle analysis is therefore needed. This is facilitated by the next two results. To carry this out we rewrite $A_2(n, K, f)$ more appropriately.

Notice that by reversibility (2.2), $M(\eta)$ may be rewritten as

$$\frac{1}{E_{n-1, K-\eta_1}[c(\eta_1)]} \frac{1}{n-1} \sum_{x=2}^{n} c(\eta_x)$$

and $A_2(n, K, f)$ as

$$E_{n,K}\left[\frac{1}{E_{n-1, K-\eta_1}[c(\eta_1)]} \left(E_{n-1, K-\eta_1}\left[f;\ \frac{1}{n-1} \sum_{x=2}^{n} c(\eta_x) \right] \right)^2 \right]. \tag{3.1}$$

We shall first estimate $A_2(n, K, f)$ for n small using asumptions (LG) and (M).

LEMMA 3.2. *Under conditions* (LG) *and* (M) *we have*, *for* $n \geq 2$,

$$A_2(n, K, f) \leq a_1^2 B_0 W(n-1) D_{n, K}(f).$$

PROOF. By the Schwarz inequality

$$\begin{aligned}&\biggl(E_{n-1, K-\eta_1}\biggl[f;\, \frac{1}{n-1}\sum_{x=2}^{n} c(\eta_x)\biggr]\biggr)^2\\&\qquad\leq E_{n-1, K-\eta_1}[f;\, f] E_{n-1, K-\eta_1}\biggl[\frac{1}{n-1}\sum_{x=2}^{n} c(\eta_x);\, \frac{1}{n-1}\sum_{x=2}^{n} c(\eta_x)\biggr].\end{aligned}$$

A simple computation exploiting Lemmas 2.1 and 2.2 shows that for every $m \geq 1$ and $L \geq 0$,

$$\begin{aligned}E_{m, L}\biggl[\frac{1}{m}\sum_{x=1}^{m} c(\eta_x);\, \frac{1}{m}\sum_{x=1}^{m} c(\eta_x)\biggr] &\leq E_{m, L}[(c(\eta_1) - E_{m, L}[c])^2]\\&\leq \frac{1}{2} B_0 E_{m, L}[c(\eta_1)(c(\eta_1 - 1) - c(\eta_1))^2].\end{aligned}$$

Now by assumption (LG), $|c(\eta_1) - c(\eta_1 - 1)| \leq a_1$. Therefore $A_2(n, K, f)$ is bounded by

$$\tfrac{1}{2} a_1^2 B_0 E_{n, K}[E_{n-1, K-\eta_1}[f;\, f]].$$

By the induction assumption, $E_{n-1, K-\eta_1}[f;\, f]$ is bounded by $W(n-1) \times D_{n-1, K-\eta_1}(f)$. Since $E_{n, K}[D_{n-1, K-\eta_1}(f)] \leq D_{n, K}(f)$, the lemma is proved. □

We now turn to the case where n is large enough that Edgeworth expansions can be used to estimate expectations involving the canonical measures. Recall the alternative formulation (3.1) for $A_2(n, K, f)$. To keep notation simple, we shall fix $m = n - 1$ and $L = K - \eta_1$ and bound the expression inside the expectation.

PROPOSITION 3.1. *For every* $\varepsilon > 0$, *there exists* n_0 *in* N *and a finite constant* $C(\varepsilon)$ *so that*

$$\frac{1}{E_{m, L}[c(\eta_1)]}\biggl(E_{m, L}\biggl[f;\, \frac{1}{m}\sum_{x=1}^{m} c(\eta_x)\biggr]\biggr)^2 \leq C(\varepsilon) D_{m, L}(f) + \frac{\varepsilon}{m} E_{m, L}[f;\, f]$$

for all L *and* $m \geq n_0$.

The proof of this result is divided into several lemmas for purposes of clarity. We first single out the case of small density.

LEMMA 3.3. *For each $\varepsilon > 0$, there exists a positive integer n_0 and a density ρ_0 such that*

$$\frac{1}{E_{m,L}[c(\eta_1)]} E_{m,L}\left[\frac{1}{m}\sum_{x=1}^{m} c(\eta_x);\ \frac{1}{m}\sum_{x=1}^{m} c(\eta_x)\right] \le \frac{\varepsilon}{m} \tag{3.2}$$

for all $m \ge n_0$ and $L/m \le \rho_0$.

PROOF. First we compute the left-hand side in (3.2) using the reversibility relation (1.4) and obtain that it is equal to

$$E_{m,L-1}[c(\eta_1)] - E_{m,L}[c(\eta_1)] + \frac{1}{m} E_{m,L-1}[c(\eta_1 + 1) - c(\eta_1)].$$

The lemma follows from Corollary 6.3. □

It remains to consider the case when the density is away from 0. Let n_0 be as specified in Corollary 6.4.

LEMMA 3.4. *For each $\rho_0 > 0$, there exists a finite constant $C(\rho_0)$ and $n_1(\rho_0) \in N$ such that the scaled square of the covariance*

$$\frac{1}{E_{m,L}[c(\eta_1)]}\left(E_{m,L}\left[f;\ \frac{1}{m}\sum_{x=1}^{m} c(\eta_x)\right]\right)^2$$
$$\le C(\rho_0)\frac{1}{m}\left\{\frac{1}{l}E_{m,L}[f;f] + (l+1)W(l+1)D_{m,L}(f)\right\}$$

for each $L/m \ge \rho_0$, $n_0 \le l < \sqrt{m}$ and $m \ge n_1$.

PROOF. Our strategy is to decompose the scaled covariance into two terms, the first of which will produce the Dirichlet form multiplied by a controlled factor; the second will yield the original variance with a small coefficient. In order to take these factors and coefficients as universal numbers, required by our iteration scheme in the previous section, we will need to use our assumptions (LG) and (M) on the rate $c(\cdot)$.

In what follows, we abbreviate L/m by ρ. For a fixed positive integer l, we divide the interval $\{1, \dots, m\}$ into $[m/l]$ adjacent intervals of length l or $l+1$. We denote by B_ι the ιth interval, by $|B_\iota|$ the total number of sites in B_ι and by y_ι the density of particles in B_ι: $y_\iota = |B_\iota|^{-1}\sum_{x\in B_\iota}\eta_x$. To keep notation simple, we denote by $E_{|B_\iota|, y_\iota}$ the expectation with respect to the canonical measure on $|B_\iota|$ sites and $y_\iota|B_\iota|$ particles.

By the Schwarz inequality, we have

$$\left(E_{m,L}\left[f;\ \frac{1}{m}\sum_{x=1}^{m} c(\eta_x)\right]\right)^2 \le 2\left(E_{m,L}\left[f;\ \frac{1}{m}\sum_{\iota=1}^{[m/l]}\sum_{x\in B_\iota}\{c(\eta_x) - E_{|B_\iota|, y_\iota}[c]\}\right]\right)^2$$
$$+ 2\left(E_{m,L}\left[f;\ \frac{1}{m}\sum_{\iota}|B_\iota|\, E_{|B_\iota|, y_\iota}[c]\right]\right)^2.$$

Let I_1 be the first term on the right-hand side and let I_2 be the second.

We may estimate the first term through the elementary inequality $2ab \le \varepsilon^{-1}a^2 + \varepsilon b^2$ and the induction assumption as follows. Taking conditional expectation with respect to y_ι, rewrite $\sqrt{(1/2)I_1}$ as

$$\begin{aligned}
&\left|\frac{1}{m}\sum_\iota |B_\iota| E_{m,L}\left[E_{|B_\iota|, y_\iota}\left[f; \frac{1}{|B_\iota|}\sum_{x\in B_\iota} c(\eta_x)\right]\right]\right| \\
&\quad \le \frac{\varepsilon}{2m}\sum_\iota |B_\iota| W(|B_\iota|) E_{m,L}[D_{|B_\iota|, y_\iota}(f)] \qquad (3.3)\\
&\qquad + \frac{\varepsilon^{-1}}{2m}\sum_\iota |B_\iota| E_{m,L}\left[\left(\frac{1}{|B_\iota|}\sum_{x\in B_\iota} c(\eta_x) - \alpha(\rho)\right)^2\right].
\end{aligned}$$

Here, the inequality follows from the induction assumption.

On the one hand, a straightforward computation shows that

$$\sum_\iota E_{m,L}[D_{|B_\iota|, y_\iota}(f)]$$

is bounded above by $D_{m,L}(f)$. In particular the first term on the right-hand side is bounded by $\varepsilon(l+1)W(l+1)(2m)^{-1}D_{m,L}(f)$. On the other hand, by the Schwarz inequality, the second term is bounded above by

$$(2\varepsilon)^{-1}E_{m,L}[(c(\eta_1) - \alpha(\rho))^2].$$

Since the density L/m is assumed to be bounded below by $\rho_0 > 0$, Corollary 6.1(c) is invoked and the previous variance is bounded by

$$E_2(2\varepsilon)^{-1}E_\rho[(c(\eta_1) - \alpha(\rho))^2] \le E_2(2\varepsilon)^{-1}a_1\alpha(\rho)$$

for some universal constant E_2 because by (5.1) the variance of $c(\eta_1)$ is dominated by $a_1\alpha(\rho)$. Therefore the second term on the right-hand side of (3.3) is dominated by $(2\varepsilon)^{-1}E_2 a_1\alpha(\rho)$. Minimizing in ε, we have that I_1 is bounded above by $C(a_1, \rho_0)(l+1)W(l+1)m^{-1}\alpha(\rho)D_{m,L}(f)$.

Finally, by Corollary 6.4 and (5.2), $E_{m,L}[c(\eta_1)]$ is bounded below by $C(\rho_0)\alpha(\rho)$ for some positive constant $C(\rho_0)$ provided $\rho \ge \rho_0$. This shows that

$$I_1 \le C(a_1, \rho_0)(l+1)W(l+1)m^{-1}E_{m,L}[c(\eta_1)]D_{m,L}(f).$$

We turn now to I_2. We may rewrite this expression as

$$2\left(E_{m,L}\left[f; \frac{1}{m}\sum_\iota |B_\iota|\left\{E_{|B_\iota|, y_\iota}[c] - \alpha(\rho) - \alpha'(\rho)\left[y_\iota - \frac{L}{m}\right]\right\}\right]\right)^2.$$

For $1 \le \iota \le [m/l]$, let $F(y_\iota) = E_{|B_\iota|, y_\iota}[c] - \alpha(\rho) - \alpha'(\rho)[y_\iota - L/m]$. By the Schwarz inequality this last expression is bounded by

$$2E_{m,L}[f; f]E_{m,L}\left[\left(\frac{1}{m}\sum_\iota |B_\iota| F(y_\iota)\right)^2\right]. \qquad (3.4)$$

To conclude the proof of the lemma it remains to show that the the second expectation is bounded by $\varepsilon m^{-1}E_{m,L}[c(\eta_1)]$ for sufficiently large l.

Assume, to keep notation simple, that $|B_\iota| = l$ for all ι. In this case the expectation is equal to

$$\frac{l}{m} E_{m,L}[F(y_1)^2] + \left(1 - \frac{l}{m}\right) E_{m,L}[F(y_1)F(y_2)].$$

Since, by assumption, the density $L/m \geq \rho_0$, we have that by Corollary 6.1(b), $E_{m,L}[F(y_1)^2]$ and $E_{m,L}[F(y_1)F(y_2)]$ are, respectively, dominated by

$$E_{L/m}[F(y_1)^2] + \frac{E(\rho_0)l}{m}\{E_{L/m}[F(y_1)^4]\}^{1/2} \tag{3.5}$$

and

$$(E_{L/m}[F(y_1)])^2 + \frac{E(\rho_0)l}{m} E_{L/m}[F(y_1)^2]. \tag{3.6}$$

Notice that $E_\rho[F(y_1)] = 0$. In particular the second variance in (3.4) is bounded by

$$\frac{E(\rho_0)l}{m} E_\rho[F^2] + \frac{E(\rho_0)l^2}{m^2}\{E_\rho[F^4]\}^{1/2}. \tag{3.7}$$

We shall consider these two terms separately.

Recall from (1.3) that $\alpha'(\rho) = \alpha(\rho)/\sigma(\alpha(\rho))^2$. We prove in Section 5 [cf. (5.2)] that $\alpha(\rho)/\sigma(\alpha(\rho))^2$ and therefore $\alpha'(\rho)$ is uniformly bounded on R_+. Moreover, we show in Corollary 6.4 that $|E_{l,y_1}[c] - \alpha(y_1)|$ is bounded by $C_1 l^{-1}\sqrt{1+\alpha(y_1)}$ for some universal constant C_1. Therefore we may bound F by $C(\|\alpha'\|_\infty, \rho_0)|l^{-1}\sum_{1\leq x\leq l}(\eta_x - \rho)| + C_1 l^{-1}\sqrt{1+\alpha(\rho)}$ and obtain that

$$E_\rho[F^4] \leq C(\|\alpha'\|_\infty, \rho_0, a_1)\frac{1}{l^2}\sigma(\alpha(\rho))^4$$

because, by Lemma 5.2, $\sigma(\alpha)^{-4}\bar{E}_\alpha[(\eta_0 - \rho(\alpha))^4]$ is bounded for α away from the origin and, by (5.2), $\sigma(\alpha)^2/\alpha$ is bounded below by a strictly positive constant and above by a finite constant. In particular the second term in (3.7) is bounded by $C(\rho_0)lm^{-2}\sigma(\alpha(\rho))^2$.

We turn now to the first term of (3.7). By the Schwarz inequality the expectation is bounded above by

$$2E_\rho[(E_{l,y_1}[c] - \alpha(y_1))^2] + 2E_\rho[(\alpha(y_1) - \alpha(\rho) - \alpha'(\rho)[y_1 - \rho])^2]. \tag{3.8}$$

Again by Corollary 6.4, we may bound the first term in the last expression by

$$\frac{C_1}{l^2}\{1 + \alpha(\rho) + E_\rho[\alpha(y_1) - \alpha(\rho)]\}.$$

Much in the same way we estimated the second term in (3.7) we can prove that this last expression is less than or equal to

$$\frac{C(\rho_0)}{l^2}\{1 + \alpha(\rho) + \sigma(\alpha(\rho))l^{-1/2}\} \leq \frac{C(\rho_0)}{l^2}\{1 + \sigma(\alpha(\rho))^2\}.$$

The second expectation in (3.8) is slightly more complicated.

Denote $l^{-1}\sum_{1\le x\le l}(\eta_x-\rho)/\sigma(\alpha(\rho))$ as Z_σ and $\alpha(y_1)-\alpha(\rho)-\alpha'(\rho)[y_1-\rho]$ as F_1. We shall decompose the second expectation in (3.8) according to the value of Z_σ. Fix some small parameter β. On the set where $|Z_\sigma|>\beta$, since $|F_1|$ is bounded by $2\|\alpha'\|_\infty\sigma|Z_\sigma|$, the expectation is less than or equal to

$$C_2\sigma(\alpha)^2E_\rho[Z_\sigma^2 1_{|Z_\sigma|>\beta}]\le C_2\frac{\sigma(\alpha)^2}{\beta^2}E_\rho[Z_\sigma^4]\le C(\rho_0)\frac{\sigma(\alpha)^2}{l^2\beta^2}$$

because $\sigma(\alpha)^{-4}\bar E_\alpha[(\eta_0-\rho(\alpha))^4]$ is bounded for α away from the origin by Lemma 5.2.

On the set $|Z_\sigma|\le\beta$, by Taylor's expansion, $F(y)$ is bounded by

$$(1/2)\sigma(\alpha(\rho))^2Z_\sigma^2\sup_{|y-\rho|\le\beta\sigma}\big|\alpha''(y)\big|.$$

We will prove below in Lemma 3.5 that for all $\rho_1>0$, $\sup_{y\ge\rho_1}\sigma(\alpha(y))|\alpha''(y)|\le C(\rho_1)$ and that

$$\sup_{\rho\ge\rho_1}\sup_{|y-\rho|\le\beta\sigma(\alpha(\rho))}\sigma(\alpha(\rho))\sigma(\alpha(y))^{-1}\le 2$$

for $\beta=\beta(\rho_1)$ small enough. In particular, on the set $|Z_\sigma|\le\beta$ the expectation of F^2 is bounded above by

$$C(\rho_0)\sigma(\alpha(\rho))^2E_\rho[Z_\sigma^4]\le C(\rho_0)\sigma(\alpha(\rho))^2l^{-2}$$

because, by Lemma 5.2, $\sigma(\alpha)^{-4}\bar E_\alpha[(\eta_0-\rho(\alpha))^4]$ is bounded for α away from the origin.

In conclusion, we have shown that the second expectation in (3.8) is bounded above by $C(\rho_0)\sigma(\alpha(\rho))^2l^{-2}$. Therefore the first expectation in (3.7) is bounded above by

$$\frac{C(\rho_0)}{ml}\{1+\sigma(\alpha(\rho))^2\}.$$

In particular, (3.7) is bounded by $C(\rho_0)(ml)^{-1}\{1+\sigma(\alpha(\rho))^2\}$ for $m\ge l^2$.

Since by Corollary 6.1(b) and (5.1), $E_{m,L}[c(\eta_1)]$ is bounded below by $C(\rho_0)\alpha(\rho)$ and since by (5.2), $\sigma(\alpha(\rho))^2/\alpha(\rho)\le C$, we have now demonstrated that I_2 is dominated by $C(\rho_0)(ml)^{-1}E_{m,L}[c(\eta_1)]\langle f;f\rangle_{m,L}$. This concludes the proof of the lemma. □

LEMMA 3.5. (a) *For all* $\rho_1>0$, *there exists* $C(\rho_1)<\infty$ *such that*

$$\sup_{\rho\ge\rho_1}\sigma(\alpha(\rho))|\,\alpha''(\rho)\,|\le C(\rho_1).$$

(b) *For all* $\rho_1>0$, *there exists a positive constant* $\beta=\beta(\rho_1)$, *so that*

$$\sup_{\rho\ge\rho_1}\sup_{|y-\rho|\le\beta\sigma(\alpha(\rho))}\frac{\sigma(\alpha(\rho))}{\sigma(\alpha(y))}\le 2.$$

PROOF. A simple computation shows that

$$\alpha''(\rho) = \frac{\{\sigma(\alpha(\rho))^2 - m_3(\alpha(\rho))\}\alpha(\rho)}{\sigma(\alpha(\rho))^6}$$

provide $m_3(\alpha(\rho))$ stands for $\bar{E}_\alpha[(\eta_0 - \rho(\alpha))^3]$. The first part of the lemma follows therefore from inequality (5.2) and Lemma 5.2.

To prove (b), notice that $\partial_\rho \sigma(\alpha(\rho)) = m_3(\alpha(\rho))/2\sigma(\alpha(\rho))^3$. By Lemma 5.2 this last quotient is bounded by $C_1(\rho_1)$ for $\rho \ge \rho_1$. In particular, on the set $|y-\rho| \le \beta\sigma(\alpha(\rho))$, $\sigma(\alpha(y))$ is at least $(1 - C_1(\rho_1)\beta)\sigma(\alpha(\rho))$. It remains to set $\beta = (2C_1(\rho_1))^{-1}$. □

4. One site spectral gaps. In this section we will examine a one-coordinate process and show that assumptions (LG) and (M) imply the one site spectral gap for this model. The dynamics of the model are defined implicitly by the Dirichlet form $D_{n,K}(H)$ acting solely on functions of the type $H = H(\eta_1)$. After a few calculations we can recover the generator:

$$\begin{aligned} D_{n,K}(H(\eta_1)) &= -E_{n,K}[H(\eta_1)(L_n H)(\eta_1)] \\ &= -\tfrac{1}{2}E_{n,K}[c(\eta_1)(H(\eta_1 - 1) - H(\eta_1))H(\eta_1)] \\ &\quad - \tfrac{1}{2}E_{n,K}[c(\eta_2)(H(\eta_1 + 1) - H(\eta_1))H(\eta_1)]. \end{aligned}$$

The second term is then computed as

$$\begin{aligned} &E_{n,K}[c(\eta_2)(H(\eta_1+1) - H(\eta_1))H(\eta_1)] \\ &\quad = E_{n,K}[E_{n,K}[c(\eta_2)|\eta_1](H(\eta_1+1) - H(\eta_1))H(\eta_1)] \\ &\quad = E_{n,K}[E_{n-1,K-\eta_1}[c(\eta_2)](H(\eta_1+1) - H(\eta_1))H(\eta_1)]. \end{aligned}$$

We conclude from these calculations that the generator specifies a biased nearest-neighbor random walk, a birth–death process X_t, on the state space $\{0, 1, \dots, K\}$ with rates

$$\begin{aligned} &(1/2)E_{n-1,K-r}[c(\cdot)] && \text{for jumps } r \to r+1, \\ &(1/2)c(r) && \text{for jumps } r \to r-1, \end{aligned}$$

and associated Dirichlet form $D^1_{n,K}$ given by

$$D^1_{n,K}(f) = (1/2)E^1_{n,K}[c(X)[f(X-1) - f(X)]^2].$$

In the same way, if instead of the canonical measures $P_{n,K}$ we consider the grand canonical measures P_ρ, we obtain a birth and death process X_t on N with rates

$$\begin{aligned} &(1/2)\alpha(\rho) && \text{for jumps } r \to r+1, \\ &(1/2)c(r) && \text{for jumps } r \to r-1, \end{aligned}$$

and associated Dirichlet form D^1_ρ given by

$$D^1_\rho(f) = (1/2)E^1_\rho[c(X)[f(X-1) - f(X)]^2].$$

Here E^1_ρ indicates expectation with respect to the one site marginal of the product measure P_ρ.

The purpose of this section is to prove a spectral gap independent of n, K and ρ for birth and death processes with the stated rates.

LEMMA 4.1. *Under assumptions* (LG) *and* (M), *there exists a universal constant $B_0 = B_0(a_1, a_2, k_0)$ such that*

$$E^1_{n,K}[(H - E^1_{n,K}[H])^2] \le B_0 D^1_{n,K}(H)$$

for all $n \ge 1$, $K \ge 1$ and H in $L^2(P^1_{n,K})$.

LEMMA 4.2. *Under assumptions* (LG) *and* (M), *there exists a universal constant $B'_0 = B'_0(a_1, a_2, k_0)$ such that*

$$E_\rho[(f(\eta_1) - E_\rho[f])^2] \le B'_0 D^1_\rho(f)$$

for all functions $f\colon N \to R$ in $L_2(P_\rho)$.

Our proof relies on the following two general propositions. The proof of the first one can be found in Lemma 3.11 and Lemma 3.12 of [12]. Consider a continuous time Markov process Y_t on some countable set $\mathscr{E}$. For a site r in $\mathscr{E}$, denote by τ^Y_r the hitting time of r for the process Y.

PROPOSITION 4.1. *Let there exist r_0 in $\mathscr{E}$ and $\lambda > 0$ such that $E_r[\exp(\lambda\tau^Y_{r_0})] < \infty$ for all r. Then the spectral gap is larger than λ.*

PROPOSITION 4.2. *Denote by L the generator of Y_t. Fix some r_0 in $\mathscr{E}$ and suppose $u, v\colon \mathscr{E} \to R_+$ are functions satisfying $Lv + uv = 0$. If $u(r) \ge \lambda$ for $r \ne r_0$ and $v(r) \ge \gamma$ for all r in $\mathscr{E}$, then $E_x[\exp(\lambda\tau^Y_{r_0})] \le v(x)/\gamma$.*

PROOF OF PROPOSITION 4.2. Define the martingale

$$m(t) = v(Y_t)\exp\left(\int_0^t u(Y_s)\,ds\right),$$

with respect to the usual σ-fields. By Doob's stopping theorem, $E_x[m(\tau^Y_{r_0} \wedge t)] \le v(x)$ for all $t \ge 0$. The result follows from the inequality $E_x[m(\tau^Y_{r_0} \wedge t)] \ge \gamma E_x[\exp\lambda(\tau^Y_{r_0} \wedge t)]$ and Fatou's lemma. □

We shall prove now a spectral gap for birth and death processes on N with death rate equal to $c(\cdot)$ and some birth rate $b(\cdot)$ that we shall assume to be Lipschitz.

From hypotheses (LG) and (M) we have that

$$c(k) - c(j) \ge a_2\left[\frac{k-j}{k_0}\right] - a_1 k_0 \ge \frac{a_2}{k_0}(k-j) - a_2 - a_1 k_0$$

for every $k \ge j$. Here $[r]$ denotes the integer part of r. Denote a_2/k_0 by J_1 and $a_2 + a_1 k_0$ by e_2.

LEMMA 4.3. *Let Y_t be a birth and death process on N with death rate $c(\cdot)$ and birth rate b: $N \to R_+$. Assume that $\sup_k |b(k+1) - b(k)| \le J_2 < J_1$. The spectral gap λ is bounded below by a strictly positive constant $C = C(J_2, a_1, a_2, k_0, c^*)$, where $c^* = \min_k c(k)$.*

Notice that Lemma 4.2 follows from this proposition since the birth rate is constant.

PROOF OF LEMMA 4.3. Denote by $D(r)$ the inward drift at r:

$$D(r) = c(r) - b(r).$$

Notice that $D(0) = -b(0) \le 0$. Let r_0 be the first integer with nonnegative inward drift: $r_0 = \min\{r \in N;\ D(r) \ge 0\}$. For $r \ge r_0$, we have that

$$D(r) = c(r) - c(r_0) + D(r_0) + b(r_0) - b(r) \ge (J_1 - J_2)(r - r_0) - e_2$$

because $D(r_0) \ge 0$ by definition of r_0. For $r \ge r_2 = r_0 + 2e_2/(J_1 - J_2)$, $D(r) \ge I_1(r - r_0)$, provided $I_1 = (J_1 - J_2)/2$. Moreover, for $r_0 \le r \le r_2$, $b(r) \le c(r_0) + I_2$ and $c(r) \ge c(r_0) - I_3$, where $I_2 = J_2(r_2 - r_0)$ and $I_3 = J_1(r_2 - r_0)$. Of course $c(r_0) - I_3$ could be negative. In this case we have the alternative lower bound $c(r) \ge c^*$.

In possession of the above estimates, it is not difficult to find functions u and v satisfying assumptions of Proposition 4.2 for some $\gamma > 0$ and some $\lambda = \lambda(J_2, a_1, a_2, k_0, c^*)$. We leave the details to the reader and just indicate that v can be chosen to be quadratic on $\{r_0, \dots, r_2\}$ and linear on $\{r_2, \dots, \infty\}$.

A similar argument applies to the set $\{0, \dots, r_0\}$. Proposition 4.1 concludes the argument. □

PROOF OF LEMMA 4.1. Set $J_2 = J_1/2$. By Proposition 4.3, there exists n_0 in N so that $|E_{n,K+1}[c(\eta_0)] - E_{n,K}[c(\eta_0)]|$ is bounded by J_2 for $n \ge n_0$. In particular, in this case Lemma 4.1 follows from Lemma 4.3. For $n \le n_0$ and $K \ge K_0$, the proof of Lemma 4.1 is similar to the proof of Lemma 4.3 and relies on the following inequality proved in Corollary 4.2:

$$E_{n,K}[c(\eta_0)] \le E_{n,K+B}[c(\eta_0)] + B_4(c)$$

for all $B \ge B_1 n$. We leave the details to the reader. Finally, since there are only a finite number of cases for $n \le n_0$ and $K \le K_0$ and since for each fixed n and K the birth and death process is a finite state ergodic Markov process, the lemma is proved. □

We conclude this section with some estimates on the birth rate $E_{n,K}[c(\eta_0)]$ that were used above.

PROPOSITION 4.3. *There exists a constant $B_3 = B_3(a_1, a_2, k_0)$ such that*

$$\left|E_{n,K+1}[c(\eta_0)] - E_{n,K}[c(\eta_0)]\right| \le \frac{B_3}{n}$$

for all $n \ge 1$ and $K \ge 0$.

We first show that the difference is bounded in absolute value by a constant independent of n and K. The proof of this statement relies on a coupling argument.

LEMMA 4.4. *There exists a constant* $B_1 = B_1(a_1, a_2, k_0)$ *such that*

$$P_{n,K} \le P_{n,K+B} \quad \textit{for all } B \ge B_1 n \textit{ and } K \textit{ in } N.$$

PROOF. To clarify the proof, we shall assume that $c(k) - c(j) \ge 0$ if $k - j \ge 2$. It is not difficult to adapt this proof to assumption (M).

Fix n, K and some positive integer B. Consider the zero range process on $\{1, \dots, n\}$ where particles jump to any site with rate $c(\cdot)$. Its generator $\tilde{L}_n$ acts on functions as

$$(\tilde{L}_n f)(\eta) = \sum_{1 \le x,\, y \le n} c(\eta_x)[f(\eta^{x,y}) - f(\eta)].$$

Notice that the canonical measures $P_{n,K}$ are ergodic and reversible for this Markov process.

Fix two configurations η and ξ, respectively, on $\Sigma_{n,K}$ and $\Sigma_{n,K+B}$ such that $\eta \le \xi$. The proof consists of finding a coupled process $(\eta(t), \xi(t))$ with three properties. We require that $\eta(0) = \eta$, $\xi(0) = \xi$, both marginals to evolve as zero-range processes with generator $\tilde{L}_n$, and that $\eta(t) \le \xi(t)$ for all $t \ge 0$. This can be easily done in the case where the jump rate c is a nondecreasing function (cf. [1]). We extend here the coupling for processes satisfying assumptions (LG) and (M).

It is enough to show that for any two configurations (η, ξ) in $\Sigma_{n,K} \times \Sigma_{n,K+B}$ such that $\eta \le \xi$, we may couple the η-particles' jumps with the ξ-particles' jumps for the order to be maintained after any possible jump.

Fix therefore two such configurations. Denote by b_0 (b_1) the total number of sites where the number of η and ξ particles differ by 0 (1). Since $\eta \le \xi$,

$$b_0 = \bigl|\{x;\ \eta(x) = \xi(x)\}\bigr| \quad \text{and} \quad b_1 = \bigl|\{x;\ \eta(x) = \xi(x) - 1\}\bigr|.$$

Since particles jump indifferently to any site and since $c(k) - c(j) \ge 0$ as long as $k \ge j + 2$, by assumption (LG), the rate at which an uncoupled η particle appears is bounded above by $b_0 b_1 a_1$.

To conclude the proof it remains to show that for $B \ge B_1 n$, the rate at which one uncoupled ξ particle in the remaining $b = n - b_0 - b_1$ sites jumps to one of the b_0 sites is bounded below by $b_0 b_1 a_1$. Denote by $u_1, \dots, u_b$ the number of uncoupled ξ particles in the remaining b sites. Notice that $\sum_{1 \le i \le b} u_i = B - b_1$. By assumption (M), the rate at which one uncoupled ξ particle jumps to one of the b_0 sites with equal numbers of η and ξ particles is bounded below by

$$\begin{aligned} b_0 \sum_{i=1}^{b} \left\{ a_2 \left[\frac{u_i}{k_0} \right] - a_1 k_0 \right\} &\ge b_0 \sum_{i=1}^{b} \left\{ a_2 \left\{ \frac{u_i}{k_0} - 1 \right\} - a_1 k_0 \right\} \\ &= \frac{a_2 b_0}{k_0}(B - b_1) - a_2 b b_0 - a_1 k_0 b b_0 \\ &\ge \frac{a_2 b_0}{k_0} B - b_0 n \left\{ a_2 + \frac{a_2}{k_0} + a_1 k_0 \right\}. \end{aligned}$$

We just have to take $B_1 = a_1 a_2^{-1} k_0(k_0+1) + k_0 + 1$. □

COROLLARY 4.1. *There exists a constant $B_2 = B_2(k_0, a_1, a_2)$ such that*

$$\left| E_{n,\,K+1}[c(\eta_0)] - E_{n,\,K}[c(\eta_0)] \right| \le B_2$$

for all $n \ge 1$ and $K \ge 0$.

PROOF. Let B_1 be the constant given by the previous lemma. Set $K_0 = B_1 n + 1$. The absolute value in the statement of the lemma is bounded by

$$\left| E_{n,\,K}[c(\eta_0)] - E_{n,\,K+K_0}[c(\eta_0)] \right| + \left| E_{n,\,K+1}[c(\eta_0)] - E_{n,\,K+K_0}[c(\eta_0)] \right|.$$

We concentrate on the first expression. The second one is estimated in the same way.

Since $P_{n,\,K} \le P_{n,\,K+K_0}$, there exists (cf. [7], Theorem II.2.4) a coupled measure $P_{n,\,K,\,K_0}$ on the product space $N^n \times N^n$ with first marginal equal to $P_{n,\,K}$, second marginal equal to $P_{n,\,K+K_0}$ and concentrated on the configurations (η, ξ) above the diagonal: $P_{n,\,K,\,K_0}\{(\eta,\xi),\ \eta \le \xi\} = 1$. With this notation, we may bound the first expression in the last formula by

$$E_{n,\,K,\,K_0}[|c(\eta_0) - c(\xi_0)|] \le a_1 E_{n,\,K,\,K_0}[|\eta_0 - \xi_0|].$$

Since the probability measure $P_{n,\,K,\,K_0}$ is ordered, the last expression is equal to

$$a_1\{E_{n,\,K+K_0}[\xi_0] - E_{n,\,K}[\eta_0]\} = a_1 K_0 n^{-1}.$$

The corollary is thus proved because $K_0 = B_1 n + 1$. □

COROLLARY 4.2. *There exists a finite constant B_4 such that*

$$E_{n,\,K}[c(\eta_0)] \le E_{n,\,K+B}[c(\eta_0)] + B_4$$

for all $B \ge B_1 n$.

PROOF. By assumption (LG) and (M), the jump rate $c(\cdot)$ may be decomposed as $c = c_1 + c_2$, where c_1 is a nondecreasing function and c_2 is a bounded function. Take for instance $c_1(k) = c([k/k_0]k_0)$. Since by Lemma 4.4, $P_{n,\,K} \le P_{n,\,K+B}$ for $B \ge B_1 n$, we have that

$$E_{n,\,K}[c(\eta_1)] \le E_{n,\,K}[c_1(\eta_1)] + \|c_2\|_\infty \le E_{n,\,K+B}[c(\eta_1)] + 2\|c_2\|_\infty. \qquad \square$$

We are now ready to prove Proposition 4.3.

PROOF OF PROPOSITION 4.3. The proof consists of three different parts. We first consider the case of a large number of sites and bounded density, then the case of a large number of sites and densities bounded away from 0 and finally the case of a small number of sites.

Fix some density $\rho_1 > 0$. By Corollary 6.4 there exists $n_0 > 0$ such that

$$|E_{n,\,K+1}[c(\eta_0)] - E_{n,\,K}[c(\eta_0)]| \le n^{-1}\Big\{\sup_{0\le\rho\le\rho_1} |\alpha'(\rho)| + 2E_3\sqrt{1+\alpha(\rho_1)}\Big\}$$

for all $n \ge n_0$ and $\rho = K/n \le \rho_1$ because $\alpha(\cdot)$ is an increasing function. Since by inequality (5.1), $\alpha'(\rho)$ [that is equal to $\alpha(\rho)/\sigma(\alpha(\rho))^2$] is bounded above by a_1, this estimate proves the proposition for $n \ge n_0$ and $\rho \le \rho_1$.

We turn now to the case $\rho \ge \rho_1$. From identity (1.4), we have that

$$E_{n,\,K}[c(\eta_0)] = \alpha(\rho)\frac{P_\rho[\sum_{x=1}^n \eta_x = K-1]}{P_\rho[\sum_{x=1}^n \eta_x = K]}.$$

We may therefore rewrite the difference $E_{n,\,K+1}[c(\eta_0)] - E_{n,\,K}[c(\eta_0)]$ as

$$\alpha(\rho)\frac{\big((\nabla P_\rho)[\sum_{x=1}^n \eta_x = K-1]\big)^2 - P_\rho[\sum_{x=1}^n \eta_x = K-1](\Delta P_\rho)[\sum_{x=1}^n \eta_x = K]}{P_\rho[\sum_{x=1}^n \eta_x = K]P_\rho[\sum_{x=1}^n \eta_x = K+1]}.$$

In this formula ∇ and Δ stand, respectively, for the discrete derivative and Laplacian and are applied to K, the total number of particles (cf. notation introduced just before Theorem 6.2). It follows from Theorem 6.2, taking $k_1 = 3$, that there exists n_0 and a constant $E = E(n_0, \rho_1)$ such that this ratio is bounded by $En^{-1}\sigma(\alpha(\rho))^{-2}$ for $n \ge n_0$ and $\rho = K/n \ge \rho_1$. This proves that the difference $E_{n,\,K+1}[c(\eta_0)] - E_{n,\,K}[c(\eta_0)]$ is bounded above by $C(\rho_1)n^{-1}$ because $\alpha(\rho)\sigma(\alpha(\rho))^{-2} \le a_1$ by inequality (5.2).

It remains to consider the case where the total number of sites is bounded, but this follows from the estimate stated in Corollary 4.1 provided we take B_3 large enough. □

5. Properties of the zero-range marginal. We prove in this section some properties of zero-range distributions used in the previous three sections. For each $k \in N$, denote by $\gamma_k(\alpha)$ the kth cumulant of the probability $\bar{P}_\alpha$; the kth cumulant is a polynomial in the normalized moments $m_i(\alpha)$ for $0 \le i \le k$, where

$$m_i(\alpha) = \bar{E}_\alpha[(\eta_0 - \rho(\alpha))^i].$$

We shall abbreviate $\gamma_2(\alpha)$ by $\sigma(\alpha)^2$.

We start proving that $\sigma(\alpha)^2\alpha^{-1}$ is bounded above by a finite constant and below by a positive constant uniformly on R_+. By a change of variables and assumption (LG) we have that

$$\begin{aligned} &E_\rho[\{c(\eta_0) - \alpha(\rho)\}^2] = \alpha(\rho)E_\rho[c(\eta_0+1) - c(\eta_0)] \le a_1\alpha(\rho), \\ &E_\rho[\{c(\eta_0) - \alpha(\rho)\}\{\eta_0 - \rho\}] = \alpha(\rho). \end{aligned} \tag{5.1}$$

Therefore, by the Schwarz inequality,

$$\alpha(\rho) = E_\rho[\{c(\eta_0) - \alpha(\rho)\}\{\eta_0 - \rho\}] \le a_1^{1/2}\sigma(\alpha(\rho))\alpha(\rho)^{1/2} \le a_1\sigma^2(\alpha(\rho)).$$

The reverse inequality follows from the one site spectral gap for the birth and death process on N associated with the grand canonical measure.

LEMMA 5.1. *Let B_0' be the constant given by Lemma 4.2. Then*

$$\frac{\sigma^2(\alpha(\rho))}{\alpha(\rho)} \le \frac{1}{2}B_0'.$$

PROOF. Apply Lemma 4.2 to the function η_1 and notice that the variance of η_1 is $\sigma^2(\alpha(\rho))$ and that its Dirichlet form is $(1/2)\alpha(\rho)$. □

In conclusion, there exist constants C_1 and C_2 depending only on a_1, a_2 and k_0 such that

$$0 < C_1 \le \frac{\sigma^2(\alpha)}{\alpha} \le C_2 < \infty \tag{5.2}$$

for all α in R_+. Notice that the zero-range processes described in Example 1.1 do not satisfy the upper bound.

The second main goal of this section is to derive some estimates required in the proof of local central limit theorems that are uniform in density. Denote by $v_\alpha(t)$ the normalized characteristic function associated with the distribution $\bar{P}_\alpha$:

$$v_\alpha(t) = \bar{E}_\alpha[\exp\{it(\eta_0 - \rho(\alpha))/\sigma(\alpha)\}].$$

In the next section we shall prove uniform Edgeworth expansions under the following set of assumptions. For each $\bar{\alpha} > 0$ and $\bar{k}$ in N, let the following statements hold:

(CL1) There exists a finite constant K_0 such that

$$\sup_{\alpha\ge\bar{\alpha}} m_{2k}(\alpha)/\sigma(\alpha)^{2k} \le K_0 \quad \text{for } 1 \le k \le \bar{k}.$$

(CL2) For every $\delta > 0$, there exists $C(\delta) < 1$ such that

$$\sup_{\alpha\ge\bar{\alpha}} \sup_{\delta\le|t|\le\pi\sigma(\alpha)} |v_\alpha(t)| \le C(\delta).$$

(CL3) There exists $\kappa > 0$ so that

$$\sup_{\alpha\ge\bar{\alpha}} \int_{|t|\le\pi\sigma(\alpha)} |v_\alpha(t)|^\kappa\, dt \le C < \infty.$$

Notice that these hypotheses are satisfied for all $\bar{\alpha} > 0$ as soon as they are satisfied by any positive parameter $\bar{\alpha}_0 > 0$. It is also a simple task to verify that both zero-range processes defined in Examples 1.1 and 1.2 satisfy (CL1)–(CL3).

We now deduce assumptions (CL1)–(CL3) from hypotheses (LG) and (M). We start with (CL1). Recall that $\gamma_k(\alpha)$ denotes the kth cumulant of the occupation variable with respect to the probability $\bar{P}_\alpha$.

LEMMA 5.2. *Assume hypotheses* (LG) *and* (M). *For all* $k \geq 1$, *there exists a finite constant* $C(k)$ *such that*

$$m_{2k}(\alpha) \leq C(k)\sigma(\alpha)^{2k}$$

for all $\alpha \geq 1$.

PROOF. To keep notation simple, denote $\rho(\alpha)$ simply by ρ. We shall prove this lemma by induction on k. Notice that by the Schwarz inequality and Lemma 4.2,

$$\begin{aligned} m_{2k}(\alpha) &= \bar{E}_\alpha[((\eta_0 - \rho)^k - m_k(\alpha) + m_k(\alpha))^2] \\ &\leq 2B_0' D_\rho^1((\eta_0 - \rho)^k) + 2C(k)^2\sigma^{2k}(\alpha). \end{aligned}$$

The Dirichlet form in the first term is equal to

$$D_\rho^1((\eta_0 - \rho)^k) = (\alpha/2)E_\rho[((\eta_0 - \rho)^k - (\eta_0 - \rho + 1)^k)^2].$$

Computing the kth power inside the expectation and applying the Schwarz inequality and the induction assumption, we obtain that the right-hand side is bounded above by

$$C(k)\alpha \sum_{j=0}^{k-1} \sigma^{2j}(\alpha).$$

On the one hand, we have by (5.2) that $\alpha \leq C_1^{-1}\sigma(\alpha)^2$ and on the other $\sigma^{2j}(\alpha) \leq C\sigma^{2(k-1)}(\alpha)$ for $0 \leq j \leq k-1$ and some finite universal constant C because $\alpha \geq 1$ and $\sigma(\alpha)^2 \geq C_1\alpha$. $\square$

We turn now to assumptions (CL2) and (CL3). For $\rho \geq 0$, define

$$p_\rho(k) = \frac{1}{Z(\alpha(\rho))} \frac{\alpha(\rho)^k}{c(1)\cdots c(k)} \quad \text{for } k \in N.$$

Most of the time we will omit the index ρ to keep notation simple. We begin with an estimate which provides a bound on the characteristic functions of the occupation variable.

LEMMA 5.3. *Under assumption* (LG), *for every* $\rho > 0$,

$$\sum_{k\geq 0} |p_\rho(k+1) - p_\rho(k)| \leq \frac{\sqrt{a_1}}{\sqrt{\alpha(\rho)}}.$$

PROOF. First note that $p(k+1)-p(k)$ is equal to $p(k)\{\alpha(\rho)c(k+1)^{-1}-1\}$. In particular the above sum is equal to

$$E_\rho\left[\left|\frac{\alpha(\rho)}{c(\eta_0+1)}-1\right|\right]=\frac{1}{\alpha(\rho)}E_\rho\left[c(\eta_0)\left|\frac{\alpha(\rho)}{c(\eta_0)}-1\right|\right].$$

By the Schwarz inequality this last expectation is bounded above by $\alpha(\rho)^{-1}E_\rho[(c(\eta_0)-\alpha)^2]^{1/2}$. By inequality (5.1), this expression is less than or equal to $a_1^{1/2}\alpha(\rho)^{-1/2}$. $\square$

Recall that we denoted v_α as the normalized characteristic function of the occupation variable under $P_{\rho(\alpha)}$. The previous lemma implies a certain decay of $v_\alpha(t)$, uniform over α.

LEMMA 5.4. *Assume hypothesis* (LG). *There exists a constant C_0 depending only on a_1, $c(1)$ and C_2 such that*

$$|v_\alpha(t)|\le\frac{C_0}{|t|}$$

for each $\alpha>0$ and $0<|t|\le\pi\sigma(\alpha)$.

PROOF. Fix $\delta<1$ and $\alpha>0$. Define $\tilde{v}_\alpha(t)=\bar{E}_\alpha[\exp\{it(\eta_0-\rho)\}]$,

$$\tilde{v}_\alpha^\delta(t)=\sum_{k\ge 0}\delta^k e^{itk}p(k)\quad\text{and}\quad S(k)=\sum_{l\ge k}\delta^l e^{itl}.$$

Notice that the absolute value of $S(k)$ is bounded by $|1-\delta\exp(it)|^{-1}$, uniformly on k. Summing by parts we write

$$\begin{aligned}\tilde{v}_\alpha^\delta(t)&=\sum_{k\ge 0}\{S(k)-S(k+1)\}\,p(k)\\&=S(0)p(0)+\sum_{k\ge 0}\{p(k+1)-p(k)\}\,S(k+1).\end{aligned}$$

Therefore,

$$|\tilde{v}_\alpha^\delta(t)|\le\frac{1}{|1-\delta e^{it}|}\left\{p(0)+\sum_{k\ge 0}|p(k+1)-p(k)|\right\}.$$

Notice that $Z(\alpha)\ge 1+\alpha c(1)^{-1}$. In particular, $Z(\alpha)\ge C\alpha^{1/2}$ for some constant C that depends on $c(1)$ only and

$$p(0)=\frac{1}{Z(\alpha)}\le\frac{C}{\sqrt{\alpha}}.$$

Hence, by the previous lemma, we have for each $\delta<1$ that

$$|\tilde{v}_\alpha^\delta(t)|\le\frac{1}{|1-\delta e^{it}|}\frac{C(a_1,c(1))}{\sqrt{\alpha}}.$$

Since $|1-e^{it}| \ge (\pi/2)|t|$ for $0 \le t \le \pi$, allowing $\delta \uparrow 1$, we obtain that

$$|\tilde{v}_\alpha(t)| \le \frac{C(a_1, c(1))}{|t|\sqrt{\alpha}}.$$

To conclude the proof, we need only comment that $v_\alpha(t) = \tilde{v}_\alpha(t\sigma^{-1})$ and that $\sigma^2\alpha^{-1}$ is bounded above by C_2 by (5.2). □

Assumption (CL3) now follows from the estimate obtained in the previous lemma and the fact that absolute value of the characteristic function is always bounded by 1. It remains to deduce hypothesis (CL2). We turn to a series of lemmas devised to bound the characteristic function $v_\alpha(t)$ near the origin. We start with a general result.

Consider a density function $f: R \to R_+$ on the real line and a cosine wave function $W: R \to [-1, 1]$ with period l and amplitude 1. Let $I = [a, b]$ be an interval and assume that the following statements hold:

ASSUMPTION (a). There is $\gamma > 0$ such that $\int_I f(x)\,dx \ge \gamma$.

ASSUMPTION (b). There is a finite constant C_3 such that $\sup_{x,\, y \in I} f(x)/f(y) \le C_3$.

ASSUMPTION (c). The period l satisfies $100l \le b - a$.

LEMMA 5.5. *Under Assumptions* (a), (b) *and* (c) *there is a strictly positive constant* $\delta = \delta(\gamma, C_3)$ *such that*

$$\int f(x)W(x)\,dx \le 1 - \delta.$$

PROOF. Since f is a density, it is enough to show that

$$\int_I f(x)(1 - W(x))\,dx \ge \delta.$$

Let E be the closed set defined by $E = \{x: 1 - W(x) \ge 1/10\}$. Since there are at least 98 full periods in I, there exist universal constants $0 < A_1, A_2 < \infty$ such that

$$A_1 < \frac{|E \cap I|}{|E^c \cap I|} < A_2. \tag{5.3}$$

In this formula, for a measurable set B, $|B|$ stands for its Lebesgue measure. It is now easy to obtain a lower bound for the previous integral. Since W is bounded above by 1 and $1 - W \ge 1/10$ on E, we have that

$$\int_I f(1 - W)\,dx \ge \int_{I\cap E} f(1 - W)\,dx \ge \tfrac{1}{10}\int_{I\cap E} f\,dx.$$

To complete the proof we replace the last integral by $\int_I f\,dx$ and take advantage of Assumption (a). In order to perform this replacement, we rely on Assumption (b) and on inequalities (5.3). Indeed, by Assumption (b),

$$\int_{I\cap E} f\,dx \ge C_3^{-1} f(a)|I\cap E|.$$

By inequality (5.3) and Assumption (b), this last expression is at least

$$C_3^{-1} A_1 f(a)|I\cap E^c| \ge C_3^{-2} A_1 \int_{I\cap E^c} f\,dx.$$

In conclusion,

$$\int_{I\cap E} f\,dx \ge \{1 + C_3^2 A_1^{-1}\}^{-1} \int_I f\,dx \ge \gamma\{1 + C_3^2 A_1^{-1}\}^{-1}.$$

This completes the proof with $\delta = 10^{-1}\gamma[1 + C_3^2 A_1^{-1}]^{-1}$. □

A version of this lemma for discrete probabilities on the integers is valid provided the amplitude l is not too small, say $l > 100$. Our intention is to apply this result to bound the characteristic function $v_\alpha(t)$ near the origin.

LEMMA 5.6. *Under assumptions* (LG) *and* (M), *for every $R_0 > 0$, there exists $\alpha_0 \ge 1$ with the following property. For every $\varepsilon > 0$, there exists a strictly positive constant $\delta = \delta(\varepsilon)$ such that*

$$\sup_{\varepsilon < |t| \le 2\pi R_0} |v_\alpha(t)| \le 1 - \delta.$$

for all $\alpha > \alpha_0$.

Before we prove this lemma, we show how to derive (CL2) from (LG) and (M).

COROLLARY 5.1. *Assumption* (CL2) *follows from hypotheses* (LG) *and* (M).

PROOF. Fix $\varepsilon > 0$. We wish to bound $\sup_{\varepsilon<|t|\le\pi\sigma} |v_\alpha(t)|$ by a constant strictly smaller than 1. In Lemma 5.4 we bounded $|v_\alpha(t)|$ by $C_0|t|^{-1}$ uniformly over the parameter α. In particular, for $|t| \ge 2C_0$, $|v_\alpha(t)| \le 1/2$ for all $\alpha \ge 1$. It remains to consider the interval $[\varepsilon, 2C_0]$. Take $R_0 = \pi^{-1} C_0$ in Lemma 5.6. By this result, there exists α_0, independent of ε, and $\delta = \delta(\varepsilon)$ strictly positive so that

$$\sup_{\varepsilon<|t|\le 2C_0} |v_\alpha(t)| \le 1 - \delta.$$

This concludes the proof. □

We conclude this section by proving Lemma 5.6.

PROOF OF LEMMA 5.6. Rewrite $v_\alpha(t)$ as $|v_\alpha(t)| \exp\{i\theta(t)\}$ for some real continuous function θ (that may depend on the parameter α). Then

$$|v_\alpha(t)| = \sum_k p(k) \cos\big(t\{k - \rho(\alpha)\}\sigma(\alpha)^{-1} - \theta(t)\big).$$

Fix t such that $\varepsilon \le |t| \le 2\pi R_0$ and let $W(x) = \cos(t\{x - \rho(\alpha)\}\sigma(\alpha)^{-1} - \theta(t))$. Denote by $l = 2\pi\sigma|t|^{-1}$ the period of $W(\cdot)$. Since $|t| \le 2\pi R_0$, the period l is bounded below by $\sigma(\alpha)R_0^{-1}$. In particular, l is bounded below by 100 provided $\alpha_0 = \alpha_0(R_0)$ is chosen large enough and $\alpha > \alpha_0$. Fix such parameter α.

Let $b_0 = \max\{50, 2R_0\}$ and define the interval I by

$$I = [\rho(\alpha) - b_0 l, \rho(\alpha) + b_0 l].$$

Assumption (c) is obviously satisfied. It is now a matter of verifying the first two assumptions. This is the subject of the following three lemmas. □

LEMMA 5.7. *There exists a strictly positive constant $\gamma = \gamma(R_0)$, independent of α, such that*

$$\sum_{k\in I} p(k) \ge \gamma$$

for all $|t| \le 2\pi R_0$.

PROOF. This follows from Chebyshev's inequality, the fact that $|t|$ is bounded above by $2\pi R_0$, and that $2R_0 \le b_0$. This argument gives $\gamma = 3/4$. □

For the next lemma we will linearly interpolate the jump rate $c(\cdot)$ to the positive real line.

LEMMA 5.8. *Assume hypotheses* (LG) *and* (M). *There exists a constant C_4 depending only on a_1, a_2 and k_0 such that*

$$|c(\rho) - \alpha(\rho)| \le C_4\sqrt{\alpha(\rho)}$$

for all $\rho > 0$.

PROOF. By the Schwarz inequality and assumption (LG),

$$|\alpha(\rho) - c(\rho)| \le E_\rho[|c(\eta_0) - c(\rho)|] \le a_1 E_\rho[|\eta_0 - \rho|] \le a_1\sigma(\alpha(\rho)).$$

Inequality (5.2) completes the proof. □

LEMMA 5.9. *Under assumptions* (LG) *and* (M), *there exists a finite constant C_5, depending only on ε, R_0, a_1, a_2 and k_0 such that*

$$\sup_{x,\, y\in I} \frac{p(x)}{p(y)} < C_5.$$

PROOF. Fix $x < y$. Since $\log(1+u) \le u$ for all $u > -1$,

$$\log \frac{p(x)}{p(y)} = \log \prod_{k=x}^{y-1} \frac{p(k)}{p(k+1)} = \sum_{k=x}^{y-1} \log\left(1 + \left\{\frac{p(k)}{p(k+1)} - 1\right\}\right)$$
$$\le \sum_{k=x}^{y-1} \left|\frac{p(k)}{p(k+1)} - 1\right|.$$

From the definition of $p(k) = p_\alpha(k)$, $p(k)/p(k+1)$ is equal to $c(k+1)\alpha^{-1}$. Therefore, since x and y belong to the interval I and the linearly interpolated function c: $R \to R_+$ is a_1-Lipschitz continuous, this last sum is bounded above by

$$\sum_{i=-b_0 l}^{b_0 l} \left|\frac{c(\rho+i)}{\alpha} - 1\right| + \frac{a_1}{\alpha} \le \frac{C_4}{\sqrt{\alpha}} + 2a_1 \sum_{i=1}^{b_0 l} \frac{i+1}{\alpha}.$$

In the last inequality, we have used the previous lemma to bound the difference $|c(\rho) - \alpha(\rho)|$ by $C_4\sqrt{\alpha}$. Since $\alpha \ge \alpha_0 \ge 1$ and $|t| \ge \varepsilon$, the last sum is bounded by $C\sigma^2(\alpha\varepsilon^2)^{-1} + C$ for some constant C which depends only on R_0 and a_1. By inequality (5.2), $\sigma^2\alpha^{-1}$ is bounded above by some constant C_2. Therefore the last sum is dominated by the expression $C_5(a_1, a_2, k_0, R_0)\varepsilon^{-2}$. A similar argument follows for $x > y$. $\square$

6. Uniform local central limit theorem. This section is devoted to the study of local central limit theorems for zero range distributions uniformly in the parameter α. As a by-product we will also derive a few results important for the induction step in the proof of Theorem 1.1.

Diaconis and Freedman [2] considered uniform Edgeworth expansions up to the second order for exponential families under slightly different assumptions, not verifiable for zero-range marginals corresponding to small values of the parameter α. More precisely, recall that $m_k(\alpha)$ and $\gamma_k(\alpha)$ denote, respectively, the kth moment and the kth cumulant of the distribution $\overline{P}_\alpha$ and $\sigma(\alpha)^2 = \gamma_2(\alpha)$. For α close to 0, a simple computation shows that $m_k(\alpha) = c(1)^{-1}\alpha + O(\alpha^2)$ and, therefore, $\gamma_k(\alpha)/\sigma(\alpha)^k$ diverges as α approaches 0 for $k \ge 3$. In particular, the Edgeworth expansion with an error uniform in the parameter α cannot be correct close to the origin. In fact, as we shall see below, these errors are of order $(n\sigma(\alpha)^2)^{-k}$.

To state the main theorem of this section we recall several classical definitions. For $m \ge 0$, denote by $H_m(x)$ the Hermite polynomial of degree m:

$$H_m(x) = (-1)^m \exp\left(\frac{x^2}{2}\right) \frac{d^m}{dx^m} \exp\left(\frac{-x^2}{2}\right) = m! \sum_{k=0}^{[m/2]} (-1)^k \frac{x^{m-2k}}{k!(m-2k)!2^k}.$$

Let $q_0(x)$ denote the density of the normalized Gaussian distribution and, for $j \ge 1$, let

$$q_j(x) = \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x^2}{2}\right) \sum H_{j+2a}(x) \prod_{m=1}^{j} \frac{1}{k_m!} \left(\frac{\gamma_{m+2}}{(m+2)!\sigma^{m+2}}\right)^{k_m},$$

where the summation is carried out over all nonnegative integer solutions of $k_1 + 2k_2 + \cdots + jk_j = j$ and $k_1 + k_2 + \cdots + k_j = a$. We are now ready to state the main result of this section.

THEOREM 6.1. (a) *For all $\alpha_0 > 0$ and $k_0 \in N$, there exist finite constants $E_0 = E_0(\alpha_0, k_0)$ and $A = A(\alpha_0, k_0)$ such that*

$$\left|\sqrt{n\sigma^2}\bar{P}_\alpha\left[\sum_{i=1}^n \eta_i = K\right] - \sum_{j=0}^{k_0-2} \frac{1}{n^{j/2}} q_j(x)\right| \le \frac{E_0}{(\sigma^2(\alpha)n)^{(k_0-1)/2}}$$

uniformly over all parameters $\alpha \le \alpha_0$ such that $\sigma^2(\alpha)n \ge A$. In this formula x stands for $(K - n\rho(\alpha))/\sigma(\alpha)\sqrt{n}$.

(b) *Assume hypotheses* (CL1)–(CL3) *with $\bar{\alpha} = \alpha_1$ and $\bar{k} = [k_1/2] + 1$. There exist a constant $E_0 = E_0(k_1, \alpha_1)$ and $n_0 = n_0(k_1, \alpha_1)$ such that*

$$\sup_\alpha\left|\sqrt{n\sigma^2}\bar{P}_\alpha\left[\sum_{i=1}^n \eta_i = K\right] - \sum_{j=0}^{k_1-2} \frac{1}{n^{j/2}} q_j(x)\right| \le \frac{E_0}{n^{(k_1-1)/2}}$$

for all $n \ge n_0$ where the supremum is taken over all $\alpha \ge \alpha_1$.

The proof of this theorem is omitted since it follows closely the classical arguments given for instance in Petrov (Theorem VII.12 of [9]). There is only the slight problem of controlling the integral I_3 in this theorem. In part (a) this is not difficult since, if $\tilde{v}_\alpha(t) = \bar{E}_\alpha[\exp\{it(X - \rho(\alpha))\}]$ denotes the characteristic function of $\eta_0 - \rho(\alpha)$ under $\bar{P}_\alpha$, we may write

$$\begin{aligned}\left|\tilde{v}_\alpha(t)\right|^2 - 1 &\le \bar{P}_\alpha[\eta_0 = 0]\bar{P}_\alpha[\eta_0 = 1]\{\cos t - 1\}\\ &\le C(\alpha_0)\alpha\{\cos t - 1\}\end{aligned}$$

for some universal constant $C(\alpha_0)$ because

$$\bar{P}_\alpha[\eta_0 = 0] = 1 + O(\alpha) \quad \text{and} \quad \bar{P}_\alpha[\eta_0 = 1] = c(1)^{-1}\alpha + O(\alpha^2).$$

These estimates permit us to bound the integral I_3.

On the other hand, for part (b), hypotheses (CL2) and (CL3) are built to take care of the integral I_3.

Denote by ∇ and by Δ, respectively, the discrete derivative and Laplacian

$$(\nabla f)(i) = f(i+1) - f(i), \qquad (\Delta f)(i) = \{f(i+1) + f(i-1) - 2f(i)\}$$

for each $f\colon Z \to R$. In the next theorem we use the notation

$$\begin{aligned}(\nabla\bar{P}_\alpha)\left[\sum_{i=1}^n \eta_i = K\right] &= \bar{P}_\alpha\left[\sum_{i=1}^n \eta_i = K+1\right] - \bar{P}_\alpha\left[\sum_{i=1}^n \eta_i = K\right],\\ (\nabla q_j)(x) &= q_j\left(\frac{K+1-n\rho(\alpha)}{\sigma(\alpha)\sqrt{n}}\right) - q_j\left(\frac{K-n\rho(\alpha)}{\sigma(\alpha)\sqrt{n}}\right)\end{aligned}$$

and similar notation with Δ replacing ∇.

THEOREM 6.2. *Under the assumptions of Theorem* 6.1(b), *there exist constants* $E_0' = E_0'(k_1, \alpha_1)$ *and* $n_0' = n_0'(k_1, \alpha_1)$ *such that*

$$\left|\sqrt{n\sigma^2}(\nabla \bar{P}_\alpha)\left[\sum_{i=1}^{n} \eta_i = K\right] - \sum_{j=0}^{k_1-2} \frac{1}{n^{j/2}}(\nabla q_j)(x)\right| \le \frac{E_0'}{\sigma(\alpha) n^{(k_1-1)/2}}$$

and

$$\left|\sqrt{n\sigma^2}(\Delta \bar{P}_\alpha)\left[\sum_{i=1}^{n} \eta_i = K\right] - \sum_{j=0}^{k_1-2} \frac{1}{n^{j/2}}(\Delta q_j)(x)\right| \le \frac{E_0'}{\sigma(\alpha)^2 n^{(k_1-1)/2}}$$

for all $n \ge n_0'$ *and all* $\alpha \ge \alpha_1$. *In these formulas the discrete derivatives and Laplacians are applied to* K.

The proof of Theorem 6.2 follows closely that of Theorem 6.1 and is therefore omitted. Notice that the estimate improves as the parameter α increases since $\sigma^2(\alpha)$ behaves as α by inequality (5.2).

We conclude this section by stating several corollaries important to the proof of the spectral gap result, Theorem 1.1.

COROLLARY 6.1. *Fix* $f\colon N^\ell \to R$ *for some fixed positive integer* ℓ.

(a) *There exists a constant* $E_1 = E_1(\alpha_0)$ *such that*

$$\begin{aligned}&\left|E_{n,K}[f] - E_\rho[f]\right| \\ &\quad\le \frac{E_1\ell}{n}\left\{\frac{1}{\sigma^2(\alpha(\rho))}E_\rho[|f - \langle f\rangle_\rho|] + \frac{1}{\sigma(\alpha(\rho))}\sqrt{E_\rho[(f - \langle f\rangle_\rho)^2]}\right\}\end{aligned}$$

uniformly over all n *and* K *such that* $n \ge 2\ell$, $K/n \le \rho(\alpha_0)$ *and* $\sigma^2(\alpha(\rho))n \ge A(\alpha_0, 3)$, *where* A *is given by Theorem* 6.1(a). *In this formula* ρ *stands for the density* K/n.

(b) *Assume hypotheses* (CL1)–(CL3) *with* $\bar{k} = 5$ *and* $\bar{\alpha} = \alpha_1$. *There exists a constant* $E_1 = E_1(\alpha_1)$ *such that*

$$\left|E_{n,K}[f] - E_\rho[f]\right| \le \frac{E_1\ell}{n}\sqrt{E_\rho[(f - \langle f\rangle_\rho)^2]}$$

uniformly over all $n \ge \max\{2\ell, n_0(3, \alpha_1)\}$ *and* K *so that* $K/n \ge \rho(\alpha_1)$.

(c) *Under the same assumptions of* (b), *there exist a universal constant* E_2 *and* $n_0 = n_0(\alpha_1)$ *so that*

$$E_{n,K}[f] \le E_2 E_{K/n}[\,|f|\,]$$

for all $n \ge n_0(\alpha_1)$ *and all* K *so that* $K/n \ge \rho(\alpha_1)$.

The proof is left to the reader. It is a simple consequence of the Edgeworth expansion up to the second order.

To keep notation simple, we will denote the variance of a function f with respect to the probability $\bar{P}_\alpha$ as $\sigma_f(\alpha)^2$. Completing the Edgeworth expansion up to the third order we obtain the following corollary.

COROLLARY 6.2. *Fix $f: N^\ell \to R$ for some fixed positive integer ℓ. There exists a constant $E_1 = E_1(\ell, \alpha_0)$ such that*

$$\left| E_{n,K}[f] - E_\rho[f] - \frac{1}{2n}\left\{ \frac{\gamma_3}{\sigma^4}\left\langle f; \sum_{i=1}^{\ell}(\eta_i - \rho)\right\rangle_\rho - \frac{1}{\sigma^2}\left\langle f; \left\{\sum_{i=1}^{\ell}(\eta_i - \rho)\right\}^2\right\rangle_\rho \right\}\right| \le \frac{E_1 \sigma \sigma_f}{(n\sigma^2)^{3/2}}$$

for all n and K such that $n \ge 2\ell$, $K/n \le \rho(\alpha_0)$ and $\sigma^2(\alpha(\rho))n \ge A(\alpha_0, 4)$, where A is given by Theorem 6.1(a). *In this formula ρ denotes the density K/n and $\langle f; g\rangle_\rho$ stands for the covariance of f and g with respect to P_ρ.*

Allowing $\ell = 1$ and $f(\eta) = c(\eta_0)$ in Corollary 6.2, we obtain the following result.

COROLLARY 6.3. *For every $\varepsilon > 0$, there exist $n_0 > 0$ and $\alpha_0 > 0$ such that*

$$\left| E_{n,K-1}[c(\eta_0)] - E_{n,K}[c(\eta_0)] + \frac{1}{n} E_{n,K-1}[c(\eta_0 + 1) - c(\eta_0)]\right| \le \frac{\varepsilon}{n}$$

for all $n \ge n_0$ and $K/n < \rho(\alpha_0)$.

PROOF. Fix some parameter $\alpha_0 > 0$ and $\varepsilon > 0$. For n large enough and K such that $\sigma^2(\alpha(\rho))n \ge C(\alpha_0, \varepsilon)$, the estimate follows from the previous corollary. On the other hand, $\sigma^2(\alpha(\rho))n \le C(\alpha_0, \varepsilon)$ implies that the total number of particles K is bounded by some constant $C_1 = C_1(\alpha_0, \varepsilon)$. It is therefore easy to complete the proof by inspection. □

With similar arguments we obtain from Corollary 6.1(a) and (b) the following result.

COROLLARY 6.4. *There exist a universal constant E_3 and $n_0 > 0$ such that*

$$|E_{n,K}[c(\eta_0)] - E_{K/n}[c(\eta_0)]| \le \frac{E_3}{n}\sqrt{1 + \alpha(\rho)}$$

for all $n \ge n_0$.

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