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# ON THE DENSITY OF THE MAXIMUM OF SMOOTH GAUSSIAN PROCESSES

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We obtain an integral formula for the density of the maximum of smooth Gaussian processes. This expression induces explicit nonasymptotic lower and upper bounds which are in general asymptotic to the density. Moreover, these bounds allow us to derive simple asymptotic formulas for the density with rate of approximation as well as accurate asymptotic bounds. In particular, in the case of stationary processes, the latter upper bound improves the well-known bound based on Rice's formula. In the case of processes with variance admitting a finite number of maxima, we refine recent results obtained by Konstant and Piterbarg in a broader context, producing the rate of approximation for suitable variants of their asymptotic formulas. Our constructive approach relies on a geometric representation of Gaussian processes involving a unit speed parameterized curve embedded in the unit sphere.

**1. Introduction.** Let $X(t)$, $t \in I = [0, T]$, be a real Gaussian process with mean 0 and continuous sample functions. Numerous papers have been devoted to the study of

$$Z = \sup_{t \in I} X(t)$$

[see the monographs by Adler (1981, 1990), Berman (1992), Leadbetter, Lindgren and Rootzen (1983), Ledoux and Talagrand (1991), and Piterbarg (1996)]. It turns out that the exact distribution of $Z$ is known for the Wiener process, the Brownian bridge $B(t)$, $B(t) - \int_0^1 B(u)\,du$ [Darling (1983)], the integrated Wiener process [see, e.g., Lachal (1991)], a class of sawtooth processes [see, e.g., Cressie (1980)] and the random cosine wave $X(t) = \xi_1 \cos(t) + \xi_2 \sin(t)$, where $\xi_1$ and $\xi_2$ are i.i.d. $\mathscr{N}(0, 1)$.

Otherwise, two directions have been mainly explored. The first one consists of deriving, under minimal restrictions, upper and lower bounds for $P\{Z > a\}$ for $a$ large enough, or first-order asymptotics for $P\{Z > a\}$ as $a \to \infty$. At this level of generality, these bounds often involve unknown constants and are not sharp enough to be used as $p$-values in statistical tests and stochastic modelization where precise estimations of $P\{Z > a\}$, lying between 0.1 and 0.01, are required. Works of the second category try to obtain precise asymptotics for $P\{Z > a\}$ under more rigid restrictions on the process (stationarity,

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smooth covariance function, ...). In our approach, we study the nonasymptotic behavior of the density $f_Z(b)$ of $Z$. We derive an integral representation for $f_Z(b)$ which induces sharp explicit upper and lower bounds for all $b > 0$. It appears that these bounds are asymptotic to the density as $b \to \infty$. Moreover, they allow us to derive simple asymptotic formulas for $f_Z(b)$ with rate of approximation as well as specially accurate asymptotic bounds (see, e.g., the examples in subsection 2.3). Note that Weber (1985) and Lifshits (1986) have studied the density of $Z$ in a broader context and they have obtained bounds involving unknown constants.

In this paper, we consider the class of centered Gaussian processes of the form

$$X(t) = \tau_X^{-1}(t) \sum_{j=1}^{n} \xi_j g_j(t), \qquad 2 \le n \le \infty, \tag{1}$$

where $\xi_j$, $j \ge 1$, are i.i.d. $\mathcal{N}(0,1)$, $\sum_{j=1}^n g_j^2(t) \equiv 1$ and the functions $\tau_X(t) > 0$ and $g_j(t)$, $j \ge 1$, are sufficiently smooth.

The existence of the Karhunen–Loève expansion for Gaussian processes with continuous sample functions ensures that this class is very large [Adler (1990)]. First, all Gaussian processes with smooth covariance function have such a representation. Second, more general processes can be approximated by processes of this class with respect to the uniform norm. Indeed, in the context of nonsmooth Gaussian processes, a classical approach consists of working with the regularized version $X_\delta(t) = \int_{-\infty}^{\infty} X(t-s)\psi_\delta(s)\,ds$, where $\psi_\delta(t) = \psi(t/\delta)/\delta$ is a smooth kernel. By letting $\delta \to 0$, $X_\delta(t)$ converges to $X(t)$ under weak assumptions [Azaïs and Florens-Zmirou (1987)]. Since $P\{\sup_{t\in I} |X(t) - X_\delta(t)| > \alpha\}$ is basically controlled by $\sup_{t\in I} \operatorname{Var}(X(t) - X_\delta(t))$, choosing $\delta$ such that $\sup_{t\in I} \operatorname{Var}(X(t) - X_\delta(t))$ is sufficiently small should allow us to transform sharp bounds for $P\{\sup_{t\in I} X_\delta(t) > a\}$ into usable bounds for $P\{Z > a\}$ with $a$ in a suitable fixed compact subset of $(0, \infty)$. Indeed, our motivation in undertaking this study was to provide good approximations for $P\{Z > a\}$ around practical values of $a$, that is, around the 0.95 quantile, for smooth processes and processes whose covariance function admits a Taylor expansion sufficiently close to that of smooth processes in the neighborhood of the diagonal. These approximations are under current research. Of course, the possibility of such approximations has no direct relation with the possible divergent first-order asymptotic expansions of the tail distributions as $a \to \infty$. For stationary Gaussian processes, the truncation of the spectral measure also leads to the form (1). See also Berman (1988) for stochastic modelizations leading to processes (1), Davies (1977) for a class of statistical tests and Konakov and Piterbarg (1983) for confidence regions in nonparametric density estimation involving quantiles of the distribution of the maximum of smooth Gaussian processes.

Berman (1988) studies the asymptotic behavior of the tail probabilities of the supremum $Z$ of processes of the form (1) with $\tau_X(t) \equiv 1$, finite $n$ and orthogonally invariant joint distribution of $(\xi_1, \dots, \xi_n)$. In the normal case, his results render the well-known expressions (Theorem 18.1, page 37):

$$P\{Z > a\} \sim L(2\pi)^{-1} \exp(-a^2/2), \qquad a \to \infty, \tag{2}$$

where $L = \int_0^T (\sum_{j=1}^n g_j'^2(t))^{1/2}\, dt$, and (Corollary 17.1, page 36)

$$P\{Z > a\} \le L(2\pi)^{-1} \exp(-a^2/2) + \int_a^\infty (2\pi)^{-1/2} \exp(-x^2/2)\, dx, \qquad a > 0. \tag{3}$$

The main term appearing in both expressions results from Rice's formula, which measures the expected number of upcrossings of a fixed level $a$ [Marcus (1977)]. In subsection 2.3, we improve the upper bound (3) for smooth stationary processes providing a higher-order expansion for $P\{Z > a\}$.

Johnstone and Siegmund (1989) consider processes of the form (1) with $\tau_X(t) \equiv 1$, finite $n$ and $(\xi_1, \dots, \xi_n)$ uniformly distributed on the unit sphere. By making use of the connection between the standard Gaussian distribution in $\mathbb{R}^n$ and the uniform distribution on the unit sphere of $\mathbb{R}^n$, we can adapt their result (Theorem 3.3, page 190) to our context. It turns out that the resulting upper bound is (3).

Sun (1993) investigates an asymptotic expansion for the tail probabilities of the maximum of smooth Gaussian random fields with unit variance. In the special case of processes, her results concern periodic processes of the form (1) with $\tau_X(t) \equiv 1$. For finite $n$, Sun obtains the asymptotic formula (2) (Theorem 3.1, page 40) as a consequence of Weyl's formula for the volume of tubes around a manifold embedded in the unit sphere. For infinite $n$, (2) still holds under additional assumptions, otherwise it becomes an upper bound (Theorems 3.2 and 3.3, page 41).

The sharpest results concerning smooth Gaussian processes are due to Piterbarg (1981, 1988) and Konstant and Piterbarg (1993) who produce very precise asymptotic formulas for $P\{Z > a\}$. In subsection 2.3, our results are compared to theirs. In particular, we provide rates of approximation for suitable variants of the asymptotic formulas given in Konstant and Piterbarg (1993).

Our approach is based on the interpretation of the functions $g_j(t)$, $j \ge 1$, as a parameterization of a curve embedded in the unit sphere of $\mathbb{R}^n$ or of the space of square summable sequences. With the canonical moving frame induced by this parameterization, we describe each level manifold $\{z = b\}$, $b \in \mathbb{R}$, of the functional

$$z = \sup_{t \in I} \tau_X^{-1}(t) \sum_{j=1}^n x_j g_j(t),$$

where $\{x_j\colon\ j \ge 1\}$ is a realization of $\{\xi_j\colon\ j \ge 1\}$, as an envelope of the family of hyperplanes $\{\tau_X(t)^{-1} \sum_{j=1}^n x_j g_j(t) = b\colon\ t \in I\}$. This technique enables us to express the density $f_Z(b)$ of $Z$ as the canonical volume of $\{z = b\}$, leading to an integral formula for $f_Z(b)$ (Theorem 1). This representation provides (under weaker assumptions due to a perturbation argument) nonasymptotic lower and upper bounds for $f_Z(b)$ and $P\{Z > a\}$ with remarkable asymptotic features. Moreover, it allows us to handle processes with quite general varying variance, to deal with the case $n = \infty$ and to manage the boundary effects with great care.

The remainder of the paper is organized as follows. Our main results are presented in Section 2 for $n < \infty$ and are extended to the infinite case in Section 3. The theorems in Sections 2 and 3 are proved in Section 4. Some related known results of differential geometry are briefly introduced in the Appendix.

NOTATION. Throughout the paper, a.e. means almost every, $\ell^2$ refers to the Hilbert space of square summable sequences, $\mathbf{x} = (x_1, x_2, \ldots)$ is an element of either $\mathbb{R}^n$ or $\ell^2$, $\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{j=1}^{n} x_j y_j$, $2 \le n \le \infty$, $\|\mathbf{x}\| = \langle \mathbf{x}, \mathbf{x} \rangle^{1/2}$, $\mathrm{Vect}(\mathbf{x}_1, \ldots, \mathbf{x}_d)$ and $\mathrm{Vect}^{\perp}(\mathbf{x}_1, \ldots, \mathbf{x}_d)$ denote the linear subspace spanned by $(\mathbf{x}_1, \ldots, \mathbf{x}_d)$ and its orthogonal, respectively, $\mathrm{Gram}(\mathbf{x}_1, \ldots, \mathbf{x}_d)$ is the determinant of the matrix $G$ with entries $G_{ij} = \langle \mathbf{a}_i, \mathbf{a}_j \rangle$ [note that $\det G = \det^2(\mathbf{x}_1, \ldots, \mathbf{x}_d)$]. $\mu_n$ is the Gaussian measure on $\mathbb{R}^n$ with density $\varphi_n(\mathbf{x}) = (2\pi)^{-n/2} \exp(-\|\mathbf{x}\|^2/2)$, $\varphi(x) = \varphi_1(x)$ and $\Phi(x) = \int_{-\infty}^{x} \varphi(y)\, dy$. By convention, $\varphi(\infty) = 0$ and $\Phi(\infty) = 1$. $\mathscr{C}^m(A)$ denotes the set of functions $A \to \mathbb{R}$ having $k$th-order continuous derivatives for $k = 1, \ldots, m$. The partial derivatives $\partial^{k+l} r(x, y)/\partial x^k \partial y^l$ where $r \in \mathscr{C}^{k+l}(A)$ are written $D_{kl} r(x, y)$. The Jacobian matrix of a differentiable mapping $\mathbf{p}$: $\mathbb{R}^n \to \mathbb{R}^n$ is denoted $D\mathbf{p}$.

## 2. Main results.

2.1. *An integral formula.* Let $X(t)$, $t \in I = [0, T]$, be a Gaussian process with mean 0 and variance $\sigma_X^2(t) > 0$, of the form

$$X(t) = \tau_X^{-1}(t)\, U(t), \qquad U(t) = \langle \boldsymbol{\Xi}, \mathbf{g}(t) \rangle, \tag{4}$$

where $\tau_X(t) = \sigma_X^{-1}(t)$, $\mathbf{g}(t) = (g_1(t), \ldots, g_n(t))$, $n \ge 2$, and $\boldsymbol{\Xi} = (\xi_1, \ldots, \xi_n)$ is a Gaussian r.v. with zero mean and identity covariance matrix. With this representation, the covariance function $r_X(t_1, t_2)$ of $X(t)$ is given by

$$r_X(t_1, t_2) = \tau_X^{-1}(t_1)\, \tau_X^{-1}(t_2)\, r_U(t_1, t_2) = \tau_X^{-1}(t_1)\, \tau_X^{-1}(t_2)\, \langle \mathbf{g}(t_1), \mathbf{g}(t_2) \rangle$$

and $D_{kl} r_U(t_1, t_2) = \langle \mathbf{g}^{(k)}(t_1), \mathbf{g}^{(l)}(t_2) \rangle$. Since $\sigma_X^2(t) = r_X(t, t) = \|\mathbf{g}(t)\|^2/\tau_X^2(t)$, $\|\mathbf{g}(t)\|^2 = r_U(t, t) \equiv 1$ and $\mathbf{g}(t)$ parameterizes a curve $\gamma$ embedded in the unit sphere $S^{n-1}$ in $\mathbb{R}^n$. Let us denote $\rho_{kl,u}(t) = D_{kl} r_U(t_1, t_2)\,|_{t_1 = t_2 = t}$. In this subsection, we assume that

CONDITION 1. $\tau_X(t)$ is in $\mathscr{C}^2(I)$ and $r_U(t_1, t_2)$ has continuous partial derivatives $D_{kl} r_U(t_1, t_2)$ for $0 \le k, l \le 2$.

CONDITION 2. $\rho_{11,u}(t) \ne 0$ for all $t \in I$.

CONDITION 3. $\{t\colon\ c_g(t) = 0\} \subset \{t\colon\ \tau_X(t) - \tau_X'(t)\rho_{12,u}(t)/\rho_{11,u}^{3/2}(t) + \tau_X''(t)/\rho_{11,u}(t) > 0\}$, where the function $c_g(t) \ge 0$ defines the geodesic curvature of $\gamma$ at the point $\mathbf{g}(t)$ (see the Appendix) and is given by $c_g^2(t) = (\rho_{11,u}(t)\rho_{22,u}(t) - \rho_{12,u}^2(t))/\rho_{11,u}^3(t) - 1$.

CONDITION 4. Whenever $\mathbf{g}(t)$, $\mathbf{g}'(t)$ and $\mathbf{g}(t')$ are linearly dependent for $t' \neq t$, $\tilde{\mathbf{g}}(t') \neq \tilde{\mathbf{g}}(t)$ where $\tilde{\mathbf{g}}(t) = \tau_X(t)\mathbf{g}(t) + \tau'_X(t)\mathbf{g}'(t)/\rho_{11,u}(t)$.

REMARK. We can relax Condition 2, allowing a finite set $I_0$ of points such that $\rho_{11,u}(t) = 0$ for $t \in I_0$. Though technical, Conditions 3 and 4 are very weak and easily checked. A sufficient standard condition is that $(U(t), U'(t), U''(t), U(t'))$ admits a joint density for all $t' \neq t$. However, Conditions 3 and 4 are required only for the derivation of an explicit formula in Theorem 1 and will not be used in the following subsections.

REMARK. If $\sigma_X(t) \equiv 1/\tau$ is constant, Condition 3 is automatically satisfied, $\tilde{\mathbf{g}}(t) = \tau\mathbf{g}(t)$ and Condition 4 means that the curve $\gamma$ has no self-intersection, that is, $t' \neq t \Rightarrow \mathbf{g}(t') \neq \mathbf{g}(t)$. In other words, $r_U(t, t') < 1$ for all $t, t' \in (0, T)$ if $\mathbf{g}(t)$ [i.e., $U(t)$] is $T$-periodic and for all $t, t' \in [0, T]$ otherwise.

We are interested in finding nonasymptotic estimates for the distribution of

$$Z = \sup_{t \in I} X(t).$$

The key idea of our approach is to transform this problem into a geometric problem concerning the standard Gaussian measure of certain convex subsets of $\mathbb{R}^n$. We obtain an integral formula for the density $f_Z$ of $Z$ which is stated in Theorem 1. The derivation of this formula which is sketched below is greatly simplified if we parameterize $\gamma$ with unit speed. This can be done without loss of generality when Condition 2 holds. Let us define the Gaussian process $Y(s)$, $s \in J$, as

$$Y(s) = \tau_Y^{-1}(s)V(s), \qquad V(s) = \langle \mathbf{\Xi}, \mathbf{f}(s) \rangle,$$

where $\tau_Y(s) = \tau_X(\lambda^{-1}(s))$, $\mathbf{f}(s) = \mathbf{g}(\lambda^{-1}(s))$ and $s = \lambda(t) = \int_0^t \rho_{11,u}^{1/2}(t')\,dt'$ defines a unit speed parameterization of $\gamma$, $J = [0, L]$ with $L = |\gamma| = \lambda(T)$. Then we have

$$Z = \sup_{t \in I} X(t) = \sup_{s \in J} Y(s).$$

The covariance function of $Y(s)$ is given by $r_Y(s_1, s_2) = r_X(\lambda^{-1}(s_1), \lambda^{-1}(s_2))$. Moreover, $\|\mathbf{f}(s)\| = \|\mathbf{f}'(s)\| \equiv 1$ for all $s \in J$. Note that, in terms of $s = \lambda(t)$, Condition 3 becomes: $\{s\colon\ c_g^2(s) = \|\mathbf{f}''(s)\|^2 - 1 = 0\} \subset \{s\colon\ \tau_Y(s) + \tau''_Y(s) > 0\}$.

For simplicity, all our results will be expressed in terms of this unit speed parameterization. However, simple transformations give the corresponding formulas expressed in the original parameterization. In practice, only the latter are used.

Our method relies on the existence of an orthonormal moving frame $(\mathbf{f}(s), \mathbf{T}(s), \mathbf{K}_1(s), \ldots, \mathbf{K}_{n-2}(s))$ of $\mathbb{R}^n$ such that the space tangent to $S^{n-1}$ at $\mathbf{f}(s)$ is spanned by $(\mathbf{T}(s), \mathbf{K}_1(s), \ldots, \mathbf{K}_{n-2}(s))$ and $\mathbf{T}(s) = \mathbf{f}'(s)$ (see the Appendix).

Let us consider the realization of $(\Omega, \mathscr{A}, P)$ as $(\mathbb{R}^n, \mathscr{B}(\mathbb{R}^n), \mu_n)$, where $\mathscr{B}(\mathbb{R}^n)$ denotes the Borel $\sigma$-field of $\mathbb{R}^n$. It follows that

$$Y(s, \mathbf{x}) = \tau_Y^{-1}(s)\langle \mathbf{x}, \mathbf{f}(s)\rangle, \qquad \mathbf{x} \in \mathbb{R}^n,$$

is a realization of the process $Y(s)$ and

$$P\{Z \le a\} = \mu_n(C_a), \qquad a \in \mathbb{R}, \tag{5}$$

where

$$C_a = \Big\{\mathbf{x} \in \mathbb{R}^n \colon \sup_{s \in J} Y(s, \mathbf{x}) \le a\Big\}.$$

The boundaries $\partial C_b$ of $C_b$, $b \le a$, partition $C_a$. Indeed, by Lemma 7 in Section 4,

$$\partial C_b = \Big\{\mathbf{x} \in \mathbb{R}^n \colon \sup_{s \in J} Y(s, \mathbf{x}) = b\Big\}.$$

This suggests that a suitable change of variable will express $\mu_n(C_a)$ as an integral over $b \le a$ of appropriate superficial measures of $\partial C_b$:

$$\mu_n(C_a) = \int_{-\infty}^{a} \psi_b(\partial C_b)\, db = \int_{-\infty}^{a} f_Z(b)\, db. \tag{6}$$

Such a decomposition can be worked out basically—for simplicity, we assume $Y(s)$ $[X(t)]$ periodic here, the aperiodic case can be treated essentially in the same way—because it is possible (see Lemma 9) to parameterize $\partial C_b$ by

$$\mathbf{p}_b(s, u) = c_1(b, s)\mathbf{f}(s) + c_2(b, s)\mathbf{T}(s) + \sum_{j=1}^{n-2} u_j \mathbf{K}_j(s),$$

where $s \in J$, $u = (u_1, \dots, u_{n-2}) \in D_b(s)$, $c_1(b, s)$ and $c_2(b, s)$ are defined in terms of $b$, $\tau_Y(s)$ and $\tau_Y'(s)$, and $D_b(s)$ is a closed convex subset of $\mathbb{R}^{n-2}$. We show in Lemmas 11 and 12 that the transformation $\mathbf{p}$: $(b, s, u) \to \mathbf{p}_b(s, u)$ is a $C^1$-diffeomorphism from an open subset of $\mathbb{R}^n$ into $\mathbb{R}^n$. By the change-of-variable formula and Fubini's theorem, we have, for all $A \in \mathscr{B}(\mathbb{R}^n)$,

$$\begin{aligned}
\mu_n(A) &= \int_{(b, s, u) \in \mathbf{p}^{-1}(A)} \varphi_n(\mathbf{p}(b, s, u)) \mathrm{Gram}^{1/2} D\mathbf{p}(b, s, u)\, db\, ds\, du \\
&= \int_{b \in \mathbb{R}} \int_{(s, u) \in \mathbf{p}_b^{-1}(A \cap \partial C_b)} \varphi_n(\mathbf{p}_b(s, u)) \mathrm{Gram}^{1/2} D\mathbf{p}(b, s, u)\, ds\, du\ db \\
&= \int_{b \in \mathbb{R}} \psi_b(A \cap \partial C_b)\, db.
\end{aligned}$$

Since $C_a \cap \partial C_b = \partial C_b$ if $b \le a$ and $C_a \cap \partial C_b = \varnothing$ otherwise, we obtain (6).

REMARK. The canonical superficial measure on $\partial C_b$ induced by $\varphi_n(\mathbf{x})$ is defined by

$$\psi_b^*(A \cap \partial C_b) = \int_{(s, u) \in \mathbf{p}_b^{-1}(A \cap \partial C_b)} \varphi_n(\mathbf{p}_b(s, u)) \mathrm{Gram}^{1/2} D\mathbf{p}_b(s, u)\, ds\, du$$

[Berger and Gostiaux (1988), page 203]. If $\sigma_Y(s) \equiv 1/\tau$ is constant, it results from Lemma 11 that $\psi_b(A \cap \partial C_b) = \tau \psi_b^*(A \cap \partial C_b)$ for all $A \in \mathscr{B}(\mathbb{R}^n)$. Otherwise, $\psi_b$ appears as a weighted version of $\psi_b^*$, with weight $\tau_Y(s)$:

$$\psi_b(A \cap \partial C_b) = \int_{(s,u) \in \mathbf{p}_b^{-1}(A \cap \partial C_b)} \varphi_n(\mathbf{p}_b(s,u)) \tau_Y(s) \mathrm{Gram}^{1/2} D\mathbf{p}_b(s,u)\, ds\, du. \tag{7}$$

The above approach leads to the following expression for $f_Z(b)$, $b \in \mathbb{R}$.

THEOREM 1. *Under Conditions* 1–4, *the density of* $Z$ *is given by*

$$\begin{aligned} f_Z(b) = \int_0^L \int_{D_b(s)} & \tau_Y(s)\big(b(\tau_Y(s) + \tau_Y''(s)) - u_1 c_g(s)\big) \\ & \times \varphi_n(\mathbf{p}(b,s,u))\, du_1 \cdots du_{n-2}\, ds + \delta_Z(b), \end{aligned} \tag{8}$$

*where*

$$\delta_Z(b) = \begin{cases} 0, & Y(s)\ L\text{-periodic}, \\ \tau_Y(0)\varphi(b\tau_Y(0))\mu_{n-1}(G_b(0)) & \\ \qquad + \tau_Y(L)\varphi(b\tau_Y(L))\mu_{n-1}(G_b(L)), & \text{otherwise}, \end{cases}$$

*with*

$$D_b(s) = \Big\{u = (u_1, \ldots, u_{n-2}) \in \mathbb{R}^{n-2}\colon \sup_{s' \in J} \tau_Y^{-1}(s') \langle \mathbf{p}(b,s,u), \mathbf{f}(s') \rangle \le b \Big\},$$

$$\mathbf{p}(b,s,u) = b(\tau_Y(s)\mathbf{f}(s) + \tau_Y'(s)\mathbf{T}(s)) + \sum_{j=1}^{n-2} u_j \mathbf{K}_j(s),$$

$$G_b(l) = \Big\{v = (v_1, \ldots, v_{n-1}) \in \mathbb{R}^{n-1}\colon \sup_{s' \in J} \tau_Y^{-1}(s') \langle \mathbf{p}_{b,l}(v), \mathbf{f}(s') \rangle \le b \Big\}$$

*and*

$$\mathbf{p}_{b,l}(v) = b\tau_Y(l)\mathbf{f}(l) + v_{n-1}\mathbf{T}(l) + \sum_{j=1}^{n-2} v_j \mathbf{K}_j(l), \qquad l = 0, L.$$

REMARK. Taylor expansions of order 1 of $\langle \mathbf{p}_{b,0}(v), \mathbf{f}(s) \rangle - b\tau_Y(s)$ [respectively $\langle \mathbf{p}_{b,L}(v), \mathbf{f}(s) \rangle - b\tau_Y(s)$] around $l$, $l = 0, L$, show that $G_b(0)$ [respectively $G_b(L)$] has Lebesgue measure 0 in $\mathbb{R}^{n-1}$ if $Y(s)$ is $L$-periodic.

2.2. *Nonasymptotic bounds.* Several nonasymptotic upper and lower bounds for $P\{Z > a\}$ have been proposed [see, e.g., Samorodnitsky (1991), and the references therein, Berman and Kôno (1989), and Weber (1989)]. However, these bounds, obtained in general in a broader context, either involve unknown constants or are too crude to be used as $p$-values in statistical tests. In this section, we provide explicit sharp bounds for $f_Z(b)$ that turn out to be asymptotic to the density as $b \to \infty$, as shown in the next section.

From the integral representation (8), we can, under weaker assumptions with the help of a perturbation argument, deduce an efficient and easily computable upper bound for $f_Z(b)$, $b \in \mathbb{R}$.

THEOREM 2. *Under Conditions* 1 *and* 2 *and for* $b \in \mathbb{R}$,

$$
\begin{aligned}
f_Z(b) &\le M(b) \\
&= \frac{b}{2\pi}\int_0^L \tau_Y(s)(\tau_Y(s)+\tau_Y''(s))\exp\Big(-\frac{b^2}{2}(\tau_Y^2(s)+{\tau_Y'}^2(s))\Big) \\
&\qquad\qquad \times \Phi\Big(\frac{b(\tau_Y(s)+\tau_Y''(s))}{c_g(s)}\Big)\,ds \\
&\quad + \frac{1}{2\pi}\int_0^L \tau_Y(s)c_g(s)\exp\Big(-\frac{b^2}{2}(\tau_Y^2(s)+{\tau_Y'}^2(s))\Big) \\
&\qquad\qquad \times \varphi\Big(\frac{b(\tau_Y(s)+\tau_Y''(s))}{c_g(s)}\Big)\,ds + \delta_M(b)
\end{aligned}
\tag{9}
$$

*where*

$$
\delta_M(b) = \begin{cases} 0, & Y(s)\ L\text{-}periodic, \\ \tau_Y(0)\varphi(b\tau_Y(0))\Phi(b\tau_Y'(0)) & \\ \quad + \tau_Y(L)\varphi(b\tau_Y(L))(1-\Phi(b\tau_Y'(L))), & otherwise. \end{cases}
$$

REMARK. Expressions in terms of $t = \lambda^{-1}(s)$ for $M(b)$ and $\delta_M(b)$ are obtained by replacing $L$ by $T$, $ds$ by $\rho_{11,u}^{1/2}(t)dt$, $c_g(s)$ by $c_g(t)$ (see the Appendix), $\tau_Y(s)$ by $\tau_X(t)$, $\tau_Y'(s)$ by $\tau_X'(t)/\rho_{11,u}^{1/2}(t)$ and $\tau_Y''(s)$ by $-\tau_X'(t)\rho_{12,u}(t)/\rho_{11,u}^{3/2}(t) + \tau_X''(t)/\rho_{11,u}(t)$.

The upper bound $M(b)$ can be used to derive a lower bound for $f_Z(b)$, $b > 0$. Indeed, the integral formula (8) can be rewritten as

$$
\begin{aligned}
f_Z(b) &= \frac{b}{2\pi}\int_0^L \tau_Y(s)(\tau_Y(s)+\tau_Y''(s)) \\
&\qquad\qquad \times \exp\Big(-\frac{b^2}{2}(\tau_Y^2(s)+{\tau_Y'}^2(s))\Big)\mu_{n-2}(D_b(s))\,ds \\
&\quad - \frac{1}{2\pi}\int_0^L \tau_Y(s)c_g(s)\exp\Big(-\frac{b^2}{2}(\tau_Y^2(s)+{\tau_Y'}^2(s))\Big) \\
&\qquad\qquad \times \int_{D_b(s)} u_1\varphi_{n-2}(u)\,du_1\cdots du_{n-2}\,ds + \delta_Z(b).
\end{aligned}
\tag{10}
$$

By the definition of $D_b(s)$ and relation (5), $\mu_{n-2}(D_b(s))$ and $\int_{D_b(s)} u_1\varphi_{n-2}(u)\,du$ can be interpreted in terms of $P\{\sup_{s'\in J} W_s(s') \le b\}$, where $W_s(s')$ is a suitable Gaussian process of the form (4) for a.e. $s \in J$. Therefore, Theorem 2 provides upper bounds for $P\{\sup_{s'\in J} W_s(s') > b\}$ and the absolute value of the second term on the right-hand side of (10). This approach requires the following assumptions:

CONDITION 5. $\tau_X(t)$ is in $\mathscr{C}^3(I)$ and $r_U(t_1,t_2)$ has continuous partial derivatives $D_{kl}r_U(t_1,t_2)$ for $0 \le k+l \le 6$.

CONDITION 6. For all $t \neq t' \in I$, the joint distribution of $(U(t), U'(t), U(t'), U'(t'))$ admits a density.

More precisely, for each $s \in \operatorname{int}(J)$,

$$\mu_{n-2}(D_b(s)) = P\Big\{\sup_{s' \in J} Y(s') \le b \mid Y(s) = b, Y'(s) = 0\Big\}$$
$$= P\Big\{\sup_{s' \in J} \tau_Y^{-1}(s')(b(\tau_Y(s)\langle \mathbf{f}(s), \mathbf{f}(s')\rangle + \tau_Y'(s)\langle \mathbf{T}(s), \mathbf{f}(s')\rangle) + \sum_{j=1}^{n-2} \omega_j \langle \mathbf{K}_j(s), \mathbf{f}(s')\rangle) \le b\Big\},$$

where $\mathbf{\Omega} = (\omega_1, \dots, \omega_{n-2})$ is a Gaussian r.v. with mean 0 and identity covariance matrix. The Gaussian process $Y(s') \mid Y(s) = b, Y'(s) = 0$ has mean given by $b\tau_Y^{-1}(s')(\tau_Y(s)\langle \mathbf{f}(s), \mathbf{f}(s')\rangle + \tau_Y'(s)\langle \mathbf{T}(s), \mathbf{f}(s')\rangle)$ and variance given by $\tau_Y^{-2}(s')\alpha_s^2(s')$ where $\alpha_s^2(s') = 1 - \langle \mathbf{f}(s), \mathbf{f}(s')\rangle^2 - \langle \mathbf{T}(s), \mathbf{f}(s')\rangle^2 > 0$ by Condition 6. Therefore,

$$\mu_{n-2}(D_b(s)) = P\Big\{\text{for all } s' \in J, \quad \sum_{j=1}^{n-2} \omega_j \langle \mathbf{K}_j(s), \mathbf{f}(s')\rangle \le b\beta_s(s')\Big\}, \tag{11}$$

where

$$\beta_s(s') = \tau_Y(s') - \tau_Y(s)\langle \mathbf{f}(s), \mathbf{f}(s')\rangle - \tau_Y'(s)\langle \mathbf{T}(s), \mathbf{f}(s')\rangle$$
$$= b^{-1}\tau_Y(s')(b - E(Y(s') \mid Y(s) = b, Y'(s) = 0)).$$

If $\beta_s(s') > 0$ for all $s' \neq s$, we show (Theorem 4) that $\mu_{n-2}(D_b(s)) \to 1$ as $b \to \infty$. Note that, for such an $s$, $\tau_Y(s) + \tau_Y''(s) \ge 0$ since $\beta_s(s + h) = (\tau_Y(s) + \tau_Y''(s))h^2/2 + o(h^2)$ as $h \to 0$. If $\beta_s(s') < 0$ for some $s' = s^* \neq s$, then $\mu_{n-2}(D_b(s)) \le \Phi(b\alpha_s^{-1}(s^*)\beta_s(s^*)) \to 0$ as $b \to \infty$ and is negligible compared to the previous case. Finally, if $\beta_s(s') \ge 0$ for all $s' \neq s$ and $\beta_s(s^*) = 0$ for some $s^* \neq s$, $\mu_{n-2}(D_b(s))$ is also small for large $b$ (see Theorem 4). Moreover, for most of the processes of interest [e.g., $\tau_Y(s)$ constant on $J$ or admitting at least one minimum in $\operatorname{int}(J)$ or a unique minimum at the boundary, say 0, and $\tau_Y'(0) = 0$], the set $J^+ = \{s\colon\ \beta_s(s') > 0 \text{ for all } s' \neq s \text{ and } \tau_Y(s) + \tau_Y''(s) > 0\}$ has positive Lebesgue measure and contains all points giving the largest contribution to $f_Z(b)$. For the case where $\tau_Y(s)$ admits a unique minimum at the boundary, say 0, and $\tau_Y'(0) \neq 0$, it can be shown that the main contribution to the density is given by $\delta_Z(b)$ (see Theorem 4) and, for the sake of brevity in the present paper, we have chosen to take 0 as the lower bound for $\delta_Z(b)$.

From the above considerations, it follows that we can restrict ourselves to the subset $J^+ \subset J$ in the elaboration of a lower bound for $f_Z(b)$, taking 0 on $J \setminus J^+$. On $J^+$, it is possible to rewrite (11) in terms of a process of the form (4) and to use Theorem 2 to provide a lower bound for $f_Z(b)$:

$$\mu_{n-2}(D_b(s)) = P\Big\{\sup_{s' \in J} W_s(s') \le b\Big\}, \qquad s \in J^+,$$

where $W_s(s') = \tau_s^{-1}(s')\langle \mathbf{\Omega}, \mathbf{k}_s(s')\rangle$, $\tau_s^{-1}(s') = \mathrm{Var}^{1/2}(W_s(s')) = \alpha_s(s')\beta_s^{-1}(s') > 0$ and $\mathbf{k}_s(s') = (k_{s,1}(s'), \dots, k_{s,n-2}(s'))$ parameterizes a curve $\gamma_s$ on the unit sphere of $\mathbb{R}^{n-2}$. This curve is the normalized orthogonal projection of $\gamma$ on $\mathrm{Vect}^\perp(\mathbf{f}(s), \mathbf{T}(s))$ and consequently $s' \to \mathbf{k}_s(s')$ is not unit speed. As in subsection 2.1, we need the autocovariance function of $\langle \mathbf{\Omega}, \mathbf{k}_s(s')\rangle$ which is given by

$$\begin{aligned} r_s(s_1', s_2') &= \langle \mathbf{k}_s(s_1'), \mathbf{k}_s(s_2')\rangle \\ &= \alpha_s^{-1}(s_1')\alpha_s^{-1}(s_2')\big(\langle \mathbf{f}(s_1'), \mathbf{f}(s_2')\rangle - \langle \mathbf{f}(s), \mathbf{f}(s_1')\rangle\langle \mathbf{f}(s), \mathbf{f}(s_2')\rangle \\ &\qquad - \langle \mathbf{T}(s), \mathbf{f}(s_1')\rangle\langle \mathbf{T}(s), \mathbf{f}(s_2')\rangle\big) \end{aligned}$$

if $s_1' \neq s$ and $s_2' \neq s$, $r_s(s, s_2') = \langle \mathbf{f}(s) + \mathbf{f}''(s), \mathbf{f}(s_2')\rangle/(c_g(s)\alpha_s(s_2'))$ and $r_s(s,s) = 1$. By Condition 5, $r_s(s_1', s_2')$ has continuous partial derivatives $D_{kl} r_s(s_1', s_2')$ for $0 \le k, l \le 2$. Let us denote $\rho_{kl,s}(s') = D_{kl} r_s(s_1', s_2')|_{s_1' = s_2' = s'}$.

In order to apply Theorem 2 to $Z_s = \sup_{s' \in J} W_s(s')$, it remains to determine $\|\mathbf{k}_s'(s')\|$ and the geodesic curvature $c_{g,s}(s')$ of $\gamma_s$. We have $\|\mathbf{k}_s'(s')\| = \rho_{11,s}^{1/2}(s') > 0$ by Condition 6 and $c_{g,s}^2(s') = (\rho_{11,s}(s')\rho_{22,s}(s') - \rho_{12,s}^2(s'))/\rho_{11,s}^3(s') - 1$.

THEOREM 3. *Under Conditions* 2, 5 *and* 6 *and for* $b > 0$,

$$\begin{aligned} f_Z(b) &\ge m(b) \\ &= \frac{b}{2\pi}\int_{J^+} \tau_Y(s)(\tau_Y(s) + \tau_Y''(s)) \\ &\qquad \times \exp\Big(-\frac{b^2}{2}(\tau_Y^2(s) + \tau_Y'^2(s))\Big)\Big(1 - \int_b^\infty M_s(b')\,db'\Big)\,ds \\ &\qquad - \frac{1}{2\pi}\int_{J^+} \tau_Y(s) c_g(s)\exp\Big(-\frac{b^2}{2}(\tau_Y^2(s) + \tau_Y'^2(s))\Big)\eta^{-1}(s) M_s(b)\,ds, \end{aligned} \tag{12}$$

*where* $0 < \eta(s) = \inf_{s' \in J} \tau_s(s') \le c_g^{-1}(s)(\tau_Y(s) + \tau_Y''(s))$ *for all* $s \in J^+$ *and*

$$\begin{aligned} M_s(b) &= \frac{b}{2\pi}\int_0^L \tau_s(s')\zeta_s(s')\exp\Big(-\frac{b^2}{2}\Big(\tau_s^2(s') + \frac{\tau_s'^2(s')}{\rho_{11,s}(s')}\Big)\Big) \\ &\qquad \times \Phi\Big(\frac{b\zeta_s(s')}{c_{g,s}(s')}\Big)\rho_{11,s}^{1/2}(s')\,ds' \\ &\qquad + \frac{1}{2\pi}\int_0^L \tau_s(s') c_{g,s}(s')\exp\Big(-\frac{b^2}{2}\Big(\tau_s^2(s') + \frac{\tau_s'^2(s')}{\rho_{11,s}(s')}\Big)\Big) \\ &\qquad \times \varphi\Big(\frac{b\zeta_s(s')}{c_{g,s}(s')}\Big)\rho_{11,s}^{1/2}(s')\,ds' + \delta_{M_s}(b), \end{aligned}$$

$$\delta_{M_s}(b) = \begin{cases} 0, & Y(s)\ L\text{-periodic}, \\ \tau_s(0)\varphi(b\tau_s(0))\Phi\Big(\dfrac{b\tau_s'(0)}{\rho_{11,s}^{1/2}(0)}\Big) & \\ \quad + \tau_s(L)\varphi(b\tau_s(L))\Big(1 - \Phi\Big(\dfrac{b\tau_s'(L)}{\rho_{11,s}^{1/2}(L)}\Big)\Big), & \text{otherwise}, \end{cases}$$

*and*

$$\zeta_s(s') = \tau_s(s') - \tau'_s(s')\rho_{12,\,s}(s')/\rho^{3/2}_{11,\,s}(s') + \tau''_s(s')/\rho_{11,\,s}(s').$$

REMARK. Expressions in terms of $t = \lambda^{-1}(s)$ are obtained by applying the transformations of Theorem 2 and by replacing $\tau_s(s')$ by $\tau_t(t') = \beta_t(t')/\alpha_t(t')$ where $\alpha^2_t(t') = 1 - r_U(t,t')^2 - D^2_{10}r_U(t,t')/\rho_{11,\,u}(t)$, $\beta_t(t') = \tau_X(t') - \tau_X(t)r_U(t,t') - \tau'_X(t)D_{10}r_U(t,t')/\rho_{11,\,u}(t)$, $\tau'_s(s')$ by $\tau'_t(t')/\rho^{1/2}_{11,\,u}(t')$, $\tau''_s(s')$ by $-\tau'_t(t')\rho_{12,\,u}(t')/\rho^{3/2}_{11,\,u}(t') + \tau''_t(t')/\rho_{11,\,u}(t')$ and $r_s(s'_1, s'_2)$ by $r_t(t'_1, t'_2)$ with similar transformations for $\rho_{kl,\,s}(s')$.

The function $M_s(b)$ involved in the expression of $m(b)$ may be too complex for practical purposes if the process $Y(s)$ is not stationary. However, by means of Laplace's formula for integral representation [De Bruijn (1962), page 65], we can derive a good approximation of $M_s(b)$ already for moderate $b$:

$$M_s(b) \approx \kappa(s)\eta(s)\varphi(b\eta(s)),$$

where $\kappa(s)$ is defined in Theorem 5 and can be numerically computed using finite differences for the derivatives involved.

2.3. *Asymptotic behavior of $f_Z(b)$, $M(b)$ and $m(b)$.* A remarkable feature of our nonasymptotic approach is that it produces naturally very fine asymptotics for $f_Z(b)$ and hence $P\{Z > a\}$. Theorem 4 states first-order asymptotic results. It shows that in general our bounds are asymptotic to $f_Z(b)$ as $b \to \infty$ and it makes explicit simple asymptotic formulas for $f_Z(b)$. In particular, the expression (13) extends the well-known asymptotic formula (14) for a class of smooth Gaussian processes with varying variance $\sigma^2_X(t)$. Moreover, it shows that the remainder term $|f_Z(b) - e(b)|$, where $e(b)$ denotes the asymptotics of $f_Z(b)$ under consideration, has a superexponential decay as $b \to \infty$. In addition, the fine first-order asymptotic results obtained by Konstant and Piterbarg (1993), Corollaries 2.2 and 2.3, in a broader context are recovered in statements (iii) and (iv).

In the case of smooth stationary nonperiodic Gaussian processes [hence $\tau_Y(s) \equiv 1$], Piterbarg (1981) provides a higher-order asymptotic expansion for $P\{Z > a\}$. Under conditions comparable to Conditions D2–D4 in Section 3, his Theorem 2.2 states that there exist constants $\rho$, $0 \le \rho < 1$, and $B$ such that

$$|P\{Z > a\} - (L(2\pi)^{-1}\exp(-a^2/2) + 1 - \Phi(a))| \le B\exp(-a^2/(1+\rho)).$$

We do not give such a precise result in the nonperiodic case [Theorem 4(ii)] but we do [Theorem 4(i)] in the periodic case (not treated by Piterbarg). However, under slightly stronger conditions than those of Theorem 2.2, Piterbarg (1981) obtains the highest-order expansion to our knowledge. His Theorem 2.3 states that there is an $L_0$ small enough such that, for all $L \le L_0$,

$$\begin{aligned} P\{Z > a\} = {} & L(2\pi)^{-1}\exp(-a^2/2) + 1 - \Phi(a) \\ & - C\,a^{-5}\exp(-a^2(1+c^2_g)/(2c^2_g))(1+o(1)) \end{aligned}$$

as $a \to \infty$ where $C = 6^{1/2} 3 L c_g^9 / (4\pi^{3/2}(\lambda_6 - 1 - c_g^2))$ and $\lambda_6$ is the sixth spectral moment of the process.

Together with our nonasymptotic results, Theorem 4 enables us to derive simple second-order asymptotic bounds for $f_Z(b)$ (Theorem 5). In the case of smooth stationary Gaussian processes (Examples 1 and 2), these bounds improve the well-known results (2) and (3). In the case of smooth Gaussian processes with a variance admitting a finite number of maxima (Example 3), we refine Corollaries 2.1 and 2.2 of Konstant and Piterbarg (1993), producing rates of decay of the remainders.

THEOREM 4. *Under the conditions of Theorem* 2 *for* $M(b)$ *and of Theorem* 3 *for* $m(b)$, *and consequently for* $f_Z(b)$,

$$\begin{aligned} f_Z(b) &= e(b)\,(1 + R(b)), \\ M(b) &\le e(b)\,(1 + R_M(b)), \qquad R_M(b) > 0, \\ m(b) &\ge e(b)\,(1 - R_m(b)), \qquad R_m(b) > 0 \end{aligned}$$

*as* $b \to \infty$, *where*:

(i) *if* $J \setminus J^+$ *has Lebesgue measure* 0 *and* $Y(s)$ *is periodic*, *or not periodic with* $\min(\tau_Y(0), \tau_Y(L)) > \inf_{s \in J} \tau_Y(s)$,

$$e(b) = \frac{b}{2\pi} \int_0^L \tau_Y(s)(\tau_Y(s) + \tau_Y''(s)) \exp\left( -\frac{b^2}{2}(\tau_Y^2(s) + \tau_Y'^2(s)) \right) ds \tag{13}$$

*and* $R(b) = R_M(b) = R_m(b) = O(\varphi(b\theta))$ *for some* $\theta > 0$;

(ii) *if* $Y(s)$ *is not periodic and* $\tau_Y(s) \equiv 1$ *is constant*,

$$e(b) = \frac{b}{2\pi} L \exp\left( -\frac{b^2}{2} \right), \tag{14}$$

$R(b) = R_M(b) = O(b^{-1})$ *and* $R_m(b) = O(\varphi(b\theta))$ *for some* $\theta > 0$;

(iii) *if* $Y(s)$ *is not periodic*, $\tau_Y(s^*) = \tau < \tau_Y(s)$ *for all* $s \ne s^*$, $s^* \in \mathrm{int}(J)$ *and* $q = \inf\{k \ge 1\colon \tau_Y^{(k)}(s^*) = \tau^{(k)} > 0\}$ *is assumed finite*, *then*

1. $e(b) = (1 + \tau/\tau'')^{1/2} \tau \varphi(b\tau)$ *and* $R(b) = R_M(b) = R_m(b) = o(1)$ *if* $q = 2$;
2. $e(b) = b^{1-2/q} (2\pi)^{-1/2} \tau^{2-1/q} (\tau^{(q)})^{-1/q} q^{-1} \Gamma(q^{-1}) \Gamma(q+1)^{1/q} \varphi(b\tau)$ *and* $R(b) = R_M(b) = R_m(b) = o(1)$ *if* $q \ge 3$;

(iv) *if* $Y(s)$ *is not periodic*, $\tau_Y(0) = \tau < \tau_Y(s)$ *for all* $s \ne 0$ *and* $q = \inf\{k \ge 1\colon \tau_Y^{(k)}(0) = \tau^{(k)} > 0\}$ *is assumed finite*, *then*

1. $e(b) = \delta_M(b)$, $R(b) = o(1)$, $R_M(b) = O(\varphi(b\theta))$ *and* $R_m(b) = 1 - O(\varphi(b\theta))$ *for some* $\theta > 0$ *if* $q = 1$;
2. $e(b) = (2^{-1}(1 + \tau/\tau'')^{1/2} + 1) \tau \varphi(b\tau)$, $R(b) = R_M(b) = o(1)$ *and* $R_m(b) = 1 - (2^{-1}(1 + \tau/\tau'')^{1/2} + 1)^{-1} - o(1)$ *if* $q = 2$;
3. $e(b) = b^{1-2/q} (2\pi)^{-1/2} \tau^{2-1/q} (\tau^{(q)})^{-1/q} q^{-1} \Gamma(q^{-1}) \Gamma(q+1)^{1/q} \varphi(b\tau)$ *and* $R(b) = R_M(b) = R_m(b) = o(1)$ *if* $q \ge 3$.

REMARK. The results stated in Theorem 4(iii) and (iv) can be easily adapted to the case where $\tau_Y(s)$ reaches its absolute minimum on a finite set of points by adding the asymptotics over these points.

Note that when $Y(s)$ reaches its maximum at the boundaries with high probability, the main contribution to the density is given by the additional term $\delta_Z(b)$. This phenomenon affects the good behavior of $m(b)$ since we have chosen to take 0 as the lower bound for $\delta_Z(b)$ for the sake of brevity. However, it would be possible to improve $m(b)$ by introducing a term $0 < \delta_m(b) \le \delta_Z(b)$ which corrects this imperfection. This subject is under current research.

THEOREM 5. *Assume that Conditions* 2, 5 *and* 6 *hold*, $J \setminus J^+$ *has Lebesgue measure* 0 *and, for a.e.* $s \in J$, *the function* $s' \to \tau_s(s')$ *reaches its infimum* $\eta(s) > 0$ *at a finite number of points* $s'_i$, $i = 1, \dots, k$, *in* $J$, *with* $\tau''_s(s'_i) > 0$ *for* $s'_i \in \operatorname{int}(J)$ *and* $\tau'_s(s'_i) \neq 0$ *or* $\tau''_s(s'_i) > 0$ *if* $s'_i = 0, L$. *Then there exists for a.e.* $s \in J$ *a positive number* $\kappa(s)$ *such that, for* $b \to \infty$,

$$f_Z(b) \le \frac{b}{2\pi} \int_0^L \tau_Y(s)(\tau_Y(s) + \tau''_Y(s)) \times \exp\biggl(-\frac{b^2}{2}(\tau_Y^2(s) + {\tau'_Y}^2(s))\biggr)\psi_u(s, b)\, ds + \delta_Z(b),$$

*where*

$$\psi_u(s, b) = 1 - \kappa(s)(1 - \Phi(b\eta(s)))(1 + o(1)) + \kappa(s)c_g(s)(b(\tau_Y(s) + \tau''_Y(s)))^{-1}\varphi(b\eta(s))(1 + o(1));$$

$$f_Z(b) \ge \frac{b}{2\pi} \int_0^L \tau_Y(s)(\tau_Y(s) + \tau''_Y(s)) \exp\biggl(-\frac{b^2}{2}(\tau_Y^2(s) + {\tau'_Y}^2(s))\biggr)\psi_l(s, b)\, ds,$$

*where*

$$\psi_l(s, b) = 1 - \kappa(s)(1 - \Phi(b\eta(s)))(1 + o(1)) - \kappa(s)c_g(s)(b(\tau_Y(s) + \tau''_Y(s)))^{-1}\varphi(b\eta(s))(1 + o(1)).$$

*The function* $\kappa(s)$ *is given by* $\kappa(s) = \sum_{i=1}^k \kappa_i(s)$, *where*:

(i) $\kappa_i(s) = (1 + \eta(s)\rho_{11,s}(s'_i)/\tau''_s(s'_i))^{1/2}$ *if* $s'_i \in \operatorname{int}(J)$,

(ii) $\kappa_i(s) = (1 + (1 + \eta(s)\rho_{11,s}(s'_i)/\tau''_s(s'_i))^{1/2})/2$ *if* $s'_i = 0, L$, $\tau'_s(s'_i) = 0$, $\tau''_s(s'_i) > 0$ *and* $Y(s)$ *is not periodic*,

(iii) $\kappa_i(s) = 1/2$ *if* $s'_i = 0, L$, $\tau'_s(s'_i) \neq 0$ *and* $Y(s)$ *is not periodic*.

REMARK. Since $1 - \Phi(b\eta(s)) \sim \varphi(b\eta(s))/b\eta(s)$ as $b \to \infty$ and $\eta(s) \le (\tau_Y(s) + \tau''_Y(s))/c_g(s)$, it follows that $\psi_u(s, b) \le 1 + o(1)$ as $b \to \infty$.

EXAMPLE 1. If $Y(s)$, satisfying the conditions of Theorem 5, is stationary and $L$-periodic, then $c_g(s) \equiv c_g$, $\tau_s(s') = \tau(s')$. Therefore, $\eta(s) \equiv \eta \le c_g^{-1}$,

$\kappa(s) \equiv \kappa$ and

$$f_Z(b) \le bL(2\pi)^{-1}\exp(-b^2/2)\big[1-\kappa(1-\Phi(b\eta))(1+o(1)) + \kappa c_g b^{-1}\varphi(b\eta)(1+o(1))\big].$$

Moreover, the function $\tau(s')$ is even and reaches a local minimum at $s'=0$ and $\tau(0)=c_g^{-1}$. If this minimum is global, it turns out that $\kappa = 3^{1/2}$. Otherwise, $\eta^{-1}-c_g>0$ and

$$f_Z(b) \le bL(2\pi)^{-1}\exp(-b^2/2)[1-\kappa\varphi(b\eta)b^{-1}(\eta^{-1}-c_g)(1+o(1))]$$

for $b$ large enough. Therefore,

$$P\{Z>a\} \le L(2\pi)^{-1}\exp(-a^2/2) - \kappa(\eta^{-1}-c_g)L(2\pi)^{-1}(1+\eta^2)^{-1/2}(1-\Phi(a(1+\eta^2)^{1/2}))(1+o(1)),$$

which improves the well-known upper bound (3) for $a$ large enough. Similarly,

$$P\{Z>a\} \ge L(2\pi)^{-1}\exp(-a^2/2) - \kappa(\eta^{-1}+c_g)L(2\pi)^{-1}(1+\eta^2)^{-1/2}(1-\Phi(a(1+\eta^2)^{1/2}))(1+o(1)).$$

EXAMPLE 2. If $Y(s)$, satisfying the conditions of Theorem 5, is stationary but not periodic, then $c_g(s)\equiv c_g$, $\tau_s(s')=\tau(s'-s)$, $\eta^{-1}(s)\ge c_g$ for all $s$ and

$$P\{Z>a\} \le L(2\pi)^{-1}\exp(-a^2/2)+1-\Phi(a)-(2\pi)^{-1}\int_0^L \kappa(s)(\eta^{-1}(s)-c_g) \times (1+\eta^2(s))^{-1/2}(1-\Phi(a(1+\eta^2(s))^{1/2}))(1+o(1))\,ds$$

for $a$ large enough, which also improves (3). Similarly,

$$P\{Z>a\} \ge L(2\pi)^{-1}\exp(-a^2/2)-(2\pi)^{-1}\int_0^L \kappa(s)(\eta^{-1}(s)+c_g) \times (1+\eta^2(s))^{-1/2}(1-\Phi(a(1+\eta^2(s))^{1/2}))(1+o(1))\,ds.$$

EXAMPLE 3. If $\tau=\tau_Y(s^*)<\tau_Y(s)$ for all $s\ne s^*$, $s^*\in \mathrm{int}(J)$, there exists $\varepsilon>0$ such that $J_\varepsilon^*=[s^*-\varepsilon, s^*+\varepsilon]\subset J^+$. We can then apply Theorem 5 to the restriction of $J$ to $J_\varepsilon^*$. Let us denote

$$e_\varepsilon(b)=\frac{b}{2\pi}\int_{s^*-\varepsilon}^{s^*+\varepsilon}\tau_Y(s)(\tau_Y(s)+\tau_Y''(s))\exp\left(-\frac{b^2}{2}(\tau_Y^2(s)+\tau_Y'^2(s))\right)ds$$

and $(\tau+\delta)^2=\inf_{s\in J\setminus J^*}(\tau_Y^2(s)+\tau_Y'^2(s))$. By assumption, $\delta>0$. From (9) and (12), we have $M(b)=e_\varepsilon(b)+\delta_M(b)+O(\varphi(b(\tau+\delta)))$ and $m(b)=e_\varepsilon(b)(1-O(\varphi(b\eta^*)))$ as $b\to\infty$, with $\eta^*=\inf_{s\in J}\eta(s)>0$. As $\min(\tau_Y(0),\tau_Y(L))>\tau$, it results that

$$f_Z(b)=e(b)(1+O(\varphi(b\theta))),\qquad b\to\infty,$$

for some $\theta>0$ since $e_\varepsilon(b)=e(b)(1+O(\varphi(b(\tau+\delta))))$ with $e(b)$ given by (13). In other words, the remaining term $|f_Z(b)-e(b)|$ has a superexponential decay. Note that in Konstant and Piterbarg (1993) it is not proved that the remaining term $|P\{Z > a\} - E(a)|$, where $E(a)$ denotes the asymptotic formula they obtain by making use of Laplace's formula, has such a superexponential decay.

**3. Extension to series representation.** The results of Section 2 can be extended to the maximum $Z$ of Gaussian processes of the form $X(t) = \tau_X^{-1}(t)U(t)$, $t \in I = [0, T]$, where $\tau_X(t) > 0$, $U(t)$ is a centered Gaussian process with unit variance and covariance function $r_U(t_1, t_2) = \langle \mathbf{g}(t_1), \mathbf{g}(t_2)\rangle = \sum_{j=1}^{\infty} g_j(t_1)g_j(t_2)$. The functions $\mathbf{g}(t) = (g_1(t), g_2(t), \ldots)$ parameterize a curve $\gamma$ embedded in the unit sphere of $\ell^2$. Let us denote $\rho_{kl,u}(t) = D_{kl}r_U(t_1, t_2)|_{t_1=t_2=t}$. We assume that

CONDITION D1. The function $\tau_X(t)$ is in $\mathscr{C}^3(I)$ and $r_U(t_1, t_2)$ has continuous partial derivatives $D_{kl}r_U(t_1, t_2)$ for $0 \le k, l \le 4$.

CONDITION D2. $\rho_{11,u}(t) \neq 0$ for all $t \in I$.

Under these conditions, we give an upper bound for the density $f_Z(b)$ of $Z = \sup_{t\in I} X(t)$. With the following assumptions we also derive a lower bound.

CONDITION D3. The function $\tau_X(t)$ is in $\mathscr{C}^4(I)$ and $r_U(t_1, t_2)$ has continuous partial derivatives $D_{kl}r_U(t_1, t_2)$ for $0 \le k + l \le 8$.

CONDITION D4. For all $t \neq t' \in I$, the joint distribution of $(U(t), U'(t), U(t'), U'(t'))$ admits a density.

The length and geodesic curvature of $\gamma$ are given by $L = \int_0^T \rho_{11,u}^{1/2}(t')\,dt'$ and $c_g^2(t) = (\rho_{11,u}(t)\rho_{22,u}(t) - \rho_{12,u}^2(t))/\rho_{11,u}^3(t) - 1$. As in Section 2, $\gamma$ has a unit speed parameterization $s = \lambda(t) = \int_0^t \rho_{11,u}^{1/2}(t')dt'$ under Condition D2. Therefore, $Z = \sup_{t\in I} X(t) = \sup_{s\in J} Y(s)$, where $Y(s) = \tau_Y^{-1}(s)V(s)$ with $\tau_Y(s) = \tau_X(\lambda^{-1}(s))$, $V(s) = U(\lambda^{-1}(s))$ a Gaussian process of variance 1 and covariance function $r_V(s_1, s_2) = \langle \mathbf{f}(s_1), \mathbf{f}(s_2)\rangle$ with $\mathbf{f}(s) = \mathbf{g}(\lambda^{-1}(s))$. The functions $\alpha_s(s')$, $\beta_s(s')$, $\tau_s(s')$, $\eta(s)$, $r_s(s_1', s_2')$, $\rho_{kl,s}(s')$, $c_{g,s}(s')$ and $\zeta_s(s')$ defined in Section 2 are also well defined in the present context and we can show the following result.

THEOREM 6. *Under Conditions* D1 *and* D2, $Z$ *has a density* $f_Z(b)$ *and* $f_Z(b) \le M(b)$ *for* $b \in \mathbb{R}$, *with* $M(b)$ *given by* (9). *Under Conditions* D2–D4, $f_Z(b) \ge m(b)$ *for* $b > 0$, *with* $m(b)$ *given by* (12). *In addition*, *Theorems* 4 *and* 5 *still hold*.

**4. Proofs of the results of Sections 2 and 3.**

4.1. *Proof of Theorem* 1. We give a detailed proof for the case $Y(s)$ periodic and sketch the straightforward adaptation for the other case. A complete proof of all results can be found in Diebolt and Posse (1995) and is available upon request.

We need a workable description of the boundary $\partial C_a$ of $C_a$ for $a \in \mathbb{R}$. Note first that $C_a$ is the intersection of the closed half spaces $\{\mathbf{x} \in \mathbb{R}^n\colon\ Y(s, \mathbf{x}) \le a\}$, $s \in J$. These half spaces have hyperplane boundaries given by

$$H(s; a) = \{\mathbf{x} \in \mathbb{R}^n\colon\ Y(s, \mathbf{x}) = a\}, \qquad s \in J. \tag{15}$$

Lemma 7 shows that the surface $\partial C_a$ is closely related to the hypersurface enveloped by the hyperplanes $H(s; a)$, $s \in J$.

LEMMA 7. *If $C_a \neq \varnothing$, $\partial C_a = \{\mathbf{x} \in \mathbb{R}^n\colon\ \sup_{s\in J} Y(s, \mathbf{x}) = a\}$.*

LEMMA 8. *$\partial C_a = \Sigma_a \cap C_a$, where $\Sigma_a = \cup_{s\in J}\Pi_a(s)$, with $\Pi_a(s)$ denoting the affine subspace of dimension $n-2$ of $\mathbb{R}^n$ defined by the equations*

$$\langle \mathbf{x}, \mathbf{f}(s)\rangle = a\tau_Y(s),$$
$$\langle \mathbf{x}, \mathbf{T}(s)\rangle = a\tau'_Y(s).$$

LEMMA 9. (i) *The hypersurface $\Sigma_a$ can be parameterized by*

$$\mathbf{p}_a(s, u) = a(\tau_Y(s)\mathbf{f}(s) + \tau'_Y(s)\mathbf{T}(s)) + \sum_{j=1}^{n-2} u_j \mathbf{K}_j(s), \tag{16}$$

*with $s \in J$, $u = (u_1, \dots, u_{n-2}) \in \mathbb{R}^{n-2}$.*

(ii) *The hypersurface $\partial C_a$ can be parameterized by $\mathbf{p}_a(s, u)$, with $s \in J$ and $u \in D_a(s)$, where $D_a(s)$ is the closed convex subset of $\mathbb{R}^{n-2}$ (possibly empty) defined by the set of inequalities $\{\sup_{s'\in J} Y(s', \mathbf{p}_a(s, u)) \le a\}$.*

LEMMA 10. *Let $s_0 \in J$ be given.*

(i) *If $u \in D_a(s_0) \neq \varnothing$, then $d(a, s_0) - u_1 c_g(s_0) \ge 0$.*

(ii) *If $u \in \operatorname{int}(D_a(s_0)) \neq \varnothing$ and $c_g(s_0) > 0$, then $d(a, s_0) - u_1 c_g(s_0) > 0$, where $d(a, s) = a(\tau_Y(s) + \tau''_Y(s))$.*

PROOF. (i) For each fixed $u \in \mathbb{R}^{n-2}$ and $s_0$, the function $h_{u,s_0}(s) = Y(s, \mathbf{p}_a(s_0, u))$, $s \in J$, is twice differentiable and $h'_{u,s_0}(s_0) = 0$. Furthermore, since $\mathbf{f}''(s) = c_g(s)\mathbf{K}(s) - \mathbf{f}(s)$ for all $s \in J$, $h''_{u,s_0}(s_0) = -(d(a, s_0) - u_1 c_g(s_0))/\tau_Y(s_0)$. If $u \in D_a(s_0) \neq \varnothing$, $h_{u,s_0}(s)$ reaches its maximum value at $s = s_0$, implying that $h''_{u,s_0}(s_0) \le 0$.

(ii) Suppose that $u \in \operatorname{int}(D_a(s_0)) \neq \varnothing$. Let us show by contradiction that $h''_{u,s_0}(s_0) < 0$. Otherwise, we would have $h''_{u,s_0}(s_0) = 0$ by (i). If $h''_{u,s_0}(s_0) = 0$, since $c_g(s_0) > 0$ and $u \in \operatorname{int}(D_a(s_0))$, we can pick $v \in D_a(s_0)$ (close enough to $u$) such that $h''_{v,s_0}(s_0) > 0$ (by taking $v_1 > u_1$), which contradicts (i). $\square$

Let us define the $C^1$-function

$$\mathbf{p}(b, s, u) = \mathbf{p}_b(s, u), \quad b \in \mathbb{R}, \quad s \in J, \quad u = (u_1, \dots, u_{n-2}) \in \mathbb{R}^{n-2}.$$

This function maps $V_a = \{(b, s, u)\colon\ b \le a, s \in J \text{ and } u \in D_b(s)\} \subset \mathbb{R}^n$ onto $C_a \subset \mathbb{R}^n$ since $\cup_{b \le a} \partial C_b = C_a$. Moreover, the restriction of $\mathbf{p}(b, s, u)$ to the open subset $V_a^0 = \{(b, s, u)\colon\ b < a,\ b \ne 0,\ s \in \operatorname{int}(J) \text{ and } u \in \operatorname{int}(D_b(s))\} \subset V_a$ maps $V_a^0$ onto a subset $C_a^0$ of $C_a$. In the following, we will assume that $a$ is such that $V_a^0 \ne \varnothing$.

LEMMA 11. (i) $|\det D\mathbf{p}(b, s, u)| = \tau_Y(s)|d(b, s) - u_1 c_g(s)|$.

(ii) *The function* $\mathbf{p}(b, s, u)$ *is a local* $C^1$*-diffeomorphism from* $V_a^0$ *onto* $C_a^0$ *and* $C_a^0$ *is an open subset of* $\mathbb{R}^n$.

PROOF. (i) follows from the fact that $\operatorname{Gram} D\mathbf{p} = \tau_Y^2(s)(d(b, s) - u_1 c_g(s))^2$ and (ii) follows from (i), Lemma 10, Condition 3 and that $\det D\mathbf{p}(b, s, u) \ne 0$ for $(b, s, u) \in V_a^0$. □

LEMMA 12. *The function* $\mathbf{p}$ *is a one-to-one mapping from* $V_a^0$ *onto* $C_a^0$.

PROOF. Using Condition 4, the proof is similar to the proof of Lemma 10(ii). □

LEMMA 13.

$$\mu_n(C_a^0) = \int_{V_a^0} \tau_Y(s)(d(b, s) - u_1 c_g(s))\varphi_n(\mathbf{p}(b, s, u))\, du_1 \cdots du_{n-2}\, ds\, db. \tag{17}$$

PROOF. According to Lemmas 11 and 12, the function $\mathbf{p}$ is a $C^1$-diffeomorphism from $V_a^0$ onto $C_a^0$. Moreover, according to Lemmas 10 and 11, $\det D\mathbf{p}(b, s, u) > 0$ for all $(b, s, u) \in V_a^0$. Then (17) results from the change of variable $\mathbf{x} = \mathbf{p}(b, s, u)$ applied to the integral $\mu_n(C_a^0) = \int_{C_a^0} \varphi_n(\mathbf{x})\, d\mathbf{x}$. □

Since $D_b(s)$ is a convex of $\mathbb{R}^{n-2}$, the Lebesgue measure in $\mathbb{R}^{n-2}$ of $\partial D_b(s)$ is 0. Therefore, the Lebesgue measure of $V_a \setminus V_a^0$ is 0 and we can replace $V_a^0$ by $V_a$ in (17). Moreover, $C_a \setminus C_a^0$ has Lebesgue measure 0 since $C_a \setminus C_a^0 = \mathbf{p}(V_a) \setminus \mathbf{p}(V_a^0) \subset \mathbf{p}(V_a \setminus V_a^0)$ and $\mathbf{p}(b, s, u)$ is a $C^1$-function from $\mathbb{R}^n$ to $\mathbb{R}^n$. Consequently, $\mu_n(C_a) = \mu_n(C_a^0)$ which, with (5) and (6), concludes the proof of Theorem 1 for $Y(s)$ periodic.

Suppose now that $Y(s)$ is not periodic. Lemma 7 still holds. In Lemma 8, we have to replace $\Sigma_a$ by $\Sigma_a' = (\cup_{s \in \operatorname{int}(J)} \Pi_a(s)) \cup H(0, a) \cup H(L, a)$, where $H(l, a)$ is defined in (15). Then $\partial C_a$ can be partitioned as $\partial C_a = \partial C_{a,\mathrm{int}} \cup \partial C_{a,0} \cup \partial C_{a,L}$, where $\partial C_{a,\mathrm{int}} = (\cup_{s \in \operatorname{int}(J)} \Pi_a(s)) \cap C_a$, $\partial C_{a,l} = H(l, a) \cap C_a$, $l = 0, L$. Lemmas 9–13 can be applied without modification to $C_{a,\mathrm{int}} = \cup_{b \le a} \partial C_{b,\mathrm{int}}$. The additional term $\delta_Z(b)$ is obtained from the boundaries $\partial C_{b,l}$, $b \le a$, $l = 0, L$. Indeed, for $l = 0, L$, $\partial C_{b,l}$ can be parameterized by $\mathbf{p}_{b,l}(v) = b\tau_Y(l)\mathbf{f}(l) + v_{n-1}\mathbf{T}(l) + \sum_{j=1}^{n-2} v_j \mathbf{K}_j(l)$ with $v \in G_b(l) = \{v \in \mathbb{R}^{n-1}\colon\ \sup_{s' \in J} \tau_Y(s')^{-1} \langle \mathbf{p}_{b,l}(v), \mathbf{f}(s')\rangle \le b\}$. Since $\operatorname{Gram} D\mathbf{p}_{b,l} = \tau_Y^2(l)$, we have $\psi_b(\partial C_{b,l}) = \int_{v \in G_b(l)} \varphi_n(\mathbf{p}_{b,l}(v))\tau_Y(l)\, dv$. □

4.2. *Proof of Theorem* 2. (i) Under Conditions 1–4, the inequality (9) is a direct consequence of Lemma 10.

(ii) To enlarge the scope of this inequality, we use the following perturbation argument. Let us consider the auxiliary Gaussian process $Y^*(s)$, $s \in J$,

$$Y^*(s) = \tau_Y^{-1}(s)V^*(s), \qquad V^*(s) = \sum_{j=n+1}^{n+k} \xi_j f_j^*(s),$$

where $\xi_j$, $j = n+1, \dots, n+k$, are independent standard Gaussian r.v.'s, independent of $\xi_j$, $j = 1, \dots, n$, $\sum_{j=n+1}^{n+k} f_j^{*2}(s) = \sum_{j=n+1}^{n+k} f_j^{*'2}(s) \equiv 1$ for $s \in J$, the functions $f_j^*(s)$ are in $\mathscr{C}^\infty(J)$, $k$ is sufficiently large to ensure that the vectors $\mathbf{f}^*(s_1)$, $\mathbf{f}^{*'}(s_1)$, $\mathbf{f}^*(s_2)$, $\mathbf{f}^{*'}(s_2)$ and $\mathbf{f}^*(s_3)$, where $\mathbf{f}^*(s) = (f_{n+1}^*(s), \dots, f_{n+k}^*(s))$, are linearly independent for all $s_1 \neq s_2$, $s_1 \neq s_3$ and $s_2 \neq s_3$ in $J$. For instance, $V^*(s) = \sum_{i=1}^p b_i(\xi_{n+2i-1}\cos(is/A) + \xi_{n+2i}\sin(is/A))$ with $b_i \neq 0$, $i = 1, \dots, p$, $\sum_{i=1}^p b_i^2 = 1$, $\sum_{i=1}^p i^2 b_i^2 = A^2$ and $L/A \neq 0 \bmod(2\pi)$. This process is stationary, nonperiodic on $J$ and its geodesic curvature $c_g^*(s)$ satisfies $c_g^*(s) \equiv c_g^* > 0$. We form the process

$$Y_\varepsilon(s) = \tau_Y^{-1}(s)(1+\varepsilon^2)^{-1/2}(V(s) + \varepsilon V^*(s)) = \tau_Y^{-1}(s) \sum_{j=1}^{n+k} \xi_j f_{\varepsilon,j}(s),$$

where $f_{\varepsilon,j}(s) = (1+\varepsilon^2)^{-1/2} f_j(s)$ for $j \le n$ and $f_{\varepsilon,j}(s) = (1+\varepsilon^2)^{-1/2}\varepsilon f_j^*(s)$ for $j > n$.

It is easily shown that $Y_\varepsilon(s)$ is a centered Gaussian process with variance $\tau_Y^{-2}(s)$ and that $\mathbf{f}_\varepsilon(s) = (f_{\varepsilon,1}(s), \dots, f_{\varepsilon,n+k}(s))$, $s \in J$, parameterizes with unit speed a curve $\gamma_\varepsilon$ whose geodesic curvature $c_{\varepsilon,g}(s) = (1+\varepsilon^2)^{-1/2}(c_g^2(s) + \varepsilon^2 c_g^{*2})^{1/2}$ is positive for $\varepsilon > 0$. Hence, $Y_\varepsilon(s)$ satisfies Condition 3 for $\varepsilon > 0$. By linear independence, it also satisfies Condition 4 for $\varepsilon > 0$. Therefore, we can apply Theorem 1 and (i) to $Y_\varepsilon(s)$, for all $\varepsilon > 0$, to obtain that $f_{Z_\varepsilon}(b) \le M_\varepsilon(b)$, where $Z_\varepsilon = \sup_{s\in J} Y_\varepsilon(s)$ and $M_\varepsilon(b)$ is given by (9) with $c_g(s)$ replaced by $c_{\varepsilon,g}(s)$. Then

$$\begin{aligned} M_\varepsilon(b) \le{}& \frac{|b|}{2\pi}\int_0^L \tau_Y(s)|\tau_Y(s) + \tau_Y''(s)| \exp\biggl(-\frac{b^2}{2}(\tau_Y^2(s) + \tau_Y'^2(s))\biggr)\, ds \\ &+ \frac{1}{2\pi}\int_0^L \tau_Y(s) c_{\varepsilon,g}(s) \exp\biggl(-\frac{b^2}{2}(\tau_Y^2(s) + \tau_Y'^2(s))\biggr)(2\pi)^{-1/2}\, ds + \delta_M(b), \end{aligned}$$

where $\delta_M(b)$ is independent of $\varepsilon$. For $|\varepsilon| \le \varepsilon_0$, the expression on the right-hand side is bounded above by $M^*(b) = O(|b|\exp(-b^2\eta_Y^2/2))$, where $\eta_Y = \inf_{s\in J} \tau_Y(s) (> 0)$.

Finally, $P(Z_\varepsilon \le a) \to P(Z \le a)$ for all $a$ as $\varepsilon \to 0$ since

$$|Z_\varepsilon - Z| \le \eta_Y^{-1}\Bigl(((1+\varepsilon^2)^{1/2} - 1)\sup_{s\in J}|V(s)| + |\varepsilon| \sup_{s\in J}|V^*(s)|\Bigr).$$

It follows that $f_{Z_\varepsilon}(b) \to f_Z(b)$ as $\varepsilon \to 0$ as in the last paragraph of the proof of Theorem 6. □

4.3. *Proof of Theorem* 3. For each $s \in J^+$, $\mu_{n-2}(D_b(s))$ in formula (10) can be interpreted as $P(Z_s \le b)$, where $Z_s = \sup_{s' \in J} W_s(s')$,

$$W_s(s') = \begin{cases} \sum_{j=1}^{n-2} \omega_j \langle \mathbf{K}_j(s), \mathbf{f}(s') \rangle / \beta_s(s'), & s' \neq s, \\ \omega_1 c_g(s) / (\tau_Y(s) + \tau_Y''(s)), & s' = s, \end{cases}$$

with $\mathbf{\Omega} = (\omega_1, \dots, \omega_{n-2})$ a Gaussian r.v. with zero mean and identity covariance matrix. With this definition, $W_s(s')$ is defined and continuous on $J$, $\operatorname{Var}(W_s(s')) = \alpha_s^2(s')/\beta_s^2(s')$ for $s' \neq s$ and $\operatorname{Var}(W_s(s)) = c_g^2(s)/(\tau_Y(s) + \tau_Y''(s))^2$. Moreover, $W_s(s')$ is of the form (4) and satisfies Conditions 1 and 2: $W_s(s') = \tau_s^{-1}(s') \langle \mathbf{\Omega}, \mathbf{k}_s(s') \rangle$, where $\tau_s(s') = (\operatorname{Var}(W_s(s')))^{-1/2}$ and $\mathbf{k}_s(s') = (k_{s,1}(s'), \dots, k_{s,n-2}(s'))$ with $k_{s,j}(s') = \langle \mathbf{K}_j(s), \mathbf{f}(s') \rangle / \alpha_s(s')$ for $s' \neq s$, $k_{s,1}(s) = 1$ and $k_{s,j}(s) = 0$, $j \ge 2$. Therefore, we can apply Theorem 2 to $Z_s$ to obtain an upper bound for $1 - \mu_{n-2}(D_b(s))$.

This interpretation can also be used to derive an upper bound for the second term in (10). Indeed, by Stokes' theorem [Berger and Gostiaux (1988), page 195]

$$\begin{aligned} \left| \int_{D_b(s)} u_1 \varphi_{n-2}(u)\, du_1 \dots du_{n-2} \right| &= \left| \oint_{\partial D_b(s)} \varphi_{n-2}(u)\, du_2 \wedge \dots \wedge du_{n-2} \right| \\ &\le \int_{\partial D_b(s)} \varphi_{n-2}\, dV, \end{aligned}$$

where $dV$ denotes the canonical volume element of the manifold $\partial D_b(s)$ [Berger and Gostiaux (1988), page 203]. A straightforward adaptation of Lemmas 7–10 yields

$$\int_{\partial D_b(s)} \varphi_{n-2}\, dV = \int_{(s'', v) \in \mathbf{p}_{b,s}^{-1}(\partial D_b(s))} \varphi_{n-2}(\mathbf{p}_{b,s}(s'', v)) \operatorname{Gram}^{1/2} D\mathbf{p}_{b,s}(s'', v)\, ds''\, dv,$$

where $s'' \in J$, $v \in \mathbb{R}^{n-4}$ and $(s'', v) \to \mathbf{p}_{b,s}(s'', v)$ defines a parameterization of $\partial D_b(s)$ analogous to (16). From (6) and (7) applied to $W_s(s')$, it follows that

$$\int_{\partial D_b(s)} \varphi_{n-2}\, dV \le \frac{1}{\inf_{s' \in J} \tau_s(s')} f_{Z_s}(b) = \eta^{-1}(s) f_{Z_s}(b),$$

where $f_{Z_s}(b)$ is the density of $Z_s$. $\square$

4.4. *Proof of Theorem* 4. (i) Assume $Y(s)$ periodic and $J \setminus J^+$ has Lebesgue measure 0. Let us denote

$$e_1(b) = \frac{b}{2\pi} \int_0^L \tau_Y(s)(\tau_Y(s) + \tau_Y''(s)) \exp\left( -\frac{b^2}{2} (\tau_Y^2(s) + {\tau_Y'}^2(s)) \right) ds.$$

By (9),

$$\begin{aligned} M(b) = e_1(b) + \frac{b}{2\pi} \int_0^L & \tau_Y(s)(\tau_Y(s) + \tau_Y''(s)) \\ & \times \exp\left( -\frac{b^2}{2} (\tau_Y^2(s) + {\tau_Y'}^2(s)) \right) H\left( \frac{b(\tau_Y(s) + \tau_Y''(s))}{c_g(s)} \right) ds, \end{aligned}$$

where $0 < H(x) = x^{-1}\varphi(x) - (1-\Phi)(x) \le x^{-3}\varphi(x)$ for all $x > 0$ and $H(\infty) = 0$. Since $c_g(s)$ is continuous and $\tau_Y(s) + \tau_Y''(s)$ is continuous and positive on $J$, it follows that $\theta_1 = \inf_{s\in J}(\tau_Y(s) + \tau_Y''(s))/c_g(s) > 0$ and $M(b) - e_1(b) \le e_1(b)(b\theta_1)^{-3}\varphi(b\theta_1)$. By (12), $m(b) = e_1(b) - A - B$, where $A$ involves $\int_b^\infty M_s(b')\,db'$ and $B$ involves $M_s(b)$. Therefore, it suffices to examine the asymptotic behavior of $M_s(b)$. Since $\eta(s) > 0$,

$$M_s(b) \le \frac{1}{2\pi}\exp\biggl(-\frac{b^2\eta^2(s)}{2}\biggr)\int_0^L \tau_s(s')(b|\zeta_s(s')| \\ + c_{g,s}(s')(2\pi)^{-1/2})\rho_{11,s}^{1/2}(s')\,ds'.$$

Using Taylor expansions of sufficient order, it can be shown that the functions $(s,s') \in J\times J \to \tau_s(s'), \tau_s'(s'), \tau_s''(s'), \rho_{kl,s}(s')$, $0 \le k,l \le 2$, and $c_{g,s}(s')$ are continuous. Therefore, their supremum over $J\times J$ is finite and, by the positivity of $\tau_s(s')$ as a function of $(s,s') \in J\times J$, $\theta_2 = \inf_{s\in J}\eta(s) = \inf_{(s,s')\in J\times J}\tau_s(s') > 0$. It follows that $M_s(b) \le C_1\, b\,\varphi(b\theta_2)$ for all $s\in J$ and all $b > 0$. Hence, $\int_b^\infty M_s(b')\,db' \le C_2\,\varphi(b\theta_2)$. Consequently, $A \le e_1(b)\,C_3\,\varphi(b\theta_2)$ and $B \le e_1(b)\,C_4\,\varphi(b\theta_2)$. From the asymptotic behavior of $M(b)$ and $m(b)$, it follows that $e_1(b) = e(b)$.

If $Y(s)$ is not periodic and $\min(\tau_Y(0), \tau_Y(L)) > \inf_{s\in J}\tau_Y(s)$,

$$\frac{\varphi(b\tau_Y(0))}{e_1(b)} \\ = C_5\biggl(\int_0^L \tau_Y(s)(\tau_Y(s)+\tau_Y''(s))\exp\biggl(\frac{b^2}{2}(\tau_Y^2(0) - \tau_Y^2(s) - {\tau_Y'}^2(s))\biggr)\,ds\biggr)^{-1}.$$

For $\varepsilon > 0$ small enough, the subset $J_\varepsilon = \{s\in J\colon \tau_Y^2(0) - \tau_Y^2(s) - {\tau_Y'}^2(s) \ge \varepsilon^2\}$ of $J$ contains a nonempty interval [around a global minimum of $\tau_Y^2(s) + {\tau_Y'}^2(s)$]. Therefore, it has positive Lebesgue measure. For such an $\varepsilon > 0$,

$$\begin{aligned}\int_0^L &\tau_Y(s)(\tau_Y(s)+\tau_Y''(s))\exp\biggl(\frac{b^2}{2}(\tau_Y^2(0) - \tau_Y^2(s) - {\tau_Y'}^2(s))\biggr)\,ds \\ &\ge \exp\biggl(\frac{b^2\varepsilon^2}{2}\biggr)\int_{J_\varepsilon}\tau_Y(s)(\tau_Y(s)+\tau_Y''(s))\,ds \\ &= C_6\,\varphi^{-1}(b\varepsilon).\end{aligned}$$

Hence, $\varphi(b\tau_Y(0)) = e_1(b)\,O(\varphi(b\theta_3))$, that is, $\delta_M(b) = e_1(b)\,O(\varphi(b\theta_3))$. Finally, from the continuity and positivity of $s\to\tau_s(0)$ and $\tau_s(L)$, $\sup_{s\in J}\delta_{M_s}(b) = e_1(b)\,O(\varphi(b\theta_4))$.

(ii) Straightforward.

(iii) Let us take $\varepsilon > 0$ such that $(s^*-\varepsilon, s^*+\varepsilon) \subset J^+$. Such an $\varepsilon$ exists since $\beta_s(s') > 0$ for $s$ close to $s^*$ and $s'\in J$. Then there exists $\delta > 0$ such that

$\tau_Y^2(s) + \tau_Y'^2(s) \geq (\tau + \delta)^2$ for $s \in (s^* - \varepsilon, s^* + \varepsilon)$ and

$$M(b) = \frac{b}{2\pi} \int_{s^*-\varepsilon}^{s^*+\varepsilon} \tau_Y(s)(\tau_Y(s) + \tau_Y''(s)) \times \exp\left(-\frac{b^2}{2}(\tau_Y^2(s) + \tau_Y'^2(s))\right) ds \left(1 + O\left(\frac{\varphi(b\theta_1)}{b^3}\right)\right) + O(b\varphi(b(\tau + \delta)))$$

[see (i)]. A similar result is obtained for $m(b)$, replacing $O(\varphi(b\theta_1)/b^3)$ by $O(\varphi(b\theta_2))$ for some $\theta_1$ and $\theta_2 > 0$. This shows that $M(b) \sim m(b) \sim f_Z(b)$. The conclusion follows by applying Laplace's formula to the main term of $M(b)$ in the above expression using a Taylor expansion of order $q$ of $\tau_Y^2(s) + \tau_Y'^2(s)$ around $s = s^*$.

(iv) Analogous to (iii). □

4.5. *Proof of Theorem* 5. This follows directly from a straightforward adaptation of the proof of Theorem 3 and the application of Laplace's formula to $M_s(b)$. □

4.6. *Proof of Theorem* 6. Lemma 14 shows that the process $X(t)$ admits the representation (1). Therefore, by truncation and renormalization, we can construct a sequence of Gaussian processes $X_n(t)$ of the form (4) which converge to $X(t)$. Moreover, under Conditions D1 and D2, $X_n(t)$ satisfies Conditions 1 and 2 for all $n$ sufficiently large. Similarly, under Conditions D2–D4, $X_n(t)$ satisfies Conditions 2, 5 and 6 for $n$ large enough. Hence, the density $f_{Z_n}(b)$ of $Z_n = \sup_{t \in I} X_n(t)$ has an upper bound $M_n(b)$ of the form (9) and a lower bound $m_n(b)$ of the form (12).

We will show that $M_n(b) \to M(b)$ for all $b$, $m_n(b) \to m(b)$ for all $b > 0$ and the sequence $\{f_{Z_n}\}$ is weakly relatively compact in $L^1(\mathbb{R})$. With an equality due to Dmitrovskii [Lifshits (1986)], this implies that $Z$ has a density $f_Z(b)$ which is the limit of $f_{Z_n}(b)$ and satisfies $m(b) \leq f_Z(b) \leq M(b)$.

LEMMA 14. *Under Condition* D1:

(i) *there exists a representation* $\tau_X^{-1}(t) \sum_{j=1}^{\infty} \xi_j g_j(t)$ *of* $X(t)$, *where the r.v.'s* $\xi_j$, $j \geq 1$, *are i.i.d.* $\mathscr{N}(0, 1)$;

(ii) *the functions* $g_j(t)$, $j \geq 1$, *are in* $\mathscr{C}^3(I)$;

(iii) $\sup_{(t_1, t_2) \in I \times I} |D_{kl} r_U(t_1, t_2) - \sum_{j=1}^{n} g_j^{(k)}(t_1) g_j^{(l)}(t_2)| \to 0$ *as* $n \to \infty$, $0 \leq k, l \leq 3$.

PROOF. (i) Straightforward.

(ii) From (i), $g_j(t) = E(U(t)\,\xi_j)$ for all $j \geq 1$ and all $t \in I$. Therefore, by Theorem 2.2.2 in Adler (1981) and the Cauchy–Schwarz inequality, $g_j^{(k)}(t) = E(\dot{U}^{(k)}(t)\,\xi_j)$ for all $0 \leq k \leq 4$, where $\dot{U}^{(k)}(t)$ denotes the $k$th quadratic mean derivative of $U(t)$.

(iii) This is a consequence of Dini's theorem and Parseval's inequality. □

Let $\mathbb{P}_n\colon \ell^2 \to \ell^2$ denote the orthogonal projection of $\ell^2$ onto $\mathbb{R}^n \times \{0,\dots\}$, defined by $\mathbb{P}_n \mathbf{x} = (x_1,\dots,x_n,0,\dots)$ for $\mathbf{x} \in \ell^2$.

LEMMA 15. (i) *Under Condition* D1, $\|\mathbb{P}_n \mathbf{g}(t)\| = 1 + \varepsilon_n(t)$ *with* $\varepsilon_n^{(k)}(t) \to 0$ *as* $n \to \infty$ *uniformly for* $t \in I$, *for* $0 \le k \le 3$;

(ii) *Under Conditions* D1 *and* D2, $\|\mathbb{P}_n \mathbf{g}'(t)\| \to \|\mathbf{g}'(t)\|$ *uniformly for* $t \in I$ *as* $n \to \infty$ *and there exists* $N_1$ *such that* $\inf_{t\in I} \|\mathbb{P}_n \mathbf{g}'(t)\| > 0$ *for all* $n \ge N_1$.

PROOF. All statements are direct consequences of Lemma 14 and the continuity and positivity of $\|\mathbf{g}'(t)\|$ in $I$. $\square$

Hence, from Lemma 15, $\mathbf{g}_n(t) = \mathbb{P}_n \mathbf{g}(t)/\|\mathbb{P}_n \mathbf{g}(t)\| = (g_{n,1}(t),\dots,g_{n,n}(t), 0,\dots)$, $t \in I$, is well defined for all $n \ge N_1$ and $\mathbf{g}_n^{(k)}(t) \to \mathbf{g}^{(k)}(t)$ uniformly for $0 \le k \le 3$. The functions $\mathbf{g}_n(t)$, $t \in I$, parameterize a curve $\gamma_n$ on the unit sphere of $\mathbb{R}^n \times \{0,\dots\} \subset \ell^2$. Moreover, there exists $N_2$ such that $\inf_{t\in I} \|\mathbf{g}_n'(t)\| > 0$ for all $n \ge N_2$. The corresponding unit speed parameterization of $\gamma_n$ is defined by $\mathbf{f}_n(\lambda_n(t)) = (f_{n,1}(\lambda_n(t)),\dots,f_{n,n}(\lambda_n(t)),0,\dots)$, where $\lambda_n(t) = \int_0^t \|\mathbf{g}_n'(t')\|\,dt'$. With this notation $Z_n = \sup_{t\in I} X_n(t) = \sup_{t\in I} Y_n(\lambda_n(t))$, where $X_n(t) = \tau_X^{-1}(t) U_n(t)$, $U_n(t) = \sum_{j=1}^n \xi_j g_{n,j}(t)$, $Y_n(\lambda_n(t)) = \tau_{Y_n}^{-1}(\lambda_n(t)) V_n(\lambda_n(t))$ and $V_n(\lambda_n(t)) = \sum_{j=1}^n \xi_j f_{n,j}(\lambda_n(t))$.

LEMMA 16. *Under Conditions* D1 *and* D2, $\lambda_n^{(k)}(t) \to \lambda^{(k)}(t)$, $(\lambda_n^{-1})^{(k)}(\lambda_n(t)) \to (\lambda^{-1})^{(k)}(\lambda(t))$, $\mathbf{f}_n^{(k)}(\lambda_n(t)) \to \mathbf{f}^{(k)}(\lambda(t))$ *all uniformly for* $t \in I$ *and for* $0 \le k \le 3$, $\tau_{Y_n}^{(l)}(\lambda_n(t)) \to \tau_Y^{(l)}(\lambda(t))$ *uniformly for* $t \in I$ *and for* $0 \le l \le 3$ *and* $c_{g,n}(\lambda_n(t)) \to c_g(\lambda(t))$ *uniformly for* $t \in I$.

PROOF. All convergences follow directly from Lemma 15 [$\mathbf{g}_n'(t) \to \mathbf{g}'(t)$ uniformly], Lemma 14 and the uniform continuity of $\tau_X(t)$ and its derivatives over $I$. $\square$

If Condition D1 is replaced by Condition D3 in Lemmas 15 and 16, all the results hold for $0 \le k \le 7$ and $0 \le l \le 4$. Let us define $\beta_{n,\lambda_n(t)}(\lambda_n(t')) = \tau_{Y_n}(\lambda_n(t')) - \tau_{Y_n}(\lambda_n(t))\langle \mathbf{f}_n(\lambda_n(t)), \mathbf{f}_n(\lambda_n(t'))\rangle - \tau_{Y_n}'(\lambda_n(t))\langle \mathbf{T}_n(\lambda_n(t)), \mathbf{f}_n(\lambda_n(t'))\rangle$.

LEMMA 17. *Under Conditions* D1 *and* D2, *for each compact subset* $K^+ \subset \lambda^{-1}(J^+)$:

(i) *there exists* $N_3$ *such that, for all* $n \ge N_3$, $\delta = \inf_{t\in K^+}(\tau_{Y_n}(\lambda_n(t)) + \tau_{Y_n}''(\lambda_n(t))) > 0$;

(ii) *there exists* $N_4$ *such that, for all* $n \ge N_4$, $\beta_{n,\lambda_n(t)}(\lambda_n(t')) > 0$ *for all* $t \in K^+$ *and* $t' \in I$.

PROOF. (i) The proof is similar to the proof of Lemma 15(ii).

(ii) By a Taylor expansion of order 3 and Lemma 16, $\beta_{n,\lambda_n(t)}(\lambda_n(t)+h) = (\tau_{Y_n}(\lambda_n(t)) + \tau_{Y_n}''(\lambda_n(t)))h^2/2 + R_n(t,h)$, where $R_n(t,h) \le C|h|^3$ for some constant $C > 0$. Moreover, by (i), we have for $n \geq N'_4$ that $\beta_{n,\lambda_n(t)}(\lambda_n(t)+h) \geq h^2|\delta/2 - C|h|| > 0$ for all $t \in K^+$ and $h \neq 0$ such that $|h| < \varepsilon = \delta/(4C)$.

Since $K_\varepsilon = \{(t,t') \in K^+ \times I\colon |t'-t| \geq \varepsilon\}$ is compact and the function $(t,t') \to \beta_{\lambda(t)}(\lambda(t'))$ is continuous and positive on $K_\varepsilon$, $\inf_{(t,t')\in K_\varepsilon} \beta_{n,\lambda_n(t)}(\lambda_n(t')) > 0$ for all $n \geq N''_4$ in view of Lemma 16. The conclusion follows for $n \geq N_4 = \max(N'_4, N''_4)$. $\square$

LEMMA 18. *Under Conditions* D1–D4, *there exists* $N_5$ *such that, for all* $n \geq N_5$ *and all* $t' \neq t \in I$, $\mathbf{f}_n(\lambda_n(t))$, $\mathbf{f}'_n(\lambda_n(t))$, $\mathbf{f}_n(\lambda_n(t'))$, $\mathbf{f}'_n(\lambda_n(t'))$ *are linearly independent.*

PROOF. First, using a Taylor expansion of sufficient order and Lemma 16, we show that the Gram determinant of this system is positive for all $t' \neq t$ sufficiently close whenever $n$ is large enough. Second, we take advantage of the continuity and positivity of the corresponding Gram determinant with $\mathbf{f}_n$ and $\mathbf{f}_n{}'$ replaced by $\mathbf{f}$ and $\mathbf{f}'$ for $|t'-t| \geq \varepsilon > 0$ and its uniform convergence $(n \to \infty)$. $\square$

LEMMA 19. *Under Condition* D1, $P\{Z_n \leq a\} \to P\{Z \leq a\}$ *for all* $a$.

PROOF. First,

$$|Z - Z_n| \leq \sup_{t\in I} |X(t) - X_n(t)| \leq \Big(\inf_{t\in I} \tau_X(t)\Big)^{-1} \sup_{t\in I} |U(t) - U_n(t)|,$$

$$\kappa_n^2 = \sup_{t\in I} \mathrm{Var}(U(t) - U_n(t)) \to 0$$

and

$$d_n^2(t_1,t_2) = E((U(t_1) - U_n(t_1)) - (U(t_2) - U_n(t_2)))^2 \leq C|t_1 - t_2|^2$$

for $n$ large enough, by Lemma 14. On the other hand, we have the following inequality due to Dmitrovskii [Lifshits (1986)]:

$$P\Big\{\sup_{t\in I} |U(t) - U_n(t)| > u\Big\} = 2\exp(-u^2/(2\kappa_n^2))\, q_n(u),$$

where $q_n(u) = 4.1\exp\big(2^{1/2}\, 6\,\Psi((\kappa_n/u)^{1/2})\big)(\kappa_n/u)^{1/2}\,(\Psi((\kappa_n/u)^{1/2}))^{1/2}$ with $\Psi(\varepsilon_n) \sim \varepsilon_n \log^{1/2}(C_1/\varepsilon_n)$ when $\varepsilon_n \to 0$ as $n \to \infty$ and $d_n(t_1,t_2) \leq C|t_1 - t_2|$. Therefore, $P\{|Z - Z_n| > u\} \to 0$ as $n \to \infty$ and $Z_n$ converges in probability, hence weakly, to $Z$. Then $P\{Z_n \leq a\} \to P\{Z \leq a\}$ for all continuity points $a$. As $P\{Z \leq a\}$ is continuous [Tsirel'son (1975)], $P\{Z_n \leq a\} \to P\{Z \leq a\}$ for all $a$. $\square$

For each compact subset $K^+ \subset \lambda^{-1}(J^+)$, let us define the restrictions $m_{K^+}(b)$ and $m_{n,K^+}(b)$ of $m(b)$ and $m_n(b)$, respectively, obtained by replacing in (12) the integral in $s$ over $J^+$ by the integral over $\lambda(K^+)$ and $\lambda_n(K^+)$, respectively. By Lebesgue's convergence theorem and Lemmas 14–18, it follows that $M_n(b) \to M(b)$ for all $b$ and $m_{n,K^+}(b) \to m_{K^+}(b)$ for all $b > 0$. Moreover,

$M_n(b) \le M^*(b)$ for $n$ large enough and $M(b) \le M^*(b)$, where $M^*(b) = C\varphi(b\theta)$ for some $C > 0$ and $\theta > 0$. Hence, the sequence $\{f_{Z_n}\}$ is weakly compact in $L^1(\mathbb{R})$ [Bourbaki (1967), pages 112–113], which means that there exist a function $f \in L^1(\mathbb{R})$ and a subsequence $\{f_{Z_{n'}}\}$ such that $\int_{-\infty}^{\infty} f_{Z_{n'}}(b)h(b)\,db \to \int_{-\infty}^{\infty} f(b)h(b)\,db$ for all $h \in L^\infty(\mathbb{R})$. By taking $h(b) = 1_{(-\infty,\,a]}(b)$, it follows that $P\{Z_{n'} \le a\} \to \int_{-\infty}^{a} f(b)\,db$ for all $a$. Therefore, by Lemma 19, $Z$ has a density $f_Z(b) = f(b)$. Moreover, the same result holds for any accumulation point of $\{f_{Z_n}\}$. Finally, $f_{Z_n}(b) \to f_Z(b)$ for a.e. $b$. By Tsirel'son (1975), $f_Z(b)$ has bounded variation on every compact interval of $\mathbb{R}$. Hence, $f_Z(b)$ has a right and a left limit at each point. If we select a left-continuous (say) version of $f_Z(b)$, it follows that $m_{K^+}(b) \le f_Z(b)$ for all $b > 0$ and $f_Z(b) \le M(b)$ for all $b$. Since the Lebesgue measure of $\lambda^{-1}(J^+)\backslash K^+$ can be made arbitrarily small, it follows that $m(b) \le f_Z(b)$ for all $b > 0$. □

## APPENDIX

Let $\mathbf{f}(s) = (f_1(s), \dots, f_n(s))$, $s \in J = [0, L]$, be a unit speed parameterization of a smooth curve $\gamma$ embedded in the unit sphere of $\mathbb{R}^n$. Therefore, $\|\mathbf{f}(s)\| = \|\mathbf{f}'(s)\| \equiv 1$, $\langle \mathbf{f}(s), \mathbf{f}''(s)\rangle \equiv -1$, $\|\mathbf{f}''(s)\| \ge 1$ for all $s \in J$ and $L = |\gamma|$ is the length of $\gamma$.

At each point $M = M(s)$ of $\gamma$, we can define the unit vector tangent to $\gamma$ $\mathbf{T} = \mathbf{T}(s) = \mathbf{f}'(s)$ and the principal normal vector $\mathbf{N} = \mathbf{N}(s) = \mathbf{f}''(s)/\|\mathbf{f}''(s)\|$. Since $\|\mathbf{T}\| = 1$, $\mathbf{N} = \mathbf{T}'/c$ is orthogonal to $\mathbf{T}$, where $c = c(s) = \|\mathbf{f}''(s)\|$ defines the curvature of $\gamma$ at the point $M = M(s)$.

For each $s \in J$, if $\mathbf{N}(s) \ne -\mathbf{f}(s)$, there exists only one unit vector $\mathbf{K}$=$\mathbf{K}(s) \in \mathrm{Vect}^\perp(\mathbf{f}, \mathbf{T})$ such that $\mathbf{K} \in \mathrm{Vect}(\mathbf{f}, \mathbf{N})$ and $\langle \mathbf{N}, \mathbf{K}\rangle = \cos\alpha > 0$. Moreover, $\mathbf{K}(s)$ is $C^1$ on each interval on which $\langle \mathbf{f}, \mathbf{N}\rangle > -1$, that is, $c \ne 1$. If $c = 1$ we can define $\mathbf{K}(s) = \mathbf{K}(s_-)$. Let us denote by $\mathbf{K}_1 = \mathbf{K}_1(s), \dots, \mathbf{K}_{n-2} = \mathbf{K}_{n-2}(s)$ an orthonormal basis of $\mathrm{Vect}^\perp(\mathbf{f}, \mathbf{T})$ such that $\mathbf{K}_1 \equiv \mathbf{K}$ and the functions $\mathbf{K}_2, \dots, \mathbf{K}_{n-2}$ are $C^1$. At each point $M = M(s)$ of $\gamma$, the moving frame $(M, \mathbf{f}, \mathbf{T}, \mathbf{K}_1, \dots, \mathbf{K}_{n-2})$ is orthonormal and the matrix of $(\mathbf{f}', \mathbf{T}', \mathbf{K}_1', \dots, \mathbf{K}_{n-2}')$ with respect to $(\mathbf{f}, \mathbf{T}, \mathbf{K}_1, \dots, \mathbf{K}_{n-2})$ is antisymmetric:

$$
\begin{array}{cc}
\begin{array}{cccccc} \mathbf{f}' & \mathbf{T}' & \mathbf{K}_1' & \mathbf{K}_2' & \cdots & \mathbf{K}_{n-2}' \end{array} & \\
\begin{pmatrix}
0 & -1 & 0 & 0 & \cdots & 0 \\
1 & 0 & -c_g & 0 & \cdots & 0 \\
0 & c_g & & & & \\
0 & 0 & & & & \\
\vdots & \vdots & & * & & \\
0 & 0 & & & &
\end{pmatrix}
&
\begin{array}{c} \mathbf{f} \\ \mathbf{T} \\ \mathbf{K}_1 \\ \mathbf{K}_2 \\ \vdots \\ \mathbf{K}_{n-2} \end{array}
\end{array}
$$

where $c_g = c_g(s) = \langle \mathbf{K}_1, \mathbf{T}'\rangle = c\langle \mathbf{K}_1, \mathbf{N}\rangle = c\cos\alpha \ge 0$ defines the geodesic curvature of $\gamma$ at the point $M = M(s)$. By the definition of $\alpha$, we have $\langle \mathbf{f}, \mathbf{N}\rangle^2 =$

$\sin^2\alpha = 1/c^2$ and then $c_g^2 = \|\mathbf{f}''\|^2 - 1$. Note that $c_g = 0$ iff $\cos\alpha = 0$, that is, $\langle \mathbf{f}, \mathbf{N}\rangle = -1$. Finally, it can be shown that $\mathbf{K} = (\mathbf{f} + \mathbf{f}'')/c_g$.

If $\mathbf{g}(t)$, $t \in I = [0, T]$, is a general parameterization of $\gamma$ with $\|\mathbf{g}'(t)\| > 0$ for all $t \in I$, the corresponding unit speed parameterization of $\gamma$ is given by $\mathbf{f}(s) = \mathbf{g}(\lambda^{-1}(s))$, $s \in J$, where $s = \lambda(t) = \int_0^t \|\mathbf{g}'(u)\|\, du$, $t \in I$, is the arc length of $\gamma$ from 0 to $t$. We have

$$\mathbf{T}(s) = \mathbf{f}'(s) = \frac{d\mathbf{f}}{ds} = \frac{d\mathbf{g}}{dt}\frac{dt}{ds} = \frac{\mathbf{g}'(t)}{\lambda'(t)} = \frac{\mathbf{g}'(t)}{\|\mathbf{g}'(t)\|},$$

$$\begin{aligned} c(s)\mathbf{N}(s) = \mathbf{f}''(s) &= \frac{d^2\mathbf{f}}{ds^2} = \frac{d}{dt}\left(\frac{\mathbf{g}'}{\|\mathbf{g}'\|}\right)\frac{dt}{ds} \\ &= \frac{1}{\|\mathbf{g}'(t)\|^2}\left(\mathbf{g}''(t) - \frac{\langle \mathbf{g}'(t), \mathbf{g}''(t)\rangle}{\|\mathbf{g}'(t)\|^2}\,\mathbf{g}'(t)\right), \end{aligned}$$

$$c_g^2(t) = \frac{\|\mathbf{g}'(t)\|^2\|\mathbf{g}''(t)\|^2 - \langle \mathbf{g}'(t), \mathbf{g}''(t)\rangle^2}{\|\mathbf{g}'(t)\|^6} - 1.$$

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