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L^p-BOUNDEDNESS OF THE OVERSHOOT IN MULTIDIMENSIONAL RENEWAL THEORY

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Let T_r be the first time a sum S_n of nondegenerate i.i.d. random variables leaves a ball of radius r in some given norm on $\mathbb{R}^d$. In the case of the Euclidean norm we completely characterize L^p-boundedness of the overshoot $\|S_{T_r}\| - r$ in terms of the underlying distribution. For more general norms we provide a similar characterization under a smoothness condition on the norm which is shown to be very nearly sharp. One of the key steps in doing this is a characterization of the possible limit laws of $S_{T_r}/\|S_{T_r}\|$ under the weaker condition $\|S_{T_r}\|/r \to_p 1$.

1. Introduction. We extend to multidimensions results from renewal theory on the boundedness of moments of the overshoot of a random walk at the exit time from a ball of radius r. Roughly speaking, the results of this paper cover the nonzero mean case; when combined with those in [4] they provide necessary and sufficient conditions for the overshoot to be bounded in L^p. The results apply to many norms in addition to the Euclidean norm, for example, all l^q-norms, where $1 < q < \infty$. We also show that the basic dichotomy that occurs when studying the overshoot is not between the mean-zero and non-mean-zero cases, but between two analytic conditions related to relative stability and asymptotic normality. These conditions are reflected probabilistically in a sharp difference in the exit behaviors of the walk.

Let $Z, Z_1, Z_2, \ldots$ be a sequence of nondegenerate, independent and identically distributed $\mathbb{R}^d$-valued random vectors, and set $S_n = \sum_{j=1}^n Z_j$. Let $\|\ \|$ be an arbitrary norm on $\mathbb{R}^d$ and define $T_r = \min\{n: \|S_n\| > r\}$. The overshoot, $\|S_{T_r}\| - r$, is one of the main objects of study in classical renewal theory. For example, in many applications of renewal theory, $d = 1$ and the $Z_i \geq 0$ represent the lifetimes of components of some device which are replaced by identical copies upon failure. The overshoot, $S_{T_r} - r$, then represents the remaining lifetime of the component in use at time r. If $d = 1$, but Z is not assumed to be nonnegative, then instead of $\|S_{T_r}\| - r$ many authors have considered the one-sided overshoot, that is, with T_r replaced by $T_r^+ = \min\{n: S_n > r\}$. We will consider only the two-sided case since it generalizes naturally to higher dimensions.

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Our interest in the multidimensional overshoot comes from the problem of trying to determine when it is bounded in L^p, that is, $\sup_{r>0} E(\|S_{T_r}\| - r)^p < \infty$. In one dimension this problem has been studied in two quite distinct cases: the first when $EZ \neq 0$ and the second when $EZ = 0$. As the following standard result suggests, the behavior of the overshoot is strikingly different in these two settings.

THEOREM 1. *Assume $d = 1$ and $p > 0$:*

$$\text{(1.1)} \qquad \text{if } E|Z|^{1+p} < \infty \text{ and } EZ \neq 0, \text{ then } \sup_{r>0} E\big(|S_{T_r}| - r\big)^p < \infty;$$

$$\text{(1.2)} \qquad \text{if } E|Z|^{2+p} < \infty \text{ and } EZ = 0, \text{ then } \sup_{r>0} E\big(|S_{T_r}| - r\big)^p < \infty.$$

This result can be derived from the corresponding one-sided version. The one-sided version is obtained from results for nonnegative walks by passing to the ladder height process; see, for example, [6] and [9]. This approach does not generalize to higher dimensions.

We found in [4] that the following condition played a crucial role in extending (1.2) to higher dimensions:

(E) The family $Y_r = S_{T_r}/\|S_{T_r}\|$ has no subsequential limit supported on a closed half-space.

For a complete description of the L^p-boundedness of moments of the overshoot we need to modify this condition slightly. Let μ be a measure supported on the unit sphere, and let V be the smallest subspace of $\mathbb{R}^d$ which contains the support of μ. We say that μ is complete if $\mu(\Gamma) > 0$ for every cone Γ, with vertex at the origin, which is open in V. Thus, for example, a measure supported on $\{(1,0),(0,1)\}$ is not complete, but one supported on $\{(-1,0),(1,0)\}$ is. We then introduce the following exit condition on the walk:

(C) every subsequential limit of the family Y_r is complete.

The following results are minor modifications of two of the main results in [4].

THEOREM 2. *Fix $p > 0$. Then the following are equivalent:*

$$\text{(1.3)} \qquad E\|Z\|^{2+p} < \infty \quad \text{and} \quad EZ = 0;$$

$$\text{(1.4)} \qquad \sup_{r>0} E\big(\|S_{T_r}\| - r\big)^p < \infty \quad \text{and} \quad \text{(C)}.$$

The result corresponding to $p = 0$ is the following theorem:

THEOREM 3. *The following are equivalent:*

$$\text{(1.5)} \qquad E\|Z\|^2 < \infty \quad \text{and} \quad EZ = 0,$$

$$\text{(1.6)} \qquad \big(\|S_{T_r}\| - r\big) \text{ is tight} \quad \text{and} \quad \text{(C)}.$$

The corresponding results (Theorems 1.2 and 1.3) of [4] are different in that (C) is replaced by (E) and the random variables are assumed genuinely d-dimensional, that is, not supported on any affine subspace. The proofs of Theorems 2 and 3 are slight modifications of those of Theorems 1.2 and 1.3 in [4].

We now turn to the main results of this paper, which cover the situation dealt with in (1.1). We need to introduce another exit condition on the walk which is in a sense at the opposite extreme from (C):

(A) every subsequential limit of the family Y_r is an atom.

We shall be concerned only with random walks for which

$$\frac{\|S_{T_r}\|}{r} \to_p 1. \tag{1.7}$$

Observe that this condition is much weaker than either L^p-boundedness or tightness of the overshoot and is presumably the weakest condition under which study of the overshoot is of interest.

It is perhaps surprising that (1.7) severely constrains the subsequential limits of Y_r. In general it is straightforward to construct examples which show that *any* given distribution on the unit sphere can arise as a subsequential limit of Y_r. However, when (1.7) holds, either (C) must hold or (A) must hold.

THEOREM 1.1. *If* $\|S_{T_r}\|/r \to_p 1$, *then either* (C) *or* (A) *holds.*

We will in fact prove a refinement of this result which gives simple necessary and sufficient analytic conditions that determine whether (C) or (A) holds. To describe these results we need to introduce some notation. For $r > 0$ let

$$\begin{aligned} G(r) &= P(\|Z\| > r), \\ K(r) &= r^{-2}E(\|Z\|^2; \|Z\| \le r), \\ M(r) &= r^{-1}E(Z; \|Z\| \le r) \end{aligned}$$

and

$$h(r) = G(r) + K(r) + \|M(r)\|.$$

THEOREM 1.2. *The following are equivalent:*

$$\frac{\|S_{T_r}\|}{r} \to_p 1 \quad \textit{and} \quad (\mathrm{A}); \tag{1.8}$$

$$\lim_{r\to\infty} \frac{\|M(r)\|}{h(r)} = 1. \tag{1.9}$$

THEOREM 1.3. *The following are equivalent*:

$$\frac{\|S_{T_r}\|}{r} \to_p 1 \quad \textit{and} \quad (\mathrm{C}); \tag{1.10}$$

$$\lim_{r\to\infty} \frac{K(r)}{h(r)} = 1. \tag{1.11}$$

These results show that study of the overshoot falls into two distinct cases, given analytically by (1.9) and (1.11), and giving rise to very distinctive exit behavior. It is also interesting to note that in Theorem 1.2 the mean term in $h(r)$ dominates [however, (1.9) does not imply $EZ \neq 0$, or even that the mean exists], while in Theorem 1.3 it is the second moment term which dominates. These conditions are related (in fact, equivalent in one dimension) to relative stability and asymptotic normality, respectively; see [3].

In conjunction with the results described above, the following result completes the characterization of L^p-boundedness of the overshoot in the case of the Euclidean norm. First recall that Z is said to belong to weak L^p, denoted WL^p, if $\sup_{r>0} r^p G(r) < \infty$.

THEOREM 1.4. *Fix $p > 0$ and assume $\| \; \|$ is the Euclidean norm. Then the following are equivalent*:

$$\|Z\| \in WL^{1+p}, \qquad EZ \neq 0 \quad \textit{and} \quad E\langle Z, EZ\rangle_+^{1+p} < \infty; \tag{1.12}$$

$$\sup_{r>0} E\big(\|S_{T_r}\| - r\big)^p < \infty \quad \textit{and} \quad (\mathrm{A}). \tag{1.13}$$

Whereas this result is stated only for the Euclidean norm, Theorems 2 and 3 are valid for arbitrary norms. The difficulty with norms that are not isotropic stems from the fact that a nonzero mean provides a preferred direction. As we will see, the behavior of the boundary of the unit ball at the point $\hat{\mu} = EZ/\|EZ\|$ plays a crucial role. It seemed to us reasonable to expect that if $\langle Z, EZ\rangle_+ \in L^{p+1}$ were replaced by $\langle Z, \nu\rangle_+ \in L^{p+1}$, where ν is the outward unit normal at $\hat{\mu}$, then Theorem 1.4 should remain true in all cases where $\hat{\mu}$ is a smooth point on the unit sphere. This turns out not to be the case. To state the result for more general norms we need to introduce the modulus of smoothness

$$\rho(h) = \sup\big\{\tfrac{1}{2}(\|z + w\| + \|z - w\| - 2) \colon \|z\| = 1, \|w\| = h\big\}.$$

As we indicate later, ρ is strictly increasing if (as we may assume) $d \geq 2$. The smoothness condition we must impose on the norm is the integrability condition (1.14) below. As we discuss in Section 5, this implies the existence of a unique outward unit normal ν at $\hat{\mu}$.

THEOREM 1.5. *Fix $p > 0$ and assume $\| \ \|$ has modulus of smoothness satisfying*

$$\int_0 \left(\frac{u}{\rho^{-1}(u)} \right)^{1+p} \frac{du}{u} < \infty. \tag{1.14}$$

Then the following are equivalent:

$$\|Z\| \in WL^{1+p}, \qquad EZ \neq 0 \quad \text{and} \quad E\langle Z, \nu \rangle_+^{1+p} < \infty; \tag{1.15}$$

$$\sup_{r>0} E\big(\|S_{T_r}\| - r\big)^p < \infty \quad \text{and} \quad (\mathrm{A}). \tag{1.16}$$

In the case of the Euclidean norm $\rho^{-1}(u) \sim u^{1/2}$ and $\nu = EZ/\|EZ\|$, so we recover Theorem 1.4. More generally for l^q-norms with $1 < q < \infty$, $\rho^{-1}(u) \sim u^{(1/q) \vee (1/2)}$. Hence the integral in (1.14) converges, and so (1.15) and (1.16) are equivalent. We think it likely that the integral test in (1.14) is sharp. That is, if the integral diverges, then (1.15) and (1.16) are not equivalent. We will give an example in which the integral barely diverges, and (1.15) and (1.16) are not equivalent; see Example 5.11. We also point out in Remark 5.9 that all that is needed in (1.14) is a local version of ρ in a neighborhood of $\hat{\mu}$. Thus, for example, we can extend Theorem 1.5 to cover the case where $\hat{\mu}$ is a smooth point on the boundary of the unit ball in the l^1- or l^∞-norm; see also Example 5.10 for the case when $\hat{\mu}$ is not a smooth point.

If the integral in (1.14) diverges, then Theorem 1.5 gives no information about the L^p-boundedness of the overshoot. We present one result which is considerably easier to prove than Theorem 1.5, and which is valid for arbitrary norms. It gives conditions which are necessary and conditions which are sufficient for (1.16).

THEOREM 1.6. *Fix $p > 0$. Then*

$$\|Z\| \in L^{1+p} \quad \text{and} \quad EZ \neq 0 \tag{1.17}$$

implies

$$\sup_{r>0} E\big(\|S_{T_r}\| - r\big)^p < \infty \quad \text{and} \quad (\mathrm{A}) \tag{1.18}$$

implies

$$\|Z\| \in WL^{1+p} \quad \text{and} \quad EZ \neq 0. \tag{1.19}$$

Theorem 3 gives necessary and sufficient conditions for tightness when (C) holds. The analogous result when (A) holds seems considerably more difficult. Again it is the problem with preferred directions. We do have the analogue of Theorem 3 in one dimension, but in higher dimensions the problem remains open. Before stating the one-dimensional result, observe that since we are in the situation where (A) holds, we may assume that (1.9) holds. In one dimension it is easy to check that this forces either $M(r)/h(r) \to 1$ or $M(r)/h(r) \to -1$. Clearly there is no harm in assuming it is the former.

THEOREM 1.7. *Assume $d = 1$ and $M(r)/h(r) \to 1$. Then the following are equivalent*:

$$(|S_{T_r}| - r) \text{ is tight}; \tag{1.20}$$

$$(S_{T_r} - r)^+ \text{ is tight}; \tag{1.21}$$

$$\int^{\infty} \frac{dF(u)}{h(u)} < \infty. \tag{1.22}$$

Given the results on L^p-boundedness of the overshoot when $\|M(r)\|/h(r) \to 1$, one would perhaps expect the condition for tightness in this setting to be slightly weaker than $Z \in L^1$. However, this is not the case, as we point out after the proof of Theorem 1.7 in Section 6. If $Z \in L^1$ and $EZ > 0$, then it is easy to check that $M(u) \sim h(u) \sim EZ/u$ and hence

$$\frac{M(r)}{h(r)} \to 1 \quad \text{and} \quad \int^{\infty} \frac{dF(u)}{h(u)} < \infty,$$

that is, $(|S_{T_r}| - r)$ is tight. If $Z \in L^1$ and $EZ = 0$ it is still possible that $M(r)/h(r) \to 1$. In this case, however, $\int^{\infty}(dF(u)/h(u))$ may or may not be finite; see Example 6.3.

The plan of the paper is as follows: Section 2 contains preliminaries and some further notation. The proofs of the results on the exit conditions (C) and (A) are given in Section 3. After some estimates on the occupation measure in Section 4, we prove the main results on L^p-boundedness of the overshoot in Section 5. Finally in Section 6 we discuss tightness of the overshoot.

2. Preliminaries. The interplay between the functions $G(r)$, $K(r)$ and $M(r)$ plays an important role in our analysis. Each of these functions is right continuous with left limits and approaches 0 as $r \to \infty$. Their behavior near $r = 0$ will not be of much importance, but we point out that $G(r) \to P(\|X\| > 0) > 0$ as $r \to 0$; hence, $h(r)$ is bounded away from 0 as $r \to 0$. Furthermore, h is strictly positive for all $r > 0$ since the same is true of $G + K$; see below.

The importance of the function h can be seen from the following estimate of Pruitt [11]:

$$ET_r \approx \frac{1}{h(r)}, \tag{2.1}$$

where $\approx$ means that the ratio of the two quantities is bounded above and below by constants independent of $r > 0$. The following estimates on the distribution of T_r can also be found in [11]:

$$P(T_r > n) \le \frac{c}{nh(r)}, \qquad P(T_r \le n) \le cnh(r), \tag{2.2}$$

where c denotes a universal constant which may change from one usage to the next. The function h also satisfies a useful doubling property:

$$h(r) \approx h(2r). \tag{2.3}$$

It is convenient to introduce $Q = G + K$. Observe that

$$r^2 Q(r) = E(\|X\| \wedge r)^2 = \int_0^r 2uG(u)\,du.$$

Since $Q(r)\downarrow$ and $r^2 Q(r)\uparrow$, it is easy to check that Q satisfies a doubling property analogous to (2.3).

The following lemma will be needed in the proof of Theorem 1.5:

LEMMA 2.1. *Assume $Z \in L^{1+\varepsilon}$ for some $\varepsilon > 0$. Then*

$$EZ \neq 0 \quad \text{if and only if} \quad \lim_{r\to\infty} \frac{\|M(r)\|}{h(r)} = 1.$$

PROOF. The implication from left to right is easy and in fact it does not require the side condition that $Z \in L^{1+\varepsilon}$. To see this, observe that $\|M(r)\| \sim \|\mu\|/r$, while

$$Q(r) = \frac{1}{r^2}\int_0^r 2uG(u)\,du = o\left(\frac{1}{r}\right)$$

since $Z \in L^1$.

For the other direction assume $\|M(r)\|/h(r) \to 1$ and $EZ = 0$. Then

$$E\frac{(\|Z\|; \|Z\| > r)}{rG(r)} \geq \frac{\|E(Z; \|Z\| > r)\|}{rG(r)} = \frac{\|E(Z; \|Z\| \leq r)\|}{rG(r)} \to \infty.$$

Hence

$$\frac{\int_r^\infty G(u)\,du}{rG(r)} \to \infty.$$

Thus by the theory of regular variation (see [3]),

$$\varphi(r) = \int_r^\infty G(u)\,du$$

is slowly varying. However, $G(u) \leq cu^{-(1+\varepsilon)}$ since $Z \in L^{1+\varepsilon}$. Hence $\varphi(r) \leq cr^{-\varepsilon}$. However, φ being slowly varying entails that $r^\varepsilon\varphi(r) \to \infty$ for all $\varepsilon > 0$, which is a contradiction. □

We will let $B(w; r) = \{z\colon \|z - w\| \leq r\}$ and use $\partial B(w; r)$ to denote its boundary. For $\xi \in \partial B(0; 1)$ and $\beta \in [0, 1]$, let

$$\Gamma(\xi, \beta) = \{z : \langle z, \xi\rangle \geq \beta\|z\|\},$$

where $\langle\ ,\ \rangle$ denotes the usual inner product on $\mathbb{R}^d$. Thus in the case of the Euclidean norm, $\Gamma(\xi, \beta)$ is the cone with vertex at the origin and having β as the cosine of the angle between ξ and any generator of the cone.

Let U_r be the occupation measure defined by

$$U_r(dx) = E\left(\sum_{j=0}^{T_r-1} I(S_j \in dx)\right).$$

The next lemma follows from essentially the same proof as Proposition 3.1 in [4].

LEMMA 2.2. *For any nonnegative Borel function* φ: $\mathbb{R}^{2d} \to \mathbb{R}$,

$$E\varphi(S_{T_r}, X_{T_r}) = \int_{B(0,r)} E(\varphi(Z+z, Z); \|Z+z\| > r) U_r(dz).$$

Of particular interest to us will be the case where

$$\varphi(w_1, w_2) = (\|w_1\| - r)^p I(w_1 - w_2 \in A),$$

where A is a Borel subset of $B(0; r)$, in which case we may rewrite the inner expectation to obtain

$$\begin{aligned} &E\Big((\|S_{T_r}\| - r)^p; S_{T_r-1} \in A\Big) \\ &\qquad = \int_A \int_0^\infty p\lambda^{p-1} P(\|z + Z\| > r + \lambda)\, d\lambda\, U_r(dz). \end{aligned} \tag{2.4}$$

Since all norms on $\mathbb{R}^d$ are equivalent, we will often use that if Z_i have finite variance, then

$$E\|S_n - ES_n\|^2 \le cnE\|Z\|^2 \tag{2.5}$$

for some constant c depending only on the norm.

3. Exit distributions. In this section we will characterize the two distinct kinds of exit behavior that can occur under (1.7). We begin by recalling that, by Theorem 3.5 of [4],

$$\frac{\|S_{T_r}\|}{r} \to_p 1 \quad \text{if and only if} \quad \frac{G(r)}{h(r)} \to 0. \tag{3.1}$$

The following result can then be understood as the analytic counterpart of the dichotomy in Theorem 1.1.

PROPOSITION 3.1. *If* $G(r)/h(r) \to 0$ *and* $\limsup_{r\to\infty} \|M(r)\|/h(r) > 0$, *then* $\|M(r)\|/h(r) \to 1$.

PROOF. For any $r < s$,

$$\begin{aligned} sM(s) &= rM(r-) + O(sG(r-)), \\ s^2K(s) &= r^2K(r-) + O(s^2G(r-)). \end{aligned}$$

Thus

$$\frac{K(s)}{h(s)} = \frac{(r/s)^2 K(r-) + O(G(r-))}{(r/s)\|M(r-)\| + (r/s)^2 K(r-) + O(G(r-))}. \tag{3.2}$$

Now there exists an $\varepsilon > 0$ and a sequence $t_k \to \infty$ such that

$$\frac{\|M(t_k)\|}{h(t_k)} \geq \varepsilon,$$

for all k. We first show

$$\liminf_{r\to\infty} \frac{\|M(r)\|}{h(r)} \geq \frac{\varepsilon}{2}. \tag{3.3}$$

If not, then for each k,

$$s_k = \inf\left\{ r > t_k : \frac{\|M(r)\|}{h(r)} \leq \frac{\varepsilon}{2} \right\} < \infty.$$

By right continuity,

$$\frac{\|M(s_k)\|}{h(s_k)} \leq \frac{\varepsilon}{2}.$$

Let

$$r_k = \sup\left\{ r < s_k : \frac{\|M(r)\|}{h(r)} \geq \varepsilon \right\}.$$

Then $t_k \leq r_k \leq s_k$ and

$$\frac{\|M(r_k -)\|}{h(r_k -)} \geq \varepsilon.$$

Also clearly

$$\frac{\|M(r)\|}{h(r)} \leq \varepsilon \quad \text{on } [r_k, s_k]. \tag{3.4}$$

Along a further subsequence if necessary, we may assume

$$\frac{s_k}{r_k} \to \lambda \in [1, \infty] \quad \text{and} \quad \frac{\|M(r_k -)\|}{h(r_k -)} \to \varepsilon' \geq \varepsilon.$$

Case 1 ($\lambda < \infty$). Letting $r = r_k$, $s = s_k$, dividing through by $h(r_k -)$ and letting $k \to \infty$ in (3.2) we obtain

$$\lim_{k\to\infty} \frac{K(s_k)}{h(s_k)} = \frac{(1/\lambda^2)(1-\varepsilon')}{(1/\lambda)\varepsilon' + (1/\lambda^2)(1-\varepsilon')},$$

where we have used that $G(r)/h(r) \to 0$ implies $G(r-)/h(r-) \to 0$. (This follows immediately from monotonicity of G and the doubling property of h.)

Thus

$$\lim_{k\to\infty} \frac{K(s_k)}{h(s_k)} = \frac{1-\varepsilon'}{\lambda\varepsilon' + (1-\varepsilon')} \le \frac{1-\varepsilon'}{\varepsilon' + (1-\varepsilon')} = 1-\varepsilon'$$

since $\lambda \ge 1$. Hence

$$\liminf_{k\to\infty} \frac{\|M(s_k)\|}{h(s_k)} \ge \varepsilon',$$

which is a contradiction.

Case 2 ($\lambda = \infty$). Then by (3.2), for any $\eta > 1$,

$$\frac{K(\eta r_k)}{h(\eta r_k)} = \frac{(1/\eta)K(r_k -)/h(r_k -) + \eta O(G(r_k -)/h(r_k -))}{\|M(r_k -)\|/h(r_k -) + 1/(\eta K(r_k -)h(r_k -)) + \eta O(G(r_k -)/h(r_k -))}$$

Thus

$$\lim_{k\to\infty} \frac{K(\eta r_k)}{h(\eta r_k)} = \frac{(1/\eta)(1-\varepsilon')}{\varepsilon' + (1/\eta)(1-\varepsilon')}.$$

By letting $\eta \to \infty$, we obtain

$$\lim_{\eta\to\infty} \liminf_{k\to\infty} \frac{\|M(\eta r_k)\|}{h(\eta r_k)} = 1.$$

However, for fixed η,

$$\limsup_{k\to\infty} \frac{\|M(\eta r_k)\|}{h(\eta r_k)} \le \varepsilon$$

by (3.4) and the fact that $\lambda = \infty$. This is again a contradiction. Thus we have shown (3.3) and consequently that

$$\liminf_{r\to\infty} \frac{\|M(r -)\|}{h(r -)} \ge \frac{\varepsilon}{2}.$$

If we now repeat this last argument, we obtain, for each $\eta > 1$,

$$\limsup_{u\to\infty} \frac{K(u)}{h(u)} = \limsup_{r\to\infty} \frac{K(\eta r)}{h(\eta r)} \le \frac{1/\eta}{\varepsilon/2},$$

by simply bounding $K(r -)/h(r -)$ above by 1 and below by 0. Now let $\eta \to \infty$ to get

$$\limsup_{u\to\infty} \frac{K(u)}{h(u)} = 0,$$

which completes the proof. □

COROLLARY 3.2. $\|S_{T_r}\|/r \to_p 1$ *if and only if one of the following conditions holds*:

$$\frac{\|M(r)\|}{h(r)} \to 1 \quad or \quad \frac{K(r)}{h(r)} \to 1.$$

We now consider the behavior of the exit position $Y_r = S_{T_r}/\|S_{T_r}\|$ under each of the two conditions above. To do this we will need the following two propositions.

PROPOSITION 3.3. *Assume* $\|M(r)\|/h(r) \to 1$ *and fix a subsequence* r_k. *Then*

$$Y_{r_k} \to_w \delta_\theta \quad iff \quad \frac{M(r_k)}{\|M(r_k)\|} \to \theta.$$

PROOF. By the usual subsequence argument it suffices to prove the if statement. For this we observe that for any N, by Doob's inequality and (2.5),

$$P\left(\|S_n - nEZI(\|Z\| \le r_k)\| > \varepsilon r_k \text{ for some } n \le \frac{N}{h(r_k)}\right)$$

$$\le P\left(\|Z_i\| > r_k \text{ for some } i \le \frac{N}{h(r_k)}\right)$$

$$+ P\left(\left\|\sum_{i=1}^{n} Z_i I(\|Z_i\| \le r_k) - nEZI(\|Z\| \le r_k)\right\| > \varepsilon r_k \text{ for some } n \le \frac{N}{h(r_k)}\right)$$

$$\le \frac{N}{h(r_k)} G(r_k) + \frac{cN}{h(r_k)} \frac{r_k^2 K(r_k)}{\varepsilon^2 r_k^2} \to 0.$$

Also by (2.2),

$$P\left(T_{r_k} > \frac{N}{h(r_k)}\right) \le \frac{c}{N}.$$

Now, for any $t > 0$ and $w \in \mathbb{R}^d$, let

$$C(w;t) = \left\{z: \inf_{\lambda > 0} \|z - \lambda w\| < t\right\}$$

be the semiinfinite cylinder of radius t with axis given by the line $\{\lambda w : \lambda > 0\}$. *Then we have shown that*

$$\liminf_{k \to \infty} P\left(S_{T_{r_k}} \in C(M(r_k); \varepsilon r_k)\right) \ge 1 - \frac{c}{N}.$$

Hence letting $N \to \infty$, *we have, for every* $\varepsilon > 0$,

$$P\left(Y_{r_k} \in C(M(r_k); \varepsilon)\right) \to 1. \qquad \square$$

Before stating the next proposition, we recall that in one dimension $K(r)/h(r) \to 1$ is equivalent to Z being in the domain of attraction of the standard normal law without centering. In higher dimensions, it is equivalent to the existence of a scalar sequence a_n such that

(3.5) S_n/a_n is tight and all subsequential limits are nondegenerate mean-zero Gaussian laws.

It may happen that some subsequential limits are not genuinely d-dimensional. That is, they could be supported on various subspaces, but they do not degenerate to 0. If the sequence a_n is chosen so that $nK(a_n) \to 1$, then (3.5) is satisfied.

PROPOSITION 3.4. *Assume $K(r)/h(r) \to 1$ and along some subsequence $Y_{r_k} \to Y$. Then Y is complete.*

PROOF. Let $n_k = [1/K(r_k)]$. Then along possibly a further subsequence,

$$\frac{S_{n_k}}{r_k} \to W,$$

where W is a nondegenerate mean-zero Gaussian random variable. Let $B(t)$ be a Brownian motion based on W and

$$B_k(t) = \frac{S_{[n_k t]} + (n_k t - [n_k t])(S_{[n_k t]+1} - S_{[n_k t]})}{r_k}$$

be the standard linear interpolation of the partial sums. Then by an invariance principle (see [4] Theorem 1), $B_k(t) \to B(t)$ in $C[0,\infty)$. It then follows that the law of Y_{r_k} converges to that of the exit distribution of $B(t)$ from $B(0;1)$. Thus Y is the harmonic measure of B on $B(0;1)$. This distribution is easily seen to be complete by arguing as in Lemma 4.2 of [4]. □

PROOF OF THEOREM 1.2. Assume (1.8). By Corollary 3.2, either $K(r)/h(r) \to 1$ or $\|M(r)\|/h(r) \to 1$. In the former case, by Proposition 3.4 it is impossible for (A) to hold; thus (1.9) holds.

Now assume (1.9). Then by Corollary 3.2, $\|S_{T_r}\|/r \to_p 1$, and by Proposition 3.3, (A) holds. □

PROOF OF THEOREM 1.3. Assume (1.10). Then, by Corollary 3.2 and Theorem 1.2, (1.11) holds.

Now assume (1.11). Then by Corollary 3.2, $\|S_{T_r}\|/r \to_p 1$, and by Proposition 3.4, (C) holds. □

PROOF OF THEOREM 1.1. This now follows immediately from Corollary 3.2 and Theorems 1.2 and 1.3. □

4. Some renewal theory. The classical renewal theorems concern the asymptotic behavior of $N_r = \sum_{j=0}^{\infty} I(\|S_j\| \le r)$ for transient random walk. We are interested in the smaller r.v. T_r, which has some properties very similar to those of N_r. (See, e.g., [6] for background.)

Recall that if $E\|Z\| < \infty$ and $\mu = EZ \neq 0$, then $\hat{\mu} = \mu/\|\mu\|$.

PROPOSITION 4.1. *If $\mu \neq 0$, then $\|S_{T_r}\|/r \to_p 1$ and $S_{T_r}/\|S_{T_r}\| \to_p \hat{\mu}$.*

PROOF. As pointed out in the proof of Lemma 2.1, when $\mu \neq 0$,

$$h(r) \sim \|M(r)\| \sim \frac{\|\mu\|}{r}, \qquad r \to \infty.$$

Thus the first statement follows from (3.1) and the second from Proposition 3.3. □

The following result may be viewed as an analogue of the elementary renewal theorem.

COROLLARY 4.2. *If $\mu \neq 0$, then*

$$\lim_{r \to \infty} \frac{ET_r}{r} = \frac{1}{\|\mu\|}. \tag{4.1}$$

PROOF. By Wald's identity,

$$\|\mu\| \left| \frac{ET_r}{r} - \frac{1}{\|\mu\|} \right| = \left\| \frac{ES_{T_r}}{r} - \mu \right\|.$$

By [4], Theorem 3.3, $\{S_{T_r}/r\}_{r \geq 1}$ is uniformly integrable. Thus, the desired result follows from Proposition 4.1. □

Recalling the occupation measure U_r defined by

$$U_r(dx) = E\left(\sum_{j=0}^{T_r - 1} I(S_j \in dx) \right),$$

we set

$$U_{r-u,r} = \int_{r-u < \|z\| \leq r} U_r(dx).$$

The latter quantity, which represents the expected occupation time of an annulus, will be important in the study of the distribution of the overshoot. Let $\hat{U}_{r-u,r}$ be the analogous quantity with T_r replaced by $+\infty$. In dimension $d = 1$ and for nonlattice random walk the asymptotic behavior of $\hat{U}_{r-u,r}$ as $r \to \infty$ is described by a variant of Blackwell's renewal theorem:

$$\lim_{r \to \infty} \hat{U}_{r-u,r} = \frac{u}{\|\mu\|}, \qquad d = 1. \tag{4.2}$$

See, for example, [6], Theorem 6.6. Bickel and Yahav [1] extend this to $d = 2$, but only if the unit sphere of $\| \ \|$ is a polygon which does not have any side parallel to μ. Without assuming more than first moments there is, apparently, no known analogous result for the general norms we consider (or even for the Euclidean norm). More information is available (see, e.g., [2]) under more restrictive hypotheses. We shall only need the following approximate result, which we state under a more general assumption than $EZ \neq 0$.

THEOREM 4.3. *Fix $\theta \in \partial B(0;1)$ and assume $Y_r \to_w \delta_\theta$. Then there exist constants $\beta_0 < 1$ and $c_1 < \infty$ such that*

$$\int_{\{z:\, r-v<\|z\|\le r,\ z\in\Gamma(\theta,\beta_0)\}} U_r(dz) \le \frac{c_1}{h(v)}, \qquad 0 \le v \le r, \tag{4.3}$$

once r is sufficiently large. If, in addition, the overshoot distribution is tight, then there are constants $c_2 > 0$ and $c_3 > 0$ such that, for every $0 < \beta < 1$,

$$\int_{\{z:\, r-v<\|z\|\le r,\ z\in\Gamma(\theta,\beta)\}} U_r(dz) \ge \frac{c_2}{h(v)}, \qquad c_3 \le v \le r, \tag{4.4}$$

provided r is sufficiently large (depending on β).

The proof of this result is similar enough to that of the corresponding results in [4] that we provide only a brief sketch.

For (4.4) one may argue as in [4], Lemma 5.2. Fix $0 < \beta < 1$. By tightness of the overshoot if $v/3$ is sufficiently large, and by $Y_r \to_w \delta_\theta$ if $r - 2v/3$ is sufficiently large, we have

$$P\left(S_{T_{r-2v/3}} \in \Gamma\left(\theta, \frac{1+\beta}{2}\right), \|S_{T_{r-2v/3}}\| \le r - \frac{v}{3}\right) \ge \frac{1}{2}. \tag{4.5}$$

Thus there exist constants c_3 and $r_0 = r_0(\beta)$ such that if $r \ge r_0$ and $c_3 \le v \le r$, then (4.5) holds. (It is important to note here that how large c_3 needs to be is independent of β since $r - 2v/3 \ge r/3$.) A little geometry shows that, for any $z \in \Gamma(\theta, (1+\beta)/2)$ with $\|z\| \in [r - 2v/3, r - v/3]$, it is possible to choose a ball centered at z which is entirely contained in $\Gamma(\theta, \beta)$ and which has radius comparable to v. Since the expected exit time from such a ball is comparable to $1/h(v)$ by (2.1) and (2.3), one may obtain the desired result by applying the strong Markov property at time $T_{r-2v/3}$. (The hypotheses $E\|Z_1\|^2 < \infty$ and $EZ = 0$ in [4], Lemma 5.2, are not needed here.)

Turning to (4.3), note that by the proof of [4], Lemma 2.5, there are $\beta_0 > 0$ and $\alpha > 0$ such that $\|z + w\| \ge \|z\| + \alpha\|w\|$ for $z, w \in \Gamma(\theta, \beta_0)$. Thus, with $\gamma = 1/\alpha$, the conditions $\|z\| > r - v$, $\|w\| \ge \gamma v$ and $z, w \in \Gamma(\theta, \beta_0)$ imply $\|z + w\| > r$. One may then argue as in [4], Proposition 5.1, with $u = v$, $\tau = \inf\{n > 0\colon r - v < \|S_n\| \le r,\ S_n \in \Gamma(\theta, \beta_0)\}$, and using $\lim_{v\to\infty} P_0(S_{T_{\gamma v}} \in \Gamma(\theta, \beta_0)) = 1$ in place of [4], Lemma 4.1.

A typical application of this result will be when $EZ \neq 0$, in which case $Y_r \to_p \hat{\mu}$ by Proposition 4.1 and $h(v) \sim \|\mu\|/v$.

5. Proof of Theorems 1.5 and 1.6. We will prove Theorems 1.5 and 1.6 simultaneously. First we require some preliminary geometric results. In particular, we will need estimates on the increments of the norm given in

Lemma 5.3 and Corollary 5.4 below.

Let $\| \;\|$ denote a fixed norm on $\mathbb{R}^d$. Recall that the modulus of smoothness of $\| \;\|$ is the function ρ defined by

$$\rho(h) = \sup\left\{\tfrac{1}{2}(\|z + w\| + \|z - w\| - 2): \|z\| = 1, \|w\| = h\right\}.$$

Also define the modulus of convexity, δ, by

$$\delta(\varepsilon) = \inf\left\{1 - \frac{\|z + w\|}{2}: \|z\| = \|w\| = 1, \|z - w\| = \varepsilon\right\}.$$

The modulus of convexity of the dual norm will be denoted δ_*. For an introduction to the properties of these functions, see [10], Section 1.e. In particular, ρ is a nondecreasing convex function such that $\rho(0) = 0$. If the dimension is at least 2, then ρ is strictly increasing. Also, ρ satisfies $\rho(2h) \le c\rho(h)$, $h \ge 0$, for some constant c. In order not to have to separate out the case when $d = 1$ (which is much easier), we will assume that $d \ge 2$. If the walk is one dimensional, we can embed it in $\mathbb{R}^2$, extend the norm to (a multiple of) the Euclidean norm on $\mathbb{R}^2$ and then use the result in this case to recover the one-dimensional result.

It is easy to check that by convexity (1.14) forces $\rho(u)/u \to 0$ as $u \downarrow 0$. Thus $\| \;\|$ is uniformly smooth, and hence by Proposition 1.2 of [10] the dual norm is uniformly convex. This means that $\delta_*(\varepsilon) > 0$ for all $\varepsilon > 0$. Hence by [10], Lemma 1.8, $\delta_*(\varepsilon)$ is strictly increasing on $[0, 2]$. Thus δ_*^{-1} exists for sufficiently small t and $\delta_*^{-1}(t) \downarrow 0$ as $t \downarrow 0$.

For each $\|\xi\| = 1$ let l_ξ^* be a linear functional that satisfies $\|l_\xi^*\| = l_\xi^*(\xi) = 1$. Then the level set $\{z: l_\xi^*(z) = 1\}$ defines a support hyperplane of the unit ball at ξ. We will call such a functional a support functional at ξ. As we have just observed, when (1.14) holds, the norm is uniformly smooth. Hence there is a unique support functional at each ξ. Of particular interest to us will be the support functional at $\hat{\mu}$, which for simplicity we denote by l^*. It then follows that $l^*(z) = (z, \nu)/|(\hat{\mu}, \nu)|$, where ν is the unique outward normal at $\hat{\mu}$ with $\|\nu\| = 1$; see Theorem 7F and the discussion following in [7].

LEMMA 5.1. *For any ξ and ζ with $\|\xi\| = \|\zeta\| = 1$,*

$$\delta_*\big(\|l_\xi^* - l_\zeta^*\|\big) \le \tfrac{1}{2}\|\xi - \zeta\|.$$

PROOF. We have

$$\begin{aligned}\delta_*\big(\|l_\xi^* - l_\zeta^*\|\big) &\le 1 - \tfrac{1}{2}\|l_\xi^* + l_\zeta^*\| \le 1 - \tfrac{1}{2}(l_\xi^* + l_\zeta^*)(\xi)\\ &= \tfrac{1}{2} - \tfrac{1}{2}l_\zeta^*(\xi) = \tfrac{1}{2}l_\zeta^*(\zeta - \xi) \le \tfrac{1}{2}\|\zeta - \xi\|.\end{aligned}$$ □

COROLLARY 5.2. *Assume $\| \;\|$ is uniformly smooth. Fix $\|\xi\| = 1$ and assume w satisfies $l^*(w) \le 0$. Then*

$$l_\xi^*(w) \le \|w\|\delta_*^{-1}\big(\tfrac{1}{2}\|\hat{\mu} - \xi\|\big), \tag{5.1}$$

provided $\|\hat{\mu} - \xi\|$ *is sufficiently small that* δ_*^{-1} *exists. If* $l^*(w) = 0$, *then*

$$|l_\xi^*(w)| \le \|w\| \delta_*^{-1}\left(\tfrac{1}{2}\|\hat{\mu} - \xi\|\right), \tag{5.2}$$

provided $\|\hat{\mu} - \xi\|$ *is sufficiently small.*

PROOF. If $l_\xi^*(w) \le 0$, then (5.1) clearly holds. If $l_\xi^*(w) \ge 0$, then since $l^*(w) \le 0$,

$$l_\xi^*(w) \le |l_\xi^*(w) - l^*(w)| \le \|w\| \, \|l_\xi^* - l^*\|.$$

Statement (5.1) now follows from Lemma 5.1. If $l^*(w) = 0$, then we can apply (5.1) to obtain

$$-l_\xi^*(w) = l_\xi^*(-w) \le \|w\| \delta_*^{-1}\left(\tfrac{1}{2}\|\hat{\mu} - \xi\|\right).$$

Together with (5.1) this proves (5.2). □

In the next lemma it is reasonable to think of the two terms on the right-hand sides of the two inequalities as measuring the increment of the norm in the tangential and normal directions, respectively.

LEMMA 5.3. *For vectors* v *and* z *we have*

$$\|v + z\| - \|z\| \le 2\|z\| \rho\left(\frac{2\|v\|}{\|z\|}\right) + |l_{z/\|z\|}^*(v)|. \tag{5.3}$$

If $\|v\| \le \frac{1}{2}\|z\|$, *we have*

$$\|v + z\| - \|z\| \le 2\|z\| \rho\left(\frac{4\|v\|}{\|z\|}\right) + [l_{z/\|z\|}^*(v)]_+. \tag{5.4}$$

PROOF. Set $\xi = z/\|z\|$. If $l_\xi^*(w) = 0$, then

$$\|w \pm z\| - \|z\| \ge l_\xi^*(z \pm w) - l_\xi^*(z) = 0.$$

Thus

$$\|w + z\| - \|z\| \le \|w + z\| + \|w - z\| - 2\|z\| \le 2\|z\| \rho(\|w\|/\|z\|). \tag{5.5}$$

To obtain (5.3), first write $v = v - \xi l_\xi^*(v) + \xi l_\xi^*(v)$, apply the triangle inequality and then (5.5) with $w = v - \xi l_\xi^*(v)$.

For (5.4), if $l_\xi^*(v) < 0$ and $\|v\| \le \frac{1}{2}\|z\|$, then $\|z\|/2 \le \|z + \xi l_\xi^*(v)\| \le \|z\|$. Thus by (5.5),

$$\begin{aligned} \|v + z\| - \|z\| &\le \|v - \xi l_\xi^*(v) + z + \xi l_\xi^*(v)\| - \|z + \xi l_\xi^*(v)\| \\ &\le 2\|z + \xi l_\xi^*(v)\| \rho\left(\frac{\|v - \xi l_\xi^*(v)\|}{\|z + \xi l_\xi^*(v)\|}\right) \\ &\le 2\|z\| \rho\left(\frac{4\|v\|}{\|z\|}\right). \end{aligned}$$

□

COROLLARY 5.4. *Assume* $\| \ \|$ *is uniformly smooth. If* $l^*(w) = 0$, *then for any* z,

$$\|w + z\| - \|z\| \le 2\|z\|\,\rho\left(\frac{2\|w\|}{\|z\|}\right) + \|w\|\delta_*^{-1}\left(\frac{1}{2}\left\|\hat{\mu} - \frac{z}{\|z\|}\right\|\right),$$

provided $\|\hat{\mu} - \xi\|$ *is sufficiently small that* δ_*^{-1} *exists. If* $l^*(w) \le 0$ *and* $\|w\| \le \frac{1}{2}\|z\|$,

$$\|w + z\| - \|z\| \le 2\|z\|\,\rho\left(\frac{4\|w\|}{\|z\|}\right) + \|w\|\delta_*^{-1}\left(\frac{1}{2}\left\|\hat{\mu} - \frac{z}{\|z\|}\right\|\right),$$

provided $\|\hat{\mu} - \xi\|$ *is sufficiently small.*

PROOF OF THEOREMS 1.5 AND 1.6. We begin with the proof of the implications (1.16) ⇒ (1.15) and (1.18) ⇒ (1.19). We start with the latter implication. To see that Z belongs to WL^{p+1}, we use a bootstrapping argument similar to the proof of [4], Theorem 1.1, page 850. By [4], Lemma 6.1,

$$\begin{aligned} E\big(\|S_{T_r}\| - r\big)^p &\ge ET_r \int_r^{2r} p\lambda^{p-1} P(\|Z\| > 3\lambda)\, d\lambda \\ &\ge (2^p - 1) ET_r r^p P(\|Z\| > 6r). \end{aligned} \tag{5.6}$$

Since ET_r is nondecreasing in r, we may conclude that $Z \in WL^p$.

Next, we claim that, in fact, $Z \in L^1$. Suppose $0 < p < 1$. By [4], Lemma 2.4, we have $\liminf_{r\to\infty}(ET_r/r^p) > 0$. Using this in (5.6), we conclude $Z \in WL^{2p}$. Iteration of this argument as in the proof of [4], Theorem 1.1, establishes the claim in this case. If $p = 1$ the claim follows, since we may use any smaller value of p in the same argument.

For $Z \in L^1$ we have $\liminf_{r\to\infty}(ET_r/r) > 0$. This follows from Corollary 4.2 if $EZ \ne 0$, and from [4], Lemma 2.4, if $EZ = 0$. Using this in (5.6) completes the proof that $Z \in WL^{1+p}$.

To see that, in fact, $EZ \ne 0$, first note that (1.18) ⇒ (1.8) ⇒ (1.9). Since we have already shown that $Z \in WL^{1+p}$, it then follows from Lemma 2.1 that $EZ \ne 0$.

This completes the proof that (1.18) ⇒ (1.19). To complete the proof of (1.16) ⇒ (1.15), we must show that under (1.14) we also have $l^*(Z)_+ \in L^{1+p}$. Actually for this we do not need the full strength of (1.14), only that $\| \ \|$ is uniformly smooth. To do this we require the following claim:

CLAIM. For any compact set K there exists a cone Γ containing $\hat{\mu}$ in its interior and a constant $R > 0$ such that the conditions $\lambda > 1$, $r > R$, $z \in \Gamma$, $r/2 \le \|z\| \le r$, $Z \in K$ and $l^*(Z) > 4(r + \lambda - \|z\|)$ imply that $\|Z + z\| > r + \lambda$.

To see how the claim is used observe that if $\Gamma_r = \{z \in \Gamma\colon r/2 \le \|z\| \le r\}$, then by (2.4) for large r,

$$\begin{aligned} E\big(\|S_{T_r}\| - r\big)^p &\ge \int_{\Gamma_r}\int_1^r p\lambda^{p-1}P(\|z+Z\| > r+\lambda, Z \in K)\,d\lambda\,U_r(dz) \\ &\ge \int_{\Gamma_r}\int_1^r p\lambda^{p-1}P(l^*(Z)_+ > 4(r+\lambda-\|z\|), Z\in K)\,d\lambda\,U_r(dz) \\ &\ge \int_{\Gamma_r}\int_{(r-\|z\|)\vee 1}^r p\lambda^{p-1}P(l^*(Z)_+ > 8\lambda, Z \in K)\,d\lambda\,U_r(dz) \\ &= \int_1^r\left(\int_{\{z\colon r-\lambda<\|z\|\le r,\ z\in\Gamma_r)} U_r(dz)\right) \\ &\qquad \times p\lambda^{p-1}P(l^*(Z)_+ > 8\lambda, Z\in K)\,d\lambda \\ &\ge \int_{c_3}^{r/2} cp\lambda^p P(l^*(Z)_+ > 8\lambda, Z \in K)\,d\lambda, \end{aligned}$$

by (4.4). (Note that the overshoot is tight since we are assuming it is bounded in L^p.) The result now follows by letting $r \to \infty$ and then $K \uparrow \mathbb{R}^d$.

To prove the claim, first note that $l_\xi^*(w)$ is jointly continuous in (ξ, w). This is because, by Lemma 5.1,

$$\begin{aligned} \big|l_\xi^*(w) - l_{\xi'}^*(w')\big| &\le \|l_\xi^* - l_{\xi'}^*\|\,\|w\| + \|l_{\xi'}^*\|\,\|w-w'\| \\ &\le \delta_*^{-1}\big(\tfrac{1}{2}\|\xi-\xi'\|\big)\|w\| + \|w-w'\|. \end{aligned} \tag{5.7}$$

Now by [5], Lemma 6.4, we have $\|w + t\xi\| - t \downarrow l_\xi^*(w)$ as $t \to \infty$, for any $\|\xi\| = 1$. Thus by Dini's theorem this convergence is uniform in ξ and w belonging to any compact set. Hence writing $Z = X + Y$, where $X = (l^*(Z)\nu)/(l^*(\nu))$ [thus $l^*(Y) = 0$], it follows that if $\|z\|$ is sufficiently large,

$$\begin{aligned} \big|\|Z+z\| - \|X+z\|\big| &\le \big|l_{z/\|z\|}^*(Z) - l_{z/\|z\|}^*(X)\big| + \frac{1}{2} \\ &= \big|l_{z/\|z\|}^*(Y)\big| + \frac{1}{2} \\ &\le \|Y\|\delta_*^{-1}\left(\frac{1}{2}\left\|\hat{\mu} - \frac{z}{\|z\|}\right\|\right) + \frac{1}{2}, \end{aligned} \tag{5.8}$$

if $\|\hat{\mu} - z/\|z\|\,\|$ is sufficiently small as in Corollary 5.2.

Next observe that $l^*(\nu) = \langle \nu, \nu\rangle/|\langle \hat{\mu}, \nu\rangle| > 0$. Thus by Lemma 5.1, if $\alpha = l^*(\nu)/2$, then $l_\xi^*(\nu) \ge \alpha$ provided $\|\xi - \hat{\mu}\|$ is sufficiently small. Hence there is a cone Γ' containing $\hat{\mu}$ such that, for all $L > 0$ and all $z \in \Gamma'$,

$$\begin{aligned} \|z + L\nu\| &\ge \big|l_{z/\|z\|}^*(z + L\nu)\big| \\ &= \big|\|z\| + Ll_{z/\|z\|}^*(\nu)\big| \\ &\ge \|z\| + \alpha L. \end{aligned} \tag{5.9}$$

Now choose $\Gamma \subseteq \Gamma'$ so that $\hat{\mu} \in \Gamma$, and so that if $Z \in K$ and $z \in \Gamma$, then

$$\|Y\| \delta_*^{-1}\left(\frac{1}{2}\left\| \hat{\mu} - \frac{z}{\|z\|}\right\|\right) < \frac{1}{2}.$$

This is possible since $\|Y\|$ is bounded and $\delta_*^{-1}(t) \downarrow 0$ as $t \downarrow 0$. Hence if $\lambda > 1$, $\|z\| \le r$, $z \in \Gamma$, $Z \in K$ and $l^*(Z) > 4(r + \lambda - \|z\|)$, we then have, by (5.9),

$$\|z + X\| - \|z\| \ge \frac{l^*(Z)}{2} > 2(r + \lambda - \|z\|) > r + \lambda - \|z\| + 1.$$

Thus, if in addition $\|z\| \ge r/2$ and r is sufficiently large, then by (5.8), $\|Z + z\| - \|z\| > r + \lambda - \|z\|$ and the claim is proved.

We now turn to the proof of (1.15) $\Rightarrow$ (1.16) and (1.17) $\Rightarrow$ (1.18). We proceed as far as possible assuming only that $\|Z\| \in WL^{1+p}$ and $EZ \ne 0$. That (A) holds follows immediately from Proposition 4.1. Let $\beta = 1 - (p^2/(p+1)^2)$. For simplicity we will assume that coordinates in $\mathbb{R}^d$ have been chosen so that $z = (x, y)$, $x \in \mathbb{R}$, $y \in \mathbb{R}^{d-1}$ with $\hat{\mu} = (1, 0)$. We will often write $|x|$ for $\|(x, 0)\|$ and $\|y\|$ for $\|(0, y)\|$. Set

$$\Delta_r = B(0; r/2) \cup \{(x, y) \in B(0; r): x > 0, \|y\| < r^\beta\}.$$

Observe that if $\|\ \|$ were the Euclidean norm, then for every integer n we would have $\|z - n\mu\| \ge \|y\|$. However, since all norms on $\mathbb{R}^d$ are comparable, it follows that, for some C depending only on the norm, $\|y\| \le C(\|z - n\mu\|)$ for every n. We claim that, for r sufficiently large and any $n \ge 0$,

$$\|z\| \le r, \qquad \|z - n\mu\| < \frac{r^\beta}{C} \quad \Rightarrow \quad z \in \Delta_r. \tag{5.10}$$

From the discussion above we only need consider the case $x \le 0$. By the triangle inequality,

$$|x - n| \le \|z - n\mu\| + \|y\| \le (1 + C^{-1}) r^\beta.$$

Combined with $x \le 0$, this forces $|x| \le (1 + C^{-1}) r^\beta$. Hence

$$\|z\| \le |x| + \|y\| \le (2 + C^{-1}) r^\beta \le \frac{r}{2}$$

for large r. Thus $z \in \Delta_r$ and the claim is proved.

LEMMA 5.5. *There is a constant c, depending only on the distribution, such that*

$$P_0(S_n \notin \Delta_r \text{ for some } n < T_r) \le c r^{-p/(p+1)}.$$

PROOF. First observe that for all $n \le 2r/\|\mu\|$ we have

$$\begin{aligned}\|nE(Z; \|Z\| \le r^\beta) - n\mu\| &= \|nE(Z; \|Z\| > r^\beta)\| \\ &\le \frac{2r^{1+\beta}}{\|\mu\|} P(\|Z\| > r^\beta) + \frac{2r}{\|\mu\|}\int_{r^\beta}^\infty P(\|Z\| > t)\, dt\end{aligned}$$

$$\leq c\frac{r^{1+\beta}}{r^{(1+p)\beta}} + cr\int_{r^\beta}^{\infty}\frac{dt}{t^{1+p}}$$
$$= o(r^\beta) \quad \text{as } r \to \infty.$$

Thus, using (5.10) with $z = S_n$, we have for large r,

$$\{S_n \notin \Delta_r \text{ some } n < T_r\}$$
$$\subset \left\{\left\|S_n - nE(Z; \|Z\| \leq r^\beta)\right\| > \frac{r^\beta}{2C} \text{ for some } n \leq \frac{2r}{\|\mu\|}\right\}.$$

However, with $\tilde{S}_n = \sum_{i=1}^n Z_i I(\|Z\| \leq r^\beta)$, by Doob's inequality and (2.5) we have

$$P\left(\left\|S_n - nE(Z; \|Z\| \leq r^\beta)\right\| > \frac{r^\beta}{2C} \text{ for some } n \leq \frac{2r}{\|\mu\|}\right)$$
$$\leq P\left(\|Z_i\| > r^\beta \text{ for some } n \leq \frac{2r}{\|\mu\|}\right)$$
$$+ P\left(\left\|\tilde{S}_n - nE(Z; \|Z\| \leq r^\beta)\right\| > \frac{r^\beta}{2C} \text{ for some } n \leq \frac{2r}{\|\mu\|}\right)$$
$$\leq \frac{2r}{\|\mu\|}G(r^\beta) + \frac{8C^2 r}{\|\mu\|}K(r^\beta) \leq \frac{cr}{\|\mu\|}Q(r^\beta)$$
$$= \frac{cr^{1-2\beta}}{\|\mu\|}\int_0^{r^\beta} 2uG(u)\,du \leq cr^{-p/(p+1)}. \qquad \square$$

LEMMA 5.6. *There are constants $A > 0$ and $r_0 > 0$ such that*

$$E_z\left(Z_{T_r}^*\right)^p \leq Ar^{p/(p+1)}, \qquad r \geq r_0,$$

for all $z \in B(0; r)$, where $Z_n^* = \max_{1 \leq i \leq n}\|Z_i\|$.

PROOF. By replacing r with $2r$ we may assume $z = 0$. Let $\varphi(r) = r^{1/(p+1)}$. Now

$$E\left(Z_{T_r}^*\right)^p = \int_0^{\varphi(r)} p\lambda^{p-1}P\left(Z_{T_r}^* > \lambda\right)d\lambda + \int_{\varphi(r)}^{\infty} p\lambda^{p-1}P\left(Z_{T_r}^* > \lambda\right)d\lambda$$
$$= \mathrm{I} + \mathrm{II}.$$

Clearly, $\mathrm{I} \leq \int_0^{\varphi(r)} p\lambda^{p-1}\,d\lambda = \varphi(r)^p$. Now, by Wald's identity and Corollary 4.2, there is r_0 so that, for $r \geq r_0$,

$$P\left(Z_{T_r}^* > \lambda\right) \leq E\left(\sum_{j=1}^{T_r} \mathrm{I}\left(\|Z_j\| > \lambda\right)\right)$$
$$= ET_r P(\|Z\| > \lambda) \leq \frac{2r}{\|\mu\|}P(\|Z\| > \lambda).$$

Thus, using $Z \in WL^{1+p}$, there is a constant c so that

$$\begin{aligned} \mathrm{II} &\le \frac{2r}{\|\mu\|} \int_{\varphi(r)}^{\infty} p\lambda^{p-1} P(\|Z\| > \lambda)\, d\lambda \\ &\le \frac{cr}{\|\mu\|} \int_{\varphi(r)}^{\infty} \lambda^{-2}\, d\lambda = \frac{cr}{\|\mu\| \varphi(r)} \\ &= \frac{cr^{p/(p+1)}}{\|\mu\|}, \qquad r > r_0. \end{aligned}$$

Combining the two estimates, we obtain the desired result. □

Now, by the strong Markov property at time $\sigma = \min\{n\colon S_n \notin \Delta_r\}$, we have

$$E\big(\|S_{T_r}\| - r\big)^p \le E_0\Big(E_{S_\sigma}(Z_{T_r}^*)^p;\, \sigma < T_r\Big) + E_0\Big(\big(\|S_{T_r}\| - r\big)^p;\, \sigma \ge T_r\Big).$$

Lemmas 5.5 and 5.6 show that the first term is bounded in r. For the second term, let $\Delta'_r = \{z \in \Delta_r\colon r/2 < \|z\| < r - 1\}$. Then by (2.4), since $\{\sigma \ge T_r\} \subset \{S_{T_r - 1} \in \Delta_r\}$,

$$\begin{aligned} &E\Big(\big(\|S_{T_r}\| - r\big)^p;\, \sigma \ge T_r\Big) \\ &\quad\le \int_{\Delta'_r} \int_0^r p\lambda^{p-1} P(\|Z + z\| > r + \lambda)\, d\lambda\, U_r(dz) \\ &\qquad + \int_{\{z\colon \|z\| \le r/2\}} \int_0^r p\lambda^{p-1} P(\|Z + z\| > r + \lambda)\, d\lambda\, U_r(dz) \\ &\qquad + \int_{\{z\colon r-1 \le \|z\| \le r\} \cap \Delta_r} \int_0^r p\lambda^{p-1} P(\|Z + z\| > r + \lambda)\, d\lambda\, U_r(dz) \\ &\qquad + \int_{\Delta_r} \int_r^{\infty} p\lambda^{p-1} P(\|Z + z\| > r + \lambda)\, d\lambda\, U_r(dz). \end{aligned}$$

The last three terms on the right-hand side may each be bounded by relatively simple arguments. Using the triangle inequality and Corollary 4.2, the second term is bounded by

$$cr \int_0^r p\lambda^{p-1} P\left(\|Z\| > \frac{r}{2} + \lambda\right) d\lambda \le cr^{p+1} P\left(\|Z\| > \frac{r}{2}\right),$$

which is $O(1)$ since $\|Z\| \in WL^{p+1}$. The third term is bounded by $A\|Z\|_p^p$, where A denotes the expected occupation time of the region $\{z\colon \|z\| \in [r-1, r]\} \cap \Delta_r$. Since for any $0 < \beta < 1$ this region is contained in $\Gamma(\hat{\mu}, \beta)$ once r is large enough, it follows immediately from (4.3) that A is bounded independent of r. Finally, by the triangle inequality, $Z \in WL^{p+1}$ and Corollary 4.2 the fourth term is bounded by

$$cr\int_r^\infty \lambda^{p-1}P(\|Z\| > \lambda)\,d\lambda \le cr\int_r^\infty \lambda^{-2}\,d\lambda = c.$$

So far we have not used the full strength of the hypotheses. To complete the proof of Theorem 1.6 we now assume $Z \in L^{1+p}$. Then the first term is bounded by

$$\begin{aligned}
&\int_{\Delta'_r}\int_0^r p\lambda^{p-1}P(\|Z\| > r + \lambda - \|z\|)\,d\lambda\,U_r(dz)\\
&\quad = \int_{\Delta'_r}\int_0^r p\lambda^{p-1}\int_{u > r+\lambda-\|z\|} P(\|Z\| \in du)\,d\lambda\,U_r(dz)\\
&\quad = \int_{\Delta'_r}\int_{u > r-\|z\|}\int_0^{[u-(r-\|z\|)]\wedge r} p\lambda^{p-1}\,d\lambda\,P(\|Z\| \in du)U_r(dz)\\
&\quad \le \int_{\Delta'_r}\int_{u > r-\|z\|} u^p P(\|Z\| \in du)U_r(dz) \qquad (5.11)\\
&\quad = \int_0^r u^p \int_{\Delta'_r \cap \{\|z\| \in (r-u, r]\}} U_r(dz)P(\|Z\| \in du)\\
&\qquad + \int_{u>r} u^p \int_{\Delta'_r} U_r(dz)P(\|Z\| \in du)\\
&\quad \le c\|Z\|_{1+p}^{1+p}
\end{aligned}$$

by Corollary 4.2 and (4.3), where we have used the fact that for any $0 < \beta < 1$ we have $\Delta'_r \subseteq \Gamma(\hat{\mu}, \beta)$ once r is large enough. (Note also $\Delta'_r \cap \{z\colon \|z\| \in [r-u, r]\} = \varnothing$ if $u < 1$.) The proof of Theorem 1.6 is now complete.

To estimate the first term under the hypotheses of Theorem 1.5, we write

$$\begin{aligned}
\{\|Z + z\| > r + \lambda\} &= \{\|Z + z\| > r + \lambda,\ l^*(Z) \ge 0\}\\
&\quad \cup \{\|Z + z\| > r + \lambda,\ l^*(Z) < 0\}\\
&= \Omega_1 \cup \Omega_2.
\end{aligned}$$

Putting $Z = X + Y$ with $X = l^*(Z)\nu/l^*(\nu)$ as before, we have

$$\|Z + z\| - \|z\| = \|X + Y + z\| - \|z\| \le \|X\| + (\|Y + z\| - \|z\|).$$

Thus

$$\Omega_1 \subset \left\{\frac{l^*(Z)}{l^*(\nu)} > \frac{r + \lambda - \|z\|}{2}\right\} \cup \left\{\|Y + z\| - \|z\| > \frac{r + \lambda - \|z\|}{2}\right\}.$$

This leads to

$$
\begin{aligned}
&\int_{\Delta'_r}\int_0^r p\lambda^{p-1}P(\|Z+z\|>r+\lambda)\,d\lambda\,U_r(dz)\\
&\quad\le \int_{\Delta'_r}\int_0^r p\lambda^{p-1}P\left(\frac{l^*(Z)}{l^*(\nu)}>\frac{r+\lambda-\|z\|}{2}\right)d\lambda\,U_r(dz)\\
&\qquad+\int_{\Delta'_r}\int_0^r p\lambda^{p-1}P\left(\|Y+z\|-\|z\|>\frac{r+\lambda-\|z\|}{2}\right)d\lambda\,U_r(dz)\\
&\qquad+\int_{\Delta'_r}\int_0^r p\lambda^{p-1}P\bigg(\|Z+z\|-\|z\|>r+\lambda-\|z\|,\\
&\qquad\qquad\qquad l^*(Z)<0,\|Z\|\le\frac{1}{2}\|z\|\bigg)\,d\lambda\,U_r(dz)\\
&\qquad+\int_{\Delta'_r}\int_0^r p\lambda^{p-1}P\left(\|Z\|\ge\frac{1}{2}\|z\|\right)d\lambda\,U_r(dz).
\end{aligned}
\tag{5.12}
$$

The last term is easily seen to be $O(1)$ as $r\to\infty$ by using Corollary 4.2 and $Z\in WL^{1+p}$. To estimate the first term, proceed exactly as (5.11) with $\|Z\|$ replaced by $2l^*(Z)/l^*(\nu)$ to obtain a bound of $c\|l^*(Z)_+\|_{p+1}^{p+1}$.

It remains to handle the second and third terms in (5.12). For this we need the following two lemmas.

LEMMA 5.7. *If* (1.14) *holds, there is a constant c such that, for all t sufficiently small,*

$$\delta_*^{-1}(t)\le c\left[\log\left(\frac{1}{t}\right)\right]^{-(1/(p+1))}.$$

PROOF. By [10], Proposition 1.e.6,

$$\delta_*(\varepsilon)\ge\sup_r\left\{\frac{r\varepsilon}{2}-\rho(r)\right\},\qquad 0<\varepsilon<2.$$

It follows from this that $\delta_*^{-1}(t)\le 2\inf_{r>0}[(t+r)/(\rho^{-1}(r))]$. However,

$$
\begin{aligned}
&\inf_{r>0}\left(\log\left(\frac{1}{t}\right)\left[\frac{t+r}{\rho^{-1}(r)}\right]^{1+p}\right)\\
&\quad=\left(\int_t^1\frac{du}{u}\right)\inf_{u>0}\left[\frac{t+u}{\rho^{-1}(u)}\right]^{1+p}\\
&\quad\le\int_t^1\left[\frac{t+u}{\rho^{-1}(u)}\right]^{1+p}\frac{du}{u}\le 2^{1+p}\int_t^1\left(\frac{u}{\rho^{-1}(u)}\right)^{1+p}\frac{du}{u}.
\end{aligned}
$$

By (1.14) the last expression is bounded as $t \to 0$, and the proof is complete. □

In the next result, only the upper bound (with $\hat{\Delta} = \Delta'_r$) will be used in the remainder of the proof of Theorem 1.5. The lower bound will be needed in Examples 5.10 and 5.11.

LEMMA 5.8. *Fix $r > 0$ and $0 < u_0 \le u_1 \le r/2$. Let $\hat{\Delta} \subset B(0;r) \setminus B(0;r/2)$ and $H(u)$, $0 < u \le r/2$, be a nonnegative, continuous, nonincreasing function. If*

$$U_r\big(\hat{\Delta} \cap \{\|z\| \in [r-u, r]\}\big) \le c_0 u, \tag{5.13}$$

for all $u_0 \le u \le u_1$, then

$$\begin{aligned}&\int_{\hat{\Delta} \cap \{\|z\| \in [r-u_1, r-u_0]\}} H(r - \|z\|) U_r(dz)\\ &\quad \le c_0 \int_{u_0}^{u_1} H(u)\,du + ET_r H\left(\frac{r}{2}\right) + c_0 u_0 H(u_0).\end{aligned} \tag{5.14}$$

If

$$U_r\big(\hat{\Delta} \cap \{\|z\| \in [r-u, r]\}\big) \ge c_1 u, \tag{5.15}$$

for all $u_0 \le u \le u_1$, then

$$\int_{\hat{\Delta}} H_r(r - \|z\|) U_r(dz) \ge c_1 \int_{u_0}^{u_1} H(u)\,du - c_1 u_1 H(u_1). \tag{5.16}$$

PROOF. We have, for any $\Delta \subset B(0;r) \setminus B(0;r/2)$,

$$\begin{aligned}&\int_{\Delta} H(r - \|z\|) U_r(dz)\\ &\quad = \int_{\Delta}\left(\int_{r-\|z\|}^{r/2} - dH(u) + H\left(\frac{r}{2}\right)\right) U_r(dz)\\ &\quad \le -\int_0^{r/2}\left(\int_{\Delta \cap \{\|z\| \in [r-u, r]\}} U_r(dz)\right) dH(u) + ET_r H\left(\frac{r}{2}\right).\end{aligned} \tag{5.17}$$

If we now let $\Delta = \hat{\Delta} \cap \{\|z\| \in [r-u_1, r-u_0]\}$, then by (5.13) the first term is bounded by

$$-c_0\int_{u_0}^{r/2}(u \wedge u_1)\,dH(u)$$
$$= c_0\int_{u_0}^{u_1}H(u)\,du + c_0(u_0H(u_0) - u_1H(u_1)) + c_0u_1\left(H(u_1) - H\left(\frac{r}{2}\right)\right)$$

after integrating by parts. This proves (5.14).

For the lower bound we let $\Delta = \hat{\Delta}$ in the equality in (5.17) and use (5.15) to obtain

$$\begin{aligned}\int_{\hat{\Delta}}H(r - \|z\|)U_r(dz) &\geq -\int_{u_0}^{u_1}\left(\int_{\hat{\Delta}\cap\{\|z\|\in[r-u,r]\}}U_r(dz)\right)dH(u)\\ &\geq -c_1\int_{u_0}^{u_1}u\,dH(u)\\ &= c_1\int_{u_0}^{u_1}H(u)\,du + c_1(u_0H(u_0) - u_1H(u_1)),\end{aligned}$$

which proves (5.16). □

Observe that if r is sufficiently large, (5.13) holds with $\hat{\Delta} = \Delta'_r$, $u_0 = 1$ and $v_1 = r/2$. This follows from (4.3) since $\Delta'_r \subset \Gamma(\hat{\mu}, \beta)$ for any $0 < \beta < 1$ if r is sufficiently large.

By the triangle inequality, for any $z = (x, y)$ having $x \geq 0$,

$$\begin{aligned}\left\|\hat{\mu} - \frac{z}{\|z\|}\right\| &= \frac{\|(\|z\| - x, 0) - (0, y)\|}{\|z\|}\\ &\leq \frac{|\|z\| - x|}{\|z\|} + \frac{\|(0, y)\|}{\|z\|}\\ &\leq 2\frac{\|y\|}{\|z\|},\end{aligned}$$

since $|\|z\| - x| = |\|z\| - |x\|| \leq \|y\|$. Thus if $z \in \Delta'_r$, then $\frac{1}{2}\|\hat{\mu} - z/\|z\|\| \leq \|y\|/\|z\| \leq 2r^{\beta-1}$. Hence, by Corollary 5.4, if $z \in \Delta'_r$ and r is sufficiently large that $\delta_*^{-1}(2r^{\beta-1})$ exists, then

$$\begin{aligned}&P\left(\|Y + z\| - \|z\| > \frac{r - \|z\| + \lambda}{2}\right)\\ &\quad\leq P\left(\|Y\| > \frac{\|z\|}{2}\rho^{-1}\left(\frac{r - \|z\| + \lambda}{8\|z\|}\right)\right) + P\left(\|Y\|\delta_*^{-1}\left(\frac{\|y\|}{\|z\|}\right) > \frac{r - \|z\| + \lambda}{4}\right)\\ &\quad\leq P\left(\|Y\| > \frac{r}{4}\rho^{-1}\left(\frac{r - \|z\| + \lambda}{8r}\right)\right) + P\left(\|Y\|\delta_*^{-1}\left(\frac{\|y\|}{\|z\|}\right) > \frac{r - \|z\| + \lambda}{4}\right).\end{aligned}$$

Note that $\|X\| = \|l^*(Z)\nu/l^*(\nu)\| \le \|Z\|/l^*(\nu)$. Thus $X \in WL^{1+p}$. Since $Y = Z - X$ and WL^{1+p} is a linear space, we may also conclude that $Y \in WL^{p+1}$. Using this fact, substitution of the above inequality into the second term in (5.12) yields two terms. The first may be written as

$$c\int_{\Delta'_r} H_r(r - \|z\|)U_r(dz), \tag{5.18}$$

where

$$H_r(u) = \int_u^{r+u} \frac{p(v-u)^{p-1}}{\rho^{-1}(v/8r)^{p+1}}\frac{dv}{r^{p+1}}.$$

To estimate (5.18) we use Lemma 5.8 with $\hat{\Delta} = \Delta'_r$, $u_0 = 1$ and $u_1 = r/2$. As mentioned after Lemma 5.8, (5.13) holds in this case. Next it is easy to check that H_r is nonincreasing and $rH_r(r/2) = O(1)$ as $r \to \infty$. Furthermore,

$$\begin{aligned} H_r(1) &= \int_0^r \frac{pv^{p-1}}{\rho^{-1}((v+1)/8r)^{p+1}}\frac{dv}{r^{p+1}} \\ &\le \int_0^1 \frac{pv^{p-1}}{\rho^{-1}(1/8r)^{p+1}}\frac{dv}{r^{p+1}} + \int_1^r \frac{pv^{p-1}}{\rho^{-1}(v/8r)^{p+1}}\frac{dv}{r^{p+1}} \\ &= \frac{1}{(r\rho^{-1}(1/8r))^{p+1}} + c\int_{1/8r}^{1/8}\left(\frac{u}{\rho^{-1}(u)}\right)^{p+1}\frac{du}{u^2 r}. \end{aligned} \tag{5.19}$$

Now as $r \to \infty$, the first term is $o(1)$ since $\rho(u)/u \to 0$, and the second term is bounded by (1.14). Thus by (5.14),

$$\begin{aligned} \int_{\Delta'_r} H_r(r - \|z\|)U_r(dz) &\le c\int_1^{r/2} du \int_u^{u+r} \frac{p(v-u)^{p-1}}{\rho^{-1}(v/8r)^{p+1}}\frac{dv}{r^{p+1}} + O(1) \\ &\le c\int_1^{r/2} dv \int_1^{v} \frac{p(v-u)^{p-1}}{\rho^{-1}(v/8r)^{p+1}}\frac{du}{r^{p+1}} \\ &\quad + c\int_{r/2}^{3r/2} dv \int_1^{r/2} \frac{p(v-u)^{p-1}}{\rho^{-1}(v/8r)^{p+1}}\frac{du}{r^{p+1}} + O(1) \\ &\le c\int_1^{3r/2} \frac{v^p}{\rho^{-1}(v/8r)^{p+1}}\frac{dv}{r^{p+1}} + O(1) \\ &= O(1), \end{aligned}$$

by (1.14). Hence (5.18) is bounded. To handle the other term, first recall that for $z \in \Delta'_r$ we have $\|y\|/\|z\| \le 2r^{\beta-1}$. Thus for large r,

$$\begin{aligned}
&P\left(\|Y\|\delta_*^{-1}\left(\frac{\|y\|}{\|z\|}\right) > \frac{r-\|z\|+\lambda}{4}\right)\\
&\quad\le \int_{\Delta'_r}\int_0^r p\lambda^{p-1}P\left(\|Y\| > \frac{r-\|z\|+\lambda}{4\delta_*^{-1}(2r^{\beta-1})}\right)d\lambda\, U_r(dz)\\
&\quad\le c\int_{\Delta'_r}\int_0^r p\lambda^{p-1}\left(\frac{\delta_*^{-1}(2r^{\beta-1})}{r-\|z\|+\lambda}\right)^{p+1} d\lambda\, U_r(dz)\\
&\quad\le c\left[\delta_*^{-1}(2r^{\beta-1})\right]^{p+1}\int_{\Delta'_r}\left(\int_0^{r-\|z\|}\frac{p\lambda^{p-1}\,d\lambda}{(r-\|z\|)^{p+1}} + \int_{r-\|z\|}^r \frac{p\lambda^{p-1}\,d\lambda}{\lambda^{p+1}}\right)U_r(dz)\\
&\quad\le c\left[\delta_*^{-1}(2r^{\beta-1})\right]^{p+1}\int_{\Delta'_r}\frac{1}{r-\|z\|}U_r(dz)\\
&\quad\le c(\log r)\left[\delta_*^{-1}(2r^{\beta-1})\right]^{p+1},
\end{aligned}$$

by Lemma 5.8. This last expression is $O(1)$ as $r \to \infty$ by Lemma 5.7. Finally, the treatment of the third term in (5.12) is essentially the same as the second. We have, by Corollary 5.4,

$$\begin{aligned}
&P\left(\|Z+z\| - \|z\| > r - \|z\| + \lambda,\ l^*(Z) < 0,\ \|Z\| \le \frac{1}{2}\|z\|\right)b\\
&\quad\le P\left(\|Z\| > \frac{\|z\|}{4}\rho^{-1}\left(\frac{r-\|z\|+\lambda}{4\|z\|}\right)\right)\\
&\qquad + P\left(\|Z\|\delta_*^{-1}\left(\frac{\|y\|}{\|z\|}\right) > \frac{r-\|z\|+\lambda}{2}\right).
\end{aligned}$$

The rest of the argument is the same with Y replaced by Z. □

REMARK 5.9. Assume $EZ = \mu \neq 0$. If two norms $\|\ \|$ and $|\ |$ agree in some open neighborhood V of $\hat{\mu}$ (i.e., the two unit balls coincide in some neighborhood of $\hat{\mu}$), then (1.16) holds for the norm $\|\ \|$ if and only if it holds for $|\ |$. This does not require that (1.14) hold. To see this, first observe that (A) holds for both norms by Proposition 4.1. Next recall that $\Delta_r = B(0; r/2) \cup \{(x, y) \in B(0; r)\colon x > 0, \|y\| < r^\beta\}$. Thus for some $\varepsilon > 0$ and large enough r we have $z \in \Delta_r$, $\|z\| > (1-\varepsilon)r$ and $\|Z\| < \varepsilon r$ imply $\|Z + z\| = |Z + z|$.

Now assume (1.16) holds for $|\ |$. Then, by Theorem 1.6, $|Z| \in WL^{1+p}$. Since all norms on $\mathbb{R}^d$ are equivalent this means $\|Z\| \in WL^{1+p}$. Let $\sigma = \min\{n\colon S_n \notin \Delta_r\}$. Then

$$\begin{aligned}
E\left(\|S_{T_r}\| - r\right)^p &= E\left(\left(\|S_{T_r}\| - r\right)^p;\ \sigma < T_r\right)\\
&\quad + E\left(\left(\|S_{T_r}\| - r\right)^p;\ \sigma \ge T_r,\ \|X_{T_r}\| \ge \varepsilon r\right)\\
&\quad + E\left(\left(\|S_{T_r}\| - r\right)^p;\ \sigma \ge T_r,\ \|X_{T_r}\| < \varepsilon r\right).
\end{aligned}$$

It follows from the strong Markov property and Lemmas 5.5 and 5.6 that the first term is bounded independently of r. Since $\{\sigma \geq T_r\} \subset \{S_{T_r - 1} \in \Delta_r\}$, it follows from Lemma 2.2 and Corollary 4.2 that the second term is bounded by

$$\int_{\Delta_r}\int_0^\infty p\lambda^{p-1}P(\|Z+z\| > r+\lambda,\ \|Z\| \geq \varepsilon r)\,d\lambda\,U_r(dz)$$

$$\leq cr^{p+1}P(\|Z\| \geq \varepsilon r) + cr\int_r^\infty p\lambda^{p-1}P(\|Z\| > \lambda)\,d\lambda,$$

which is bounded since $Z \in WL^{p+1}$. Similarly the third term is bounded by

$$\begin{aligned}&\int_{\Delta_r}\int_0^\infty p\lambda^{p-1}P(\|Z+z\| > r+\lambda,\ \|Z\| < \varepsilon r)\,d\lambda\,U_r(dz)\\ &\qquad\leq \int_{\Delta_r}\int_0^\infty p\lambda^{p-1}P(|Z+z| > r+\lambda)\,d\lambda\,U_r(dz)\\ &\qquad\leq E\big(|S_{T_r}| - r\big)^p.\end{aligned}$$

This shows that (1.16) holds for $\|\ \|$. Since the roles of the norms can be reversed, the implication in the other direction follows immediately.

As an example of the usefulness of this remark, observe that if two norms agree in a neighborhood of $\hat{\mu}$, it is clear that (1.15) holds for one norm if and only if it holds for the other. Thus if one of the norms satisfies (1.14), then (1.15) and (1.16) are equivalent for both norms. For example, Theorem 1.5 does not apply to the l^∞- or the l^1-norm even when $\hat{\mu}$ is a smooth point. However, if we smooth the unit ball away from $\hat{\mu}$ so that (1.14) holds, and this can be done for these norms, then we see that (1.15) and (1.16) are equivalent in this case. If $\hat{\mu}$ is not a smooth point, it is easy to construct examples where (1.15) and (1.16) are not equivalent for the natural choice of ν.

EXAMPLE 5.10. We consider two-dimensional space with the l^1-norm. That is, if $z = (x, y) \in \mathbb{R}^2$, then $\|z\| = |x| + |y|$. The random walk is based upon random variables having the same distribution as $Z = (1, Y)$, where Y is real-valued and symmetric with $P(|Y| > r) = r^{-(p+1)}$, $r \geq 1$. Thus (1.15) is satisfied with $\nu = (1, 0)$. Let $A_r = \{z\colon r/2 < \|z\| \leq r-1,\ x > 0\}$. Note that, by symmetry, if $z \in A_r$,

$$\begin{aligned}P(\|Z+z\| > r+\lambda) &= P(|Y+y| + x + 1 > r+\lambda)\\ &\geq P(|Y+y| + x > r+\lambda,\ Yy \geq 0)\\ &= P(|Y| + |y| + x > r+\lambda,\ Yy \geq 0)\\ &\geq \tfrac{1}{2}P(|Y| > r+\lambda-\|z\|).\end{aligned}$$

Since $EZ = (1, 0)$ and $Z \in L^{1+q}$ for any $0 < q < p$, it follows from Theorem 1.6 that the overshoot is tight. Hence (4.4) holds with $h(v) \sim 1/v$. Then by

Lemma 2.2, for large r,

$$\begin{aligned}
E\big(\|S_{T_r}\| - r\big)^p &= \int_{\|z\|\le r}\int_0^\infty p\lambda^{p-1}P(\|Z+z\| > r+\lambda)\, d\lambda\, U_r(dz) \\
&\ge \frac{p}{2}\int_{A_r}\int_0^{r-\|z\|}\lambda^{p-1}P(|Y| > r+\lambda-\|z\|)\, d\lambda\, U_r(dz) \\
&\ge c\int_{A_r}\int_0^{r-\|z\|}\frac{\lambda^{p-1}}{(r-\|z\|)^{p+1}}\, d\lambda\, U_r(dz) \\
&\ge c\int_{A_r}\frac{U_r(dz)}{r-\|z\|} \\
&\ge c\log r
\end{aligned}$$

by Theorem 4.3 and Lemma 5.8. Hence the overshoot is not bounded in L^p.

The following example is considerably more delicate and shows that the localized version of the integrability condition (1.14) on the norm, as described above in Remark 5.9, is very nearly sharp. In particular, we show that for many norms where the integral in (1.14) barely diverges, there do exist random variables satisfying (1.15), but the overshoot is not bounded in L^p.

EXAMPLE 5.11. Let $d = 2$ and fix $0 < p < 1$ and $0 < \varepsilon < 1/2$. Assume that, for $1 - \varepsilon < x \le 1$, $\partial B(0;1)$ is symmetric about the x-axis and $\partial B(0;1) \cap \{y \ge 0\}$ is given by a function $y = f(x)$, where $f(1) = 0$ and

$$f(1-\xi) = \xi|\log\xi|^{1/(1+p)}L(\xi), \tag{5.20}$$

for $0 < \xi < \varepsilon$, where $L(\xi)\downarrow 0$ as $\xi\downarrow 0$ and L is sufficiently regular. For definiteness we will take $L(\xi) = |\log_n \xi|^{-\gamma}$, where $\log_n$ is the nth iterate of the $|\log|$ function, with $\gamma > 0$ and $n \ge 2$. It is clear that f is C^1 on $(1-\varepsilon, 1)$, $|f'|$ is bounded away from 0 and $f'(x)\downarrow -\infty$ as $x \uparrow 1$. Let $g(\xi) = |f'(1-\xi)|$. Note that g is decreasing on $(0,\varepsilon)$ by convexity of f. Further

$$\xi g(\xi) \approx |f(1-\xi)|, \qquad \xi \in (0,\varepsilon), \tag{5.21}$$

and

$$\xi^\eta g(\xi) \to 0 \quad \text{for all } \eta > 0. \tag{5.22}$$

Note that the form of the unit ball near $(1,0)$ implies that if $z = (x,y)$ with $\|z\| \le 1$ and $x > 1-\varepsilon$, then $\|z\| \ge x$. We will further assume that $\partial B(0;1)$ is smooth enough away from $(1,0)$ that ρ is determined by its behavior in a sufficiently small neighborhood of $(1,0)$. Some geometry then shows that the supremum defining $\rho(u)$ is comparable to the value obtained by setting $z = (1,0)$ and taking w tangent to $\partial B(0;1)$ at $(1,0)$. It is then almost immediate that $\rho^{-1}(u) \approx f(1-u)$. In particular, by choice of f, the integral in (1.14) just diverges. Here is a brief sketch of how to see that $\rho(u)$

can be computed as claimed. By assumption,

$$\rho(h) = \sup\{\tfrac{1}{2}(\|z + w\| + \|z - w\| - 2) \colon \|z\| = 1,\ z \in \partial B_\delta,\ \|w\| = h\},$$

for some $\delta > 0$ sufficiently small, where $\partial B_\delta = \partial B(0;1) \cap B((0,1);\delta)$. For each $z \in \partial B_\delta$, let e_z be a tangent vector to the unit ball at z with Euclidean length 1. If $w = \alpha z + \beta e_z$, set $\|w\|_z = |\alpha| + |\beta|$. It is then easy to check, by first comparing with the Euclidean norm, that there are constants c_1 and c_2 independent of $z \in \partial B_\delta$ such that

$$c_1\|w\|_z \le \|w\| \le c_2\|w\|_z. \tag{5.23}$$

Define

$$\rho_T(t) = \sup\{\tfrac{1}{2}(\|z + te_z\| + \|z - te_z\| - 2) \colon z \in \partial B_\delta\}.$$

We claim that, for some $c > 0$,

$$c^{-1}\rho_T(t) \le \rho(t) \le c\rho_T(t), \qquad 0 < t \le 1. \tag{5.24}$$

The first inequality follows easily from (5.23) and the doubling property of ρ. To prove the second, fix $z \in \partial B_\delta$ and any vector w. We can write $w/\|w\|_z = \alpha z + \beta e_z$, where $|\alpha| + |\beta| = 1$. Then

$$\begin{aligned}\left\|z + t\frac{w}{\|w\|_z}\right\| + \left\|z - t\frac{w}{\|w\|_z}\right\| - 2 &\le |\alpha|\left(\left\|z + \frac{\alpha}{|\alpha|}tz\right\| + \left\|z - \frac{\alpha}{|\alpha|}tz\right\| - 2\right)\\ &\quad + |\beta|\left(\left\|z + \frac{\beta}{|\beta|}te_z\right\| + \left\|z - \frac{\beta}{|\beta|}te_z\right\| - 2\right)\\ &= |\beta|\left(\left\|z + \frac{\beta}{|\beta|}te_z\right\| + \left\|z - \frac{\beta}{|\beta|}te_z\right\| - 2\right)\\ &\le \left(\left\|z + \frac{\beta}{|\beta|}te_z\right\| + \left\|z - \frac{\beta}{|\beta|}te_z\right\| - 2\right),\end{aligned}$$

where we interpret these equations in the obvious way if either $\alpha = 0$ or $\beta = 0$. The desired result now follows easily from (5.23) and the doubling property of ρ. With (5.24) at hand, it is now a fairly straightforward calculus exercise to see that $\rho(u)$ can be computed as claimed.

Let $Z = (X, Y)$, where $X \equiv 1$ and Y has a symmetric stable distribution of index $p + 1$. Thus $EZ = (1, 0)$. Recall that the tail of Y satisfies $P(|Y| > y) \sim cy^{-(1+p)}$; thus (1.15) is satisfied. We now proceed to show that the overshoot is not bounded in L^p.

Observe that $\partial B(0;r)$ is given by $y = rf(x/r)$ if $(1 - \varepsilon)r < x \le r$. Thus if $\lambda \le \varepsilon r$, $\|z\| \le r$, $y \ge 0$ and $(1 - \varepsilon^2)r < x \le r$, then

$$Y > (r + \lambda)f\left(\frac{x}{r + \lambda}\right) - \|z\|f\left(\frac{x}{\|z\|}\right) \quad \Rightarrow \quad \|Z + z\| > r + \lambda.$$

Now by (5.21), the mean-value theorem and monotonicity of g, we have

$$
\begin{aligned}
(r+\lambda) f\left(\frac{x}{r+\lambda}\right)-\|z\| f\left(\frac{x}{\|z\|}\right) &= (r+\lambda-\|z\|) f\left(\frac{x}{r+\lambda}\right) \\
&\quad + \|z\|\left(f\left(\frac{x}{r+\lambda}\right)-f\left(\frac{x}{\|z\|}\right)\right) \\
&\leq (r+\lambda-\|z\|)\left(c g\left(\frac{r+\lambda-x}{r+\lambda}\right)+g\left(\frac{\|z\|-x}{\|z\|}\right)\right) \\
&\leq c(r+\lambda-\|z\|) g\left(\frac{\|z\|-x}{\|z\|}\right).
\end{aligned}
$$

Thus with $\Delta_r^{\varepsilon}=\{z \in B(0;r): x>(1-\varepsilon^2) r,\ \|z\| \leq r-1\}$ we have

$$
\begin{aligned}
E\left(\|S_{T_r}\|-r\right)^p &\geq \int_{\Delta_r^{\varepsilon}} \int_0^{\varepsilon r} p \lambda^{p-1} P(\|Z+z\|>r+\lambda)\, d\lambda\, U_r(dz) \\
&\geq \int_{\Delta_r^{\varepsilon}} \int_0^{\varepsilon r} p \lambda^{p-1} P\left(Y>c(r+\lambda-\|z\|) g\left(\frac{\|z\|-x}{\|z\|}\right)\right) d\lambda\, U_r(dz) \\
&\geq c \int_{\Delta_r^{\varepsilon}} \int_0^{r-\|z\|} \frac{p \lambda^{p-1}\, d\lambda}{(r+\lambda-\|z\|)^{p+1} g((\|z\|-x)/\|z\|)^{p+1}} U_r(dz) \\
&\geq c \int_{\Delta_r^{\varepsilon}} \frac{U_r(dz)}{(r-\|z\|) g((\|z\|-x)/\|z\|)^{p+1}} \\
&\geq c \int_{\Delta_r^{*}} \frac{U_r(dz)}{(r-\|z\|) g((r-\|z\|)/r)^{p+1}},
\end{aligned}
$$

where $\Delta_r^{*}=\{z \in \Delta_r^{\varepsilon}: r-\|z\| \leq\|z\|-x\}$. Now, for $z \in \Delta_r^{\varepsilon}$,

$$
z \in \Delta_r^{*} \quad \Leftrightarrow \quad x \leq 2\|z\|-r \quad \Leftrightarrow \quad |y| \geq\|z\| f\left(\frac{2\|z\|-r}{\|z\|}\right).
$$

Since

$$
\begin{aligned}
\|z\| f\left(\frac{2\|z\|-r}{\|z\|}\right) &= \|z\| f\left(1-\frac{r-\|z\|}{\|z\|}\right) \\
&\leq c(r-\|z\|) g\left(\frac{r-\|z\|}{r}\right),
\end{aligned}
$$

by (5.21), it follows that $\Delta_r^{*} \supset\{z \in \Delta_r^{\varepsilon}: |y| \geq c(r-\|z\|) g((r-\|z\|)/r)\}$.

Claim. Fix $0<\alpha<1/(p+1)$. There exist constants c_1 and c_2 such that $U_r(\Delta_r^{*} \cap\{r-u \leq\|z\| \leq r\}) \geq c_1 u$ for all $c_2 \leq u \leq r^{\alpha}$ provided r is sufficiently large.

Since $Z \in L^{1+q}$ for any $q < p$, it follows from Theorem 1.6 that the overshoot is tight. Hence by Theorem 4.3, for some c_1 and c_2,

$$U_r(\Delta_r^\varepsilon \cap \{r - u < \|z\| \le r\}) \ge c_1 u$$

if $c_2 \le u \le r^\alpha$, provided r is sufficiently large. Thus it suffices to show

$$U_r((\Delta_r^\varepsilon \setminus \Delta_r^*) \cap \{r - u < \|z\| \le r\}) = o(u)$$

uniformly in $c_2 \le u \le r^\alpha$. Let $V_n = \sum_{i=1}^n Y_i$. Observe that if $z \in (\Delta_r^\varepsilon \setminus \Delta_r^*)$, then $x > 2\|z\| - r$ and $y < c(r - \|z\|)g((r - \|z\|)/r)$. Since $vg(v/r)$ is comparable to an increasing function by (5.21), it follows by the nature of the distribution of Z, that, in order for $S_k \in (\Delta_r^\varepsilon \setminus \Delta_r^*) \cap \{r - u < \|z\| \le r\}$, it must be that $r - 2u \le k \le r$ and $|V_k| \le cug(u/r)$. Thus by the scaling property of stable random walks, if $c_2 \le u \le r^\alpha$ and r is sufficiently large,

$$\begin{aligned} U_r((\Delta_r^\varepsilon \setminus \Delta_r^*) \cap \{r - u < \|z\| \le r\}) &\le \sum_{k=r-2u}^{r} P\left(|V_k| \le cug\left(\frac{u}{r}\right)\right) \\ &= \sum_{k=r-2u}^{r} P\left(|Y_1| \le \frac{cug(u/r)}{k^{1/(1+p)}}\right) \\ &\le 2uP\left(|Y_1| \le \frac{2cug(u/r)}{r^{1/(1+p)}}\right). \end{aligned}$$

Now Y has a bounded continuous density. Thus

$$\begin{aligned} 2uP\left(|Y_1| \le \frac{2cug(u/r)}{r^{1/(1+p)}}\right) &\le \frac{cu^2 g(u/r)}{r^{1/(1+p)}} \\ &\le cur^{\alpha-(1/(p+1))}g(r^{\alpha-1}) = o(u) \end{aligned}$$

uniformly in $c_2 \le u \le r^\alpha$ by (5.22).

We now use (5.16) with $\Delta = \Delta_r^*$, $u_0 = c_2$, $u_1 = r^\alpha$ and

$$H_r(u) = \frac{1}{ug(u/r)^{p+1}}.$$

Note that $r^\alpha H_r(r^\alpha) \to 0$ as $r \to \infty$. Thus by (5.16),

$$\begin{aligned} \int_{\Delta_r^*} \frac{U_r(dz)}{(r - \|z\|)g((r - \|z\|)/r)^{p+1}} &\ge c_1 \int_{c_2}^{r^\alpha} \frac{dv}{vg(v/r)^{p+1}} - o(1) \\ &= c_1 \int_{c_2/r}^{r^{\alpha-1}} \frac{dv}{vg(v)^{p+1}} - o(1) \to \infty \end{aligned}$$

as $r \to \infty$ by choice of f. Thus the overshoot is not bounded in L^p.

A similar example works if $p \ge 1$. Just choose Y to have tail $P(|Y| > r) = r^{-(p+1)}$ and use the central limit theorem instead of the scaling property of stable random walks.

6. Tightness. When (C) holds, a necessary and sufficient condition for tightness of the overshoot follows from Theorem 3, namely, $E\|Z\|^2 < \infty$ and $EZ = 0$. The analogous problem when (A) holds seems to be considerably more complicated because there are preferred directions, namely, the directions given by the subsequential limits of $M(r)/\|M(r)\|$. (For example, when $EZ \neq 0$ there is only one preferred direction which is given by the mean.) This is an important issue even in one dimension.

LEMMA 6.1. *Assume $P(S_{T_r} > 0) \to 1$. Then there is a constant $c_2 > 0$ such that*

$$\int_{r-v<z\le r} U_r(dz) \le \frac{c_2}{h(v)}, \qquad 0 \le v \le 2r,$$

provided r is sufficiently large. If, in addition, $(S_{T_r} - r)^+$ is tight, then there exist constants $c_1 > 0$ and $v_0 > 0$ such that

$$\int_{r-v<z\le r} U_r(dz) \ge \frac{c_1}{h(v)}, \qquad v_0 \le v \le 2r,$$

provided r is sufficiently large.

For $v \le r$ this is just a restatement of Theorem 4.3. The extension to $r < v \le 2r$ is obtained by combining this with (2.1) and (2.3).

PROOF OF THEOREM 1.7. Since $M(r)/h(r) \to 1$ implies $P(S_{T_r} > 0) \to 1$ by Theorem 1.1 of [5], the equivalence (1.20) and (1.21) is clear. Now, by Lemma 2.2,

$$\begin{aligned} P\big((S_{T_r} - r)^+ > \lambda\big) &= \int_{|z|\le r} P(Z + z > r + \lambda) U_r(dz) \\ &= \int_{-r}^{r} \int_{u>r+\lambda-z} dF(u) U_r(dz) \\ &= \int_{\lambda<u\le\lambda+2r} \int_{r+\lambda-u<z\le r} U_r(dz)\, dF(u) \\ &\quad + \int_{u>\lambda+2r} \int_{-r\le z\le r} U_r(dz)\, dF(u) \\ &= \mathrm{I} + \mathrm{II}. \end{aligned}$$

For the second term we have

$$\begin{aligned} \mathrm{II} &= G^+(\lambda + 2r) ET_r \\ &\le \frac{cG^+(r)}{h(r)} \to 0, \end{aligned}$$

as $r \to \infty$. For the first term, by Lemma 6.1, if r is sufficiently large,

$$\frac{c_1}{h(u-\lambda)} \le \int_{r+\lambda-u<z\le r} U_r(dz) \le \frac{c_2}{h(u-\lambda)},$$

where the second inequality holds assuming only $P(S_{T_r} > 0) \to 1$, while the first also requires tightness and $u \ge \lambda + v_0$. Hence letting $r \to \infty$ and then $\lambda \to \infty$, we see that $(S_{T_r} - r)^+$ is tight if and only if

$$\int_{\lambda+v_0}^{\infty} \frac{dF(u)}{h(u-\lambda)} \to 0, \tag{6.1}$$

as $\lambda \to \infty$. It thus remains to show (6.1) is equivalent to (1.22). First by (2.3), if $\lambda \ge v_0$,

$$\int_{\lambda+v_0}^{\infty} \frac{dF(u)}{h(u-\lambda)} \ge c\int_{2\lambda}^{\infty} \frac{dF(u)}{h(u)}.$$

Hence (6.1) implies (1.22). Conversely, assume (1.22). Let $\xi_\lambda \in [0, \lambda]$ satisfy $h(\xi_\lambda) \le 2\inf_{0\le v\le\lambda} h(v)$. Then

$$\begin{aligned}\int_{\lambda+v_0}^{\infty} \frac{dF(u)}{h(u-\lambda)} &\le \int_{\lambda}^{2\lambda} \frac{dF(u)}{h(u-\lambda)} + c\int_{2\lambda}^{\infty} \frac{dF(u)}{h(u)}\\ &\le 2\frac{G^+(\lambda)}{h(\xi_\lambda)} + c\int_{2\lambda}^{\infty} \frac{dF(u)}{h(u)}\\ &\le 2\frac{G^+(\xi_\lambda)}{h(\xi_\lambda)} + c\int_{2\lambda}^{\infty} \frac{dF(u)}{h(u)} \to 0\end{aligned}$$

since $\xi_\lambda \to \infty$ as $\lambda \to \infty$. □

Given the results in Section 5 on L^p-boundedness of the overshoot when $\|M(r)\|/h(r) \to 1$, one would perhaps expect the condition for tightness in this setting to be slightly weaker than $Z \in L^1$. However, this is not the case. As mentioned in the Introduction it is easy to check that if $d = 1$, then $\|M(r)\|/h(r) \to 1$ implies either $M(r)/h(r) \to 1$ or $M(r)/h(r) \to -1$. Clearly we may assume it is the former. Then by the previous result, tightness of the overshoot forces $\int^\infty (dF(u)/M(u)) < \infty$. Since $M(u) \le M^+(u)$, where $M^+(u) = u^{-1}E(Z^+; Z^+ \le u)$, we then have $\int^\infty (dF(u)/M^+(u)) < \infty$. By Lemma 2.2 of [4] applied to $(Z^+)^{1/2}$ (see below), this forces $Z^+ \in L^1$. If $Z^- \notin L^1$, then $S_n \to -\infty$ a.s. and hence $P(S_{T_r} > 0) \to 0$, which contradicts $M(r)/h(r) \to 1$. Thus $Z \in L^1$. [We would like to correct the statement of Lemma 2.2 in [4]; it should read $E((K(\|X\|))^{-1}; \|X\| > c) < \infty$ for some c if and only if $E\|X\|^2 < \infty$; this is essentially what is proved there.]

If $Z \in L^1$ and $EZ > 0$, then it is easy to check as in the proof of Lemma 2.1 that $M(u) \sim h(u) \sim EZ/u$ and hence $M(r)/h(r) \to 1$ and

$$\int^{\infty} \frac{dF(u)}{M(u)} < \infty,$$

that is, $(|S_{T_r}| - r)$ is tight. If $Z \in L^1$ and $EZ = 0$ it is still possible that $M(r)/h(r) \to 1$; however, in this case $\int^{\infty}(dF(u)/M(u))$ need not be finite.

EXAMPLE 6.2. Let Z have distribution given by

$$P(Z > u) = \frac{1}{u(\log u)^2} \quad \text{and} \quad P(Z < -u) = \frac{1}{u(\log u)^2} + \frac{1}{u(\log u)^{2+p}}$$

for large u, where $-1 < p < 1$, with the remaining mass an atom positioned so that $EZ = 0$. Then it is easy to check that

$$G(r) \sim K(r) \sim \frac{1}{r(\log r)^{2 \wedge (2+p)}},$$

$$M(r) \sim \frac{1}{(p+1)r(\log r)^{1+p}}.$$

Thus $M(r)/h(r) \to 1$. However, $\int^{\infty}(dF(u)/h(u)) < \infty$ only for $-1 < p < 0$.

REFERENCES

[1] BICKEL, P. J. and YAHAV, J. A. (1965). Renewal theory in the plane. *Ann. Math. Statist.* **36** 964–955.

[2] DONEY, R. A. (1966). The renewal theorem in higher dimensions. *Proc. London Math. Soc.* **16** 669–684.

[3] FELLER, W. (1971). *An Introduction to Probability Theory and Its Applications* **2**, 2nd ed. Wiley, New York.

[4] GRIFFIN, P. S. and MCCONNELL, T. R. (1992). On the position of a random walk at the time of first exit from a sphere. *Ann. Probab.* **20** 825–854.

[5] GRIFFIN, P. S. and MCCONNELL, T. R. (1994). Gambler's ruin and the first exit position of random walk from large spheres. *Ann. Probab.* **22** 1429–1472.

[6] GUT, A. (1988). *Stopped Random Walks.* Springer, New York.

[7] HOLMES, R. B. (1975). *Geometric Functional Analysis and Its Applications.* Springer, New York.

[8] KUELBS, J. (1973). The invariance principle for Banach space valued random variables. *J. Multivariate Anal.* **3** 161–172.

[9] LAI, T. L. (1976). Asymptotic moments of random walks with applications to ladder variables and renewal theory. *Ann. Probab.* **4** 51–66.

[10] LINDENSTRAUSS, J. and TSAFRIRI, L. (1979). *Classical Banach Spaces II.* Springer, New York.

[11] PRUITT, W. E. (1981). The growth of random walks and Lévy processes. *Ann. Probab.* **9** 948–956.

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