RANDOM WALKS AND HARMONIC FUNCTIONS ON INFINITE PLANAR GRAPHS USING SQUARE TILINGS

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We study a wide class of transient planar graphs, through a geometric model given by a square tiling of a cylinder. For many graphs, the geometric boundary of the tiling is a circle and is easy to describe in general. The simple random walk on the graph converges (with probability 1) to a point in the geometric boundary. We obtain information on the harmonic measure and estimates on the rate of convergence. This allows us to extend results we previously proved for triangulations of a disk.

1. Introduction. In this paper, we continue our study of harmonic functions on planar graphs initiated in [1]. Here, we will focus on the simple random walk and the Dirichlet problem. In [1] we proved that any bounded valence transient planar graph admits nonconstant bounded Dirichlet harmonic functions. This was then extended to a wider family of graphs and manifolds. The key idea in [1] was to look for a useful geometric representation of the graph. This was done via circle packings. Further assuming that G is (the 1-skeleton of) a triangulation of a disk, we were able to solve the Dirichlet problem for the circle packing representation of G. Here, using a different geometric model for the graph, we can solve a similar Dirichlet problem for planar graphs which are not triangulations of a disk. In particular, our results apply to bounded valence, planar graphs, with one transient end.

We apply an infinite version of a theorem of Brooks, Smith, Stone and Tutte [2], to get a square tiling representation of the graph. The squares in the tiling will be indexed by edges of G. It turns out that the square tiling is natural and useful in studying the random walk on the graph. The tiling will take place in a cylinder $C \times [0, 1]$, where $C = \mathbb{R}/\eta\mathbb{Z}$, for some η to be determined. For many graphs, the geometric boundary of the corresponding tiling will be the circle $C = C \times \{1\}$. In general, it will be contained in $C \times \{1\} \cup C_0 \times [0, 1)$, where $C_0 \subset C$ has Lebesgue measure zero. The simple random walk on G, realized on the square tiling, converges with probability 1 to a point on $C \times \{1\}$. Moreover, the harmonic measure is absolutely continuous with respect to Lebesgue measure on $C \times \{1\}$, with Radon–Nikodym derivative bounded from above and below. Thus, we will be able to solve the Dirichlet problem for L^1 functions on $C \times \{1\}$.
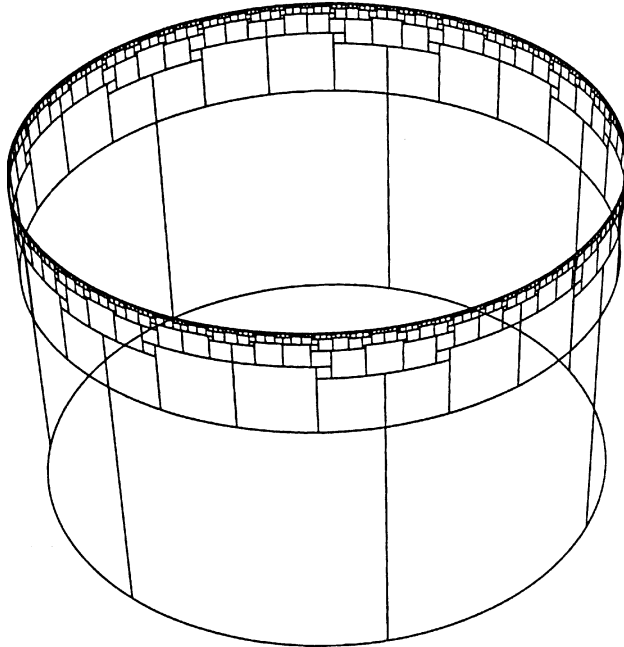
FIG. 1.

The tiling plays the same role as conformal uniformization does for planar domains. In fact, the proof of its existence illustrates parallels with the continuous theory. In a way, it is a discrete analogue of Riemann's mapping theorem. (The circle packing theorem provides another such analogue.)

The tiling depends on the choice of a base vertex in G, but essentially on nothing more. The harmonic measure for the simple random walk starting at the base is actually equal to Lebesgue measure on $C \times \{1\}$. We suspect that $C \times \{1\}$ is a geometric realization of the Poisson boundary of the graph.

We will mainly consider uniquely absorbing planar graphs (since that is the widest class of graphs for which the current techniques apply).

DEFINITION. Let $G = (V, E)$ be a connected, locally finite, transient, planar graph embedded in $\mathbb{R}^2$. A set W of vertices in G is *absorbing* if with positive probability a random walk on G visits $V - W$ only finitely many times. G is said to be *uniquely absorbing* if for every finite subgraph $G_0 \subset G$ there is exactly one connected component D of $\mathbb{R}^2 - G_0$ such that $V \cap D$ is absorbing.

We shall now clarify what we mean by an embedding of G in $\mathbb{R}^2$. Given a graph G, we let $|G|$ denote the metric space constructed as follows. Start with V, and for every edge $e \in E$, with vertices v, u, say, glue the endpoints of an isometric copy I_e of the interval $[0, 1]$ to the two vertices v, u. Let $|G|$ be the

union $\bigcup_e I_e \cup V$ modulo the identifications, with the path metric. An *embedding* of G in the plane is a 1–1 continuous map $f: |G| \to \mathbb{R}^2$. Note that a graph G is planar iff G admits an embedding in $\mathbb{R}^2$.

REMARK 1. The property of being uniquely absorbing depends on the embedding of G; that is, a graph may have two embeddings in $\mathbb{R}^2$ so that it is uniquely absorbing in one embedding, but not in the other.

REMARK 2. Recall that G *has one end* if for every finite subgraph $G_0 \subset G$ there is exactly one infinite connected component of $G - G_0$. We shall say that G is *end convergent* if for every such G_0 there is exactly one absorbing component of $G - G_0$. Suppose, as above, that G is connected, locally finite, transient and embedded in $\mathbb{R}^2$. With these assumptions, the following easy implications are valid: G is a triangulation of an open disk $\Rightarrow$ G has one end $\Rightarrow$ G is end convergent $\Rightarrow$ G is uniquely absorbing. Conversely, if G is a triangulation of a plane domain that is uniquely absorbing, then G is end convergent.

The following several sections will be devoted to a more thorough study of harmonic functions on uniquely absorbing planar graphs. As a by-product, another proof that any planar, bounded valence, transient, graph has nonconstant, harmonic, Dirichlet functions is obtained. The next section contains some background and notation. The finite and infinite square tiling theorems are the main topic of Sections 3 and 4. The rest of the paper studies the simple random walk on G, realized on the tiling. This is used then to solve the Dirichlet problem for uniquely absorbing graphs.

It is interesting to contrast the circle packing and square tilings model of a graph. The latter is a linear creature, hence simpler and more closely related to random walks and harmonic functions, as will be clear subsequently. The circle packing theorem was more suitable in [1], mainly because one can produce a packing where none of the disks degenerates to a point. This has the virtue of providing a more faithful geometric model of the graph. With square tilings, it is unavoidable in general that some of the squares degenerate to points. Another advantage of the circle packing theory is that it is inherently Möbius invariant, as the analytic theory.

Another motivation for invoking square tilings is the possible generalization to nonreversible random walks on planar graphs. Recently, R. Kenyon extended the finite square tiling theorem to a trapezoid tiling theorem. Under some assumptions, it is possible to produce an infinite variant of Kenyon's tilings along the lines of the tiling theorem that follows. The probabilistic interpretation of this has the potential to extend our work to the setup of nonreversible random walks. We are currently pursuing this direction with R. Kenyon and Y. Peres.

Finally, for more background and related work, see [1], Woess's survey [6] and Soardi's recent book [5].

2. Notations and terminology. Let $G = (V, E)$ be a graph. G may have more than one edge between two vertices, but there is no reason to allow loops. The set of vertices incident with an edge e will be denoted ∂e. This is always a subset of V that contains two vertices.

The graphs we shall consider will be connected and locally finite. The latter means that the number of edges incident with any particular vertex is finite.

Initially, the graph G is unoriented, but for notational reasons we also consider oriented edges. An *oriented* (or *directed*) edge $\vec{e}$ is a pair (e, v), where $e \in E$ and $v \in \partial e$. For such an oriented edge, the vertex v is called the *initial* vertex and is also denoted $\vec{e}(0)$, and the other vertex is the *terminal* vertex and will be denoted $\vec{e}(1)$. We also let $|\vec{e}|$ denote the original unoriented e and $-\vec{e}$ will denote the other orientation of $|\vec{e}|$.

The set of all oriented edges will usually be denoted $\vec{E}$. The collection of edges incident with a vertex v will be denoted $E(v)$ and the set of $e \in \vec{E}$ with $e(0) = v$ will be denoted $\vec{E}(v)$. The *valence*, or *degree*, of a vertex v is just the cardinality of $E(v)$. G has *bounded valence* if there is a finite upper bound for the degrees of its vertices.

Let $f: V \to \mathbb{R}$ be some function. Then df is the function $df: \vec{E} \to \mathbb{R}$ defined by

$$df(e) = f(e(1)) - f(e(0)).$$

We also define the gradient of f to be equal to df:

$$\nabla f(e) = df(e).$$

A function $j: \vec{E} \to \mathbb{R}$ is a *flow* on G if it satisfies

$$j(-e) = -j(e)$$

for every $e \in \vec{E}$. For example, for any $f: V \to \mathbb{R}$, df is a flow. The *divergence* of a flow j is the function $\operatorname{div} j: V \to \mathbb{R}$ defined by

$$\operatorname{div} j(v) = \sum_{e \in \vec{E}(v)} j(e).$$

If $\operatorname{div} j = 0$, then j is *divergence-free*. A *flow with source v* is a flow j such that $\operatorname{div} j(u) = 0$ for every $u \neq v$. We shall say that a j is a flow with source v and sink u when $\operatorname{div} j(w) = 0$ for $w \neq v, u$ and $\operatorname{div} j(v) \geq 0 \geq \operatorname{div} j(u)$. If j is a flow with source v and possibly a sink, the *flux* of j is defined as its divergence at v, flux $j = \operatorname{div} j(v)$.

For an $f: V \to \mathbb{R}$, we set

$$\Delta f = \operatorname{div} \nabla f.$$

Then $\Delta f: V \to \mathbb{R}$ is known as the *discrete laplacian* of f. If $\Delta f = 0$, then f is *harmonic*, while if $\Delta f = 0$ on a subset $V' \subset V$, we say that f is harmonic in V'. Equivalently, f is harmonic iff its value at any $v \in V$ is equal to the average of the values at the neighbors of v.

For a flow j and an $e \in E$ we let $|j(e)|$ denote $|j(\vec{e})|$, where $\vec{e}$ is any orientation of e. The norm of a flow j is defined by

$$\|j\|^2 = \tfrac{1}{2} \sum_{e \in \vec{E}} j(e)^2 = \sum_{e \in E} |j(e)|^2.$$

The collection of all flows with finite norm is a Hilbert space with the inner product given by

$$j \cdot i = \tfrac{1}{2} \sum_{e \in \vec{E}} j(e)i(e).$$

The *Dirichlet energy* of a function $f: V \to \mathbb{R}$ is defined by

$$\mathscr{D}(f) = \|df\|^2.$$

A *Dirichlet function* is an $f: V \to \mathbb{R}$ with $\mathscr{D}(f) < \infty$. The space of all Dirichlet functions on G is denoted $D(G)$. If we choose some base vertex $v_0 \in V$, then $D(G)$ becomes a Hilbert space with inner product given by

$$f \cdot g = f(v_0)g(v_0) + df \cdot dg.$$

The topology of $D(G)$ does not depend on the choice of v_0. Let $D_0(G)$ denote the closure of the set of all finitely supported functions in G and let $HD(G)$ denote the harmonic functions in $D(G)$.

The *simple random walk* on a locally finite graph $G = (V, E)$ starting at a vertex v_0 is the Markov process $(v(1), v(2), \ldots)$ on V such that $v(1) = v_0$ and the transition probability from a vertex v to a vertex u is equal to the number of edges joining v to u divided by the degree of v. A connected graph G is said to be *transient* if there is a positive probability that a simple random walk that starts at a vertex v_0 will never visit v_0 again. It is easy to see that this does not depend on the initial vertex v_0. A nontransient graph is *recurrent*.

3. The finite tiling theorem. Brooks, Smith, Stone and Tutte [2] have shown how to associate a square tiling of a rectangle to a triple (G, v_0, v_1), where G is a finite connected planar graph, and v_0, v_1 are two distinct vertices that belong to the same face of G. In this section we shall associate a square tiling of a cylinder to a triple (G, v_0, v_1), where G is a finite connected planar graph, and v_0, v_1 are any two distinct vertices. This is not a significant generalization of the result of [2]. Nevertheless, we have decided to include a complete proof, since we found the proof in [2] somewhat hard to follow and because we want to present the proof in a way that stresses the analogy to complex analysis.

One curious thing about this tiling theorem is that the squares in the tiling correspond to edges in the graph. There is a similar tiling theorem (from which the tiling theorem of [2] follows), where vertices in the graph correspond to rectangles in the tiling with specified aspect ratios; see [3] and [4].

THEOREM 3.1 (cylinder tiling essentially [2]). *Let $G = (V, E)$ be a planar connected finite graph, embedded in the plane and let $v_0, v_1 \in V$ be distinct. For $v \in V$ let $h(v)$ denote the probability that a random walk that starts at v will reach v_1 before reaching v_0. For $v \in V$ let $\vec{E}^+(v)$ denote the set of directed*

edges $e \in \vec{E}(v)$ such that $h(e(1)) \geq h(e(0)) = h(v)$ and let $\vec{E}^-(v)$ denote the set of directed edges $e \in \vec{E}(v)$ such that $h(e(1)) \leq h(v)$. Set

$$\eta = \sum_{e \in \vec{E}(v_0)} h(e(1))$$

and let C be the circle of length η, $C = \mathbb{R}/\eta\mathbb{Z}$. Then there is a square tiling S of the cylinder $C \times [0,1]$ with the following properties.

(i) *The squares in the tiling are indexed by the edges of G, $S = (S_e: e \in E)$.*

(ii) *The interiors of the squares are disjoint and the union of the squares is $C \times [0,1]$. (That is, S is a tiling of $C \times [0,1]$.)*

(iii) *When $e \in E$, $\partial e = \{v, u\}$ and $h(v) \geq h(u)$, the square S_e has the form $S_e = I_e \times [h(u), h(v)]$, where I_e is a closed subarc of C with length $h(v) - h(u)$. If $\vec{e}$ is an orientation of $e \in E$, we set $I_{\vec{e}} = I_e$, $S_{\vec{e}} = S_e$.*

(iv) *For every $v \in V$, set*

$$I_v = \bigcup_{e \in E(v)} I_e.$$

Then I_v is connected and

$$I_v = \bigcup_{e \in \vec{E}^+(v)} I_e = \bigcup_{e \in \vec{E}^-(v)} I_e \quad \text{for } v \in V - \{v_0, v_1\}.$$

We are using a relaxed definition for "square": We allow S_e to be a single point (this happens when h takes the same value on both vertices of e) and we allow S_e to be of the form $C \times I$, where I is an interval of length η in $[0,1]$ (this happens when v_0 and v_1 are in different connected components of $G - \{e\}$).

The geometric meaning of (4) is that for each $v \in V - \{v_1\}$ the segment $I_v \times \{h(v)\} \subset C \times [0,1]$ is the union of the bottom edges of the squares S_e satisfying $S_e \subset C \times [h(v), 1]$ and is equal to the union of the top edges of the squares satisfying $S_e \subset C \times [0, h(v)]$, if $v \in V - \{v_0\}$.

The tiling S will depend on the embedding of G in the plane and the choice of v_0, v_1, but essentially on nothing more than that, and will prove very useful later on.

One can give the following alternative definition for h: It is the only function satisfying $h(v_0) = 0$, $h(v_1) = 1$, which is harmonic in $V - \{v_0, v_1\}$.

PROOF OF THEOREM 3.1. Note that

$$(3.1) \qquad \sum_{\{e \in \vec{E}: e(0) = v\}} dh(e) = 0 \quad \text{for } v \in V - \{v_0, v_1\}.$$

This follows from the fact that h is harmonic in $V - \{v_0, v_1\}$. In other words, dh is a flow with source v_0 and sink v_1. The *flux* of dh, the total flow from v_0, is equal to η:

$$(3.2) \qquad \eta = \sum_{e \in \vec{E}(v_0)} dh(e).$$

Also note that $\sum_{v \in V} \sum_{e \in \vec{E}(v)} dh(e) = 0$, Since every oriented edge appears once in the sum, and the contributions of the two orientations of an unoriented edge cancel each other. Therefore, (3.1) and (3.2) imply

$$(3.3) \qquad \eta = - \sum_{e \in \vec{E}(v_1)} dh(e).$$

Let $G^* = (V^*, E^*)$ denote the dual graph of G. The set of directed edges of G^* will be denoted by $\vec{E}^*$. If $e \in \vec{E}$, there is a unique edge e^* of G^* that crosses e. Moreover, there is a unique orientation of e^* such that when walking along e^* in the given orientation you find e crossing from "left to right." We refrain from giving the technical definition here. Let $* e$ denote the edge e^* with this particular orientation.

Now define $* dh : \vec{E}^* \to \mathbb{R}$ by setting $* dh(* e) = dh(e)$.

LEMMA 3.2. *Let* $\gamma^* = (e_1^*, e_2^*, \ldots, e_n^*)$ *be a closed path in* G^*. *Then*

$$(3.4) \qquad \sum_{j=1}^{n} * dh(e_j^*) = k\eta$$

for some integer k.

That γ^* is a closed path means that $e_j^*(1) = e_{j+1}^*(0)$ holds for $j = 1, 2, \ldots,$ $n - 1$ and $e_n^*(1) = e_1^*(0)$.

PROOF OF LEMMA 3.2. First suppose that γ^* is a simple closed path; that is, $e_j^*(1) \neq e_k^*(1)$ for distinct j, k and when $n = 2$ the edges of γ^* are not the two orientations of one unoriented edge of G^*. Let D be the bounded connected component of $\mathbb{R}^2 - \gamma^*$. Consider the sum

$$\sum_{v \in V \cap D} \sum_{e \in \vec{E}(v)} dh(e).$$

The contribution of an unoriented edge $e \in E$ to this sum is zero if both vertices of e are outside D. It is also zero if both vertices of e are inside D, since the two terms corresponding to the different orientations of e cancel each other. The contribution of an edge $e \in E$ with vertices v, u such that $v \in D$, $u \notin D$ to the sum is $h(u) - h(v)$. Therefore, we have

$$(3.5) \qquad \sum_{j=1}^{n} * dh(e_j^*) = \varepsilon \sum_{v \in V \cap D} \sum_{e \in \vec{E}(v)} dh(e),$$

where $\varepsilon = 1$, if γ^* is oriented positively around D, and $\varepsilon = -1$, if γ^* is oriented negatively around D. Now (3.4) follows from (3.1), (3.2), (3.3) and (3.5).

In general, γ^* can always be written as a disjoint union of simple closed paths and of paths of length 2 consisting of a single unoriented edge of G^* traversed in both directions. We have seen that the statement of the lemma holds for simple closed paths, and it clearly holds for the other closed paths of length 2. The general case follows. $\square$

PROOF OF THEOREM 3.1 (continued). Let C be the circle $C = \mathbb{R}/\eta\mathbb{Z}$. The lemma allows us to define a function $\theta: V^* \to C$ as follows. Choose some arbitrary $v_0^* \in V^*$. For any $v^* \in V^*$ let $\gamma^* = (e_1^*, e_2^*, \ldots, e_n^*)$ be a path from v_0^* to v^* and set

$$\theta(v^*) = \sum_{j=1}^n * dh(e_j^*) \mod(\eta).$$

The lemma implies that the value $\theta(v^*)$ is independent of the choice of γ^*. Note that for any edge $e \in \vec{E}$ we have

$$dh(e) \equiv \theta(*e(1)) - \theta(*e(0)) \mod(\eta).$$

For any $e \in E$ we define the arc $I_e \subset C$ as follows. Let $\vec{e}$ be e oriented so that $h(\vec{e}(1)) \geq h(\vec{e}(0))$ and let $I_e \subset C$ be the positively oriented arc of length $dh(\vec{e})$ whose initial point is $\theta(*\vec{e}(0))$. This defines the arcs, and the collection $S = (S_e: e \in E)$ is then defined as in Theorem 3.1(iii).

To see that these are "squares" we must verify that $dh(e) \leq \eta$ for every $e \in \vec{E}$. Indeed, choose $t \in [0,1)$ and let V_t be the set of $v \in V$ with $h(v) \leq t$. From (3.1) and (3.2), we get

$$\sum_{v \in V_t} \sum_{e \in \vec{E}(v)} dh(e) = \eta.$$

This is also equal to $\sum_e dh(e)$, where the sum extends over all $e \in \vec{E}$ with $e(0) \in V_t$ and $e(1) \notin V_t$. The terms in this last sum are all positive, so we have $dh(e) \leq \eta$, when $h(e(0)) < t < h(e(1))$. Since this is valid for every $t \in [0,1)$, we get $dh(e) \leq \eta$, as required.

We shall now prove (2); that is, that S is a tiling of $C \times [0,1]$. Let x be some point in $C - \{\theta(v^*): v^* \in V^*\}$ and let $\tilde{x} \in \mathbb{R}$ satisfy $\tilde{x} \equiv x \mod(\eta)$. Consider some vertex $v \in V - \{v_0, v_1\}$ and let $e_0, e_1, \ldots, e_{n-1}$ be the edges emanating from v in counterclockwise order. For any integer j let $e_j = e_{j'}$, where $j' = j \mod(n)$. Also set $f_j = (*e_j)(1)$; that is, $f_j \in V^*$ corresponds to the face of $\mathbb{R}^2 - G$ that lies t othe left of the edge e_j. Let $\theta_0 \in \mathbb{R}$ be such that $\theta_0 \equiv \theta(f_0) \mod(\eta)$ and define inductively $\theta_{j+1} = \theta_j + dh(e_{j+1})$. Clearly, $\theta_j \equiv \theta(f_j) \mod(\eta)$. Equation (3.1) implies that $\theta_n = \theta_0$. Consequently, the number of j in the range $\{0, \ldots, n-1\}$ such that $\theta_j < \tilde{x} < \theta_{j+1}$ is equal to the number of j in this range such that $\theta_j > \tilde{x} > \theta_{j+1}$. (Because $x \neq \theta_j$.) This implies that the number of $e \in \vec{E}^+(v)$ with $x \in I_e$ is equal to the number of $e \in \vec{E}^-(v)$ with the same property. Since this is valid for any $v \in V - \{v_0, v_1\}$, it easily follows that the number of squares S_e that contain the point (x, t) is independent of t, when t is in the range $[0, 1] - \{h(v): v \in V\}$.

On the other hand, if we replace the above v with v_0, then the sequence θ_j would be monotone nondecreasing and $\theta_n = \eta + \theta_0$. Consequently, for $t \in [0, 1]$ sufficiently close to 0, there is precisely one $e \in E(v_0)$ such that $(x, t) \in S_e$. Hence for every $(x, t) \in (C - \{\theta(v^*): v^* \in V^*\}) \times ([0, 1] - \{h(v): v \in V\})$ there is precisely one square in S that contains it. This shows that S is a tiling of $C \times [0, 1]$.

The remaining claims of the theorem now easily follow. $\square$

4. The infinite tiling theorem. We now generalize the tiling theorem to infinite, transient, planar, uniquely absorbing graphs.

THEOREM 4.1 (infinite cylinder tiling). *Let $G = (V, E)$ be a planar, connected, transient graph, embedded in the plane, which is uniquely absorbing. Let v_0 be a vertex in G and, for $v \in V$, let $h(v)$ denote the probability that a random walk that starts at v will never reach v_0. For $v \in V$ let $\vec{E}^+(v)$ denote the set of edges $e \in \vec{E}(v)$ such that $h(e(1)) \geq h(v)$ and let $\vec{E}^-(v)$ denote the set of edges $e \in \vec{E}(v)$ such that $h(e(1)) \leq h(v)$. Let $\eta = \sum_{e \in \vec{E}(v_0)} h(e(1))$ and let C be the circle of length η, $C = \mathbb{R}/\eta\mathbb{Z}$. Then there is a square tiling S of the cylinder $C \times [0, 1)$ with the following properties.*

(i) *The squares in the tiling are indexed by the edges of G, $S = (S_e: e \in E)$.*

(ii) *The interiors of the squares are disjoint.*

(iii) *When $e \in E$, $\partial e = \{v, u\}$ and $h(v) \geq h(u)$, the square S_e has the form $S_e = I_e \times [h(u), h(v)]$, where I_e is a closed subarc of C with length $h(v) - h(u)$. If $\vec{e}$ is an orientation of $e \in E$, we set $I_{\vec{e}} = I_e$, $S_{\vec{e}} = S_e$.*

(iv) *For every $v \in V$, set*

$$I_v = \bigcup_{e \in E(v)} I_e.$$

Then I_v is connected and

$$I_v = \bigcup_{e \in \vec{E}^+(v)} I_e = \bigcup_{e \in \vec{E}^-(v)} I_e \quad \text{for } v \in V - \{v_0\}.$$

(v) *For almost every $x \in C$, for every $t \in [0, 1)$, the segment $\{x\} \times [0, t]$ is contained in the union of finitely many squares in S.*

The claim that S is a tiling of $C \times [0, 1)$ merely means that the interiors of the squares are disjoint and that the union of all the squares in S is dense in $C \times [0, 1)$. It follows from (5) that each component of the complement of the tiling is a vertical line segment.

Figure 1 shows the tiling corresponding to the septagonal traingulation, where each vertex has seven neighbors.

We will need the following result, which is a special case of [6], Theorem 8.1.

LEMMA 4.2. *Let $G = (V, E)$ be a locally finite connected transient graph and let $v_0 \in V$. Let $h(v)$ denote the probability that a random walk starting at $v \in V$ will never reach v_0. Then dh is the optimal flow with source v_0 on G; that is, if $i: \vec{E} \to \mathbb{R}$ is a flow with source v_0 on G, then*

(4.1)
$$\|i\| \geq \frac{|\text{flux}(i)|}{\text{flux}(dh)} \|dh\|.$$

Moreover, we have

$$(4.2) \qquad\qquad \mathrm{flux}(dh) = \|dh\|^2.$$

We do not include a proof of the lemma, but remark that it may be proved by first establishing an analogous statement for flows with a source and a sink on a finite graph, then taking an exhaustion of G by finite graphs and deducing the infinite case from the finite case.

PROOF OF THEOREM 4.1. For positive integers n, let V^n denote the set of vertices of G with distance at most n from v_0. Let $G^n = (V^n, E^n)$ denote the restriction of G to V^n. Since G^n is finite and G is uniquely absorbing, there is a unique connected component D^n of $\mathbb{R}^2 - G^n$ such that $V \cap D^n$ is absorbing. Construct a new planar graph $G'^n = (V'^n, E'^n)$, as follows. Put a new vertex v_1^n in D^n. Let $V'^n = V^n \cup v_1^n$. For every edge $e \in E$ that connects a vertex v in V^n to a vertex in $V \cap D^n$ put an edge e' from v to v_1^n and let E'^n be the union of E^n and these new edges e'. It is clear that the embedding of G^n can be extended to an embedding of $G'^n = (V'^n, E'^n)$.

Now let $h^{(n)}(v)$ denote the probability that a random walk on G'^n that starts at v will reach v_1^n before reaching v_0. Apply the finite cylinder tiling theorem (Theorem 3.1) to each G'^n, to yield tilings $S^n = (S_e^n: e \in E'^n)$ of $(\mathbb{R}/\eta^{(n)}\mathbb{Z}) \times [0,1]$, say. It is easy to verify that $h^{(n)}(v) \to h(v)$ as $n \to \infty$ for every $v \in V$ (in fact, the convergence is monotone). Clearly, $\eta^{(n)} \to \eta$ and the fact that G is transient insures that $\eta > 0$. Pick a subsequence of $n = 1, 2, \ldots$ so that the coordinates of each square S_e^n, $e \in E$ converge as $n \to \infty$ in this subsequence. (Actually, if we normalize the tilings S^n by an appropriate rotation of $C^{(n)}$, then it is not necessary to take a subsequence; the squares would converge.) Let $S = (S_e: e \in E)$ be the limit configuration in $C \times [0,1]$, where $C = \mathbb{R}/\eta\mathbb{Z}$.

First, it is clear that each S_e is a square, in our broad use of this term. Moreover, each S_e is in $C \times [0,1)$, because $h(v) < 1$ for each $v \in V$. All the other claims in the theorem are immediate consequences of the corresponding statements for the tilings S^n, except for (5) [and the fact that the tiling is dense in $C \times [0,1)$, which follows from (5)].

In the proof of (5), we shall use Lemma 4.2. For every $x \in C$ there is a maximal $t(x) \in [0,1]$ such that for every $t \in [0, t(x))$ the segment $\{x\} \times [0,t]$ is contained in finitely many of the squares S_e.

We define $i: \vec{E} \to \mathbb{R}$ as follows. Let $e \in E$ and let $\vec{e}$ be an orientation of e such that $h(\vec{e}(1)) \geq h(\vec{e}(0))$. Set $i(\vec{e})$ to be the measure of the set of $x \in C$ such that the segment $\{x\} \times [0, t(x))$ intersects S_e and set $i(-\vec{e}) = -i(\vec{e})$. It follows from (2) and (4) that i is a flow on G with source v_0. Observe that $\mathrm{flux}(i) = \mathrm{length}(C) = \mathrm{flux}(dh)$. Therefore, (4.1) gives $\|i\| \geq \|dh\|$. However, we clearly have $|i(e)| \leq |dh(e)|$ for every $e \in \vec{E}$, and $i(e)$ has always the same sign as $dh(e)$. Hence, $i(e) = dh(e)$ for every $e \in \vec{E}$.

Let $A \subset C \times [0, 1)$ be the set of all $(x, t) \in C \times [0, 1)$ such that $t < t(x)$. Note that for each $e \in E$ the are of $A \cap S_e$ is equal to $|dh(e)||i(e)|$. Therefore, we have

$$(4.3) \qquad \int_{x \in C} t(x)\, dx = \text{area}(A) = \sum_{e \in E} \text{area}(A \cap S_e)$$

$$= \sum_{e \in E} |dh(e)||i(e)| = \|dh\|^2.$$

Now, recall that $\|dh\|^2 = \text{flux}(dh) = \eta$, by (4.2). So the integral in (4.3) is equal to η, which is the length of C. However, $t(x) \le 1$ everywhere, so we get $t(x) = 1$ for almost every $x \in C$. This proves (5). $\square$

5. Existence of harmonic functions. Throughout this section, we shall let G, v_0, h, η, C, S and I_v be as in Theorem 4.1. In particular, we will assume that G is a bounded valence, uniquely absorbing, planar graph. We shall use Ω to denote the least upper bound for the degrees of the vertices of G, and shall assume that $\Omega < \infty$.

NOTATIONS. Let C' be the set of all $x \in C$ such that for every $t \in (0, 1)$ the interval $\{x\} \times [0, t]$ intersects only finitely many squares in the tiling S and x is not in the boundary of any I_e for $e \in E$. Note that C' has full measure in C, by Theorem 4.1(v).

Given $x \in C'$ and $t \in (0, 1)$, there is precisely one directed edge $e \in \vec{E}$ such that $(x, t) \in S_e$ and $h(e(0)) < t \le h(e(1))$. In other words, (x, t) is in the square S_e, but not on its bottom edge. We will let $v(x, t)$ denote the vertex $e(1)$.

For every $v \in V$ we let x_v be some arbitrary point in I_v.

THEOREM 5.1. *Let $\phi\colon C \to R$ be a continuous function on C. Then there is a bounded harmonic function $g\colon V \to \mathbb{R}$ that satisfies*

$$\lim_{t \to 1} g(v(x, t)) = \phi(x)$$

for every $x \in C'$. The limit is uniform in x. If ϕ is Lipschitz, then g has finite Dirichlet energy.

The proof of this theorem will be completed in Section 7.

LEMMA 5.2. *Let $\phi\colon C \to \mathbb{R}$ be a Lipschitz function with Lipschitz constant $M < \infty$. Define $\hat{\phi}(v) = \phi(x_v)$ for $v \in V$. Then the Dirichlet energy of $\hat{\phi}$ satisfies*

$$\mathscr{D}(\hat{\phi}) \le 4\Omega^2 M^2 \eta < \infty.$$

PROOF. Let e be some edge of G with vertices v, u. Recall that I_v and I_u both contain I_e. Hence $I_v \cap I_u \neq \varnothing$ and we have

$$
\begin{aligned}
|\hat{\phi}(v) - \hat{\phi}(u)|^2 &= |\phi(x_v) - \phi(x_u)|^2 \\
&\leq M^2 d(x_v, x_u)^2 \\
&\leq M^2 \big(\mathrm{length}(I_v) + \mathrm{length}(I_u)\big)^2 \\
&\leq 2M^2 \big(\mathrm{length}(I_v)^2 + \mathrm{length}(I_u)^2\big).
\end{aligned}
$$

Since every vertex v belongs to at most Ω edges, we get

(5.1)
$$
\begin{aligned}
\mathscr{D}(\hat{\phi}) &= \tfrac{1}{2} \sum_{e \in \vec{E}} |\hat{\phi}(e(1)) - \hat{\phi}(e(0))|^2 \\
&\leq \sum_{v \in V} 4\Omega M^2 \, \mathrm{length}(I_v)^2.
\end{aligned}
$$

The length of I_v is equal to the sum of $\mathrm{length}(I_e)$ over all $e \in \vec{E}^+(v)$, where $\vec{E}^+(v)$ is as in the tiling Theorems 3.1 and 4.1. Hence

$$
\begin{aligned}
\mathrm{length}(I_v)^2 &= \bigg(\sum_{e \in \vec{E}(v)} \mathrm{length}(I_e) \bigg)^2 \\
&\leq |\vec{E}(v)| \sum_{e \in \vec{E}(v)} \mathrm{length}(I_e)^2 \\
&\leq \Omega \sum_{e \in \vec{E}(v)} \mathrm{length}(I_e)^2.
\end{aligned}
$$

This with (5.1) gives

$$
\begin{aligned}
\mathscr{D}(\hat{\phi}) &\leq 4\Omega^2 M^2 \sum_{e \in E} \mathrm{length}(I_e)^2 \\
&= 4\Omega^2 M^2 \sum_{e \in E} \mathrm{area}(S_e) \\
&\leq 4\Omega^2 M^2 \, \mathrm{area}(C \times [0,1)) = 4\Omega^2 M^2 \eta
\end{aligned}
$$

and the proof is complete. $\square$

We are now ready to prove the following weak form of Theorem 5.1.

LEMMA 5.3.   *Let $\phi \colon C \to R$ be a Lipschitz function on C. Then there is a bounded harmonic function $f \colon V \to \mathbb{R}$ with finite Dirichlet energy such that $\lim_{t \to 1} f(v(x,t)) = \phi(x)$ in L^1.*

PROOF.   Define $\hat{\phi}(v) = \phi(x_v)$ as in Lemma 5.2. From that lemma we know that $\hat{\phi}$ has finite Dirichlet energy. Let $s \in (0,1)$ and let V_s denote the set of vertices $v \in V$ such that $h(v) < s$. It is easy to see that in the collection of all functions $g \colon V \to \mathbb{R}$ that are equal to $\hat{\phi}$ outside of V_s there is a (unique)

function with least Dirichlet energy. Denote this minimizer by g_s. From the fact that g_s is a minimizer, it readily follows that g_s is harmonic in V_s, that $\mathscr{D}(g_s) \leq D(\hat{\phi})$ and that $\sup_v |g_s(v)| \leq \sup_v |\hat{\phi}(v)| \leq \sup_x |\phi(x)|$.

Fix some $t \in (0, 1)$. For every $x \in C'$ let γ_x denote the set of $e \in \vec{E}$ such that $t < h(e(0)) \leq h(e(1))$ and $x \in I_e$. Note that γ_x are the edges of a directed path that starts at $v(x, t)$ and contains all $v(x, \tau)$ with $\tau \in (t, 1)$. Since for every $\delta > 0$ there are at most finitely many v with $\text{length}(I_v) > \delta$, it follows from the continuity of ϕ that $\lim_{\tau \to 1} g_s(v(x, \tau)) = \lim_{\tau \to 1} \hat{\phi}(v(x, \tau)) = \phi(x)$ holds for every $x \in C'$. We can calculate

$$\int_{x \in C'} |\phi(x) - g_s(v(x, t))| \, dx$$

(5.2)
$$= \int_{x \in C'} |\lim_{\tau \to 1} g_s(v(x, \tau)) - g_s(v(x, t))| \, dx$$

$$\leq \int_{x \in C'} \sum_{e \in \gamma_x} |dg_s(e)| \, dx.$$

Now consider any edge e, oriented so that $h(e(0)) \leq h(e(1))$. If $h(e(0)) < t$, then $e \notin \gamma_x$ for every $x \in C'$. On the other hand, if $h(e(0)) \geq t$, the set of $x \in C'$ such that $e \in \gamma_x$ is contained in I_e. Therefore, the contribution of such an edge to the last integral in (5.2) is at most $\text{length}(I_e)|dg_s(e)|$. Let $m(e) = 0$ if $h(e(0)) < t$ or $h(e(1)) < t$, and $m(e) = \text{length}(I_e)$, otherwise. Then we can rewrite (5.2) as

$$\int_{x \in C'} |\phi(x) - g_s(v(x, t))| \, dx$$

(5.3)
$$\leq \sum_{e \in E} m(e)|dg_s(e)|$$

$$\leq \sqrt{\sum_{e \in E} m(e)^2} \sqrt{\sum_{e \in E} |dg_s(e)|^2}$$

$$\leq \sqrt{\text{area}(C \times [t, 1))} \sqrt{\mathscr{D}(g_s)} \leq \sqrt{\eta(1-t)\mathscr{D}(\hat{\phi})}.$$

Since for every $s \in (0, 1)$ and every $v \in V$ we have $|g_s(v)| \leq \sup_x |\phi(x)|$, there is some sequence of $s_1, s_2, \ldots$ in $(0, 1)$ tending to 1 so that $f(v) = \lim_n g_{s_n}(v)$ exists for every $v \in V$. Clearly, f is harmonic and satisfies $\mathscr{D}(f) \leq \mathscr{D}(\hat{\phi}) < \infty$. Moreover, since the functions g_s are uniformly bounded and $f(v(x, t)) = \lim_n g_{s_n}(v(x, t))$ pointwise, we have

$$\lim_n \int_{x \in C'} |g_{s_n}(v(x, t)) - f(v(x, t))| \, dx = 0.$$

This and (5.3) gives

$$\int_{x \in C'} |\phi(x) - f(v(x, t))| \, dx \leq \sqrt{\eta(1 - t)\mathscr{D}(\hat{\phi})},$$

which proves the lemma. $\square$

We now have the following theorem:

THEOREM 5.4. *Let $f\colon V \to \mathbb{R}$ have finite Dirichlet energy. Then the limit $\lim_{t \to 1} f(v(x, t))$ exists in $L^1(C)$.*

PROOF. The arguments in the proof of Lemma 5.3 show that the L^1 distance from $f(v(x, t))$ to $f(v(x, t'))$ (considered as functions of x) tends to zero as $|t - t'| \to 0$. $\square$

6. Behavior of the random walk.

We use the assumptions and notations of the previous section.

THEOREM 6.1. *Let $(v(0), v(1), \dots)$ be a simple random walk on G starting at any vertex $v(0) = v$. Then with probability 1 the limit $x = \lim_n x_{v(n)}$ exists.*

We will also get quite explicit estimates for the rate of convergence of $x_{v(n)}$ in terms of the speed at which $h(x_{v(n)}) \to 1$.

LEMMA 6.2. *Let $A \subsetneqq C$ be an arc in C and let L be the length of the complementary arc $C - A$. Let V_A be the set of vertices $v \in V$ such that $x_v \in A$. Let $p, t \in (0, 1)$ and let $Q(A, t, p)$ be the set of $x \in C' - A$ such that for every $t' \in [t, 1)$ the probability that the random walk that starts at $v(x, t')$ will ever reach V_A is smaller than p. Then*

$$\text{length}(Q(A, t, p)) \geq L - 6\sqrt{\Omega/p}\,\sqrt[4]{\eta L(1 - t)}\,.$$

Here "length" means one-dimensional Lebesgue measure. The point is that the length of $Q(A, t, p)$ tends to L as $t \to 1$.

PROOF OF LEMMA 6.2. Let $s \in (t, 1)$ be close to 1. As before, let V_s denote the set of vertices $v \in V$ such that $h(v) < s$. Let $\varepsilon > 0$ be some number to be determined later. For $x \in C$ let $\phi(x)$ be the minimum of ε and the distance from x to A, and set $\hat{\phi}(v) = \phi(x_v)$ for $v \in V$. Since ϕ has Lipschitz constant 1, Lemma 5.2 gives

$$\mathscr{D}(\hat{\phi}) \leq 4\Omega^2 \eta.$$

Let g_s be the function that is equal to $\hat{\phi}$ outside of $V_s - V_A$ and has least Dirichlet energy subject to this requirement. Then g_s is harmonic in $V_s - V_a$. Moreover,

(6.1) $$\mathscr{D}(g_s) \leq \mathscr{D}(\hat{\phi}) \leq 4\Omega^2 \eta.$$

Let B be the set of points $x \in C'$ such that $\phi(x) = \varepsilon$. Using the method of proof of Lemma 5.3, it can easily be established that

$$\int_{x \in B} \sup\{|\varepsilon - g_s(v(x, t'))| \colon t' \in [t, 1)\}\, dx$$

$$\leq \sqrt{\text{area}(B \times [t, 1))}\,\sqrt{\mathscr{D}(g_s)} = \sqrt{(L - 2\varepsilon)(1 - t)}\,\sqrt{\mathscr{D}(g_s)}\,.$$

This and (6.1) give

$$(6.2) \qquad \int_{x \in B} \sup\{|\varepsilon - g_s(v(x, t'))|: t' \in [t, 1)\}\, dx \leq 2\Omega\sqrt{\eta L(1 - t)}\,.$$

Consider some $x \in B - Q(A, t, p)$. There is some $t'(x) \in [t, 1)$ such that the probability that the random walk which starts at $v(x, t'(x))$ will reach V_A is at least p. Therefore, there is some $s(x) < 1$ such that with probability at least $p/2$ this random walk will reach V_A before it gets to $V - V_{s(x)}$. For any $s > s(x)$, we then have $g_s(v(x, t'(x))) \leq \varepsilon(1 - p/2)$, because $g_s \leq \varepsilon$ everywhere, g_s is harmonic in $V_s - V_A$ and $g_s = 0$ on V_A.

Let $Z(A, t, p, s)$ denote the set o $x \in B - Q(A, t, p)$ such that $s(x) < s$. [We can make sure that $Z(A, t, p, s)$ is measurable by, say, choosing $s(x)$ minimal.] The previous paragraph shows that $\sup\{|\epsilon - g_s(v(x, t'))|: t' \in [t, 1)\} \geq \epsilon p/2$ for $x \in Z(A, t, p, s)$. Therefore, using (6.2), we find that

$$\text{length}(Z(A, t, p, s)) \leq 4\Omega\sqrt{\eta L(1 - t)}\,\varepsilon^{-1}p^{-1}.$$

However, $B - Q(A, t, p) = \bigcup_{s < 1} Z(A, t, p, s)$ and this union is monotone. Consequently,

$$\begin{aligned}
\text{length}(Q(A, t, p)) &\geq \text{length}(B) - \lim_{s \to 1} \text{length}(Z(A, t, p, s)) \\
&\geq \text{length}(B) - 4\Omega\sqrt{\eta L(1 - t)}\,\varepsilon^{-1}p^{-1} \\
&= L - 2\varepsilon - 4\Omega\sqrt{\eta L(1 - t)}\,\varepsilon^{-1}p^{-1}.
\end{aligned}$$

Choosing $\varepsilon = \sqrt{\Omega/p}\,\sqrt[4]{\eta L(1 - t)}$ now establishes the lemma. $\square$

LEMMA 6.3. *Let* $p \in (0, 1)$, $u \in V$ *and let* $\xi = (v(0), v(1), \ldots)$ *be a random walk that starts at* $v(0) = u$. *Then*

$$\text{Prob}\left(\sup_n d(x_{v(n)}, x_u) > L/2\right) \leq p,$$

where

$$L = \frac{64\sqrt[3]{\Omega^2 \eta(1 - h(u))}}{p}.$$

PROOF. Let A be the arc of C consisting of all points x such that $d(x, x_u) \geq L/2$ and let V_A be the set of vertices $v \in V$ such that $x_v \in A$. Let $t = 1 - 2(1 - h(u))/p$. As before, V_t will denote the set of $v \in V$ with $h(v) < t$.

Let E_0 be the event that the walk ξ reaches V_A and let E_1 be the event that ξ reaches V_t. The probability that a random walk starting at a vertex v will reach v_0 is equal to $1 - h(v)$. Therefore, the probability that ξ will reach

v_0 conditioned on the event E_1 is at least $1 - t = 2(1 - h(u))/p$. So we have

$$1 - h(u) = \text{Prob}(\xi \text{ reaches } v_0)$$
$$\geq \text{Prob}(\xi \text{ reaches } v_0 | E_1)\text{Prob}(E_1)$$
$$\geq 2\frac{1 - h(u)}{p}\text{Prob}(E_1).$$

Therefore,

(6.3) $\text{Prob}(E_1) \leq p/2.$

Using the notations and conclusion of Lemma 6.2 (our usage of L is compatible with the lemma), we have

$$\text{length}(Q(A, t, p/2)) > L - 9\sqrt{\Omega/p}\sqrt[4]{\eta L(1 - t)}.$$

With the definitions of L and t, this gives

$$\text{length}(Q(A, t, p/2)) > L/2.$$

This implies that $Q(A, t, p/2)$ intersects any half of the arc $\{x \in C: d(x, x_u) < L/2\}$. Consequently, there are $x_1, x_2 \in Q(A, t, p/2)$ such that $\{x_1, x_2\}$ separates x_u from A in C.

Let E_2 be the event there is an $n = 0, 1, 2, \ldots$ such that (i) $v(n') \notin V_A$ for all $n' = 0, 1, \ldots, n - 1$, (ii) $v(n) \notin V_t$ and (iii) $I_{v(n)}$ contains x_1 or x_2. Since $I_{v(j)} \cap I_{v(j+1)} \neq \varnothing$ for every $j = 0, 1, 2, \ldots$ and $\{x_1, x_2\}$ separates x_u from A, we have $E_0 - E_1 \subset E_2$. So we may estimate

$$\text{Prob}(E_0) \leq \text{Prob}(E_0 - E_1) + \text{Prob}(E_1)$$
$$= \text{Prob}((E_0 - E_1) \cap E_2) + \text{Prob}(E_1)$$
$$\leq \text{Prob}(E_0 | E_2) + p/2.$$

In the last inequality we have used (6.3). Since $x_1, x_2 \in Q(A, t, p/2)$, the probability that a random walk that starts at any vertex of the form $v(x_j, t')$ with $j = 1, 2$ and $t' \geq t$ will ever reach a vertex v with $x_v \in A$ is at most $p/2$. If $x_j \in I_{v(n)}$, then $v(n) = v(x_j, h(v(n)))$. Hence $\text{Prob}(E_0 | E_2) \leq p/2$. Thus we have $\text{Prob}(E_0) \leq p$, proving the lemma. □

LEMMA 6.4. *Let $G = (V, E)$ be a connected transient graph and let $v_0 \in V$ be some vertex. For $v \in V$ let $h(v)$ denote the probability that the random walk that starts at v will never reach v_0 and let $(v(0), v(1), \ldots)$ be a random walk on G that starts at an arbitrary vertex $v(0)$. Then $\lim_n h(v(n)) = 1$ with probability 1.*

PROOF. Let $p(n)$ be the probability that $v(j) = v_0$ for some $j \geq n$. For every n there is an n' such that the probability that $v(j) = v_0$ for some $j = n, n + 1, \ldots, n'$ is greater than $p(n)/2$. Set $n_0 = 0$ and inductively $n_{j+1} = n'_j + 1$. The walk has a probability of at least $p(n_j)/2$ to visit v_0 at some time $t \in \{n_j, n_j + 1, \ldots n_{j+1} - 1\}$. Hence the expected number of visits of the walk

to v_0 is at least $\sum_j p(n_j)/2$. Because G is transient, this must be finite. Hence $p(n_j) \to 0$ as $j \to \infty$. Since $p(n)$ is monotone decreasing, it follows that $p(n) \to 0$ as $n \to \infty$.

Now note that $p(n)$ is the expected value of $1 - h(v(n))$. Since $h \le 1$ always, we get from $p(n) \to 0$ that $h(v(n)) \to 1$ in L^1. This implies the lemma. $\square$

We will now see that Theorem 6.1 follows immediately from Lemmas 6.3 and 6.4.

PROOF OF THEOREM 6.1.   Let $\varepsilon > 0$. Let n be the least integer such that $h(v(n)) > 1 - \varepsilon^6$. Lemma 6.4 tells us that with probability 1 such as n exists. Lemma 6.3 with $p = \varepsilon$ and $u = v(n)$ tells us that

$$\text{Prob}\left(\sup_{n' > n} d(x_{v(n)}, x_{v(n')}) > 32\sqrt[3]{\Omega^2 \eta}\, \varepsilon \right) \le \varepsilon.$$

Hence with probability at least $1 - \varepsilon$, we have

$$\limsup_{m,k \to \infty} d(x_{v(m)}, x_{v(k)}) \le 64\sqrt[3]{\Omega^2 \eta}\, \varepsilon.$$

Since this holds for every ε, it follows that $\limsup_{m,k \to \infty} d(x_{v(m)}, x_{v(k)}) = 0$ with probability 1, which implies that $\lim_n x_{v(n)}$ exists with probability 1. $\square$

7. More about harmonic functions.

DEFINITION.   Given a $v \in V$, we let μ_v denote the hitting measure on C. In other words, μ_v is the measure that assigns to each Borel subset $A \subset C$ the probability that for a random walk $(v(0), v(1), v(2), \dots)$ that starts at $v(0) = v$, we shall have $\lim_n x_{v(n)} \in A$.

LEMMA AND DEFINITION 7.1.   *Let $\phi: C \to R$ be a function in $L^1(\mu_{v_0})$. Define*

$$P(\phi)(v) = \int_C \phi(x)\, d\mu_v(x).$$

Then $P(\phi): V \to R$ is harmonic in V. The function $P(\phi)$ will be called the Poisson integral of ϕ.

We shall see subsequently that μ_{v_0} is equal to Lebesgue meaure on C. Therefore, $L^1(\mu_{v_0}) = L^1(C)$.

PROOF OF LEMMA 7.1.   Let $v, v' \in V$. Let p be the probability that a random walk that starts at v will reach v'. Clearly, $p > 0$. For any measurable $A \subset C$ we have $\mu_v(A) \ge p\mu_{v'}(A)$. This together with the corresponding opposite inequality imply that a function is in $L^1(\mu_v)$ if and only if it is in $L^1(\mu_{v'})$. Hence $P(\phi)$ is well defined. If $u_1, \dots, u_k$ are the neighbors of a vertex u, then clearly μ_u is the average of $\mu_{u_1}, \dots, \mu_{u_k}$. This shows that $P(\phi)$ is harmonic. $\square$

PROOF OF THEOREM 5.1.   Take $g = P(\phi)$. It is an easy consequence from Lemma 6.3 that $g(v(x, t)) \to \phi(x)$ as $t \to 1$ uniformly in $x \in C'$.

Now suppose that ϕ is Lipschitz. Recall our construction of the function $\hat{\phi}$: $V \to \mathbb{R}$, the functions g_s and their limit f in the proof of Lemma 5.3. There we have shown that f has finite Dirichlet energy. Now note that $|g_s(v) - P(\phi)(v)|$ is small for $v \notin V_s$, if s is close to 1. (Again, this follows from Lemma 6.3.) Hence, $|g_s(v) - P(\phi)(v)|$ is small for every $v \in V$, for s close to 1. This gives then $f(v) = P(\phi)(v) = g(v)$, so g has finite Dirichlet energy. $\square$

The following proposition shows that in a certain limited sense the harmonic function with prescribed boundary values is unique.

PROPOSITION 7.2.   *Let* $\phi: C \to R$ *be continuous and let* $f: V \to R$ *be harmonic. Suppose that*

$$\phi(x) = \lim_{t \to 1} f(v(x, t))$$

uniformly for $x \in C'$. *Then* $f = P(\phi)$.

PROOF.   First, since ϕ is bounded, $\phi \in L^1(\mu_{v_0})$ and $P(\phi)$ exists.

Let $\varepsilon > 0$. From the uniform convergence of $f(v(x, t))$ to ϕ we know that if $\delta > 0$ is chosen sufficiently small, then

$$|f(v) - \phi(x_v)| < \varepsilon/2$$

holds for every $v \in V$ satisfying $h(v) > 1 - \delta$. It is also easy to see that Lemma 6.3 and the continuity of ϕ imply that

$$|P(\phi)(v) - \phi(x_v)| < \varepsilon/2$$

holds for every $v \in V$ satisfying $h(v) > 1 - \delta$, if δ is sufficiently small. Hence $|f(v) - P(\phi)(v)| < \varepsilon$ is true for all v with $1 - h(v)$ sufficiently small. Now Lemma 6.4 shows that $|f(v) - P(\phi)(v)| < \varepsilon$ for every $v \in V$. Since $\varepsilon > 0$ is arbitrary, this completes the proof. $\square$

THEOREM 7.3. μ_{v_0} *is equal to Lebesgue measure on* C.

PROOF.   Let $\phi: C \to R$ be a Lipschitz function and let $f = P(\phi)$. We know from the proof of Theorem 5.1 that f has finite Dirichlet energy; that is, $\|\nabla f\|$ is finite. Since f is harmonic, ∇f is a divergence-free flow on G. For $t \in R$, $\nabla h + t \nabla f$ is a flow on G with source v_0 and flux$(\nabla h + t \nabla f) = $ flux(∇h). However, we know from Lemma 4.2 that ∇h is the least energy flow on G with source v_0 and flux(∇h). Therefore

(7.1) $$0 = \frac{d}{dt}\Big|_{t=0} \|\nabla h + t \nabla f\|^2 = 2 \nabla h \cdot \nabla f.$$

We now make an argument similar to one given in the proof of Lemma 5.3. For each $x \in C'$, let β_x denote the set of $e \in \vec{E}$ so that $e(0) \le e(1)$ and S_e intersects the segment $\{x\} \times [0, 1)$. Then β_x is the collection of edges of a directed path that starts at v_0 and contains every $v(x, t)$.

We compute,

$$f(v_0) - \int_{x \in C} \phi(x)\, dx = \int_{x \in C'} \left(f(v_0) - \lim_{t \to 1} f(v(x, t)) \right) dx$$

(7.2)

$$= \int_{x \in C'} \left(\sum_{e \in \beta_x} df(e) \right) dx.$$

Now we want to see that the last expression is absolutely convergent. Consider,

(7.3)
$$\int_{x \in C'} \left(\sum_{e \in \beta_x} |df(e)| \right) dx.$$

We may examine the contribution of any edge $e \in \vec{E}$ to this expression. Suppose that $h(e(0)) \le h(e(1))$. Then the set of $x \in C'$ such that $e \in \beta_x$ is exactly $C' \cap I_e$, which has length $|dh(e)|$. Hence, (7.3) is equal to $|\nabla h| \cdot |\nabla f| < \infty$. This means that the last expression in (7.2) is absolutely convergent, and the same kind of argument shows that it is equal to $\nabla h \cdot \nabla f$, which we know is zero from (7.1). Hence we get

$$\int_{x \in C} \phi(x)\, d\mu_{v_0} = P(\phi)(v_0) = f(v_0) = \int_{x \in C} \phi(x)\, dx.$$

Since this is true for every Lipschitz $\phi: C \to \mathbb{R}$, the theorem follows. $\square$

QUESTION 7.4. Is C a realization of the Poisson boundary of G?

8. When G is not end convergent. Theorem 5.1 shows that on any transient, bounded valence, planar, uniquely absorbing graph G, there are nonconstant, bounded, harmonic, Dirichlet functions. In [1], we have proved this without the assumption that G is uniquely absorbing. This latter assumption is easy to get rid of, because of the following theorem, which is known (see [5], page 79).

THEOREM 8.1. *Let G be a bounded valence connected graph, which is transient but not end convergent. Then there are nonconstant, bounded harmonic functions on G with finite Dirichlet energy.*

PROOF. Since G is not end convergent, there is a finite set $K \subset V$ such that there is more than one absorbing component of $G - K$. This implies that one can write $V - K$ as a disjoint union $V - K = A \cup B$, such that A and B are absorbing and the vertices of each component of $G - K$ are contained either in A or in B. With probability 1 a random walk on G will visit K only finitely many times. Hence it is either eventually in A or eventually in B. For any $v \in V$ let $h(v)$ be the probability that a random walk that starts at v will eventually stay in A. It is clear that h is a bounded nonconstant harmonic function. We shall show that it also has finite energy.

Let V_n be the set of vertices in V with distance at most n to K. Let $y_n: V \to \mathbb{R}$ be the function that is 0 on $B - V_n$, 1 in $A - V_n$ and harmonic in V_n.

Because with probability 1 the random walk visits K finitely many times, it is not difficult to see that $h_n \to h$ pointwise as $n \to \infty$. This implies that

$$(8.1) \qquad \mathscr{D}(h) \le \limsup_n \mathscr{D}(h_n).$$

Now

$$(8.2) \qquad \begin{aligned} \mathscr{D}(h_n) &= \tfrac{1}{2} \sum_{e \in \vec{E}} dh_n(e)\big(h_n(e(1)) - h_n(e(0))\big) \\ &= -\sum_{v \in V} \sum_{e \in \vec{E}(v)} dh_n(e) h_n(v) \\ &= -\sum_{v \in V} h_n(v) \operatorname{div} dh_n(v) = -\sum_{v \in A - V_n} \operatorname{div} dh_n(v). \end{aligned}$$

Since K disconnects A from B, and dh_n has finite support, it follows that

$$(8.3) \qquad -\sum_{v \in A} \operatorname{div} dh_n(v) = \sum_{e \in \vec{E}(A, K)} dh_n(e),$$

where $\vec{E}(A, K)$ denotes the set of all $e \in \vec{E}$ with $e(0) \in A$ and $e(1) \in K$. However, $\operatorname{div} dh_n = 0$ in V_n, so it follows from (8.2) and (8.3) that

$$\mathscr{D}(h_n) \le \sum_{e \in \vec{E}(A, K)} |dh_n(e)| \le \sum_{v \in K} \sum_{e \in \vec{E}(v)} |dh_n(e)|.$$

Taking the lim sup and using (8.1), we get

$$\mathscr{D}(h) \le \sum_{v \in K} \sum_{e \in \vec{E}(v)} |dh(e)| < \infty,$$

which establishes the claim. $\square$

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