The Annals of Probability
1996, Vol. 24, No. 2, 916–931

A LIAPOUNOV BOUND FOR SOLUTIONS OF THE POISSON EQUATION

BY PETER W. GLYNN[1] AND SEAN P. MEYN[2]

Stanford University and University of Illinois

In this paper we consider ψ-irreducible Markov processes evolving in discrete or continuous time on a general state space. We develop a Liapounov function criterion that permits one to obtain explicit bounds on the solution to the Poisson equation and, in particular, obtain conditions under which the solution is square integrable.

These results are applied to obtain sufficient conditions that guarantee the validity of a functional central limit theorem for the Markov process. As a second consequence of the bounds obtained, a perturbation theory for Markov processes is developed which gives conditions under which both the solution to the Poisson equation and the invariant probability for the process are continuous functions of its transition kernel. The techniques are illustrated with applications to queueing theory and autoregressive processes.

1. Introduction. In this paper we develop a "Foster's criterion," or "Liapounov function," approach to obtaining finite-valued solutions $\hat{g}$ to the *Poisson equation*, which in discrete time may be written as

$$\overline{g} = \hat{g} - P\hat{g}, \tag{1}$$

where g is a given real-valued function on the state space, $\overline{g} = g - \pi(g)$, and π is an invariant probability. In the special case where $g \equiv 0$, solutions to the Poisson equation are precisely *harmonic functions*. In general, if $\hat{g}_1$ and $\hat{g}_2$ are two solutions to the Poisson equation, then the difference $\hat{g}_1 - \hat{g}_2$ is harmonic.

For continuous-time processes, the Poisson equation becomes

$$\overline{g} = -\widetilde{\mathscr{A}}\hat{g}, \tag{2}$$

where $\widetilde{\mathscr{A}}$ is the extended generator of the Markov process $\mathbf{\Phi}$, formally defined in (11) and (12) below.

The Poisson equation and the general potential theory of positive kernels are developed in the seminal work of Neveu [28], Revuz [32] and Constantinescu and Cornea [7]. The reader is referred to Nummelin [30] for some of the most current results on the Poisson equation, to whom we owe much.

Received September 1993; revised August 1995.

[1]Research supported in part by the U.S. Army Research Office under Contract DAAL-03-kk-K-0063, and by a Natural Sciences of Engineering Research Council of Canada grant.

[2]Research supported in part by NSF Grants ECS-89-10088 and ECS-92-16487, and by University of Illinois Research Board Grant 1-2-69637.

AMS 1991 *subject classifications*. 68M20, 60J10.

Key words and phrases. Markov chain, Markov process, Poisson equation, Liapounov function, Foster's criterion, functional central limit theorem, perturbation theory.

The solution $\hat{g}$ to the Poisson equation (1) is fundamental to the analysis of the additive functional

$$S_n = \sum_{k=0}^{n-1} \overline{g}(\Phi_k), \qquad S_0 = 0, \tag{3}$$

and it is equally valuable in studying the analogous additive functional in continuous time. The principal observation that underlies the analysis of S_n is that the behavior of such an additive functional is closely related to that of a certain martingale. Specifically, let

$$M_n = \hat{g}(\Phi_n) + S_n, \qquad n \in \mathbb{Z}_+. \tag{4}$$

We note that since $\hat{g}$ solves (1),

$$\begin{aligned} M_n &= \hat{g}(x) + \sum_{k=1}^{n} [\hat{g}(\Phi_k) - P\hat{g}\,(\Phi_{k-1})] \\ &= \hat{g}(x) + \sum_{k=1}^{n} [\hat{g}(\Phi_k) - \mathsf{E}(\hat{g}(\Phi_k) \mid \mathscr{F}_{k-1})], \end{aligned}$$

where $\mathscr{F}_n = \sigma(\Phi_0, \dots, \Phi_n)$. Hence, under suitable integrability conditions on $\hat{g}$, the adapted process $\mathbf{M} = \{(M_n, \mathscr{F}_n)\colon n \geq 1\}$ is a square integrable martingale.

It is evident that the asymptotic behavior of $(S_n\colon n \geq 0)$ will typically mimic that of the martingale $\mathbf{M}$. In particular, laws of large numbers, central limit theorems and laws of the iterated logarithm can often be derived for $(S_n\colon n \geq 0)$ by applying appropriate martingale theorems. This approach is taken in Maigret [17] and Duflo [10] to obtain a functional central limit theorem (FCLT) for Markov chains. In a related work, Kurtz [15] considers chains arising in models found in polymer chemistry. Bhattacharaya [5] considers the Poisson equation in continuous time to derive an FCLT and functional law of the iterated logarithm for ergodic Markov processes.

Using a more classical approach based on the existence of atoms for a split chain, Meyn and Tweedie [23] derive a central limit theorem and law of the iterated logorithm for geometrically ergodic chains. The Liapounov function approach of [23] is similar to that of the present paper, but the results reported here require far milder assumptions than geometric ergodicity.

In order to successfully apply the Poisson equation to obtain an FCLT for a given chain, one must basically show that the martingale difference sequence $\Delta_n = \hat{g}(\Phi_n) - (P\hat{g})(\Phi_{n-1})$ is appropriately square integrable. The Liapounov function criterion developed in this paper permits one to verify that the solution $\hat{g} \in L^2(\pi)$, where π is the (unique) invariant measure of $\mathbf{\Phi}$. This guarantees that the martingale differences are well behaved, so that the above-mentioned FCLT's apply.

Before proceeding, we note that the solution $\hat{g}$ is often unique in a certain sense. The reader is referred to Shwartz and Makowski [36] for further results in a discrete-time–discrete-space setting.

PROPOSITION 1.1. *Suppose that* $\mathbf{\Phi}$ *is an ergodic Markov chain with unique invariant probability* π, *with discrete- or continuous-time parameter, and suppose that* $\hat{g}$ *and* $\hat{g}_{\bullet}$ *are two solutions to the Poisson equation with* $\pi(|\hat{g}|+|\hat{g}_{\bullet}|) < \infty$. *Then, for some constant* c, $\hat{g}(x) = c + \hat{g}_{\bullet}(x)$ *for a.e.* $x \in \mathsf{X}$ $[\pi]$.

PROOF. The proofs are similar, so we will consider only the continuous-time case. We have already remarked that $h := \hat{g} - \hat{g}_{\bullet}$ is harmonic.

We apply the ergodic theorem for Markov chains: to do so, we consider the skeleton chain Φ_n and the function $H = \int_0^1 P^s h\, ds$. From the Poisson equation we have that $P^t h = h$ for all t. From the ergodic theorem

$$n^{-1} \int_0^n P^t h\, dt = n^{-1} \sum_{k=0}^{n-1} (P^k H)(x) \to \pi(h)$$

a.e. $[\pi]$ (cf. [11]). Hence $h = \pi(h)$ a.e. $[\pi]$. □

2. Discrete-time processes. In this paper we consider a Markov process $\mathbf{\Phi} = \{\Phi_t\colon\ t \in T\}$ where $T = \mathbb{R}_+$ or $\mathbb{Z}_+$, evolving on a locally compact separable metric space X, whose Borel σ-algebra shall be denoted by $\mathscr{B}(\mathsf{X})$. We use P_μ and E_μ to denote probabilities and expectations conditional on Φ_0 having distribution μ, and P_x and E_x when μ is concentrated at x. In the discrete-time setting the conditions on the state space can be relaxed somewhat, but it is convenient here to have the same set of assumptions for continuous- and discrete-time processes.

In this section we will develop our main results for discrete-time processes. In Section 3 we show how the results may be extended to the continuous-time case.

A set $\boldsymbol{\alpha} \in \mathscr{B}^+(\mathsf{X})$ is called an *atom* if transitions from distinct points in $\boldsymbol{\alpha}$ are identical: $P(x, \cdot\,) = P(y, \cdot\,)$, $x,\ y \in \boldsymbol{\alpha}$. When an atom exists, a solution to the Poisson equation (1) can easily be found. Define the function $\hat{g}$ by

$$\hat{g}(x) = \mathsf{E}_x\left[\sum_{k=0}^{\sigma_\alpha} \overline{g}(\Phi_k)\right] = \sum_{k=0}^{\infty} (I_{\alpha^c} P)^k \overline{g} \tag{5}$$

when this is well defined. When $\mathbf{\Phi}$ is positive Harris [25] and $\pi(\boldsymbol{\alpha}) > 0$, we have

$$\int_\alpha \pi(dx) \mathsf{E}_x\left[\sum_{k=1}^{\tau_\alpha} |g(\Phi_k)|\right] = \int_\alpha \pi(dx) \mathsf{E}_x\left[\sum_{k=0}^{\tau_\alpha - 1} |g(\Phi_k)|\right] = \pi(|g|).$$

Hence, when $\pi(|g|) < \infty$, the expression (5) is well defined on $\boldsymbol{\alpha}$, and in fact $\hat{g}$ is defined and finite a.e. $[\pi]$. It follows from the second equality in (5) that $\hat{g}$ solves the Poisson equation (1).

Even when an atom does not exist, by considering a *split chain* one can construct an atom $\check{\boldsymbol{\alpha}}$ on the split state space. Define then the kernel $G_{s,\,\nu}$ for functions f and states x by

$$G_{s,\nu}(x, f) = \check{\mathsf{E}}_{\delta_x^*}\left[\sum_{k=0}^{\sigma_{\check{\alpha}}} f(\check{\Phi}_k)\right]. \tag{6}$$

Then the function $\hat{g}(x) = G_{s,\nu}(x, \overline{g})$ solves the Poisson equation and is finite a.e. if $\pi(|g|) < \infty$ (see [29] and [30]).

The split chain is defined through a generalization of atoms known as *petite sets*. Let a be a probability distribution on $\mathbb{Z}_+$, and let K_a denote the Markov transition function $K_a = \sum_{i=0}^{\infty} a(i)P^i$. A set $C \subset \mathsf{X}$ is called ν_a-petite, where ν_a is a nontrivial measure on $\mathscr{B}(\mathsf{X})$, if, for the distribution a on $\mathbb{Z}_+$,

$$K_a(x, A) \ge \nu_a(A), \qquad x \in C,\ A \in \mathscr{B}(\mathsf{X}).$$

The distribution a is called the *sampling distribution* for the petite set C. If the particular measure ν_a is unimportant, the prefix will be omitted so that C is simply called *petite*.

Much of the development to follow is concerned with verifying a recurrence condition for a Markov chain known as f-regularity. Let $f \ge 1$ be a real-valued function on X, and suppose that, for some finite-valued function V_0 and any $B \in \mathscr{B}^+(\mathsf{X})$, there exists $c(B) < \infty$ such that

$$\mathsf{E}_x\left[\sum_{k=0}^{\tau_B - 1} f(\Phi_k)\right] \le V_0(x) + c(B), \qquad x \in \mathsf{X}. \tag{7}$$

Then the chain is called *f-regular* (*with bounding function* V_0). This definition is apparently stronger than similar notions of f-regularity given in [29] or [25], but we will find that all of these definitions are essentially equivalent.

This condition may be verified by establishing a drift property for the chain toward a single petite set.

> For a function $f\colon \mathsf{X} \to [1, \infty)$, a petite set $C \in \mathscr{B}(\mathsf{X})$, a constant $b < \infty$ and a function $V\colon \mathsf{X} \to [0, \infty)$,

$$PV(x) \le V(x) - f(x) + b\mathbb{1}_C(x), \qquad x \in \mathsf{X}. \tag{8}$$

The power of (8) largely comes from the following result.

THEOREM 2.1 (Comparison theorem). *Suppose that the bound*

$$PV(x) \le V(x) - f(x) + s(x), \qquad x \in \mathsf{X},$$

is satisfied. Then, for each $x \in \mathsf{X}$, $N \in \mathbb{Z}_+$ *and any stopping time* τ, *we have*

$$\sum_{k=0}^{N} \mathsf{E}_x[f(\Phi_k)] \le V(x) + \sum_{k=0}^{N} \mathsf{E}_x[s(\Phi_k)],$$

$$\mathsf{E}_x\left[\sum_{k=0}^{\tau-1} f(\Phi_k)\right] \le V(x) + \mathsf{E}_x\left[\sum_{k=0}^{\tau-1} s(\Phi_k)\right].$$

PROOF. By Dynkin's formula (cf. [23]) and the bound on PV, we have

$$0 \le \mathsf{E}_x[V(\Phi_{\tau^n})] \le V(x) + \mathsf{E}_x\Bigg[\sum_{i=1}^{\tau^n}(s_{i-1}(\Phi_{i-1}) - [f_{i-1}(\Phi_{i-1}) \wedge N])\Bigg],$$

where $\tau^n = \min(\tau,\ n,\ \min(k\colon\ V(\Phi_k) \ge n))$. Hence, by adding the finite term

$$\mathsf{E}_x\Bigg[\sum_{k=1}^{\tau^n}[f(\Phi_{k-1}) \wedge N]\Bigg]$$

to each side, we get

$$\mathsf{E}_x\Bigg[\sum_{k=1}^{\tau^n}[f(\Phi_{k-1}) \wedge N]\Bigg] \le \mathsf{E}_x\Bigg[\sum_{k=1}^{\tau^n} s(\Phi_{k-1})\Bigg] \le \mathsf{E}_x\Bigg[\sum_{k=1}^{\tau} s(\Phi_{k-1})\Bigg].$$

Letting $n \to \infty$ and then $N \to \infty$ gives the result by the monotone convergence theorem. □

We now characterize f-regularity using (8).

THEOREM 2.2. *If* (8) *holds, then*

(i) *the Markov chain* $\mathbf{\Phi}$ *is positive Harris recurrent with invariant probability* π;

(ii) $\pi(f) < \infty$;

(iii) *for any* $B \in \mathscr{B}^+(\mathsf{X})$ *there exists* $c(B) < \infty$ *such that*

$$\sum_{k=0}^{\infty}(PI_{B^c})^k f(x) = \mathsf{E}_x\Bigg[\sum_{k=0}^{\tau_B-1} f(\Phi_k)\Bigg] \le V(x) + c(B),$$

so that $\mathbf{\Phi}$ *is* f*-regular with bounding function* V.

PROOF. Results (i) and (ii) follow easily from (iii) and the structure of π in terms of mean occupancy times: see Theorem 10.0.1 of [25].

To prove (iii), suppose that (8) holds. By Theorem 2.1, the strong Markov property and the bound

$$\mathbb{1}_C(x) \le \psi_a(B)^{-1}K_a(x,\ B),$$

which follows from the fact that C is ψ_a-petite for some ψ_a, we have, for any $B \in \mathscr{B}^+(\mathsf{X})$, $x \in \mathsf{X}$,

$$\begin{aligned}\mathsf{E}_x\Bigg[\sum_{k=0}^{\tau_B-1} f(\Phi_k)\Bigg] &\le V(x) + b\mathsf{E}_x\Bigg[\sum_{k=0}^{\tau_B-1} \mathbb{1}_C(\Phi_k)\Bigg]\\ &\le V(x) + b\mathsf{E}_x\Bigg[\sum_{k=0}^{\tau_B-1} \psi_a(B)^{-1}K_a(\Phi_k,\ B)\Bigg]\end{aligned}$$

$$= V(x) + b\psi_a(B)^{-1} \sum_{i=0}^{\infty} a_i \mathsf{E}_x\left[\sum_{k=0}^{\tau_B - 1} P^i(\Phi_k, B)\right]$$

$$= V(x) + b\psi_a(B)^{-1} \sum_{i=0}^{\infty} a_i \mathsf{E}_x\left[\sum_{k=0}^{\tau_B - 1} \mathbb{1}_B(\Phi_{k+i})\right]$$

$$\le V(x) + b\psi_a(B)^{-1} \sum_{i=0}^{\infty} i a_i.$$

Since we can choose a so that $m_a = \sum_{i=0}^{\infty} i a_i < \infty$, result (iii) follows with $c(B) = b\psi_a(B)^{-1} m_a$. □

We may now present our main result which gives a specific bound on the *fundamental kernel* $Z := (I - P + \Pi)^{-1}$, where the kernel Π is defined as $\Pi(x, \cdot) \equiv \pi(\cdot)$. For any function g on X, the function $\hat{g} := Z\overline{g}$ is a solution to the Poisson equation whenever the inverse is well defined.

Let $h \ge 1$ be a finite-valued function on X, and let L_h^∞ denote the vector space of all measurable functions g on X such that $|g(x)|/h(x)$ is bounded in x. This vector space is a Banach space with the associated norm

$$|g|_h := \sup_{x \in \mathsf{X}} \frac{|g(x)|}{h(x)}.$$

We now give uniform bounds on solutions to the Poisson equation whenever a solution to (8) exists.

THEOREM 2.3. *Suppose that $\mathbf{\Phi}$ is f-regular, so that* (8) *holds with V everywhere finite, $f \ge 1$ and C petite. Then the fundamental kernel Z is a bounded linear transformation from L_f^∞ to L_h^∞ with $h = V + 1$. That is, for some $c_0 < \infty$ and any $|g| \le f$, the Poisson equation* (1) *admits a solution $\hat{g}$ satisfying the bound $|\hat{g}| \le c_0(V + 1)$.*

PROOF. First consider the strongly aperiodic case. We can assume without loss of generality that the function V is bounded on the set S used in the definition of strong aperiodicity (for details, see [25], page 118).

For any function h on X we can define a "split" function also denoted h on the split state space $\check{\mathsf{X}}$ which is identical to h on the two copies X_0 and X_1 of X. Similarly, we let $A \subset \check{\mathsf{X}}$ denote the "split set" $A = A_0 \cup A_1$.

Since V is bounded on S, it is straightforward to check that when (8) holds we have the following bound for the split chain:

$$\check{P}V(x_i) \le V(x_i) - f(x_i) + d\,\mathbb{1}_{C \cup S}(x_i), \qquad x_i \in \check{\mathsf{X}},$$

where d is a finite constant and C is the petite set used in (8).

Hence we can apply Theorem 2.2(iii) to the split chain to find that, for some constant c,

$$\check{\mathsf{E}}_{x_i}\left[\sum_{k=0}^{\tau_{\check{\alpha}} - 1} f(\check{\Phi}_k)\right] \le V(x_i) + c, \qquad x_i \in \check{\mathsf{X}}.$$

We then have the desired solution to the Poisson equation: for any $|g| \le f$, we let $\hat{g}(x) = G_{s,\nu}(x, \overline{g})$, which is defined in (6). Then $\hat{g}$ solves the Poisson equation with $\hat{g}(x) \le c(V(x)+1)$, for a possibly larger constant c.

In the general case it is convenient to consider the K_{a_ε}-chain, which is always strongly aperiodic when $\mathbf{\Phi}$ is ψ-irreducible. We first show that the K_{a_ε}-chain satisfies a version of (8) with the same function f and a scaled version of the function V used in the theorem. We will on two occasions apply the identity

$$K_{a_\varepsilon} = \varepsilon K_{a_\varepsilon} P + (1-\varepsilon)I, \tag{9}$$

whose derivation is straightforward. Hence, by (8) for the kernel P,

$$K_{a_\varepsilon} V \le \varepsilon K_{a_\varepsilon}(V - f + b\mathbb{1}_C) + (1-\varepsilon)V.$$

Since $f \le (1-\varepsilon)^{-1} K_{a_\varepsilon} f$, it follows that with V_ε equal to a suitable constant multiple of V we have, for some b',

$$K_{a_\varepsilon} V_\varepsilon \le V_\varepsilon - f + b' K_{a_\varepsilon} \mathbb{1}_C.$$

Since C is petite for $\mathbf{\Phi}$ and hence also for the K_{a_ε}-chain, the set $C_n := \{x\colon K_{a_\varepsilon}(x, C) \ge 1/n\}$ is petite for the K_{a_ε}-chain for all n. Note that $C \subseteq C_n$ for n sufficiently large. Scaling V_ε as necessary, we may choose n and b_ε so large that

$$K_{a_\varepsilon} V_\varepsilon \le V_\varepsilon - f + b_\varepsilon \mathbb{1}_{C_n}.$$

Thus the K_{a_ε}-chain is f-regular. By strong aperiodicity there exists a constant $c_\varepsilon < \infty$ such that, for any $|g| \le f$, we have a solution $\hat{g}_\varepsilon$ to the Poisson equation

$$K_{a_\varepsilon} \hat{g}_\varepsilon = \hat{g}_\varepsilon - \overline{g}$$

satisfying $|\hat{g}_\varepsilon| \le V + c_\varepsilon$.

To complete the proof, let

$$\hat{g} := \frac{\varepsilon}{1-\varepsilon} K_{a_\varepsilon} \hat{g}_\varepsilon = \frac{\varepsilon}{1-\varepsilon}(\hat{g}_\varepsilon - \overline{g}).$$

Writing (9) in the form

$$\frac{\varepsilon}{1-\varepsilon} P K_{a_\varepsilon} = \frac{1}{1-\varepsilon} K_{a_\varepsilon} - I,$$

we have, by applying both sides to $\hat{g}_\varepsilon$,

$$P\hat{g} = \varepsilon^{-1}\hat{g} - \hat{g}_\varepsilon = \varepsilon^{-1}\hat{g} - (\varepsilon^{-1} - 1)\hat{g} - \overline{g} = \hat{g} - \overline{g},$$

so that the Poisson equation is satisfied. □

An important special case occurs when V is a constant multiple of f. In this case (8) may be written as

$$PV \le \lambda V + b\mathbb{1}_C, \tag{10}$$

where $\lambda < 1$. Aperiodic chains for which (10) hold are called *V-uniformly ergodic* in [25]. When V is bounded from above and below, the inequality (10) is equivalent to uniform ergodicity as it is usually defined (see [32], [29] and [25]).

By Theorem 2.3 we see that when (10) holds the fundamental kernel $Z = (I - P + \Pi)^{-1}$ is a bounded linear transformation from L_V^∞ to itself. This provides another important consequence of V-uniform ergodicity which is especially valuable in analyzing perturbations of the chain. We will return to this after we consider the continuous-time case.

3. Continuous-time processes. We now show how the discrete-time results developed thus far may be "lifted" through the resolvent chain to obtain analogous results for continuous-time stochastic processes. We assume that $\mathbf{\Phi}$ is a Borel right process (cf. Sharpe [35]) so that, in particular, $\mathbf{\Phi}$ has the strong Markov property. We also assume that the escape time for the process is infinite. The reader is refered to [26] and [27] for the relevant theory of Harris recurrence in continuous time.

For a measurable set A we let

$$\tau_A = \inf\{t \geq 0\colon\ \Phi_t \in A\}, \qquad \eta_A = \int_0^\infty \mathbb{1}\{\Phi_t \in A\}\, dt.$$

The kernel K_a is defined exactly as in discrete time, where now a is a probability on $\mathbb{R}_+$. When a is an exponential distribution with unit mean, we let R denote the kernel K_a: this is the usual *resolvent* for the process.

We denote by $D(\widetilde{\mathscr{A}})$ the set of all functions $V\colon \mathsf{X} \times \mathbb{R}_+ \to \mathbb{R}$ for which there exists a measurable function $U\colon \mathsf{X} \to \mathbb{R}$ such that, for each $x \in \mathsf{X}$, $t > 0$,

$$P^t V(x) = V(x) + \int_0^t P^s U(x)\, ds, \tag{11}$$

$$\int_0^t P^s |U|(x)\, ds < \infty. \tag{12}$$

We write $\widetilde{\mathscr{A}} V := U$ and call $\widetilde{\mathscr{A}}$ the *extended generator* of the process $\mathbf{\Phi}$. The function U is essentially unique (see Davis [8], page 32, for a discussion).

Davis [8] requires only a local martingale property in his definition of the extended generator. If this weaker condition holds, then the results below are still valid, although some additional steps are required in the proofs. This more complicated situation may be treated as in [27].

We say that a function $h\colon \mathsf{X} \to \mathbb{R}$ is in the *domain of* R if $R(x, |h|)$ is finite for all $x \in \mathsf{X}$. From the definition (11) it is immediate that we have the identity

$$R\widetilde{\mathscr{A}} h = (R - I)h \tag{13}$$

for any h in the domain of R. The following result, which together with (13) states that R and $\widetilde{\mathscr{A}}$ commute, is central to obtaining solutions to the Poisson equation. For a proof see Down, Meyn and Tweedie [9].

LEMMA 3.1. *The extended generator satisfies the following identity for any h in the domain of R:*

$$\widetilde{\mathscr{A}} R h = (R - I)h. \tag{14}$$

Equation (14) is a continuous-time analog of (9), which may be written as

$$(P - I)\frac{\varepsilon}{1-\varepsilon} K_{a_\varepsilon} = K_{a_\varepsilon} - I,$$

and the identity (14) will be applied exactly as in the proof of Theorem 2.3 to obtain a solution to the Poisson equation in this continuous-time context.

The following Foster–Liapounov drift condition is taken from [27]. It is entirely analogous to (8), and will yield analogous results.

For a function $f\colon \mathsf{X} \to [1, \infty)$, a petite set $C \in \mathscr{B}(\mathsf{X})$, a constant $b < \infty$ and a function $V\colon \mathsf{X} \to [0, \infty)$,

$$\widetilde{\mathscr{A}} V(x) \le -f(x) + b\mathbb{1}_C(x), \qquad x \in \mathsf{X}. \tag{15}$$

THEOREM 3.2. *Suppose that $\mathbf{\Phi}$ is ψ-irreducible and that* (15) *holds with V everywhere finite, $f \ge 1$ and C petite. Then $\mathbf{\Phi}$ is positive Harris recurrent with $\pi(f) < \infty$. For some $c_0 < \infty$ and any $|g| \le f$, the Poisson equation* (2) *admits a solution $\hat{g}$ satisfying the bound $|\hat{g}| \le c_0(V + 1)$.*

PROOF. From (13) and the bound (15),

$$RV \le V - Rf + bR\mathbb{1}_C. \tag{16}$$

Since C is petite, the set $C_n = \{x\colon R\mathbb{1}_C(x) > 1/n\}$ is also petite for any n. Hence, as in the proof of Theorem 2.3, we see that, for a constant multiple V_0 of V and some n, b' sufficiently large,

$$RV_0 \le V_0 - Rf + b'\mathbb{1}_{C_n}.$$

Applying Theorem 2.2(iii), we obtain the bound, for any $B \in \mathscr{B}^+(\mathsf{X})$,

$$\sum_{k=0}^{\infty} (RI_{C^c})^k Rf \le V_0 + c(B),$$

and, adding f to both sides, we obtain

$$\sum_{k=0}^{\infty} (RI_{C^c})^k f \le V_0 + f + c(B).$$

Hence the R-chain is f-regular with bounding function $V_0 + f$, and consequently $\pi(f) < \infty$.

From f-regularity of the resolvent chain, we may apply the discrete-time result, Theorem 2.3, to conclude that, for some $c_1 < \infty$ and any $g \le f$, there exists a function $\hat{g}_1$ satisfying the bound $|\hat{g}_1| \le c_1(V + f + 1)$ such that

$$\hat{g}_1 - R\hat{g}_1 = \overline{g}.$$

We now let $\hat{g} = R\hat{g}_1$, and apply (16) to obtain the bound

$$|\hat{g}| \le R|\hat{g}_1| \le c_1(V + V + b + 1) \le c_0(V + 1),$$

where $c_0 = c_1(3 + b)$. Applying (14), we have that the Poisson equation is satisfied:

$$\widetilde{\mathscr{A}}\hat{g} = \widetilde{\mathscr{A}}R\hat{g}_1 = (R - I)\hat{g}_1 = -\overline{g},$$

which is the desired equality. □

4. Applications. In the remainder of the paper we describe several applications of our main results.

4.1. *The functional central limit theorem.* We first present several versions of the functional central limit theorem (FCLT) for Markov chains and processes; we begin with the discrete-time case. For such models, the FCLT concerns the process obtained by interpolating the values of $\overline{g}(\Phi_k)$:

$$Z_n(t) = \frac{1}{\sqrt{n}}\left(\sum_{k=0}^{[nt]} \overline{g}(\Phi_k)\right), \qquad t \in \mathbb{R}_+.$$

Typically, in applications, the moment condition on V used in Theorem 4.1 will be obtained through an application of Theorem 2.2, which gives necessary and sufficient conditions for V^2-regularity and hence a sufficient condition for the finiteness of $\pi(V^2)$.

THEOREM 4.1. *If the chain is f-regular with bounding function V and if $\pi(V^2) < \infty$, then for any $|g| \le f$ there exists a constant $0 \le \gamma_g < \infty$ such that $Z_n \Rightarrow \gamma_g B$, P_μ-weakly, as $n \to \infty$ in $D[0, 1]$ for any initial distribution μ.*

Furthermore, the constant γ_g^2 can be defined as $\gamma_g^2 = \pi(\hat{g}^2 - \{P\hat{g}\}^2)$, where $\hat{g}$ is the solution to the Poisson equation given in Theorem 2.3.

PROOF. This basically follows from Maigret [17]. Our theorem actually requires a slightly strengthened version of her result, because Theorem 4.1 asserts that the weak convergence holds for any initial distribution μ.

This extension is given in the continuous-time case in the proof of Theorem 4.3 below. To avoid serious repetition, we omit the proof of the discrete-time result. □

The generalization of the FCLT to arbitrary initial conditions can also be carried out by applying the shift coupling property of positive Harris recurrent Markov processes. This is established for discrete-time processes in Aldous and Thorrison [1].

An important special case occurs when (10) holds, in which case we obtain a simpler criterion for the FCLT. Taking square roots in (10), we obtain by Jensen's inequality

$$PV^{1/2} \le \lambda^{1/2} V^{1/2} + b^{1/2} \mathbb{1}_C.$$

Since by (10) we have that $\pi(V) < \infty$, the following corollary to Theorem 4.1 is immediate.

THEOREM 4.2. *If* (10) *holds for some* $V \ge 1$, $\lambda < 1$ *and some petite set* C, *then for any* $g^2 \le V$ *there exists a constant* $0 \le \gamma_g < \infty$ *such that* $Z_n \Rightarrow \gamma_g B$, P_μ-*weakly, as* $n \to \infty$ *in* $D[0,1]$ *for any initial distribution* μ.

The constant γ_g^2 *can be defined as* $\gamma_g^2 = \pi(\hat{g}^2 - \{P\hat{g}\}^2)$, *where* $\hat{g}$ *is the solution to the Poisson equation given in Theorem* 2.3.

The analogous continuous-time results may be obtained by applying the main result of Bhattacharaya [5]. We now consider the sequence of stochastic processes $\{\mathbf{Z}_n\}$ defined for each n by

$$Z_n(t) = \frac{1}{\sqrt{n}} \left(\int_0^{nt} \overline{g}(\Phi_s)\, ds \right), \qquad t \in \mathbb{R}_+.$$

THEOREM 4.3. *If* (15) *holds and if* $\pi(V^2) < \infty$, *then for any* $|g| \le f$ *there exists a constant* $0 \le \gamma_g < \infty$ *such that* $Z_n \Rightarrow \gamma_g B$, P_μ-*weakly, as* $n \to \infty$ *in* $D[0,1]$ *for any initial distribution* μ.

Furthermore, the constant γ_g^2 *can be defined as* $\gamma_g^2 = 2\int \hat{g}(x)\overline{g}(x)\,\pi(dx)$, *where* $\hat{g}$ *is the solution to the Poisson equation given in Theorem* 2.3.

PROOF. In the special case where $\mathbf{\Phi}$ is stationary, or when a skeleton of the process is Harris ergodic, this follows from Theorems 2.1 and 2.6 of [5] together with Theorem 3.2. Note that we do not know if $\pi(g^2)$ is finite, a condition of Theorem 2.1 of [5]. However, by (16) and the square integrability of V, we do know that $\pi((R|g|)^2) < \infty$, and this is in fact enough to obtain the FCLT using the proof of Bhattacharaya's Theorem 2.1.

We now show how this result can be generalized to arbitrary initial conditions $\Phi_0 = x \in \mathsf{X}$; the result for an arbitrary initial distribution μ follows upon integrating over all initial conditions with respect to μ.

Recall that from Theorem 3.2 the process $\mathbf{\Phi}$ is positive Harris recurrent under the assumptions of Theorem 4.3. To show that the FCLT holds for arbitrary initial conditions under the assumptions of Theorem 4.3, we will apply the following two well-known consequences of Harris recurrence. First of all, the law of large numbers holds for any initial condition $x \in \mathsf{X}$ and any positive random variable H on sample space:

$$\frac{1}{T}\int_0^T \theta^s H\, ds \to \mathsf{E}_\pi[H] \quad \text{a.s. } [\mathsf{P}_x]. \tag{17}$$

Second, the resolvent is Harris ergodic:

$$\|R^n(x, \cdot) - \pi\| \to 0, \qquad x \in \mathsf{X}. \tag{18}$$

This is a consequence of the fact that not only is the resolvent chain with transition function $R = \int e^{-t} P^t \, dt$ positive Harris recurrent when $\mathbf{\Phi}$ has this property, but the resolvent chain is necessarily also aperiodic since the measures $R(x, \cdot)$ and $\sum_{k=1}^{\infty} 2^{-k} R^k(x, \cdot)$ are equivalent for each x.

For any r and any n we let

$$Z_{n,r}(t) := \theta^r Z_n(t) = \frac{1}{\sqrt{n}} \left(\int_r^{nt+r} \bar{g}(\Phi_s) \, ds \right), \qquad t \in \mathbb{R}_+.$$

Using the law of large numbers (17), one may show that

$$\sup_{0 \le t \le 1} |Z_n(t) - Z_{n,r}(t)| \to 0, \qquad n \to \infty \text{ a.s.} \tag{19}$$

Letting ϕ be a bounded, continuous linear functional on $C[0, 1]$, it follows from (19) that we have

$$|\mathsf{E}_x[\phi(Z_n(t))] - \mathsf{E}_x[\phi(Z_{n,r}(t))]| \to 0, \qquad n \to \infty.$$

By the Markov property this limit may be expressed as

$$\left| \mathsf{E}_x[\phi(Z_n(t))] - \int P^r(x, dy) \mathsf{E}_y[\phi(Z_n(t))] \right| \to 0, \qquad n \to \infty.$$

Integrating over $r \ge 0$, it follows by dominated convergence that, for any $m \ge 1$,

$$\left| \mathsf{E}_x[\phi(Z_n(t))] - \int R^m(x, dy) \mathsf{E}_y[\phi(Z_n(t))] \right| \to 0, \qquad n \to \infty,$$

where R denotes the resolvent. We can now apply (18). Let $\varepsilon > 0$, and choose m so large that $\|R^m - \pi\| < \varepsilon$. Then the limit above with this m implies that

$$\limsup_{n \to \infty} |\mathsf{E}_x[\phi(Z_n(t))] - \mathsf{E}_\pi[\phi(Z_n(t))]| \le \varepsilon |\phi|_\infty.$$

Since by Theorems 3.2 and 2.1 of [5] the FCLT holds when $\Phi \sim \pi$, this establishes the FCLT when $\Phi_0 = x$. $\square$

When the exponential drift

$$\widetilde{\mathscr{A}} V \le -cV + b \mathbb{1}_C \tag{20}$$

holds for some $V \ge 1$, $c > 0$, then we obtain a simpler criterion for the FCLT, just as in the discrete-time case.

We can use (16) to obtain, for some $\lambda < 1$, $b < \infty$ and a petite set C, the bound

$$RV \le \lambda V + b \mathbb{1}_C.$$

We then have, by Jensen's inequality, $RV^{1/2} \le \lambda^{1/2} V^{1/2} + b^{1/2} \mathbb{1}_C$, and then, following the proof of Theorem 3.2, we have, for some constant c_0, that the

Poisson equation (2) can be solved for any g satisfying $g^2 \le V$, and the solution satisfies the bound $\hat{g}^2 \le c_0 V$. We immediately obtain from Theorem 4.3 the following result.

THEOREM 4.4. *If* (20) *holds, then for any* $g^2 \le V$ *there exists a constant* $0 \le \gamma_g < \infty$ *such that* $Z_n \Rightarrow \gamma_g B$, P_μ*-weakly, as* $n \to \infty$ *in* $D[0,1]$ *for any initial distribution* μ.

The constant γ_g^2 *can be defined as* $\gamma_g^2 = 2\int \hat{g}(x)\overline{g}(x)\,\pi(dx)$, *where* $\hat{g}$ *is the solution to the Poisson equation.*

In Section 4.3 we consider several models where moment conditions on the disturbance process may be given explicitly to ensure that (8) or (20) holds so that we can establish the FCLT.

4.2. *Perturbations of Markov processes.* The smoothness of solutions to the Poisson equation is frequently assumed in applications to averaging and diffusion approximations and, in particular, in establishing the convergence of adaptive estimation algorithms of the stochastic approximation type (see [4] and [19]). Furthermore, well-behaved solutions are required in the theory of Markov decision processes (see [18] and [33]). An extensive bibliography of applications may be found in [36].

Suppose that $\{P_\theta\colon \theta \in \Theta\}$ is a family of Markov transition functions, where Θ denotes some open subset of Euclidean space. Assume that each of the corresponding Markov chains is ψ_θ-irreducible and that, for some $\theta_0 \in \Theta$, the chain with transition function P_{θ_0} satisfies the drift criterion (10). When the P_{θ_0}-chain is aperiodic, this means that the chain is V-uniformly ergodic.

We assume that $P_\theta \to P_{\theta_0}$ as $\theta \to \theta_0$ in the induced operator norm $\||\cdot\||_V$, defined as

$$\||P_{\theta_0} - \Pi\||_V := \sup_{\substack{h \in L_V^\infty \\ |h|_V = 1}} |(P_{\theta_0} - P_\theta)h|_V.$$

Since each of the kernels is ψ_θ-irreducible, it may be shown that the drift criterion (10) holds and that the set C used in the drift criterion is petite for the kernel P_θ for each θ in some open ball containing θ_0. Assume that this is the case for all $\theta \in \Theta$, and let $\{Z_\theta\}$ denote the corresponding collection of fundamental kernels and $\{\pi_\theta\}$ the associated invariant probabilities. From Theorem 2.3 we know that each of the kernels $\{Z_\theta, P_\theta, \Pi_\theta\colon \theta \in \Theta\}$ is a bounded linear transformation from L_V^∞ to itself.

Following [34], we define

$$U_{\theta_0,\theta} = [P_\theta - P_{\theta_0}]Z_{\theta_0}, \tag{21}$$

$$H_{\theta_0,\theta} = [I - U_{\theta_0,\theta}]^{-1}. \tag{22}$$

The first kernel is well defined since each of the kernels on the right-hand side of the defining equation maps L_V^∞ to itself. The inverse in the definition of $H_{\theta_0,\theta}$ is well defined for θ sufficiently close to θ_0.

A straightforward generalization of Theorem 2 of [34] shows that

$$\pi_\theta = \pi_{\theta_0} H_{\theta_0,\theta}, \tag{23}$$

$$Z_\theta = Z_{\theta_0} H_{\theta_0,\theta} - \Pi_{\theta_0} H_{\theta_0,\theta} U_{\theta_0,\theta} Z_{\theta_0} H_{\theta_0,\theta}, \tag{24}$$

where Π_{θ_0} is the Markov transition function defined as $\Pi_{\theta_0}(x, A) = \pi_{\theta_0}(A)$, $x \in \mathsf{X}$, $A \in \mathscr{B}(\mathsf{X})$. Hence the invariant probabilities converge in the V-total variation norm, and whenever $g \in L_V^\infty$ the solutions $\hat{g}_\theta = Z_\theta g$ to the Poisson equation converge in norm in L_V^∞ as $\theta \to \theta_0$.

4.3. *Specific models.*

Random walks and queues. Consider the random walk on a half-line given by $\Phi_n = [\Phi_{n-1} + W_n]^+$. We will assume also that the increment distribution Γ has a finite fifth moment. The chain $\mathbf{\Phi}$ can be viewed as the waiting-time sequence of a single-server queueing system (see Asmussen [2]). It is well known that if $\mathsf{E}[|W_1|] < \infty$, then it is necessary and sufficient that $\mathsf{E}[W_1] < 0$ in order that $\mathbf{\Phi}$ be a positive recurrent Harris chain. Furthermore, since $K_{a_\varepsilon}(x, \{0\})$ is positive everywhere and bounded from below on compacta, all compact sets are petite, so that $\mathbf{\Phi}$ is a T-chain (cf. [23]).

Let $f_p(x) = x^p + 1$ and $V_p(x) = cx^{p+1}$ with $c > 0$. We have that (8) holds for some c, and $p = 1, 4$: by Theorem 2.2 the chain is f_4-regular, and hence, by definition, the chain is simultaneously f_1-regular and V_1^2-regular. We see from Theorem 4.1 that the FCLT holds for any g satisfying $|g| \le f_1$. In particular, on setting $g(x) = x$ we see that the FCLT holds for $\mathbf{\Phi}$ itself. For further discussion see Glynn [12].

These results also hold for some network models. See, for example, Meyn and Down [20] and Kumar and Meyn [14], where V-uniform ergodicity of generalized Jackson networks and certain re-entrant lines is established. In these papers, the Liapounov function V may be taken as the exponential of a norm.

Linear state space models. Consider the linear state space model which is defined by an $n \times n$ matrix F and an $n \times p$ matrix G such that, for each $k \in \mathbb{Z}_+$, the random variables X_k and W_k take values in $\mathbb{R}^n$ and $\mathbb{R}^p$, respectively, and satisfy inductively, for $k \in \mathbb{Z}_+$,

$$X_{k+1} = FX_k + GW_{k+1},$$

where $X_0 \in \mathbb{R}^n$ is arbitrary. The random variables $\{W_k\}$ are independent and identically distributed (i.i.d.) and are independent of X_0, with common distribution $\Gamma(A) = \mathsf{P}(W_j \in A)$ having finite zero mean and finite covariance $\Sigma_W = \mathsf{E}[WW^\top]$.

We assume that Γ is nonsingular with respect to Lebesgue measure, so that, in particular, $\Sigma_W > 0$, and we assume that the controllability matrix $[F^{n-1}G \mid \cdots \mid FG \mid G]$ has rank n. In addition, we assume that the eigenvalues of F lie in the open unit disk in $\mathbb{C}$. Under these conditions, it follows as in [3] that every compact set is petite and that the chain $\mathbf{X}$ is positive Harris.

To construct a Liapounov function for the process, let M denote the solution to the *Liapounov equation*

$$F^\top MF = M - I.$$

This is possible because of the eigenvalue condition imposed on F (cf. Caines [6] and Duflo [10]). Letting $V_0(x) = x^\top Mx$, we have the bound

$$PV_0(x) = V_0(x) - x^\top x + \mathrm{trace}(M^{1/2}G\Sigma_W G^\top M^{1/2}) \le \lambda V_0(x) + L$$

for some $\lambda < 1$ and $L < \infty$. It then follows that (10) holds for the chain with V a constant multiple of $\sqrt{V_0}$, so that the process is V-uniformly ergodic. Since $\pi(V_0)$ and hence also $\pi(V^2)$ is finite, the FCLT holds for any $|g(x)| \le c(|x|+1)$.

Clearly the approach used here may be extended to other models with a basically linear structure. A class of bilinear models is shown to be V-uniformly ergodic in [22], and adaptive control models are treated in [21]. Hence the FCLT holds for the processes of interest in each of these models.

Acknowledgments. This paper was begun while the first author was visiting the Department of Mathematics and Statistics at Carleton University, Ottawa, Canada. He wishes to thank the department for its hospitality. We are grateful to Adam Shwartz for numerous useful comments on an earlier draft of the manuscript.

REFERENCES

[1] ALDOUS, D. J. and THORISSON, H. (1993). Shift-coupling. *Stochastic Process Appl.* **44** 1–4.

[2] ASMUSSEN, S. (1987). *Applied Probability and Queues*. Wiley, New York.

[3] ATHREYA, K. B. and PANTULA, S. G. (1986). Mixing properties of Harris chains and autoregressive processes. *J. Appl. Probab.* **23** 880–892.

[4] BENVENISTE, A., METIVIER, M. and PRIOURET, P. (1990). *Adaptive Algorithms and Stochastic Approximations*. *Applications of Mathematics* **22**. Springer, Berlin. (Translated from the French by Stephen S. Wilson.)

[5] BHATTACHARAYA, R. N. (1982). On the functional central limit theorem and the law of the iterated logarithm for Markov processes. *Z. Wahrsch. Verw. Gebiete* **60** 185–201.

[6] CAINES, P. (1988). *Linear Stochastic Systems*. Wiley, New York.

[7] CONSTANTINESCU, C. and CORNEA, A. (1972). *Potential Theory on Harmonic Spaces*. Springer, Berlin.

[8] DAVIS, M. H. A. (1993). *Markov Models and Optimization*. Chapman and Hall, London.

[9] DOWN, D., MEYN, S. P. and TWEEDIE, R. L. (1993). Geometric and uniform ergodicity of Markov processes. *Ann. Probab.* To appear.

[10] DUFLO, M. (1990). *Méthodes Récursives Aléatoires*. Masson, Paris.

[11] FOGUEL, S. R. (1969). *The Ergodic Theory of Markov Processes*. Van Nostrand Reinhold, New York.

[12] GLYNN, P. W. (1994). Poisson's equation for the recurrent $M/G/1$ queue. *Adv. in Appl. Probab.* **26** 1044–1062.

[13] HALL, P. and HEYDE, C. C. (1980). *Martingale Limit Theory and Its Application*. Academic, New York.

[14] KUMAR, P. R. and MEYN, S. P. (1995). Stability of queueing networks and scheduling policies *IEEE Trans. Automat. Control* **40** 251–260.

[15] KURTZ, T. G. (1981). The central limit theorem for Markov chains. *Ann. Probab.* **9** 557–560.

[16] KUSHNER, H. J. (1967). *Stochastic Stability and Control*. Academic, New York.

[17] MAIGRET, N. (1978). Théorème de limite centrale pour une chaîne de Markov récurrente Harris positive. *Ann. Inst. H. Poincaré Probab. Statist.* **14** 425–440.
[18] MAKOWSKI, A. and SHWARTZ, A. (1992). Stochastic approximations and adaptive control of a discrete-time single-server network with random routing. *SIAM J. Control Optim.* **30**.
[19] METIVIER, M. and PRIOURET, P. (1987). Théorèmes de convergence presque sure pour une classe d'algorithmes stochastiques a pas décroissants. *Probab. Theory Related Fields* **74** 403–428.
[20] MEYN, S. P. and DOWN, D. (1994). Stability of generalized Jackson networks. *Ann. Appl. Probab.* **4** 124–148.
[21] MEYN, S. P. and GUO, L. (1992). Stability, convergence, and performance of an adaptive control algorithm applied to a randomly varying system. *IEEE Trans. Automat. Control* **37** 535–540.
[22] MEYN, S. P. and GUO, L. (1993). Geometric ergodicity of a bilinear time series model. *J. Time Ser. Anal.* **14** 93–108.
[23] MEYN, S. P. and TWEEDIE, R. L. (1992). Stability of Markovian processes I. Discrete time chains. *Adv. in Appl. Probab.* **24** 542–574.
[24] MEYN, S. P. and TWEEDIE, R. L. (1993). Generalized resolvents and Harris recurrence of Markov processes. *Contemp. Math.* **149** 227–250.
[25] MEYN, S. P. and TWEEDIE, R. L. (1993). *Markov Chains and Stochastic Stability*. Springer, London.
[26] MEYN, S. P. and TWEEDIE, R. L. (1993). Stability of Markovian processes II. Continuous time processes and sampled chains. *Adv. in Appl. Probab.* **25** 487–517.
[27] MEYN, S. P. and TWEEDIE, R. L. (1993). Stability of Markovian processes III. Foster–Lyapunov criteria for continuous time processes. *Adv. in Appl. Probab.* **25** 518–548.
[28] NEVEU, J. (1972). Potentiel Markovien récurrent des chaînes de Harris. *Ann. Inst. Fourier (Grenoble)* **22** 7–130.
[29] NUMMELIN, E. (1984). *General Irreducible Markov Chains and Non-Negative Operators*. Cambridge Univ. Press.
[30] NUMMELIN, E. (1991). On the Poisson equation in the potential theory of a single kernel. *Math. Scand.* **68** 59–82.
[31] NUMMELIN, E. and TUOMINEN, P. (1982). Geometric ergodicity of Harris recurrent Markov chains with applications to renewal theory. *Stochastic Process Appl.* **12** 187–202.
[32] REVUZ, D. (1984). *Markov Chains*, 2nd ed. North-Holland, Amsterdam.
[33] ROSS, S. M. (1984). *Introduction to Stochastic Dynamic Programming*. Academic, New York.
[34] SCHWEITZER, P. J. (1968). Perturbation theory and finite Markov chains. *J. Appl. Probab.* **5** 401–403.
[35] SHARPE, M. (1988). *General Theory of Markov Processes*. Academic, New York.
[36] SHWARTZ, A. and MAKOWSKI, A. (1991). On the Poisson equation for Markov chains: existence of solutions and parameter dependence. Technical report, Dept. Electrical Engineering, Technion—Israel Institute of Technology.

DEPARTMENT OF OPERATIONS RESEARCH
STANFORD UNIVERSITY
STANFORD, CALIFORNIA 94305-4022
E-MAIL: glynn@leland.stanford.edu

UNIVERSITY OF ILLINOIS AT
URBANA–CHAMPAIGN
COORDINATED SCIENCE LABORATORY
COLLEGE OF ENGINEERING
1308 WEST MAIN STREET
URBANA, ILLINOIS 61801-2307
E-MAIL: s-meyn@uiuc.edu