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# THE SUPPORT OF MEASURE-VALUED BRANCHING PROCESSES IN A RANDOM ENVIRONMENT

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We consider the one-dimensional catalytic branching process introduced by Dawson and Fleischmann, which is a modification of the super-Brownian motion. The catalysts are given by a nonnegative infinitely divisible random measure with independent increments. We give sufficient conditions for the global support of the process to be compact, and sufficient conditions for noncompact global support. Since the catalytic process is related to the heat equation, compact support may be surprising. On the other hand, the super-Brownian motion has compact global support. We find that all nonnegative stable random measures lead to compact global support, and we give an example of a very rarified Lévy process which leads to noncompact global support.

## 1. Introduction and statement of main results.

1.1. *Motivation.* Let $Y(t, dx)$ be the measure-valued branching process sometimes called the super-Brownian motion. This process is described in the recent surveys of Dawson (1993) and Dynkin (1994). The properties of the support of $Y$ have aroused considerable interest. For example, in two or more dimensions, the support has fractional Hausdorff dimension. Also, as shown by Iscoe (1988), if $Y(0, dx)$ has compact support, then $Y(t, dx)$ has compact support for all $t > 0$. This property is unexpected, since at least heuristically $Y$ can be related to the heat equation with a noise term. To be specific, if we assume (perhaps falsely) that $Y(t, dx) = y(t, x)\,dx$ for some random function $y(t, x)$, then $y$ would formally satisfy

$$y_t = \tfrac{1}{2}\Delta y + (2\gamma y)^{1/2}\dot{W},$$
$$y(0, x) = \frac{Y(0, dx)}{dx},$$

where $\gamma > 0$ and $\dot{W} = \dot{W}(t, x)$ is space–time white noise. Of course, we do not expect the heat equation to have solutions of compact support.

While most probabilists have focused on the spatially homogeneous measure-valued branching processes, Dawson and Fleischmann (1991) have

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introduced a measure-valued branching model $X(t,dx)$ in which birth and death occur only at certain points. This model is called the *catalytic branching process* or *super-Brownian motion in a catalytic medium*, because one can imagine that catalysts are located at the points where branching occurs. Catalytic branching processes can have very different properties than ordinary measure-valued branching processes. For example, in Dawson and Fleischmann (1995) it is shown that for certain kinds of catalysts the random measure $X(t,dx)$ has a density even in higher dimensions. On the other hand, the usual measure-valued branching process has a density only in one dimension.

In this paper, we examine the question of compact support for the branching catalytic process. We consider catalysts given by an infinitely divisible random measure with independent increments. We focus on the one-dimensional case, so that such a random measure may be considered as the derivative of a nondecreasing Lévy process. While our choice is not guided by specific applications, we believe that our model gives some idea of the range of phenomena which can occur. Our methods depend on estimating solutions of elliptic equations with random coefficients, and we were unable to extend our method to higher dimensions. Allowing the catalysts to move would have involved the study of random parabolic equations, which were even further beyond our reach. Nonlinear parabolic and elliptic equations are a familiar tool in the field, and Iscoe's (1988) work on the support of measure-valued branching processes depended on the study of a nonlinear elliptic equation.

1.2. *Super-Brownian motion in a catalytic medium.* The purpose of this section is to describe the basic process. We begin by giving the rigorous definition of the random catalytic medium. Let $\mathscr{B}$ denote the Borel subsets of $\mathbb{R}$ and $\mathscr{B}_+$ the nonnegative Borel measurable functions. Let $\mathscr{C}_+$ denote the space of nonnegative continuous functions $\psi$ on $\mathbb{R}$ and $\mathscr{C}^2$ the space of twice continuously differentiable functions. Let $L$ denote an infinitely divisible random measure with independent increments on $\mathbb{R}$ with Laplace functional

$$\begin{aligned} &-\log E\left\{\exp\left(-\int_{\mathbb{R}} f(x)L(dx)\right)\right\} \\ (1.1)\qquad &= \int_{\mathbb{R}}\int_0^{\infty}\big(1-\exp(-\lambda f(x))\big)\nu(d\lambda)\,dx, \qquad f\in\mathscr{B}_+, \end{aligned}$$

where $\nu$ is a measure on $(0,\infty)$ which satisfies $\int_0^{\infty}\min(\lambda,1)\nu(d\lambda)<\infty$. Under this condition $L$ is almost surely a locally finite random measure which we subsequently refer to as the compound Poisson random measure with associated Lévy measure $\nu$ and denote its probability law by $Q_\nu$. The stable random measure of index $\alpha\in(0,1)$ has Lévy measure

$$\nu_\alpha(d\lambda) = \frac{c_\alpha d\lambda}{\lambda^{1+\alpha}}1(\lambda>0) \tag{1.2}$$

and some normalizing constant $c_\alpha$. Compound Poisson random measures are almost surely pure atomic. If the associated Lévy measure is finite, then the atoms are isolated. [Refer to Kallenberg (1983) for characterization and basic properties of infinitely divisible random measures.] Also note that, for any $a \in \mathbb{R}$, $\{L([a, a+t)) \colon t \geq 0\}$ is a nondecreasing process with independent increments.

Before turning to the construction of the catalytic branching process, let us briefly recall the characterization of ordinary super-Brownian motion. Let $\mathscr{M}_F(\mathbb{R})$ denote the finite Borel measures on $\mathbb{R}$ with the topology of weak convergence. Super-Brownian motion is a continuous $\mathscr{M}_F(\mathbb{R})$-valued Markov process with transition Laplace functional

$$E_{Y(0)}\left\{\exp\left[-\int_{-\infty}^{\infty}\varphi(x)Y(t,dx)\right]\right\} = \exp\left[-\int_{-\infty}^{\infty}u(t,x)Y(0,dx)\right], \tag{1.3}$$

where $u(t,x)$ satisfies

$$u_t = \tfrac{1}{2}u_{xx} - \gamma u^2,$$
$$u(0,x) = \varphi(x) \in \mathscr{B}_+.$$

In the above, $\gamma$ is a positive constant which represents the branching rate.

We next turn to the construction of the super-Brownian motion in $\mathbb{R}$ in which the branching rate is not constant but is determined by a fixed locally finite random measure $L$ given by (1.1). In fact it suffices to construct the process for a typical realization of the medium. From an intuitive viewpoint $X(t,dx)$ consists of infinitesimal Brownian particles undergoing critical branching. The branching rate is controlled by the measure $L$. If a particle is at a point where $L$ is large, its branching rate is high. If $L = 0$ on a set $A$, then branching does not occur there. Heuristically, if we imagine that the densities $\gamma(x) = L(dx)/dx$ and $r(t,x) = X(t,dx)/dx$ exist (but they may not), this process would satisfy the equation

$$r_t = \tfrac{1}{2}r_{xx} + (2r\gamma(x))^{1/2}\dot{W},$$
$$r(0,x)\,dx = X(0,dx),$$

where $\dot{W}$ is space–time white noise.

Still formally, the corresponding measure-valued branching process $X(t,dx)$ with probability law denoted by $P^L_{X(0)}$ would be given by a Laplace transition function as in (1.3) except that the log-Laplace function $u(t,x)$ would satisfy

$$u_t = \frac{1}{2}u_{xx} - u^2\frac{L(dx)}{dx}, \tag{1.4}$$
$$u(0,x) = \varphi(x) \in \mathscr{B}_+.$$

Of course, when $L$ is given by (1.1), $L(dx)/dx$ (which we sometimes will write as $\dot{L}$) is a singular term involving delta functions, but as in Dawson and Fleischmann (1992) we consider (1.4) as shorthand for the integral equation

$$u(t,x) = \int_{-\infty}^{\infty} p(t,x,y)\varphi(y)\,dy - \int_0^t\int_{-\infty}^{\infty} p(t-s,x,y)u^2(s,y)L(dy)\,ds, \tag{1.5}$$

where $p(t,x,y)$ is the fundamental solution of the heat equation $u_t = (1/2)u_{xx}$ on $\mathbb{R}$. A solution of (1.5) is called a *mild solution* of (1.4).

In fact, under the additional assumptions that

$$\int_{-\infty}^{\infty} \exp(-cx^2)L(dx) < \infty, \qquad \forall\, c > 0,$$

and that $\phi$ belongs to an appropriate class of nonnegative continuous functions, existence and uniqueness of mild solutions of (1.4) which are continuous in $t$ and $x$ are established in Dawson and Fleischmann [(1992), Section 2] and the catalytic branching process with this log-Laplace function is obtained in Dawson and Fleischmann (1991).

An alternative approach, which is employed for example in Dawson and Fleischmann (1994), is to use Dynkin's general construction [cf. Dynkin (1991, 1994)] in which the branching rate is given by an admissible Brownian additive functional. In particular, if $\tilde{L}$ is a finite measure on $\mathbb{R}$ and $\ell_{t,x}(w)$ denotes the local time of the Brownian motion $w$, then the additive functional $\kappa_{\tilde{L}}(w,t) = \int \ell_{t,x}(w)\tilde{L}(dx)$ is admissible. The existence of the corresponding measure-valued branching process $X(t,dx)$ follows from Dynkin [(1994), Theorem 3.1]. In addition, according to Dynkin [(1994), Theorem 3.2] the process $X(t,dx)$ almost surely has right-continuous paths.

However, for our purposes it is convenient to employ some modification of these constructions. Although technically our construction is not contained in the previously mentioned references, it involves only ideas and methods which appear in them, and for this reason we will simply give an outline of the construction of the modified process we consider.

The main idea is to construct the basic process as the a.s. limit of an increasing sequence of $\mathscr{M}_F(\mathbb{R})$-valued processes defined on a common probability space. In order to do so, we first construct, for each $K \in \mathbb{N}$, an $\mathscr{M}(E_K)$-valued process $\tilde{X}_K(t)$, where $E_K := \bigcup_{n=1}^{K}\{n\} \times (-n,n)$. We consider the Markov process $w_K$ in $E_K$ which, starting at $(n,x)$, $x \in (-n,n)$, is defined by $w_K(t) = (\{n\}, w(t))$, $0 \le t < \tau_n$, $w_K(\tau_n) = (\{n+1\}, w(\tau_n))$, where $\tau_n := \inf\{t\colon\ w(t) = \pm n\}$ and $w$ is a standard Brownian motion starting at $x$. Finally, the process $w_K$ dies at time $\tau_K$. Consider the random measure on $E_K$ defined by $L_K(\{n\} \times (a,b)) = L((-n,n) \cap (a,b))$, $n \le K$, and the admissible additive functional $\kappa_{L_K}(w_K,t) := \int \ell_{t,y}(w_K)L_K(dy)$. The resulting superprocess is denoted by $\tilde{X}_K(t)$.

Given a measure $\mu \in \mathscr{M}_F(\mathbb{R})$, we take as the initial measure for $\tilde{X}_K$,

$$\text{for } n \ge 1, \qquad \tilde{X}_K(0, \{n\} \times B) := \mu(B \cap [n-1, n) \cup (-n, -n+1]).$$

Note that if $K' > K$, then the law of $\tilde{X}_{K'}$ restricted to $E_K$ is identical to the law of $\tilde{X}_K$. Thus the laws $P^{L,K}_{X(0)}$ of $\tilde{X}_K$ form a consistent family whose projective limit yields the probability law of an $\mathscr{M}(E_\infty)$-valued process $\tilde{X}_\infty$, where $E_\infty := \bigcup_{n=1}^{\infty}\{n\} \times (-n,n)$.

We then define the increasing sequence of $\mathscr{M}_F((-K,K))$-valued processes: $X_K(t,B) := \sum_{n=1}^{K} \tilde{X}_\infty(t, \{n\} \times B)$.

It can be verified that the log-Laplace function for the process $X_K(t)$ satisfies

$$u_K(t,x) = \int_{-K}^{K} p_K(t,x,y)\varphi(y)\,dy - \int_0^t \int_{-K}^{K} p_K(t-s,x,y)u_K^2(s,y)L(dy)\,ds$$

and

$$E(X_K(t,B)) = \int_{-K}^{K}\int_B p_K(t,x,y)\mu(dx)\,dy,$$

where $p_K(t,x,y)$ denotes the fundamental solution of the heat equation with Dirichlet boundary conditions on $(-K,K)$. Moreover, a modification of the arguments of Dawson and Fleischmann (1992) imply that, when $\phi$ is continuous with support in $[-K,K]$, this equation has a unique solution which is jointly continuous in $t$ and $x$ and measurable in $L$. Finally, a standard argument shows that $E(X_K(t,B)) = \int_B \int_{-K}^{K} p_K(t,x,y)\mu(dx)\,dy$.

We then define the $\mathscr{M}_F(\mathbb{R})$-valued process with initial measure $\mu$ by

$$X(t,dx) := \lim_{K\to\infty} X_K(t,dx).$$

The process $X(t,dx)$ is the *super-Brownian motion in the catalytic medium* $L$.

We extend $p_K(t,\cdot,\cdot)$ to $\mathbb{R}\times\mathbb{R}$ by setting $p_K(t,x,y)=0$ if $x$ or $y \notin (-K,K)$. Then $p_K(t,\cdot,\cdot)\uparrow p(t,\cdot,\cdot)$ and, by the monotone convergence theorem,

$$E(X(t,B)) = \int_{-\infty}^{\infty}\int_B p(t,x,y)\mu(dx)\,dy.$$

Since the sequence $X_K(t,\cdot)$ is increasing in $K$, so is the associated sequence of log-Laplace functions $u_K(t,\cdot)$. By the monotone convergence theorem the log-Laplace function of $X(t)$ is given by $u(t,x) := \lim_{K\to\infty} u_K(t,x)$. Finally applying the monotone convergence theorem again we obtain

$$\begin{aligned} u(t,x) &= \lim_{K\to\infty} u_K(t,x) \\ &= \lim_{K\to\infty}\int_{-K}^{K} p_K(t,x,y)\varphi(y)\,dy \\ &\quad - \lim_{K\to\infty}\int_0^t\int_{-K}^{K} p_K(t-s,x,y)u_K^2(s,y)L(dy)\,ds \\ &= \int_{-\infty}^{\infty} p(t,x,y)\varphi(y)\,dy - \int_0^t\int_{-\infty}^{\infty} p(t-s,x,y)u^2(s,y)L(dy)\,ds. \end{aligned}$$

Note that this constuction only requires the local finiteness of $L$, but that we do not obtain (nor do we require) the uniqueness of the solution to (1.5) nor the right continuity of $X(t)$.

The weighted occupation time process for super-Brownian motion was introduced by Iscoe (1988). Since the process $X_K$ introduced above is right continuous and $X$ is the increasing limit of the $X_K$, they are measurable and the occupation time processes $\int_0^t X_K(s,\cdot)\,ds$ and $\int_0^t X(s,\cdot)\,ds$ are well defined. Let $X(0) \in \mathscr{M}_F(\mathbb{R})$ have support in $(-K, K)$, let $\psi \in \mathscr{C}_+$ have support in $(-K, K)$ and let $\theta > 0$. Following the arguments in the proof of Theorem 3.1 in Iscoe (1986), one can check that the corresponding Laplace functional is

$$\begin{aligned} &E^L_{X(0)}\left\{\exp\left[-\theta\int_0^t\int_{-\infty}^{\infty}\psi(x)X_K(s,dx)\,ds\right]\right\} \\ &\quad = \exp\left[-\int_{-\infty}^{\infty}v_K(\theta\psi;t,x)X(0,dx)\right], \end{aligned} \tag{1.6}$$

where $v_K(\theta\psi; t, x)$ is the solution of

$$\begin{aligned} u(t,x) &= 0 \quad [\text{for } x \in (-K,K)^c] \\ &= \theta\int_0^t\int_{-K}^{K}p_K(t-s,x,y)\psi(y)\,dy\,ds \\ &\quad - \int_0^t\int_{-K}^{K}p_K(t-s,x,y)u^2(s,y)L(dy)\,ds \quad [\text{for } x \in (-K,K)]. \end{aligned} \tag{1.7}$$

Let $0 < x < K$. Letting $X(0) = \delta_x$ in (1.6), it follows that $v_K(\theta\psi, t, x)$ is nonnegative and monotone increasing in both $t$ and $\psi$. Note that $v_K(\theta\psi, t, x) \le \sup_{t,x}\int_0^t\int_{-K}^{K} p_K(t-s,x,y)\theta\psi(y)\,dy\,ds < \infty$.

1.3. *The global support of X.* Given $\mu \in \mathscr{M}_F(\mathbb{R})$, let $\mathrm{supp}(\mu)$ denote the closed support of $\mu$. The *global support* of a measure-valued process $X(\cdot)$, $\mathrm{Gsupp}(X)$, is defined to be the closure of $\bigcup_{t\ge 0}\mathrm{supp}[X(t,dx)]$. Let $L$ be a fixed locally finite measure on $\mathbb{R}$. In this section we relate the question of compact global support for the super-Brownian motion in the catalytic medium $L$ to a nonlinear singular elliptic boundary value problem. In the next section (Section 1.4) these results will be applied to the case in which $L$ is a typical realization of a random catalytic medium.

Before stating the next result, let us recall some basic facts from the theory of distributions which can be found, for example, in Schwartz [(1966), Chapter 2, Section 4]. A distribution on $\mathbb{R}$ (or any open interval) whose second derivative (in the sense of distributions) is a locally finite measure (either signed or nonnegative) is a continuous function of bounded variation on every finite interval. Moreover, if its second derivative is a nonnegative measure, then (i) it is a continuous, convex function and (ii) its first derivative exists in the usual sense except possibly at a countable set of points, and it is an increasing function having left and right limits at every point.

A solution to the boundary value problem

$$\begin{aligned} &\frac{1}{2}v_{xx} = v^2(x)\frac{L(dx)}{dx} \quad \text{for } x \in (a_1, a_2), \\ &v(a_1) = \beta_1, \qquad v(a_2) = \beta_2, \end{aligned} \tag{1.8}$$

is a continuous convex function $v$, defined on $[a_1, a_2]$, which has the required boundary values and such that, for every $a_1 \leq x_0 \leq x_0 + x \leq a_2$,

$$v(x_0 + x) = v(x_0) + v'(x_0+)x + 2\int_{x_0}^{x_0+x} ds \int_{x_0}^{s} v^2(t)L(dt). \tag{1.9}$$

THEOREM A. *Assume that* $\operatorname{supp}(X(0)) \subset [a_1, a_2] \subset (-K, K)$.

(a) *There exist positive sequences* $\beta_{1,n} \uparrow \infty$, $\beta_{2,n} \uparrow \infty$, *such that for each* $n$ *the boundary value problem* (1.8) *has a unique solution* $v(\beta_{1,n}, \beta_{2,n}, x)$ *with* $\beta_1 = \beta_{1,n}$ *and* $\beta_2 = \beta_{2,n}$.

(b) *Given any sequence of functions* $v(\beta_{1,n}, \beta_{2,n}, x)$ *satisfying the conditions of* (a),

$$\begin{aligned} &P^L_{X(0)}\{\mathrm{Gsupp}[X] \subset [a_1, a_2]\} \\ &\quad = P^L_{X(0)}\{\operatorname{supp}[X(t, dx)] \cap [a_1, a_2]^c = \varnothing \text{ for all } t \geq 0\} \\ &\quad = \lim_{n\to\infty} \exp\left[-\int_{a_1}^{a_2} v(\beta_{1,n}, \beta_{2,n}, x) X(0, dx)\right]. \end{aligned} \tag{1.10}$$

PROOF. Letting $\psi_n \in \mathscr{C}_+ \uparrow 1_{(-K,K)} \cdot 1_{[a_1,a_2]^c}$ and then $t \to \infty$ in (1.6), we obtain

$$\begin{aligned} &E^L_{X(0)}\left\{\exp\left[-\theta \int_0^\infty X_K(s, [a_1, a_2]^c)\, ds\right]\right\} \\ &\quad = \exp\left[-\int_{-\infty}^{\infty} v_{K,a_1,a_2}(\theta, x) X(0, dx)\right], \end{aligned} \tag{1.11}$$

where

$$v_{K,a_1,a_2}(\theta, x) = \lim_{t\to\infty} v_{K,a_1,a_2}(\theta, t, x),$$

$$v_{K,a_1,a_2}(\theta, t, x) = \lim_{n\to\infty} v_K(\theta\psi_n, t, x).$$

Note that the function $v_{K,a_1,a_2}(\theta, t, x)$ satisfies (1.7) with $\psi = 1_{[a_1,a_2]^c}$ (by the monotone convergence theorem).

We will show that the second distribution derivative $v_{xx}$ of $v_{K,a_1,a_2}(\theta, x)$ is a signed measure and it satisfies the equation

$$\frac{1}{2} v_{xx} = v^2(x) \frac{L(dx)}{dx} - \theta 1_{[a_1,a_2]^c}(x). \tag{1.12}$$

This implies that $v_{K,a_1,a_2}(\theta, x)$ is continuous (cf. remarks made immediately before the statement of Theorem A).

To obtain (1.12), let $\phi \in \mathscr{C}$ have support in $(-K, K)$. We will show that, uniformly for small $h > 0$,

$$\lim_{t\to\infty} \frac{1}{h}\left[\int v_{K,a_1,a_2}(\theta, t+h, x)\phi(x)\, dx - \int v_{K,a_1,a_2}(\theta, t, x)\phi(x)\, dx\right] = 0.$$

Because $v_{K,a_1,a_2}(\theta,t+h,x) - v_{K,a_1,a_2}(\theta,t,x) \geq 0$, it suffices to prove this for $\phi \geq 0$. Then using the latter fact and (1.7) with the roles of $s$ and $t-s$ interchanged, we obtain

$$\begin{aligned}
0 &\leq \lim_{t\to\infty} \frac{1}{h}\left[\int v_{K,a_1,a_2}(\theta,t+h,x)\phi(x)\,dx - \int v_{K,a_1,a_2}(\theta,t,x)\phi(x)\,dx\right] \\
&\leq \lim_{t\to\infty} \frac{1}{h}\left\{\int_t^{t+h}\int\left[\int p_K(s,x,y)\theta 1_{[a_1,a_2]^c}(y)\,dy\right.\right. \\
&\qquad\left. - \int p_K(s,x,y)v^2_{K,a_1,a_2}(t+h-s,y)L(dy)\right]\phi(x)\,dx\,ds \\
&\qquad + \int_0^t\int\int p_K(s,x,y)\big[v^2_{K,a_1,a_2}(t-s,y) \\
&\qquad\left. - v^2_{K,a_1,a_2}(t+h-s,y)\big]\phi(x)\,dx\,L(dy)\,ds\right\} \\
&\leq \lim_{t\to\infty} 2K\theta\sup_x\left[\phi(x)\int p_K(t,x,y)\,dy\right] \\
&= 0.
\end{aligned}$$

On the other hand, if $\phi \in \mathscr{C}^2$, then

$$\begin{aligned}
&\lim_{t\to\infty}\frac{1}{h}\int\left[v_{K,a_1,a_2}(\theta,t+h,x)\phi(x)\,dx - \int v_{K,a_1,a_2}(\theta,t,x)\phi(x)\,dx\right] \\
&= \lim_{t\to\infty}\left\{\int \frac{T_h^K\phi(x)-\phi(x)}{h} v_{K,a_1,a_2}(\theta,t,x)\,dx\right. \\
&\qquad + \frac{1}{h}\int_0^h\int\left[\int p_K(h-s,x,y)\theta 1_{[a_1,a_2]^c}(y)\,dy\right. \\
&\qquad\left.\left. - \int p_K(h-s,x,y)v^2_{K,a_1,a_2}(\theta,t+s,y)L(dy)\right]\phi(x)\,dx\,ds\right\} \\
&= \int \frac{T_h^K\phi(x)-\phi(x)}{h} v_{K,a_1,a_2}(\theta,x)\,dx \\
&\qquad + \frac{1}{h}\left\{\int_0^h ds\int dx\,\phi(x)\left[\int p_K(h-s,x,y)\theta 1_{[a_1,a_2]^c}(y)\,dy\right.\right. \\
&\qquad\left.\left. - \int p_K(h-s,x,y)v^2_{K,a_1,a_2}(\theta,y)L(dy)\right]\right\},
\end{aligned}$$

where $\{T_h^K\colon h \geq 0\}$ denotes the semigroup of the killed Brownian motion. The first identity follows from (1.7), and the last limit follows by the monotone convergence theorem.

Take $\phi \in \mathscr{C}^2$ with support in $(-K, K)$. Then letting $h \downarrow 0$, it follows that

$$\int \tfrac{1}{2}\phi_{xx}(x) v_{K,a_1,a_2}(\theta, x)\,dx + \theta \int 1_{[a_1,a_2]^c}(x)\phi(x)\,dx - \int v_{K,a_1,a_2}(\theta, x)^2 \phi(x) L(dx) = 0.$$

However, this implies that the second distribution derivative of $v_{K,a_1,a_2}$ is a (possibly signed) measure and that the left and right limits of the first derivative of $v(x) = v_{K,a_1,a_2}(\theta, x)$ satisfy

$$\frac{dv}{dx}(x\pm) = 2\int_{x_0}^{x\pm} v^2(y) L(dy) - 2\theta \int_{x_0}^{x\pm} 1_{[a_1,a_2]^c}(y)\,dy + \text{const}, \quad \text{for } x \in (-K, K) \tag{1.13}$$

[because any two primitives of a distribution differ by a constant; cf. Schwartz (1966), Chapter 2, Section 4]. Integrating again (i.e., taking the second primitive) we obtain (1.9). Note that $\lim_{\theta\to\infty} v_{K,a_1,a_2}(\theta, t, a_i) = \infty$ since, for example, $P^L_{\delta_{a_1}}(\int_0^t X_K(s, (a_1 - 1, a_1))\,ds > 0) = 1$ for any $t > 0$. [The latter is verified by calculating the first two moments of $\int_0^t X_K(s, (a_1 - 1, a_1))\,ds$, Chebyshev's inequality and a Borel–Cantelli argument.] This implies that $\lim_{\theta\to\infty} v_{K,a_1,a_2}(\theta, a_i) = \infty$.

Since $t \to X_K(t)$ is right continuous and the map $\mu \to \text{supp}(\mu)$ is lower semicontinuous [cf. Dawson (1993), Theorem 9.3.1.2], the event $\text{supp}(X_K(t)) \cap [a_1, a_2]^c = \varnothing$ $(\forall\, t \geq 0)$ is measurable. However, the set $\text{supp}(X(t)) \cap [a_1, a_2]^c = \varnothing$ $(\forall\, t \geq 0) = \bigcap_{K=1}^{\infty} \text{supp}(X_K(t)) \cap [a_1, a_2]^c = \varnothing$ $(\forall\, t \geq 0)$ and hence is also measurable and

$$\begin{aligned}
&P^L_{X(0)}\{\text{supp}[X(t, dx)] \cap [a_1, a_2]^c = \varnothing \text{ for all } t \geq 0\} \\
&\quad = \lim_{K\to\infty} P^L_{X(0)}\{\text{supp}[X_K(t, dx)] \cap [a_1, a_2]^c = \varnothing \text{ for all } t \geq 0\} \\
&\quad = \lim_{K\to\infty} P^L_{X(0)}\left\{\int_0^\infty X_K(s, [a_1, a_2]^c)\,ds = 0\right\} \quad \text{(by right continuity)} \\
&\quad = \lim_{K\to\infty} \lim_{\theta\to\infty} \exp\left[-\int_{-\infty}^{\infty} v_{K,a_1,a_2}(\theta, x) X(0, dx)\right] \quad \text{[by (1.11)]} \\
&\quad = \lim_{n\to\infty} \exp\left[-\int_{a_1}^{a_2} v(\beta_{1,n}, \beta_{2,n}, x) X(0, dx)\right],
\end{aligned} \tag{1.14}$$

where $\beta_{1,n} = v_{K,a_1,a_2}(\theta_n, a_1)$ and $\beta_{2,n} = v_{K,a_1,a_2}(\theta_n, a_2)$ with $\theta_n \uparrow \infty$. The fact that $\beta_{1,n} \uparrow \infty$, $\beta_{2,n} \uparrow \infty$ follows from the fact that $v(\beta_1, \beta_2, x)$ is increasing in $\beta_1, \beta_2$ (which will be established in Lemma 2.2) and the fact that $\lim_{\theta\to\infty} v_{K,a_1,a_2}(\theta, a_i) = \infty$ [which was explained in the comments following (1.13)]. The fact that (1.14) is satisfied for any such sequence also follows from the fact that $v(\beta_1, \beta_2, x)$ is increasing in $\beta_1, \beta_2$. □

COROLLARY A. (a) *Let $\mu$ have support in $[x_1, x_2]$. In order that*

$$P^L_\mu\{\text{Gsupp}(X) \textit{ is compact}\} = 1,$$

*it suffices to find, for every* $\varepsilon > 0$, $x_3 = x_3(\varepsilon) < x_1$, $x_4 = x_4(\varepsilon) > x_2$ *and a nonnegative solution* $v(x)$, $x \in (x_3, x_4)$, *to* (1.9) *which satisfies the following*:

(i) $\sup_{x\in[x_1,x_2]} v(x) \le \varepsilon$;
(ii) $\lim_{x\to x_3} v(x) = +\infty$;
(iii) $\lim_{x\to x_4} v(x) = +\infty$.

(b) *Let* $\mu(\mathbb{R}) > 0$. *In order that*

$$P^L_\mu\{\text{Gsupp}(X) \textit{ is compact}\} = 0,$$

*it suffices to show that, for any* $-\infty < a_1 < a_2 < \infty$,

$$\sup_{\beta_1,\beta_2} \inf_{x\in[a_1,a_2]} v(\beta_1, \beta_2, x) = +\infty.$$

PROOF. (a) Let $v(\beta_{1,n}, \beta_{2,n}, \cdot)$ be defined as in the statement of Theorem A, but with $a_1 = x_3$, $a_2 = x_4$. Then, for each $n$, $v(\beta_{1,n}, \beta_{2,n}, x) \le v(x)$ by (2) and (3). If $\text{supp}(\mu) \subset [x_1, x_2]$ and $m$ is a positive integer, then, by (1.14),

$$\begin{aligned} &P^L_\mu\{G\,\text{supp}(X) \not\subset [x_3(2^{-m}), x_4(2^{-m})]\} \\ &\quad = 1 - \lim_{n\to\infty} \exp\left[-\int_{x_1}^{x_2} v(\beta_{1,n}, \beta_{2,n}, x)\mu(dx)\right] \\ &\quad \le 1 - \exp\left[\int_{x_1}^{x_2} -v(x)\mu(dx)\right] \\ &\quad \le 1 - \exp\left[-2^{-m}\mu([x_1, x_2])\right]. \end{aligned}$$

By the Borel–Cantelli lemma it follows that

$$P^L_\mu\{\text{Gsupp}(X) \subset [x_3(2^{-m}), x_4(2^{-m})] \text{ for some } m\} = 1.$$

(b) This follows immediately from (1.10). □

1.4. *Statement of the main results.* We now turn to the case in which the medium $L$ is a typical realization of a compound Poisson random measure with associated Lévy measure $\nu$. In particular, we let $X(t, dx)$ be a super-Brownian motion in this random catalytic medium. As a consequence of the basic construction, given a measure $\mu$,

$$\mathscr{C}G(\mu) := \{L\colon\ P^L_\mu\{\text{Gsupp}(X) \text{ is compact}\} = 1\}$$

is a measurable set and in this section we will investigate $Q_\nu(\mathscr{C}G(\mu))$ when $Q_\nu$ is the probability law of a compound Poisson random measure whose Lévy measure $\nu$ has certain properties. We first give sufficient conditions on the Lévy measure $\nu(dx)$ which imply $P^L_\mu\{\text{Gsupp}(X) \text{ is compact}\} = 1$ provided that $\mu$ has compact support for almost every realization of the catalytic medium. We then find sufficient conditions for noncompact global support, that is, $P^L_\mu\{\text{Gsupp}(X) \text{ is compact}\} = 0$. As a corollary, we can show that if $L(dx)$ arises from a stable random measure, then $X$ has compact global support with probability 1. The measure $L(dx)$ must be very rarified for $X$ to have noncompact global support, but we show that this can happen even if the atoms of $L(dx)$ are dense in $\mathbb{R}$. The intuition is as follows. If $L(dx)$ is a fairly uniform measure, then $X(t,dx)$ is similar to the usual measure-valued branching process, which has compact support. On the other hand, if $L(dx)$ is rarified, then $X(t,dx)$ is similar to the measure-valued process with no branching, which satisfies the heat equation. Of course, solutions to the heat equations have noncompact global support.

Let

$$S(t) = \sup\left\{x > 0\colon\ \nu([x,\infty)) \geq \frac{1}{t}\right\},$$

$$T(t) = \inf\left\{x > 0\colon\ \nu([x,\infty)) < \frac{1}{2t}\right\}$$

and

$$I(t) = \int_0^{T(t)} x\nu(dx).$$

Of course, $S(t) = T(t/2)$ if $\nu$ has no atoms.

THEOREM 1. *Let $\nu$ be a measure on $(0,\infty)$ such that there exist a sequence $\{b_n\}_{n\in\mathbb{Z}}$ and a constant $c_0 > 0$ satisfying the following conditions*:

(i) $b_n/b_{n+1} \geq c_0$ *for all* $n \in \mathbb{Z}$;
(ii) $\sum_{n=-N}^{\infty} 2^n/b_n < \infty$ *for all* $N > 0$;
(iii) $(2^{2n}/b_{n+1})S(2^n/b_n z) > c_0$ *for all* $n \in \mathbb{Z}$ *and* $z \in (0,1)$.

*Then, with $Q_\nu$-probability* 1, $P^L_{X(0)}(X(\cdot)$ *has compact global support*$) = 1$ *for every initial measure* $X(0,dx)$ *having compact support.*

To state the next theorem, we need to define functions $H_n(z)$, which will be the basis for a discrete dynamical system. Let

$$H_n(z) = \frac{b_n}{b_{n+1}}z + \frac{2^{2n+4}}{b_{n+1}} I\left(\frac{2^{n+1}}{b_n z}\right).$$

Assume that there exist positive constants $N_0$ and $K$ such that if $n > N_0$ and $z > K$, then

$$H_n(z) \leq \frac{z}{2}. \tag{1.15}$$

THEOREM 2. *Let $\nu$ be a measure on $(0,\infty)$ such that there exists a strictly positive sequence $b_1, b_2, \ldots$ satisfying $b_n/b_{n+1} < 1/2$, for large $n$, and $\sum_{n=1}^{\infty} 2^n/b_n = \infty$ and such that the functions $H_n(z)$ satisfy* (1.15). *Then, with $Q_\nu$-probability* 1, $P^L_{X(0)}$(*global support of* $X(\cdot)$ *is compact*$) = 0$ *for every initial measure satisfying* $X(0,\mathbb{R}) > 0$.

In view of Corollary A, the proofs of Theorems 1 and 2 will be reduced to establishing certain analytical properties of the solutions $v$ to (1.8) for a typical realization of the catalytic medium. Here is a thumbnail sketch of this analysis of (1.8): We pretend that both $v$ and $v'$ are constant on intervals $(x_n, x_{n+1}]$ on which $v$ approximately doubles. Also, we pretend that $L((x_n, x_{n+1}])$ is approximately equal to its "average value." With these ansatzes, $(v(x), v'(x))$ becomes a dynamical system whose behavior we can analyze. In fact, $S(t)$ and $T(t)$ are involved in the definition of the "average value" of $L((x_n, x_{n+1}])$. Of course, $L((x_n, x_{n+1}])$ may not have an expectation. The numbers $b_n$ arise from scaling the dyamical system. We will show that $v'(x_n)/b_n$ approaches a limit, or at least is bounded in the appropriate direction. From this fact, we can decide whether $\lim_{n\to\infty} x_n = \infty$. If $\lim_{n\to\infty} x_n = \infty$, then $v(x)$ does not reach $\infty$ for finite values of $x$, and if $\lim_{n\to\infty} x_n < \infty$, then $v(x) = \infty$ for some $x < \infty$.

The following corollaries are immediate consequences of Theorems 1 and 2.

COROLLARY 1. *Let $L$ be a stable random measure of index $\alpha \in (0,1)$. Then, with $Q_{\nu_\alpha}$-probability 1, $P^L_{X(0)}(X(\cdot)$ has compact global support$) = 1$ for every initial measure $X(0)$ having compact support.*

PROOF. By the conditions on $L$, the Lévy measure $\nu_\alpha$ is given by (1.2). Then

$$S(t) = ct^{1/\alpha}.$$

If we let

$$\delta = \frac{\alpha}{\alpha+1},$$
$$b_n = 2^{n(1+\delta)},$$

then the conditions of Theorem 1 are satisfied. □

COROLLARY 2. *Suppose that $L$ has Lévy measure $\nu(dx) = (1/x)1\ (0 < x \leq 1)$ or that $\nu(dx)$ is a finite measure. Then, with $Q_\nu$-probability 1, $P^L_{X(0)}$(global support of $X(\cdot)$ is compact$) = 0$ for every initial measure satisfying $X(0, \mathbb{R}) > 0$.*

PROOF. If $\nu(dx)$ is a finite measure, then the atoms of $L(dx)$ form a discrete set and one directly verifies the statement of Lemma 2.8 below, namely, that solutions of (1.9) are finite for all $x \in \mathbb{R}$.

Now consider the case $\nu(dx) = (1/x)1\ (x \leq 1)$. We compute that

$$T(t) = \exp\left(-\frac{1}{2t}\right)$$

and

$$I(t) = T(t).$$

If we let

$$b_n = n2^n,$$

then

$$H_n(z) = \frac{n}{n+1}\frac{z}{2} + 2^3 \exp\left[n\left(\log 2 - \frac{z}{4} - \frac{\log(n+1)}{n}\right)\right].$$

Since

$$2^3 \exp\left[n\left(\log 2 - \frac{z}{4} - \frac{\log(n+1)}{n}\right)\right] \le \frac{1}{n+1}\frac{z}{2}$$

for large $z$, the assumptions of Theorem 2 are satisfied. □

## 2. Proofs of the theorems.

PROOF OF THEOREM 1. First we prove Theorem 1, the case of compact support. As mentioned in the Introduction, the proofs of the theorems are reduced to verifying the hypotheses of Corollary A for a typical realization of $L$. In turn, this involves the study of solutions of the boundary value problem (1.8). We aim to construct positive convex solutions solutions $v(x) = v_n(x)$ which are $\infty$ outside of some compact set and such that $\lim_{n\to\infty} v_n(x) = 0$ uniformly on a given compact interval. We can then apply Corollary A.

Our analysis of (1.8) on a bounded interval takes advantage of the fact that we are working in one dimension. We may regard $x$ as a time variable and build up $v(x)$ starting from $v(0)$ and $v'(0+)$. Moreover, since we are interested in convex positive solutions, we can construct such a solution starting from a point at which it assumes its minimum value. At such a point either the first derivative $v'(0)$ exists and is 0 [if $L(0) = 0$] or $v'(0-) \le 0$ and $v'(0+) \ge 0$ [if $L(0) > 0$]. Without loss of generality we can assume that this point is 0 and restrict our attention to the half-line $x > 0$. We divide the half-line into small intervals, on which $v(x)$ does not increase very much. For each interval, then, the term $v^2(x)\dot{L}(x)$ from (1.8) is almost equal to $c^2\dot{L}(x)$, for some constant $c$. If we replace the former term by the latter, (1.8) is no longer a nonlinear equation and it is much easier to analyze.

Our first task is to prove existence and uniqueness for (1.8), on the region where the solution is finite. We consider (1.8) for $x \ge 0$ with the following initial conditions [the same argument would apply to $x \le 0$ with $v'(0-) = \alpha \le 0$]:

$$v(0) = \beta > 0,$$
$$v'(0+) = \alpha \ge 0. \tag{2.1}$$

We will show, even for small $\beta > 0$, that $v(x) = \infty$ for large values of $|x|$.

We suppose that the random measure $L$ is fixed and set up the notation

$$\frac{L(dx)}{dx} = \sum_{i=1}^{\infty} c_i \delta(x - s_i).$$

Note that each $c_i > 0$ and that $L(dx)$ is a locally finite measure.

Starting with (1.9) and using the initial conditions (2.1), we obtain

$$v(x) = \beta + \alpha x + 2\int_0^x ds \int_0^s v^2(t) L(dt) \tag{2.2}$$

or

$$v(x) = \beta + \alpha x + 2\sum_{i=1}^{\infty} (x - s_i)^+ c_i v^2(s_i), \tag{2.3}$$

where $(x)^+ = \max\{x, 0\}$.

Since $v \to v^2$ is locally Lipschitz, using a standard argument we obtain existence and uniqueness of local solutions of (2.3). It is clear that if $v(x)$ is a continuous solution of (2.3) for $0 \le x \le \bar{x}$, then $v(x)$ is nondecreasing, and hence

$$\beta + \alpha x + \beta^2 \rho(x) \le v(x) \le \beta + \alpha x + v(x)^2 \rho(x), \tag{2.4}$$

where

$$\rho(x) = 2\sum_{i=1}^{\infty} (x - s_i)^+ c_i.$$

Note that $\rho(x)$ is Lipschitz continuous, nondecreasing and satisfies $\rho(0) = 0$ and $\rho(x) \le 2c_0 x$ for $x \in [0, K]$ and for some $c_0 = c_0(K) > 0$. If $\rho(x) > 0$, then the upper inequality in (2.4) yields that either

$$v(x) \le \frac{2(\beta + \alpha x)}{1 + \sqrt{1 - 4\rho(x)(\beta + \alpha x)}} = \frac{1 - \sqrt{1 - 4\rho(x)(\beta + \alpha x)}}{2\rho(x)}$$

or

$$v(x) \ge \frac{1 + \sqrt{1 - 4\rho(x)(\beta + \alpha x)}}{2\rho(x)}.$$

However, the second of these two possible inequalities is inconsistent with the initial condition and, therefore, we conclude that the solution $v(x)$ satisfies the first one. Now, for each $\alpha$, $\beta$ in (2.1), let $x_0 > 0$ be the largest value of $x$ such that $1 - 4\rho(x)(\beta + \alpha x) \ge 1/4$. We thus have the following lemma.

LEMMA 2.1. *For any* $\alpha \ge 0$, $\beta > 0$, (1.8) *with initial conditions* (2.1) *has a continuous solution in* $[0, x_0]$, *for some* $x_0 = x_0(\alpha, \beta) > 0$, *and*

$$\beta + \alpha x \le v(x) \le \frac{2(\beta + \alpha x)}{1 + \sqrt{1 - 4\rho(x)(\beta + \alpha x)}}, \qquad x \in [0, x_0].$$

LEMMA 2.2. (a) *Let* $u(x)$ *and* $v(x)$, $0 \le x \le x_0$, *be two solutions of* (1.8) *with the same initial values* $\alpha, \beta$. *Let* $\beta \le u, v \le M$ *for some* $M > 0$. *Then* $u \equiv v$ *on* $[0, x_0]$.

(b) *A nonnegative solution* $v(\beta_1, \beta_2, x)$ *to the two-point boundary value problem* (1.8) (*if it exists*) *is unique and, for fixed* $x$, $v(\beta_1, \beta_2, x)$ *is an increasing function of* $\beta_1$ *and* $\beta_2$.

PROOF. (a) Let $\|\cdot\|_A$ denote the supremum norm on the set $A$. Since $u$ and $v$ satisfy (1.8) with the same initial conditions $\alpha, \beta$, we have

$$\|u - v\|_{[0,t_0]} = 2\left\|\sum_{i=1}^{\infty}(x - s_i)^+ c_i(u(s_i) + v(s_i))(u(s_i) - v(s_i))\right\|_{[0,t_0]}$$
$$\leq 2M\rho(t_0)\|u - v\|_{[0,t_0]}.$$

Now, let $t_0 = (1/8Mc_0)$. We conclude first that $\|u - v\|_{[0,t_0]} \leq (1/2)\|u - v\|_{[0,t_0]}$ and, therefore, $u \equiv v$ on $[0, t_0]$. Next, suppose that

$$u \equiv v \quad \text{on } [0, kt_0],\ k \geq 1. \tag{2.5}$$

Then, by (2.5),

$$\|u - v\|_{[kt_0,(k+1)t_0]} = 2\left\|\sum_{i=1}^{\infty}(x - s_i)^+ c_i(u^2(s_i) - v^2(s_i))\right\|_{[kt_0,(k+1)t_0]}$$
$$= 2\left\|\sum_{i:\ s_i > kt_0}(x - s_i)^+ c_i(u^2(s_i) - v^2(s_i))\right\|_{[kt_0,(k+1)t_0]}$$
$$\leq 4M\left(\sum_{i:\ s_i > kt_0}(x - s_i)^+ c_i\right)\|u - v\|_{[kt_0,(k+1)t_0]}$$
$$\leq 4Mt_0c_0\|u - v\|_{[kt_0,(k+1)t_0]}$$

and thus $u \equiv v$ on $[kt_0, (k+1)t_0]$. Therefore, $u \equiv v$ on $[0, x_0]$.

(b) First observe that if $v_i(x_0 + x) \geq 0$, $i = 1, 2$, satisfy

$$v_i(x + x_0) = \beta + \alpha_i x + 2\int_{x_0}^{x} ds \int_{x_0}^{s} v_i^2(t)L(dt),$$

for $0 \leq x \leq x_1$ with $\alpha_1 > \alpha_2$, then $v_1(x_0 + x) > v_2(x_0 + x)$ for $0 < x \leq x_1$. Indeed, let $D(x) = v_1(x) - v_2(x)$. Then $D(x)$ satisfies

$$D(x + x_0) = (\alpha_1 - \alpha_2)x + 2\int_{x_0}^{x} ds \int_{x_0}^{s} D(t)(v_1(t) + v_2(t))L(dt).$$

First we claim that $D(x + x_0) > 0$ for $0 < x < \delta$, for some $\delta$ small enough. Indeed $(\alpha_1 - \alpha_2)x$ increases linearly in $x$, while

$$\left|\int_{x_0}^{x} ds \int_{x_0}^{s} D(t)(v_1(t) + v_2(t))L(dt)\right|$$

is $o(x)$ for small $x$.

Now we deal with large values of $x$. Note that $D(x)$ is continuous. Suppose that $D(x_0 + x) > 0$, for $0 < x < \bar{x}$, but that $D(x_0 + \bar{x}) \leq 0$. Setting $x = \bar{x}$ in the above integral equation, we would obtain a contradiction. This proves the observation.

Now assume that we have two solutions $v(\beta_{1,i}, \beta_{2,i}, x)$, $i = 1, 2$, to (1.8) with $\beta_{1,1} > \beta_{1,2}$ and $\beta_{2,1} \geq \beta_{2,2}$ (the argument if $\beta_{1,1} \geq \beta_{1,2}$ and $\beta_{2,1} > \beta_{2,2}$ is similar). We claim that $v(\beta_{1,1}, \beta_{2,1}, x) > v(\beta_{1,2}, \beta_{2,2}, x)\ \forall\ x \in (a_1, a_2)$.

Otherwise, by continuity there exists a point $x'$ at which $v(\beta_{1,1}, \beta_{2,1}, x') = v(\beta_{1,2}, \beta_{2,2}, x')$ and $v'(\beta_{1,1}, \beta_{2,1}, x'-) \le v'(\beta_{1,2}, \beta_{2,2}, x'-)$. However, from (1.9),

$$v'(\beta_{1,1}, \beta_{2,1}, x'+) - v'(\beta_{1,1}, \beta_{2,1}, x'-) = v'(\beta_{1,2}, \beta_{2,2}, x'+) - v'(\beta_{1,2}, \beta_{2,2}, x'-),$$

so that

$$v'(\beta_{1,1}, \beta_{2,1}, x'+) \le v'(\beta_{1,2}, \beta_{2,2}, x'+).$$

In view of the observation given at the beginning of the proof of part (b), this leads to a contradiction to $v(\beta_{1,1}, \beta_{2,1}, a_2) > v(\beta_{1,2}, \beta_{2,2}, a_2)$.

To verify uniqueness, consider two solutions $v_1$ and $v_2$ to (1.8). If either $v_1'(a_1+) > v_2'(a_1+)$ or $v_1'(a_1+) < v_2'(a_1+)$, the above observation leads to a contradiction to $v_1(a_2) = v_2(a_2)$. Finally, if $v_1'(a_1+) = v_2'(a_1+)$, then uniqueness follows from part (a). □

LEMMA 2.3. *If $u$ is a solution of* (1.8) *with initial condition* (2.1) *on* $[0, x_0]$ *such that $\beta \le u \le M$, then $u$ is Lipschitz with Lipschitz norm bounded by* $\alpha + 2c_0(x_0)M^2$.

PROOF.

$$\begin{aligned}|u(x) - u(y)| &= \left|\alpha(x-y) + 2\sum_{i=1}^{\infty}[(x-s_i)^+ - (y-s_i)^+]c_i u^2(s_i)\right| \\ &\le (\alpha + 2c_0 M^2)|x-y|.\end{aligned}$$ □

Lemmas 2.1, 2.2 and 2.3 imply the following theorem.

THEOREM 3. *For any $\alpha \ge 0$, $\beta > 0$,* (1.8) *and* (2.1) *have a unique locally Lipschitz solution. The solution can be extended until it reaches $\infty$.*

PROOF OF THEOREM 1 (continued). Given a bounded interval $[k_1, k_2]$, in Lemma 2.7 we will exhibit a sequence of functions $v_N(x)$ satisfying (1.8), such that $v_N(x) \to 0$ uniformly on $[k_1, k_2]$ and such that $v_N(x) = \infty$ for large values of $|x|$.

Fix $N > 1$ and suppose that $v$ satisfies (1.8) with

$$v_N(0) = 2^{-N},$$
$$v_N'(0) = 0.$$

For ease of notation, we will subsequently drop the subscript on $v_N$. Our first goal is to show the following lemma.

LEMMA 2.4. *Let $v(x)$ be defined as above and let the assumptions of Theorem* 1 *be satisfied. Then, with $Q_\nu$-probability* 1, *$v(x) = \infty$ off of a compact interval.*

PROOF. Since $v_N'(0) = 0$, by symmetry we can restrict our attention to $x \in [0, \infty)$. The case $x \in (-\infty, 0]$ follows from the same argument, after replacing $x$ by $-x$. Note that $v(x)$ is nondecreasing on $[0, \infty)$.

Our first task is to define several new objects. These are easier to work with than $v(x)$ itself, and we will be able to use comparison methods to gain information about $v(x)$. We define a function $u(x)$ for $x \geq 0$, such that $u(x) \leq v(x)$.

For $n \geq -N$, $m \geq 0$, we define sequences $x_n$, $\bar{x}_{n,m}$, $y_n$, $\bar{u}_{m,n}$ and $z_n$ by induction. To start the induction, choose $x_{-N}$ such that $v'(x_{-N}+) > 0$. Then, let $y_{-N} = v'(x_{-N}+)$ and let $z_{-N} = y_{-N}/b_{-N}$. For the following, we need only define $u(x)$ for $x \geq x_{-N}$. To begin, let $u(x_{-N}) = 2^{-N}$ and let $u'(x_{-N}+) = v'(x_{-N}+)$.

Assume that we have defined $x_n, y_n, z_n$ such that the following hold:

(i) $u(x_n) = 2^n \leq v(x_n)$;
(ii) $y_n = u'(x_n+)$;
(iii) $z_n = y_n/b_n$;
(iv) $u(x) \leq v(x)$ for $0 \leq x \leq x_n$;
(v) $u'(x+) \leq v'(x+)$ for $0 \leq x \leq x_n$;
(vi) $L((x_n, x_n + 2^n/(b_n z_n)]) \geq S(2^n/(b_n z_n))$ if $n > -N$.

Here, $u'(x+)$ denotes the right-hand derivative of $u$ at $x$.

For $m \geq 0$, let

$$\begin{aligned} \bar{x}_{n,m} &= x_n + m\frac{2^n}{u'(x_n+)} \\ &= x_n + m\frac{2^n}{y_n} \\ &= x_n + m\frac{2^n}{b_n z_n}. \end{aligned}$$

For $m \geq 0$ and $x \in [\bar{x}_{n,m}, \bar{x}_{n,m+1})$, we define $\bar{u}_{n,m}(x)$ to be the solution of

$$\begin{aligned} \bar{u}_{n,m}(\bar{x}_{n,m}) &= u(x_n) = 2^n, \\ \bar{u}_{n,m}'(x+) &= u'(x_n+) = y_n. \end{aligned}$$

Of course, this means that $\bar{u}_{n,m}(x) = 2^n + (x - \bar{x}_{n,m})y_n$ for $x \in [\bar{x}_{n,m}, \bar{x}_{n,m+1})$.

Finally, let $M = M(n)$ be the first integer $m > 1$ such that

$$L((\bar{x}_{n,m-1}, \bar{x}_{n,m}]) \geq S\left(\frac{2^n}{b_n z_n}\right). \tag{2.6}$$

Since the intervals $(\bar{x}_{n,m-1}, \bar{x}_{n,m}]$ have equal length for different values of $m$ and since $L((\bar{x}_{n,m-1}, \bar{x}_{n,m}])$ can take arbitrarily large values with positive probability, we see that $M(n) < \infty$ with probability 1. Next, let

$$x_{n+1} = \bar{x}_{n,M(n)},$$
$$u(x) = \bar{u}_{n,m}(x) \quad \text{for } x_{n,m-1} < x \le x_{n,m},\ 1 \le m \le M(n),$$
$$u'(x_{n+1}+) = u'(x_n+) + 2^{2n} S\left(\frac{2^n}{b_n z_n}\right).$$

Note that the definitions of $y_{n+1} = u'(x_{n+1}+)$ and $z_n = y_n/b_n$ and the assumptions of Theorem 1 imply that

$$\begin{aligned} z_{n+1} &= \frac{b_n}{b_{n+1}} z_n + \frac{2^{2n}}{b_{n+1}} S\left(\frac{2^n}{b_n z_n}\right) \\ &\ge c_1 z_n + c_2 1 \quad (0 < z_n < 1) \\ &\ge c_3. \end{aligned} \tag{2.7}$$

LEMMA 2.5. *For $x_n \le x \le x_{n+1}$, we have the following*:

(i) $u(x) \le v(x)$;
(ii) $u'(x+) \le v'(x+)$.

*Furthermore,*

$$x_{n+1} - x_n = M(n) \frac{2^n}{b_n z_n}.$$

*Thus, if we show that $u(x) = \infty$ for some $x$, then we conclude that $v(x) = \infty$.*

PROOF. Here is a proof by induction. The claim about $x_{n+1} - x_n$ follows from the definitions. Since $v'(x)$ is nondecreasing for $x \ge 0$, claim (i) of the lemma easily follows. Claim (ii) also follows easily, except at $x = x_{n+1} = \bar{x}_{n,M(n)}$. Let us show that

$$u'(x_{n+1}+) \le v'(x_{n+1}+).$$

By the definition of $u'(x_{n+1}+)$ and the portions of Lemma 2.5 which we have already proved, we find

$$\begin{aligned} v'(x_{n+1}+) &= v'(x_n+) + \int_{(x_n, x_{n+1}]} v(x)^2 L(dx) \\ &\ge y_n + \int_{(\bar{x}_{n,M(n)-1}, \bar{x}_{n,M(n)}]} \bar{u}_{n,M(n)-1}(x)^2 L(dx) \\ &\ge y_n + \bar{u}^2_{n,M(n)-1}(\bar{x}_{n,M(n)-1}) L((\bar{x}_{n,M(n)-1}, \bar{x}_{n,M(n)}]) \\ &\ge y_n + 2^{2n} L\left(\left(\bar{x}_{n,M(n)-1}, \bar{x}_{n,M(n)-1} + \frac{2^n}{b_n z_n}\right]\right) \\ &\ge y_n + 2^{2n} S\left(\frac{2^n}{b_n z_n}\right) \\ &= u'(x_{n+1}+). \end{aligned}$$

This proves Lemma 2.5. □

We note the following fact.

LEMMA 2.6. *There exists an i.i.d. sequence of geometrically distributed random variables* $\{G_n\}_{n=-N}^{\infty}$ *with parameter* $p = e^{-1}$, *such that with probability* 1, $M(n) \le G_n$ *for each* $n \ge -N$.

PROOF. By the strong Markov property and the independent increments property for the Lévy process $U(x) = L([0,x])$, it suffices to show that, for $m \ge 1$,

$$Q_\nu\left\{L((\bar{x}_{n,m-1}, \bar{x}_{n,m}]) \ge S\left(\frac{2^n}{b_n z_n}\right)\right\} \ge 1 - e^{-1}. \tag{2.8}$$

Recall that $\bar{x}_{n,m} - \bar{x}_{n,m-1} = 2^n/b_n z_n$. By the definition of $S(t)$ given in the Introduction, we conclude that

$$\begin{aligned}(\bar{x}_{n,m} - \bar{x}_{n,m-1})\nu\left(S\left(\frac{2^n}{b_n z_n}\right),\infty\right) &= \frac{2^n}{b_n z_n}\nu\left(S\left(\frac{2^n}{b_n z_n}\right),\infty\right)\\ &\ge 1.\end{aligned} \tag{2.9}$$

Now we use the interpretation of $L((a,b])$ as a compound Poisson random variable. Relation (2.9) implies that $L((\bar{x}_{n,m-1}, \bar{x}_{n,m}])$ has atoms of size greater than or equal to $S(2^n/(b_n z_n))$ with intensity at least 1. Using the Poisson probability distribution, the chance of at least one atom of this size is at least $1 - e^{-1}$. This proves (2.8). □

Now we return to the proof of Lemma 2.4. By (2.7) and Lemma 2.6, and using Lemma 2.5, we have that

$$\begin{aligned}\lim_{n\to\infty} x_n &= x_{-N} + \sum_{n=-N}^{\infty} [x_{n+1} - x_n]\\ &\le x_{-N} + \sum_{n=-N}^{\infty} M(n)\frac{2^n}{b_n z_n}\\ &\le x_{-N} + \sum_{n=-N}^{\infty} G_n \frac{2^{n+1}}{y_n}\\ &\le x_{-N} + \sum_{n=-N}^{\infty} G_n \frac{2^{n+1}}{c_0 b_n}.\end{aligned} \tag{2.10}$$

By the assumptions of Theorem 1, we know that

$$\sum_{n=0}^{\infty} \frac{2^{n+1}}{c_0 b_n} < \infty$$

and, therefore,

$$E\left[\sum_{n=0}^{\infty} G_n \frac{2^{n+1}}{c_0 b_n}\right] < \infty.$$

It therefore follows that

$$\begin{aligned}\lim_{n\to\infty} x_n &\le x_{-N} + \sum_{n=0}^{\infty} G_n \frac{2^{n+1}}{c_0 b_n}\\ &< \infty\end{aligned}$$

with $Q_\nu$-probability 1. This proves Lemma 2.4. □

At this point, we need a further lemma about the solutions of (1.8). Let $v(\varepsilon, x) = v(x)$ satisfy (1.8) with initial conditions $v(0) = \varepsilon$, $v'(0) = 0$. For simplicity, we will sometimes drop the dependence of $v(\varepsilon, x)$ on $\varepsilon$ and write $v(x)$.

LEMMA 2.7. *For $Q_\nu$ almost every $L$, $v(\varepsilon, x)$ tends to* 0 *uniformly on compact intervals in $x$ as $\varepsilon \downarrow 0$.*

PROOF. Let $K > 0$. By (1.8), $v(x)$ is convex in $x$ and, since $v'(\varepsilon, 0) = 0$, we have that

$$\sup_{-K \le x \le K} v(x) = \max[v(-K), v(K)].$$

To prove Lemma 2.7, it suffices to show that

$$\lim_{\varepsilon \downarrow 0} v(\varepsilon, K) = 0$$

with probability 1, since the same argument would show that

$$\lim_{\varepsilon \downarrow 0} v(\varepsilon, -K) = 0$$

with probability 1.

First, suppose that

$$\tfrac{1}{2}\bar{v}''(x) = (\bar{v}(x) \wedge (2\varepsilon))^2 \dot{L}$$

and that $\bar{v}(0) = v(0) = \varepsilon$, $\bar{v}'(0) = 0$. Then, using the convexity of $\bar{v}$, we find that

$$\sup_{0 \le x \le K} \bar{v}'(x) = \bar{v}'(K) \le 4\varepsilon^2 L((0, K]).$$

Then,

$$\begin{aligned}\bar{v}(K) &\le \varepsilon + K \cdot 4\varepsilon^2 L((0, K])\\ &\le 2\varepsilon\end{aligned}$$

for $\varepsilon$ sufficiently small. Thus, for $0 \le x \le K$, $\bar{v}(\varepsilon, x) \le 2\varepsilon$ for $\varepsilon$ small enough. In this case, $\bar{v}(x) \wedge 2\varepsilon = \bar{v}(x)$ and thus $\bar{v}(x)$ satisfies (1.8) for $0 \le x \le K$ and for $\varepsilon$ sufficiently small. Thus, for $\varepsilon$ sufficiently small, we have

$$v(\varepsilon, x) \le 2\varepsilon.$$

Hence,

$$\lim_{\varepsilon\downarrow 0}\sup_{-K\le x\le K} v(\varepsilon,x) = \lim_{\varepsilon\downarrow 0}\max[v(\varepsilon,-K),v(\varepsilon,K)]$$
$$\le \lim_{\varepsilon\downarrow 0} 2\varepsilon$$
$$= 0.$$

This proves Lemma 2.7. □

The proof of Theorem 1 now follows since Lemma 2.7 verifies the hypothesis of Corollary A. □

PROOF OF THEOREM 2. The proof of Theorem 2 follows immediately from part (b) of Corollary A and Lemma 2.8 below. The argument follows the same lines as in the proof of Theorem 1, except that our estimates will usually go in the opposite direction. To obtain the appropriate probability estimates, we use Markov's inequality.

LEMMA 2.8. *Assume that* $\nu$ *satisfies the conditions of Theorem* 2.

(a) *Let* $v(x)$ *satisfy* (1.8) *with initial conditions* $v(0)=\beta>0$, $v'(0+)=\alpha\ge 0$. *Then, with* $Q_\nu$*-probability* 1, $v(x)<\infty$ *for all* $x\in\mathbb{R}$.

(b) *Let* $v(\beta_1,\beta_2,\cdot)$ *be as in Theorem* A. *Then for* $Q_\nu$*-a.e.* $L$ *and any* $-\infty<a_1<a_2<\infty$,

$$\sup_{\beta_1,\beta_2}\inf_{x\in[a_1,a_2]} v(\beta_1,\beta_2,x)=+\infty.$$

PROOF. To see that (b) follows from (a) assume that there exist sequences $\beta_{1,n},\beta_{2,n}$ and $x_n\to x_\infty\in[a_1,a_2]$ such that

$$\lim_{n\to\infty} v(\beta_{1,n},\beta_{2,n},x_n)=\sup_{\beta_1,\beta_2}\inf_{x\in[a_1,a_2]} v(\beta_1,\beta_2,x)=c<+\infty.$$

However, by (a) there exists a solution of (1.8) such that $v(x_\infty)>c$, but is bounded on $[a_1,a_2]$. Together with Lemma 2.2(b), this yields a contradiction. □

Again, we prove Lemma 2.8(a) for $x>0$, since the case of $x<0$ follows by symmetry. The arguments below show that, given any $\alpha\ge 0$ and $\beta>0$, the solution of (2.2) remains finite on $(0,\infty)$. In order to simplify the notation, we set $\beta=1$ and $\alpha=0$. Our definitions are then similar to the previous case. We define sequences $x_n, y_n, z_n$, this time for $n\ge 0$, and a function $u(x)\ge v(x)$, $u'(x+)\ge v'(x+)$ for $x\ge 0$. We let $y_n=u'(x_n+)$ and $z_n=y_n/b_n$.

In the proof of Theorem 1 [see (2.6)], we waited until $L((\bar{x}_{n,m-1},\bar{x}_{n,m}])$ was large enough and then chose our new variable $x_{n+1}$. The result was a function $u(x)\le v(x)$. For the proof of Theorem 2, we seek a function $u(x)\ge v(x)$. For this purpose, we hope that $L((x_n,x_n+2^n/(z_nb_n)])$ is rather small. If it is small, we define $x_{n+1}$ such that $u(x_{n+1})=2^{n+1}$. If $L((x_n,x_n+2^n/(z_nb_n)])$

happens to be too large, we must let $x_{n+1}$ be smaller than usual, such that $x_{n+1} < x_n + 2^n/(z_n b_n)$. To be precise, suppose that $u(x)$ satisfies (1.8) for $x \geq x_n$, with initial conditions given by $u(x_n)$ and $u'(x_n+)$, and choose $x_{n+1}$ to be the smallest number $x > x_n$ and such that either of the following holds:

1. $u(x) = 2^{n+1}$;
2. $u'(x+) = b_{n+1} \max[K, H_n(z_n)]$.

For such an $x_{n+1}$, we readjust the initial conditions on $u(x)$ at $x = x_{n+1}$, such that

$$u(x_{n+1}) = 2^{n+1},$$
$$u'(x_{n+1}+) = \max[K, H_n(z_n)].$$

For such a sequence $x_n$, we must show that with probability 1, $\lim_{n\to\infty} x_n = \infty$.

Here are the details. To begin with, we seek inequalities for $x_{n+1} - x_n$ and $y_{n+1} - y_n$ similar to those used in the proof of Theorem 1. First, since $u(x)$ is convex and thus $u'(x)$ is nondecreasing, and by the definition of $x_{n+1}$,

$$(x_{n+1} - x_n)u'(x_n+) \leq u(x_{n+1}-) - u(x_n) \leq 2^{n+1}$$

and, therefore,

$$\begin{aligned} x_{n+1} - x_n &\leq \frac{2^{n+1}}{u'(x_n+)} \\ &= \frac{2^{n+1}}{y_n}. \end{aligned} \tag{2.11}$$

Let $A_n$ be the event that $u(x_{n+1}-) = 2^{n+1}$. If $A_n$ holds, then

$$\begin{aligned} x_{n+1} - x_n &\geq \frac{2^n}{u'(x_{n+1}+)} \\ &= \frac{2^n}{y_{n+1}} \\ &= \frac{2^n}{b_{n+1} z_{n+1}}. \end{aligned} \tag{2.12}$$

Therefore, arguing as in the proof of Theorem 1, but using the opposite inequality, we find that, whether or not $A_n$ holds,

$$\begin{aligned} u'(x_{n+1}-) - u'(x_n+) &= \int_{(x_n, x_{n+1}]} u^2(x) L(dx) \\ &\leq u^2(x_{n+1}) L((x_n, x_{n+1}]) \\ &\leq 2^{2(n+1)} L\left(\left(x_n, x_n + \frac{2^{n+1}}{u'(x_n+)}\right]\right) \\ &= 2^{2(n+1)} L\left(\left(x_n, x_n + \frac{2^{n+1}}{y_n}\right]\right) \\ &= 2^{2(n+1)} L\left(\left(x_n, x_n + \frac{2^{n+1}}{b_n z_n}\right]\right). \end{aligned} \tag{2.13}$$

Here we record a lemma about the size of $L$. Note that $L((x_n, x_n + t])$ and $L((0, t])$ are equal in distribution.

LEMMA 2.9. *Under the assumptions of Theorem* 2,

$$Q_\nu\big\{L((0,t]) > s\big\} \le \frac{I(t)}{s} + 1 - e^{-0.5},$$

*for* $s, t > 0$.

PROOF. Let $\hat{L}$ be the compound Poisson random measure obtained from $L$ by omitting the atoms of mass greater than $T(t)$. Note that $L((0,t]) - \hat{L}((0,t])$ is a compound Poisson random variable with finite Lévy measure $t\nu(dx)1[x \ge T(t)]$. Let $F$ be the event that $L((0,t]) - \hat{L}((0,t]) = 0$. From the above, and by the definition of $T(t)$,

$$Q_\nu\{F\} = \exp\big(-t\nu([T(t),\infty))\big) \ge \exp(-0.5).$$

Now let $G$ be the event that $\hat{L}((0,t]) \ge s$. By the definition of $I(t)$ and by Markov's inequality,

$$Q_\nu\{G\} \le \frac{E(\hat{L}((0,t]))}{s}.$$

Since $F$ and $G$ are independent, we may put these two estimates together, to obtain

$$Q_\nu\big\{L((0,t]) > s\big\} \le Q_\nu\{F^c\} + Q_\nu\{G\}.$$

This proves Lemma 2.9. □

Next, note that

$$\frac{u'(x_{n+1}-)}{b_n} \le \frac{b_n}{b_{n+1}} z_n + \frac{2^{2n+2}}{b_{n+1}} L\bigg(\bigg(x_n, x_n + \frac{2^{n+1}}{b_n z_n}\bigg]\bigg).$$

Let $L_n = L((x_n, x_n + 2^{n+1}/z_n b_n])$ and let $B_n$ be the event that

$$L_n < 4I\bigg(\frac{2^{n+1}}{z_n b_n}\bigg).$$

If $B_n$ occurs, then using the definition of $H_n$ we see that

$$\frac{u'(x_{n+1}-)}{b_n} < H_n(z_n).$$

Thus, if $B_n$ occurs, condition 2 in the definition of $u, u'$ must fail, and so condition 1 must occur and, therefore, $A_n$ occurs. We have shown that $B_n \subset A_n$. Let $R_n = 1(B_n)$.

Now let $\mathscr{F}_n$ denote the $\sigma$-field generated by $L([0,x])$ for $0 \le x \le x_n$. By the independent increments property of Lévy processes, the conditional distribution of $L_n$ given $\mathscr{F}_n$ is the same as the distribution of $L((0, 2^n/(z_n b_n)])$. Using Lemma 2.9, we have

$$Q_\nu\{B_n | \mathscr{F}_n\} \ge e^{-0.5} - \frac{I(2^{n+1}/b_n z_n)}{4I(2^{n+1}/b_n z_n)} \\ \ge 0.1. \tag{2.14}$$

For future use, we let $\delta = 0.1$.

We wish to show that, for $n$ large enough,

$$z_{n+1} < M_0 \tag{2.15}$$

for some constant $M_0$ not depending on $n$. In that case, we would have $y_{n+1} \le M_0 b_{n+1}$.

To prove (2.15), we consider the discrete dynamical system obtained from the action of the $H_n$. Let

$$\overline{H}_n(z) = \max[K, H_n(z)].$$

LEMMA 2.10. *For $0 < i < j$, let $H_{i,j}(z)$ denote the composition*

$$H_{i,j}(z) = \overline{H}_i\big(\overline{H}_{i+1} \cdots \overline{H}_j(z)\big).$$

*Given $M_0 > K$ there exists a constant $N_1 > 0$ depending only on $N_0$, $K$ and $z$ such that if $j - i > N_1$, then*

$$H_{i,j}(z) < M_0.$$

PROOF. By the definition of $\overline{H}_n$, it follows that $\overline{H}_j(z) \ge K$ for all values of $i$. Now, requirement (1.13) on $H_n(z)$ states that if $n$ is large enough and $z > K$, then

$$H_n(z) \le \frac{z}{2}.$$

Thus, if $n$ is large enough and $z > 2K$, each application of $\overline{H}_n$ decreases $z$ by a factor of 2. Thus, if $j - i$ is large enough, we will have $H_{i,j}(z) < 2K$. This completes the proof of Lemma 2.10. □

Now we can finish the proof of Theorem 2. First note that, by our definition, $u'(x_{n+1}+) = b_{n+1}\overline{H}_n(z_n)$ and, therefore, $z_{n+1} = \overline{H}_n(z_n)$. Now Lemma 2.10 shows that there exists a constant $N > 0$ depending only on $z_1$ such that $z_n \le M_0$ for all $n \ge N$. Second, we wish to use the lower bound (2.12), and we can do so if $A_n$ holds. Recalling that $R_n = 1(B_n)$ and $B_n \subset A_n$, we find

$$\begin{aligned}\lim_{n\to\infty} x_n &= \sum_{n=0}^{\infty}(x_{n+1} - x_n) + x_0 \\ &\geq \sum_{n=1}^{\infty} R_n \frac{2^{n+1}}{b_n z_n} \\ &\geq \sum_{n=N}^{\infty} R_n \frac{2^{n+1}}{b_n M_0}.\end{aligned}$$

However, by the assumptions of Theorem 2, we know that

$$\sum_{n=N}^{\infty} \frac{2^{n+1}}{b_n M_0} = \infty. \tag{2.16}$$

Therefore, we need only take into account the random variables $R_n$. Unfortunately, these are not independent.

Recall that, by (2.14), $Q_\nu\{B_n | \mathscr{F}_n\} \geq \delta = 0.1$. Now, by enlarging the probability space if necessary, we may choose $\mathscr{F}_{n+1}$-measurable events $\overline{B}_n \subset B_n$ such that $Q_\nu\{\overline{B}_n | \mathscr{F}_n\} = \delta$. Because $\overline{B}_1, \dots, \overline{B}_n$ are $\mathscr{F}_n$-measurable, we conclude that $\{\overline{B}_n\}$ is a sequence of $Q_\nu$-independent events, each with $Q_\nu$-probability $\delta$. Let

$$\overline{R}_n = 1(\overline{B}_n). \tag{2.17}$$

Our immediate goal is to show the following lemma.

LEMMA 2.11. *With $Q_\nu$-probability* 1,

$$\sum_{n=1}^{\infty} \overline{R}_n \frac{2^n}{b_n} = \infty. \tag{2.18}$$

PROOF. By the assumptions of Theorem 2, we know that

$$\sum_{n=1}^{\infty} \frac{2^n}{b_n} = \infty. \tag{2.19}$$

For ease of notation, let

$$a_n = \frac{2^n}{b_n}.$$

Without loss of generality, we may assume that the terms $a_n$ are arranged in decreasing order. Indeed, if there are infinitely many terms $a_n > \varepsilon$ for any $\varepsilon > 0$, then we immediately deduce that (2.18) holds with probability 1. Choose the nondecreasing subsequence $\{n_k\}_{k=1}^{\infty}$ such that if $n_k \leq n < n_{k+1}$, then $3^{-k} \geq a_n > 3^{-(k+1)}$. Furthermore, let $k(i)$ be an enumeration of those indices $k \geq 1$ such that $n_{k+1} - n_k \geq 2^k$. Now, (2.16) implies that

$$\sum_{i=1}^{\infty} 3^{-k(i)}(n_{k(i)+1} - n_{k(i)}) = \infty. \tag{2.20}$$

Let $m(i)$ be the number of indices $n$ such that $n_{k(i)} \le n < n_{k(i)+1}$ and $\overline{R}_n = 1$. Since $n_{k(i)+1} - n_{k(i)} \ge 2^{k(i)} \ge 2^i$, by the weak law of large numbers we find

$$Q_\nu\left\{\frac{m(i)}{n_{k(i)+1} - n_{k(i)}} < \frac{\delta}{2}\right\} \le \frac{\delta(1-\delta)}{n_{k(i)+1} - n_{k(i)}} \Big/ \frac{\delta^2}{4} \le \frac{4(1-\delta)}{\delta 2^i}.$$

The Borel–Cantelli lemma now implies that, with probability 1,

$$\frac{m(i)}{n_{k(i)+1} - n_{k(i)}} > \frac{\delta}{2} \tag{2.21}$$

except for a finite number of indices $i$. Now redefine the sequence $k(i)$ by dropping those indices $k(i)$ for which (2.21) fails. Using (2.20) and (2.21), we find that

$$\begin{aligned} \sum_{n=1}^{\infty} \overline{R}_n \frac{2^n}{b_n} &\ge \sum_{i=1}^{\infty} m(i) 3^{-k(i)} \\ &> \frac{\delta}{2} \sum_{i=1}^{\infty} (n_{k(i)+1} - n_{k(i)}) 3^{-k(i)} \\ &= \infty. \end{aligned} \tag{2.22}$$

So finally, (2.22) implies that

$$\begin{aligned} \lim_{n\to\infty} x_n &= \sum_{n=1}^{\infty} (x_{n+1} - x_n) + x_1 \\ &\ge c \sum_{n=1}^{\infty} \overline{R}_n \frac{2^n}{b_n} + x_1 \\ &= \infty. \end{aligned}$$

This completes the proof of Lemma 2.11 and also Lemma 2.8(a). □

We now complete the proof of Lemma 2.8(b). Assume the contrary, namely, $\sup_{\beta_1,\beta_2} \inf_{x\in[a_1,a_2]} v(\beta_1, \beta_2, x) = \beta^* < +\infty$. Then there exist sequences $\beta_{1,n}, \beta_{2,n} \uparrow \infty$ and $x_n \to x^* \in [a_1, a_2]$, $v(\beta_{1,n}, \beta_{2,n}, x_n) \to \beta^*$ and $v(\beta_{1,n}, \beta_{2,n}, x)$ assumes its minimum at $x_n$. However, by Lemma 2.8(a) there exists a solution $v$ with $v(x^*) = \beta^* + 1$ and $v'(x^*) = 0$ which remains bounded on $[a_1, a_2]$. However, for large $n$, $\beta_{1,n} > v(a_1)$ and $\beta_{2,n} > v(a_2)$, but together with $v(x^*) > \inf_{x\in[a_1,a_2]} v(\beta_{1,n}, \beta_{2,n}, x)$ this yields a contradiction by Lemma 2.2(b).

This completes the proof of Theorem 2. □

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