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GEOMETRIC AND SYMMETRY PROPERTIES OF A NONDEGENERATE DIFFUSION PROCESS

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A diffusion process with smooth and nondegenerate elliptic infinitesimal generator on a manifold M induces a Riemannian metric g on M. This paper discusses in detail different symmetry properties of such a diffusion by geometric methods. Partial differential equations associated with the generator are studied likewise. With an eye to modelling and applications to filtering, relationships between symmetries of deterministic systems and symmetries of diffusion processes are delineated. The incidence of a stochastic framework on the properties of an original deterministic system are then illustrated in different examples. The construction of a diffusion process with given symmetries is also addressed and resulting geometric problems are raised.

1. Introduction. When dealing with a diffusion process, one is generally given its infinitesimal generator in the first place. On the other hand, there are practical problems where one is first given a vector field describing the dynamics of a deterministic system and then has to model the latter as a diffusion process. For instance, faced with a dynamical estimation problem (tracking of a target, orbit determination,...), one often designs a stochastic framework to formulate the original question as a filtering problem where the state has to be recovered from at best partial observations. This is usually done by introducing stochastic differential equations to represent the evolution of the state: one adds a stochastic term to the natural deterministic dynamics of the state. Most of the time, this stochastic term is the increment of a Brownian motion weighted by a noise parameter because very little is known about the perturbations. Poorly known for poorly known, we raise the following issue. To what extent can this stochastic term be chosen in order to simplify the resulting filtering problem? In addressing this problem, we had to focus attention upon the symmetry properties of diffusion processes.

The set of planar motions which keep a geometric figure invariant form a group, the *symmetry group* of the figure (square, triangle, circle,...). It measures the degree of symmetry of the figure and may help reconstitute it from one of its parts. In the case of an algebraic equation, a symmetry group or *invariance group* consists of transformations of the base space which

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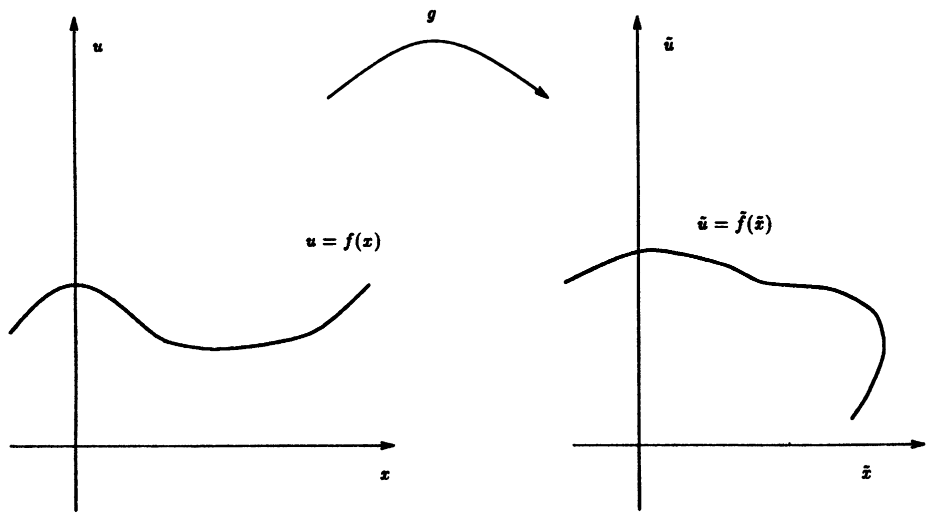

FIG. 1. *Action of a diffeomorphism on a function.*

permute solutions (this is one of the basic concepts of Galois theory). In some cases, knowledge of such a group may help solve the equation, as in the classical example of the "bisquared" equation for which $x^4 + bx^2 + c = 0$ if and only if $z = x^2$ and $z^2 + bz + c = 0$. In the case of ordinary differential equations, it was pointed out by S. Lie that all the special techniques for solving certain classes of ordinary differential equations (ODE's) have their origin in a general method related to the existence of a continuous invariance group for these ODE's (see the introduction of [16]). Basically, this (local) group consists of geometric transformations of the product space "independent variables" $\times$ "dependent variables," and its action on functions consists in transforming their graph as in Figure 1; these transformed graphs are graphs of solutions of the original ODE. Continuous groups present the advantage of being characterized by certain computational algorithms. It is indeed a crucial point in Lie theory that the nonlinear conditions expressing the invariance of a system of ODE's under a group of transformations may, in the case of continuous groups, be replaced by *linear* conditions, which are equivalent but simpler. These latter conditions reflect the infinitesimal invariance of the system under the action of the infinitesimal generators of the group. All this theory can be extended to partial differential equations (PDE's) [16, 17].

In the stochastic case, symmetry properties are also used in different contexts. The symmetries of the Laplacian on $\mathbb{R}^n$ are of great help for basic properties of Brownian motion. For a given Markov process, functions of this process that remain Markov can be captured via an analysis of the symmetry group as done by Glover and Mitro [10] (see also [8, 9]). The diffusion processes having the maximal symmetry properties are characterized by Liao [15]. In the context of PDE's, second-order linear equations have been studied by Ovsjannikov [17], especially the elliptic case with geometric tools, while Rosencrans [19, 20] contributed to the parabolic case (see also [3]).

In this paper, we focus on the case of diffusion processes with (time-independent) infinitesimal generator $\mathscr{L}$ on a manifold M of dimension n, when $\mathscr{L}$ is smooth (or analytic) and nondegenerate elliptic. We discuss systematically and in detail different symmetry properties of such a diffusion by geometric methods. Partial differential equations associated with the generator are studied likewise. Moreover, with an eye to modelling and applications to filtering, we try to delineate the relationships between symmetries of deterministic systems and symmetries of diffusion processes. The incidence of a stochastic framework on the properties of an original deterministic system is then illustrated in different examples.

In Section 2, we recall how a natural Riemannian metric g on M can be associated with $\mathscr{L}$ so that $\mathscr{L} = \frac{1}{2}\Delta_g + B + c$, where Δ_g is the Laplacian on the Riemannian manifold (M, g), B is a vector field on M and c is a smooth function on M.

In Section 3, we study how this geometric framework, namely, (M, g) and B, is modified when a time change and a diffeomorphism ϕ: $M \hookrightarrow M$ are applied to a diffusion (ξ) with generator $\mathscr{L}$. This allows us to characterize the diffusion processes which can be transformed into Brownian motions by diffeomorphism and time change.

Following [15], we recall that a diffeomorphism ϕ: $M \hookrightarrow M$ is said to be an invariance transformation of the process (ξ) if the process $(\phi(\xi))$, starting from x, is identical in law with (ξ), starting from $\phi(x)$, and it is said to be a symmetry transformation of the process (ξ) if the process $(\phi(\xi))$, starting from x, is identical in law with (ξ), starting from $\phi(x)$, after a time change. These transformations form groups, called invariance group and symmetry group, and we characterize them in terms of the geometry of the Riemannian manifold (M, g) in Section 4. We also extend these definitions and results to time-dependent transformations. The Lie-algebraic aspect is developed in Section 5.

Various partial differential equations are associated with the operator $\mathscr{L}$, such as $\mathscr{L}f = 0$, $\mathscr{L}^*f = 0$, $\partial_t f - \mathscr{L}f = 0$, $\partial_t f - \mathscr{L}^* f = 0$, $\partial_t f + \mathscr{L}f = 0$ and $\partial_t f + \mathscr{L}^* f = 0$. In Section 6, we study transformations of the extended spaces $M \times \mathbb{R}$ or $\mathbb{R} \times M \times \mathbb{R}$ which leave the set of solutions of the PDE's invariant. The analysis is done for general elliptic operators. That is, $\mathscr{L}1$ is not necessarily zero.

We close by discussing in Section 7 several applications with an emphasis on how the choice of a diffusion process to represent a dynamic system can affect the symmetries of the original system. Specific discussions on gradient vector fields and on practical filtering problems are also given. The geometric problems raised by the construction of a diffusion process with given symmetries are outlined in these last cases.

2. A Riemannian geometric framework. Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a filtered probability space, satisfying the usual conditions. Also, let the differential operator $\mathscr{L}$ be written in a given coordinate system $x_1, \dots, x_n$ on a manifold M of dimension n as

$$\mathscr{L} = \frac{1}{2} \sum_{i,j=1}^{n} a^{ij}(x) \frac{\partial^2}{\partial x_i \, \partial x_j} + \sum_{i=1}^{n} b^i(x) \frac{\partial}{\partial x_i} + c(x). \tag{1}$$

To speak of a diffusion process with infinitesimal generator $\mathscr{L}$, we follow [12], page 202.

Let $M' = M \cup \{\delta\}$, where δ is a terminal point. By convention, any smooth f on M extends to M' by $f(\delta) = \delta$ and any transformation ϕ from M to M extends from M' to M' by $\phi(\delta) = \delta$.

Let (ξ) be a family $(\xi_x)_{x \in M}$ of M'-valued, $\mathscr{F}$-adapted stochastic processes such that the following hold:

1. a.s., $\xi_x(0) = x$.
2. a.s., there exists $\zeta(\omega) \in [0, +\infty]$ such that: (a) $t \in [0, \zeta(\omega)) \mapsto \xi_x(t)$ is continuous; (b) $\xi_x(t) = \delta$ for $t \geq \zeta$.
3. For all smooth functions f on M,

$$M^f(t) = f(\xi_x(t)) - f(\xi_x(0)) - \int_0^t (\mathscr{L}f)(\xi_x(s)) \, ds$$

is a local martingale.

The process (ξ_x) is said to be an *$\mathscr{L}$-diffusion starting at x*. When the filtration is not specified, it is that generated by the diffusion.

We make the following assumptions (which are easily seen to be independent of the coordinate system).

ASSUMPTION 2.1. The operator $\mathscr{L}$ is assumed to be smooth [the functions $a^{ij}(\cdot)$, $b^i(\cdot)$ and $c(\cdot)$ are smooth] and nondegenerate elliptic [the symmetric matrix $(a^{ij}(x))_{i,j=1,\dots,n}$ is positive definite for all $x \in M$].

Thanks to this assumption, it is well known that we can introduce a Riemannian metric g on M as follows (see [17], [12] and [15]).

LEMMA 2.2. *If* $(a_{ij}(x))_{i,j=1,\dots,n}$ *denotes the inverse matrix of* $(a^{ij}(x))_{i,j=1,\dots,n}$, *then*

$$g = \sum_{i,j=1}^{n} a_{ij}(x) \, dx_i \, dx_j \tag{2}$$

defines a Riemannian metric g on M. Moreover, if Δ_g *is the Laplace–Beltrami operator* (*Laplacian*) *on the Riemannian manifold* (M, g), *then* $\mathscr{L}$ *can be written*

$$\mathscr{L} = \tfrac{1}{2}\Delta_g + B + c, \tag{3}$$

where B is a smooth vector field on M.

REMARK 2.1. The vector field B depends not only on $b^1, \dots, b^n$ in (1), but also on a^{ij}, $i, j = 1, \dots, n$.

DEFINITION 2.3. We shall denote by $g = \mathrm{met}(\mathscr{L})$ the Riemannian metric g on M associated with $\mathscr{L}$. The operator $\mathscr{L}$ is said to be an *intrinsic Laplacian* [15] if $\mathscr{L} = \frac{1}{2}\Delta_g$, that is, if $B = 0$ in (3).

REMARK 2.2. We recall that if g_0 is a Riemannian metric on M, the diffusion (ξ) is said to be a Brownian motion on (M, g_0) if $\mathscr{L} = \frac{1}{2}\Delta_{g_0}$ (see [6]). Note that, with this definition, a usual Brownian motion on $\mathbb{R}^n$ killed at a stopping time still is a Brownian motion.

It can be easily proved by writing Δ_g in local coordinates that there exists a metric g_0 such that the diffusion (ξ) is a Brownian motion on (M, g_0) if and only if $\mathscr{L}$ is an intrinsic Laplacian [and then necessarily $g_0 = \mathrm{met}(\mathscr{L})$].

The symmetry properties of the diffusion (ξ) will be shown to be related to certain geometric objects of the Riemannian manifold (M, g). This is why we review the necessary mathematical background (our references are [13] and [1]).

DEFINITION 2.4. Let T be an r-form [$(0, r)$ tensor field], let $X, X_1, \dots, X_r$ be vector fields and let f be a smooth function on M. The *Lie derivation* (of tensors) L_X is characterized by the relations

$$
\begin{aligned}
&L_X f = Xf = \langle df, X \rangle, \\
(4)\qquad &L_X X_1 = [X, X_1], \\
&L_X(T(X_1, \dots, X_r)) = (L_X T)(X_1, \dots, X_r) + \sum_{i=1}^{r} T(X_1, \dots, L_X X_i, \dots, X_r).
\end{aligned}
$$

The *inner product* i_X maps r-forms into $(r-1)$-forms. It is defined by

$$(5)\qquad (i_X T)(X_2, \dots, X_r) = T(X, X_2, \dots, X_r)$$

and is related to the Lie derivation by the Cartan formula

$$(6)\qquad L_X = d \circ i_X + i_X \circ d.$$

DEFINITION 2.5. Let X, Y and Z be vector fields [$(1, 0)$ tensor fields], let ω be a 1-form [$(0, 1)$ tensor field] and let f be a smooth function on M.

(i) D_z denotes the covariant derivation, and A_Z denotes the derivation $A_Z = L_Z - D_Z$.

(ii) A_Z induces a $(1, 1)$ tensor field by $A_Z X = -D_X Z$ whose adjoint A_Z^* is defined by $g(A_Z^* X, Y) = g(X, A_Z Y)$.

(iii) Ω_g is the *volume form* on (M, g).

(iv) The *divergence* of X is the function $\mathrm{div}_g\, X$, which satisfies $L_X \Omega_g = (\mathrm{div}_g\, X)\Omega_g$.

(v) $Z^\flat$ is the 1-form defined by duality by $Z^\flat(X) = g(Z, X)$.

(vi) $\omega^{\#}$ is the vector field defined by duality by $g(\omega^{\#}, X) = \omega(X)$.

(vii) The *gradient* of f is the vector field $\nabla_g f = (df)^{\#}$, such that $g(\nabla_g f, X) = Xf$.

(viii) The *Laplacian* (or Laplace–Beltrami operator) Δ_g is given by $\Delta_g f = \operatorname{div}_g(\nabla_g f)$.

Moreover, we will often refer to geometric identities stated in the Appendix.

3. Diffeomorphisms and time changes. Brownian motions. In this section, we study how the diffusion (ξ) is transformed into another diffusion by state diffeomorphism and time change. To this end, we need to introduce additional tools. We denote by $D(M)$ the *group of (global) diffeomorphisms of* M and by $\mathscr{X}(M)$ the *algebra of vector fields on* M.

DEFINITION 3.1. Let $\phi \in D(M)$, $X \in \mathscr{X}(M)$ and $f \in C^{\infty}(M)$. Let T be a $(0,r)$ tensor field, let $X_1, \dots, X_r$ be vector fields and let P be any differential operator on M. We define the following:

$$\phi^*(f) = f \circ \phi \quad \text{and} \quad \phi_*(f) = f \circ \phi^{-1}; \tag{7}$$

$$\phi_*(X) \cdot f = \phi_*(X \cdot \phi^*(f)) \quad \text{and} \quad \phi^*(X) \cdot f = \phi^*(X \cdot \phi_*(f)); \tag{8}$$

$$\phi_*(T)(X_1, \dots, X_r) = \phi_*(T(\phi^*(X_1), \dots, \phi^*(X_r))); \tag{9}$$

$$P^{\phi} \cdot f = \phi_*(P \cdot \phi^*(f)). \tag{10}$$

The following definitions and lemmas about time changes are well known [15, 18].

DEFINITION 3.2. Let α: $M \hookrightarrow]0, \infty[$ be a smooth function. Then $A_t = \int_0^t \alpha(\xi(s))\, ds$ is an additive functional of (ξ), having an inverse τ_t from $[0, A_\zeta)$ to $[0, \zeta)$ which is said to be the *time change with density* α for the process (ξ).

LEMMA 3.3. *Let* $\phi \in F(M)$ *and* α: $M \hookrightarrow]0, +\infty[$ *be a smooth function. If the time change* τ_t *has density* α, *then* $(\phi(\xi(\tau_t)))$ *defines a diffusion process with infinitesimal generator* $\mathscr{L}'$ *given by*

$$\mathscr{L}' = \left(\frac{1}{\alpha}\mathscr{L}\right)^{\phi} = \frac{1}{\phi_*(\alpha)}\mathscr{L}^{\phi} = \frac{1}{\alpha \circ \phi^{-1}}\mathscr{L}^{\phi}. \tag{11}$$

The following proposition describes how the geometric framework presented in the previous section is transformed under diffeomorphisms and time changes.

PROPOSITION 3.4. *Under the assumptions of Lemma* 3.3, *if* $g = \operatorname{met}(\mathscr{L})$ *and* $\mathscr{L}$ *is given by* (3), *then* $\mathscr{L}' = \frac{1}{2}\Delta_{g'} + B' + c'$, *where*

$$g' = \operatorname{met}(\mathscr{L}') = \phi_*(\alpha g) = \phi_*(\alpha)\phi_*(g) = (\alpha \circ \phi^{-1})\phi_*(g),$$

$$B' = \phi_*\left(\frac{1}{\alpha}B - \frac{1}{\alpha}\frac{n-2}{4}\nabla_g \log \alpha\right)$$

$$(12) \qquad = \phi_*\left(\frac{1}{\alpha}B\right) - \frac{n-2}{4}\nabla_{g'} \log(\alpha \circ \phi^{-1}),$$

$$c' = \phi_*\left(\frac{1}{\alpha}c\right) = \frac{c \circ \phi^{-1}}{\alpha \circ \phi^{-1}}.$$

PROOF. In Lemma 2.2, it is implicit that the coefficients a^{ij} are components of a contravariant tensor whose covariant components are a_{ij}. This implies that under the change of coordinates induced by ϕ, $\mathscr{L}$ becomes $\mathscr{L}^\phi$ and g becomes $\phi_*(g)$. Moreover, since $(\alpha(x)a_{ij}(x))_{i,j=1,\dots,n}$ is the inverse matrix of $(1/(\alpha(x))a^{ij}(x))_{i,j=1,\dots,n}$, then $\operatorname{met}((1/\alpha)\mathscr{L}) = \alpha \operatorname{met}(\mathscr{L})$. Combining both transformations, we get the expressions in (12) for g'.

Moreover, by (123), (127) and (120) in Lemma A.1 in the Appendix, we have

$$\Delta_{g'} = \Delta^\phi_{\alpha g} = \left(\frac{1}{\alpha}\Delta_g\right)^\phi + \frac{n-2}{2}\phi_*\left(\frac{1}{\alpha}\nabla_g(\log \alpha)\right)$$
$$= \left(\frac{1}{\alpha}\Delta_g\right)^\phi + \frac{n-2}{2}\nabla_{g'}\left(\log(\alpha \circ \phi^{-1})\right).$$

We also have $c' = \mathscr{L}'1$ and the expression for B' follows from

$$B' = \mathscr{L}' - \frac{1}{2}\Delta_{g'} - c' = \left(\frac{1}{2\alpha}\Delta_g\right)^\phi + \left(\frac{1}{\alpha}B\right)^\phi - \frac{1}{2}\Delta_{g'}. \qquad \square$$

This geometric decomposition of the generator $\mathscr{L}$ makes it possible to characterize the diffusion processes which can be transformed into Brownian motions by diffeomorphism and time change. The specific case of the dimension $n = 2$ can be noted here (and in the sequel).

PROPOSITION 3.5. *Let* $n \neq 2$. *The three following assertions are equivalent.*

(i) *There exists a metric* g_0 *and a time change* τ_t *with density* α *such that* $(\xi(\tau_t))$ *is a Brownian motion on* (M, g_0).

(ii) *There exists a metric* g_0, *a diffeomorphism* ϕ *and a time change* τ_t *with density* α *such that* $(\phi(\xi(\tau_t)))$ *is a Brownian motion on* (M, g_0).

(iii) *In* (3), $c = 0$ *and the vector field* B *is a gradient vector field* $B = \nabla_g \varphi$ *for the metric* $g = \operatorname{met}(\mathscr{L})$.

In this last case, if $B = \nabla_g \varphi$, *the time change density is given by* $\alpha = \exp(4/(n-2)\varphi)$.

PROOF. Clearly (i) implies (ii). If (ii) is satisfied and $\mathscr{L}'$ denotes the infinitesimal generator of the diffusion $\phi(\xi(\tau_t))$, we have, on the one hand,

$$\mathscr{L}' = \tfrac{1}{2}\Delta_{g_0}$$

and, on the other hand, by Proposition 3.4 and Lemma 3.3,

$$\mathscr{L}' = \frac{1}{2}\Delta_{g'} + \phi_*\left(\frac{1}{\alpha}B - \frac{n-2}{4\alpha}\nabla_g \log \alpha\right) + \phi_*(c)$$

$$\text{with } g' = \text{met}(\mathscr{L}') = \phi_*(\alpha\, \text{met}(\mathscr{L})) = \phi_*(\alpha g).$$

Then, necessarily, $g_0 = \text{met}(\mathscr{L}') = g'$ by Remark 2.2, $B - \frac{1}{4}(n-2)\nabla_g \log \alpha = 0$ and $c = 0$ so that (iii) is proven.

If (iii) is satisfied with $B = \nabla_g \varphi$, the time change density given by $\alpha = \exp(4/(n-2)\varphi)$ is such that the infinitesimal generator of the diffusion $(x(\tau_t))$ is $\mathscr{L}' = \Delta_{\alpha g}$ by (12). Hence, (i) is proven with the metric $g_0 = \alpha g$. □

When $M = \mathbb{R}^n$, we can characterize the diffusion processes which can be transformed into the usual Brownian motion up to a stopping time (see Remark 2.2) by diffeomorphism and time change.

PROPOSITION 3.6. *Let $M = \mathbb{R}^n$ with $n \neq 2$. There exists a diffeomorphism ϕ and a time change τ_t with density α such that the process $(\phi(\xi(\tau_t)))$ is a usual Brownian motion on $\mathbb{R}^n$ if and only if*:

(i) *In* (3), $c = 0$ *and the vector field B is a gradient vector field $B = \nabla_g \varphi$ for the metric* $g = \text{met}(\mathscr{L})$.

(ii) *There exists $\phi \in D(M)$ such that*

$$\phi_*(g) = 2\exp(-4/(n-2)\varphi \circ \phi^{-1}) g_{\mathbb{R}^n},$$

where $g_{\mathbb{R}^n}$ is the usual flat metric on $\mathbb{R}^n$.

In particular, the metric $g = \text{met}(\mathscr{L})$ *is necessarily globally conformally equivalent to the flat metric on $\mathbb{R}^n$.*

PROOF. With the notation of Proposition 3.4, $(\phi(\xi(\tau_t)))$ is a usual Brownian motion on $\mathbb{R}^n$ if and only if $\mathscr{L}' = \frac{1}{2}\Delta$ (where Δ is the Laplacian for the flat metric $g_{\mathbb{R}^n}$ on $\mathbb{R}^n$) if and only if

$$g_{\mathbb{R}^n} = \alpha \circ \phi^{-1}\phi_*(g) \quad \text{and} \quad 0 = B - \frac{n-2}{4}\nabla_g \log \alpha.$$

Denoting $\varphi = \frac{1}{4}(n-2)\log \alpha$, this completes the proof. □

REMARK 3.1. If a diffusion process on $\mathbb{R}^n$ can be transformed into the usual Brownian motion by diffeomorphism and time change, this implies necessary conditions on the curvature tensor of the metric $g = \text{met}(\mathscr{L})$. Since g must be globally conformally equivalent to the flat metric on $\mathbb{R}^n$, such conditions follow from geometric results which can be found in [7], page 152.

EXAMPLE 3.1. On $M = \mathbb{R}^n \setminus \{0\}$, consider the deterministic system

$$\dot{x} = \frac{x}{\|x\|^2} \tag{13}$$

whose trajectories are rays described from $\|x(0)\|$ to $+\infty$ [since $d\|x(t)\|^2 = 2\,dt$]. Let us add a small noise to (13) in the form

$$dx_t = \frac{x_t}{\|x_t\|^2}\,dt + \sqrt{\varepsilon}\,dv_t, \qquad x_0 = x, \tag{14}$$

where (v_t) is a Brownian motion on $\mathbb{R}^n$. Now, if we kill x_t once it hits 0, this defines a diffusion (ξ) with generator

$$\mathscr{L} = \frac{\varepsilon}{2}\Delta + \frac{x}{\|x\|^2} = \frac{\varepsilon}{2}\Delta + \nabla\log\|x\|. \tag{15}$$

Here, Δ and ∇ are the Laplacian and the gradient for the flat metric $g_{\mathbb{R}^n}$ on $\mathbb{R}^n$ and $x/\|x\|^2$ denotes the vector field whose action on a smooth function f is given by $(1/\|x\|^2)\langle df(x), x\rangle$. By (124) and (127), we also have

$$\mathscr{L} = \frac{1}{2}\Delta_g + \frac{1}{\varepsilon}\nabla_g\log\|x\|, \quad \text{where } g = \frac{1}{\varepsilon}g_{\mathbb{R}^n}. \tag{16}$$

Assume that $n \neq 2$ and let $\alpha(x) = \|x\|^{4/((n-2)\varepsilon)}$. By Proposition 3.5, (ξ) can be transformed into a Brownian motion on $(M, \alpha g_{\mathbb{R}^n})$ by the time change with density α.

For $n = 2$, analogous propositions exist and can be proved in the same way.

PROPOSITION 3.7. *Let $n = 2$. The three following assertions are equivalent.*

(i) *There exists a metric g_0 and a time change τ_t with density α such that $(\xi(\tau_t))$ is a Brownian motion on (M, g_0).*

(ii) *There exists a metric g_0, a diffeomorphism ϕ and a time change τ_t with density α such that $(\phi(\xi(\tau_t)))$ is a Brownian motion on (M, g_0).*

(iii) *The vector field B and the function c in* (3) *are both zero. That is, the diffusion (ξ) is a Brownian motion on (M, g), where $g = \mathrm{met}(\mathscr{L})$.*

If the diffusion process (ξ) is a Brownian motion, it may be noted that g is *locally* conformally flat by the existence theorem of isothermal coordinates [5]. Therefore, there exists a *local* diffeomorphism ϕ on an open subset W such that $\phi(\xi)$ is the restriction of a usual Brownian motion on $\phi(W)$.

For the *global* case, when $M = \mathbb{R}^2$, here is the characterization of the diffusion processes which can be transformed into a usual Brownian motion by diffeomorphism and time change.

PROPOSITION 3.8. *There exists a diffeomorphism ϕ and a time change τ_t with density α such that the process $(\phi(\xi(\tau_t)))$ is a usual Brownian motion on $\mathbb{R}^2$ if and only if*:

(i) *The vector field B and the function c in* (3) *are both zero.*
(ii) *There exists $\phi \in F(M)$ and $\beta \in C^\infty(M)$, $\beta > 0$, such that $\phi_*(g) = \beta g_{\mathbb{R}^2}$.*

In particular, the metric $g = \mathrm{met}(\mathscr{L})$ is necessarily globally conformally equivalent to the flat metric on $\mathbb{R}^2$ and the process (ξ) is a Brownian motion on $(M, \mathrm{met}(\mathscr{L}))$.

Actually, this result is not far reaching since it is shown in Example 4 in [15] that such global diffeomorphisms only form the group generated by Euclidean motions and dilatations on $\mathbb{R}^2$.

4. Invariance group and symmetry group. In this section, we give an extensive description in Riemannian geometric terms of the so-called invariance group and symmetry group of the diffusion (ξ). This extends to general diffusions the results of Liao [15] for Brownian motions (diffusions with intrinsic infinitesimal generator). What is more, the case of time-dependent transformations is treated.

4.1. *The time-independent case.* To begin with, we recall some properties and definitions which may be found in [15].

DEFINITION 4.1. The *invariance group* of (ξ) consists of diffeomorphisms $\phi \in D(M)$ such that the process $(\phi(\xi))$, starting from x, is identical in law with the process (ξ), starting from $\phi(x)$.

Since $(\phi(\xi))$ is a diffusion process with infinitesimal generator $\mathscr{L}^\phi$ (see Lemma 3.3), this invariance group coincides with

$$\mathrm{Inv}(\mathscr{L}) = \{\phi \in D(M) | \mathscr{L}^\phi = \mathscr{L}\}. \tag{17}$$

DEFINITION 4.2. The *symmetry group* of (ξ) consists of diffeomorphisms $\phi \in D(M)$ such that the process $(\phi(\xi))$, starting from x, is identical in law with the process (ξ), starting from $\phi(x)$, after a time change (with density).

Since $(\xi(\tau_t))$ is a diffusion process with infinitesimal generator $(1/\alpha)\mathscr{L}$ (see Lemma 3.3), the symmetry group of (ξ) coincides with

$$\mathrm{Sym}(\mathscr{L}) = \{\phi \in D(M) | \exists\, \beta \in C^\infty(M),\ \beta > 0,\ \mathscr{L}^\phi = \beta\mathscr{L}\}. \tag{18}$$

REMARK 4.1. We restrict ourselves to time changes with density although this hypothesis can result from topological assumptions [see [15], where Liao considers the case of a Lie transformation group contained in the symmetry group of (ξ)].

In the next proposition, we show how the symmetry properties of the diffusion (ξ) are related to the following geometric groups of the Riemannian manifold (M, g).

DEFINITION 4.3. We denote by $I_g(M) \subset D(M)$ the group of *isometries* of (M, g), namely,

$$I_g(M) = \{\phi \in D(M) | \phi_*(g) = g\}. \tag{19}$$

We denote by $H_g(M) \subset D(M)$ the group of *homothetic transformations* of (M, g), namely,

$$H_g(M) = \{\phi \in D(M) | \exists\, \lambda \in]0, +\infty[,\ \phi_*(g) = \lambda g\}. \tag{20}$$

We denote by $C_g(M) \subset D(M)$ the group of *conformal transformations* of (M, g), namely,

$$C_g(M) = \{\phi \in D(M) | \exists\, \beta \in C^\infty(M),\ \beta > 0,\ \phi_*(g) = \beta g\}. \tag{21}$$

PROPOSITION 4.4. *The invariance and symmetry groups of the diffusion* (ξ) *are related to these geometric groups as follows*:

$$\begin{aligned} \mathrm{Inv}(\mathscr{L}) &= \{\phi \in I_g(M) | \phi_*(B) = B \text{ and } \phi_*(c) = c\}, \\ \mathrm{Sym}(\mathscr{L}) &= \Big\{\phi \in C_g(M) | \phi_*(g) = \beta g \text{ and} \\ &\qquad \beta\phi_*(B) + \frac{n-2}{4} \nabla_g \log \beta = B \text{ and } \beta\phi_*(c) = c\Big\}. \end{aligned} \tag{22}$$

PROOF. By Proposition 3.4, it can be seen that $\mathrm{Inv}(\mathscr{L})$ consists of $\phi \in D(M)$ such that $\phi_*(g) = g$, $\phi_*(B) = B$ and $\phi_*(c) = c$.

On the other hand, $\mathrm{Sym}(\mathscr{L})$ consists of $\phi \in D(M)$ such that there exists $\alpha(x) > 0$ with $\phi_*(\alpha g) = g$, $\phi_*((1/\alpha)B) - \frac{1}{4}(n-2)\nabla_g \log(\alpha \circ \rho^{-1}) = B$ and $\phi_*((1/\alpha)c) = c$. Denoting $\beta = 1/\phi_*(\alpha)$, the second identity is thus proved. □

This last proposition makes it possible to describe $\mathrm{Inv}(\mathscr{L})$ and $\mathrm{Sym}(\mathscr{L})$ more precisely in the following cases.

EXAMPLE 4.1. If $\mathscr{L} = \frac{1}{2}\Delta_g$, that is, if $\mathscr{L}$ is an intrinsic Laplacian (see Definition 2.3), then

$$\begin{aligned} \mathrm{Inv}(\mathscr{L}) &= I_g(M), \\ \mathrm{Sym}(\mathscr{L}) &= C_g(M) \quad (\text{if } n = 2) \\ &= H_g(M) \quad (\text{if } n \neq 2). \end{aligned} \tag{23}$$

When M is compact or (M, g) is complete and nonflat, then $H_g(M) = I_g(M)$ (see [13]). In any of these cases, we can conclude as in [15] that $\mathrm{Sym}(\mathscr{L}) = \mathrm{Inv}(\mathscr{L})$ when $n \neq 2$.

EXAMPLE 4.2. If there exists $\varphi \in C^\infty(M)$ such that $\mathscr{L} = \frac{1}{2}\Delta_g + \nabla_g \varphi$, then

$$\begin{aligned} \mathrm{Inv}(\mathscr{L}) &= \{\phi \in I_g(M) | \phi_*(\varphi) = \varphi + \text{constant}\}, \\ \mathrm{Sym}(\mathscr{L}) &= \{\phi \in C_g(M) | \phi_*(g) = \beta g \text{ and} \\ &\qquad \phi_*(\varphi) + \tfrac{1}{4}(n-2)\log \beta = \varphi + \text{constant}\}. \end{aligned} \tag{24}$$

This is indeed a consequence of Proposition 4.4 and of the formula

$$\beta\phi_*(\nabla_g \varphi) = \beta\nabla_{\phi_*(g)}\phi_*(\varphi) = \beta\nabla_{\beta g}\phi_*(\varphi) = \nabla_g \phi_*(\varphi),$$

which follows from (120) and (124) in Lemma A.1.

4.2. *The time-dependent case.* We denote by $D_{\text{td}}(M)$ the *set of mappings* $\psi\colon \mathbb{R}\times M \hookrightarrow M$ such that, for all $t\in\mathbb{R}$, $\psi(t,\cdot)$ is a diffeomorphism of M. We also denote by $\mathscr{X}_{\text{td}}(M)$ the *algebra of time-dependent vector fields on M.*

DEFINITION 4.5. For $\psi \in D_{\text{td}}(M)$ and $f\in C^\infty(M)$, we define the following:

(i) The mapping $\bar{\psi}\in D(\mathbb{R}\times M)$ by

$$\bar{\psi}\colon (t,x)\mapsto (t,\psi(t,x)). \tag{25}$$

(ii) The function $\bar{f}\in C^\infty(\mathbb{R}\times M)$ by

$$\bar{f}\colon (t,x)\mapsto f(x). \tag{26}$$

(iii) The time-dependent infinitesimal generator $\mathscr{L}^\psi$ on M by

$$\mathscr{L}^\psi g(t,x) = \left(\frac{\partial}{\partial t} + \mathscr{L}\right)^{\bar{\psi}} \cdot \bar{g}(t,x), \qquad \forall\, g\in C^\infty(M). \tag{27}$$

DEFINITION 4.6. The mapping $\psi\in D_{\text{td}}(M)$ is said to be a *td-invariance transformation* of the process (ξ) if the process $(\psi(t,\xi(t)))$, with (ξ) starting from x, is identical in law with (ξ), starting from $\psi(0,x)$.

The set of td-invariance transformations coincides with

$$\text{Inv}_{\text{td}}(\mathscr{L}) = \{\psi\in D_{\text{td}}(M) | \mathscr{L}^\psi = \mathscr{L}\}. \tag{28}$$

DEFINITION 4.7. $\psi\in D_{\text{td}}(M)$ is said to be a *td-invariance transformation* of the process (ξ) if the process $(\psi(t,\xi(t)))$, with (ξ) starting from x, is identical in law with (ξ), starting from $\psi(0,x)$, after a time change (with density).

The set of td-invariance transformations coincides with

$$\text{Sym}_{\text{td}}(\mathscr{L}) = \{\psi\in D_{\text{td}}(M) | \exists\, \beta\in C^\infty(\mathbb{R}\times M),\ \beta>0,\ \mathscr{L}^\psi = \beta\mathscr{L}\}. \tag{29}$$

The sets $\text{Inv}_{\text{td}}(\mathscr{L})$ and $\text{Sym}_{\text{td}}(\mathscr{L})$ can be characterized by way of the following definition.

DEFINITION 4.8. For $\psi\in D_{\text{td}}(M)$ and $t\in\mathbb{R}$, we denote by ψ_t the diffeomorphism $\psi(t,\cdot)$ of M.

With this notation, we have

$$\mathscr{L}^\psi = \left(\frac{\partial}{\partial t} + \mathscr{L}\right)^{\psi_t} = \left(\frac{\partial}{\partial t}\right)^{\psi_t} + \mathscr{L}^{\psi_t}, \tag{30}$$

where $(\partial/\partial t)^{\psi_t}$ is a first-order operator identified with a time-dependent vector field. Thanks to this remark, the following proposition can be proved as Proposition 4.4 in the time-independent case.

PROPOSITION 4.9. *We have*

$$\mathrm{Inv}_{\mathrm{td}}(\mathscr{L}) = \Big\{ \omega \in D_{\mathrm{td}}(M) | \forall\, t,\ \psi_t \in I_g(M) \text{ and } \left(\frac{\partial}{\partial t}\right)^{\psi_t} + (\psi_t)_*(B) = B \text{ and } (\psi_t)_*(c) = c \Big\},$$

$$\text{(31)}\quad \mathrm{Sym}_{\mathrm{td}}(\mathscr{L}) = \Big\{ \psi \in D_{\mathrm{td}}(M) | \forall\, t,\ \psi_t \in C_g(M) \text{ and } (\psi_t)_*(g) = \beta_t g \text{ and } \left(\frac{\partial}{\partial t}\right)^{\psi_t} + \beta_t(\psi_t)_*(B) + \frac{n-2}{4}\nabla_g \log \beta_t \in B \text{ and } \beta_t(\psi_t)_*(c) = c \Big\}.$$

5. Invariance algebra and symmetry algebra. In this section, we focus on the infinitesimal generators of the groups defined above. The results of this section could thereby be obtained as corollaries of Propositions 4.4 and 4.9. However, we choose a more direct approach with the advantage of step-by-step introduction of some material which will also be useful in the next section. What is more, infinitesimal tools allow precise insights into the subject, especially in the time-dependent case.

5.1. *The time-independent case.* A vector field on M usually generates a *local* one-parameter group of transformations on M. On the other hand, the diffusion (ξ) has values in the whole manifold M (or rather on M'). This explains why, in the following definition, we shall only consider proper vector fields, which generate *global* one-parameter groups.

DEFINITION 5.1. The vector field X on M is said to be an *invariance infinitesimal transformation* of the process (ξ) if:

(i) X is a proper vector field;
(ii) the global flow Φ_s^X that X generates on M belongs to the invariance group of (ξ) for all s.

Thus, invariance infinitesimal transformations are the infinitesimal counterparts of the state transformations of the invariance group. They do not usually form a Lie algebra, because of the "proper" assumption, while the following sets do.

LEMMA 5.2. *The following set of smooth vector fields is a Lie algebra (for the Lie bracket) called the invariance algebra of* $\mathscr{L}$:

(32) $\mathscr{I}nv(\mathscr{L}) = \{X \in \mathscr{X}(M) | [\mathscr{L}, X] = 0\} = \{X \in \mathscr{X}(M) | \mathrm{ad}_{\mathscr{L}}(X) = 0\},$

where we recall that

$$\begin{aligned} \mathrm{ad}_P(X) &= [P, X] = PX - XP, \\ \mathrm{ad}_P^{k+1}(X) &= \mathrm{ad}_P(\mathrm{ad}_P^k(X)), \qquad \forall\, k \in \mathbb{N}, \end{aligned} \tag{33}$$

for any linear differential operator P.

PROOF. For any linear differential operator P, we have

$$\mathrm{ad}_P([X_1, X_2]) = [X_1, \mathrm{ad}_P(X_2)] - [X_2, \mathrm{ad}_P(X_1)]. \tag{34}$$

This proves the lemma. □

REMARK 5.1. The Lie algebra $\mathscr{I}nv(\mathscr{L})$ (script font) is the infinitesimal counterpart of the invariance group $\mathrm{Inv}(\mathscr{L})$ (Roman font) in Definition 4.1.

We now show that the set of invariance infinitesimal transformations is included in $\mathscr{I}nv(\mathscr{L})$ and that both sets coincide when $\mathscr{I}nv(\mathscr{L})$ contains nothing but proper vector fields.

PROPOSITION 5.3. *Every invariance infinitesimal transformation of* (ξ) *belongs to* $\mathscr{I}nv(\mathscr{L})$. *Every proper vector field of* $\mathscr{I}nv(\mathscr{L})$ *is an invariance infinitesimal transformation of* (ξ).

PROOF. If X is an invariance infinitesimal transformation of (ξ), then, by Definition 4.1, we have $\mathscr{L}^{\Phi_s^X} = \mathscr{L}$, for all s. Therefore, by Lemma A.4 in the Appendix, we have

$$\forall\, f \in C^\infty(M), \qquad 0 = \frac{d}{ds}\bigg|_{s=0} \left(\mathscr{L}^{\Phi_s^X} f\right)(x) = [\mathscr{L}, X] f(x) \tag{35}$$

so that $X \in \mathscr{I}nv(\mathscr{L})$. On the other hand, if $X \in \mathscr{I}nv(\mathscr{L})$, then

$$\begin{aligned} \frac{d}{ds}\bigg|_{s=r} \left(\mathscr{L}^{\Phi_s^X} f\right)(x) &= \frac{d}{ds}\bigg|_{s=0} \left(\mathscr{L}^{\Phi_{r+s}^X} f\right)(x) \\ &= \frac{d}{ds}\bigg|_{s=0} \left(\left(\mathscr{L}^{\Phi_r^X}\right)^{\Phi_s^X} f\right)(x) \quad \left(\text{since } \Phi_{r+s}^X = \Phi_r^X \circ \Phi_s^X\right) \\ &= \left[\mathscr{L}^{\Phi_r^X}, X\right] f(x) \quad [\text{by } (137)] \\ &= \left[\mathscr{L}^{\Phi_r^X}, X^{\Phi_r^X}\right] f(x) \\ &= [\mathscr{L}, X]^{\Phi_r^X} f(x) = 0 \end{aligned}$$

so that $\mathscr{L}^{\Phi_s^X} f = \mathscr{L}^{\Phi_0^X} f = \mathscr{L} f$ and X is an invariance infinitesimal transformation of (ξ). □

What we have just done extends itself to the case of the symmetry group.

DEFINITION 5.4. The vector field X on M is said to be a *symmetry infinitesimal transformation* of (ξ) if:

(i) X is a proper vector field;

(ii) the global flow Φ_s^X that X generates on M belongs to the symmetry group of (ξ) for all s.

The *symmetry algebra* of $\mathscr{L}$ is the following algebra of smooth vector fields on M:

$$(36) \qquad \mathscr{S}\!ym(\mathscr{L}) = \{X \in \mathscr{X}(M) | \exists\, \rho \in C^\infty(M), [\mathscr{L}, X] = \rho\mathscr{L}\}.$$

Note here that there is no condition on the sign of ρ (because the previous definition deals with tangent objects) while there is one on β in (18) (because Definition 4.2 deals with state transformations). As for Lemma 5.2, $\mathscr{S}\!ym(\mathscr{L})$ can easily be shown to be a Lie algebra.

REMARK 5.2. The Lie algebra $\mathscr{S}\!ym(\mathscr{L})$ (script font) is the infinitesimal counterpart of the symmetry group $\mathrm{Sym}(\mathscr{L})$ (Roman font) in Definition 4.2.

LEMMA 5.5. *Every symmetry infinitesimal transformation of (ξ) belongs to $\mathscr{S}\!ym(\mathscr{L})$. Every proper vector field of $\mathscr{S}\!ym(\mathscr{L})$ is a symmetry infinitesimal transformation of (ξ).*

PROOF. If X is a symmetry transformation of (ξ), then, by Definition 4.2, there exists a smooth function $\beta(s, x)$ such that $\mathscr{L}^{\Phi_s^X} = \beta(s, \cdot)\mathscr{L}$, for all s. Therefore, by Lemma A.4 in the Appendix, we have

$$(37) \quad \forall\, f \in C^\infty(M), \qquad \frac{\partial \beta}{\partial s}(0, \cdot)\mathscr{L}f(x) = \frac{d}{ds}\bigg|_{s=0}(\mathscr{L}^{\Phi_s^X}f)(x) = [\mathscr{L}, X]f(x)$$

so that $X \in \mathscr{S}\!ym(\mathscr{L})$. On the other hand, if $X \in \mathscr{S}\!ym(\mathscr{L})$, then

$$\begin{aligned}\frac{d}{ds}\bigg|_{s=r}(\mathscr{L}^{\Phi_s^X}f)(x) &= [\mathscr{L}, X]^{\Phi_r^X} f(x) \quad [\text{by } (137)]\\ &= (\rho\mathscr{L})^{\Phi_r^X} f(x)\\ &= (\rho \circ \Phi_{-r}^X)\mathscr{L}^{\Phi_r^X} f(x).\end{aligned}$$

This linear differential equation in $r \to \mathscr{L}^{\Phi_r^X} f(x)$ has a solution of the form

$$\mathscr{L}^{\Phi_r^X} f(x) = \beta(r, x)\mathscr{L}^{\Phi_0^X} f(x) = \beta(r, x)\mathscr{L}f(x),$$

where $\beta(r, x)$, being an exponential, is positive. This proves that X is a symmetry infinitesimal transformation of (ξ). □

Now we shall show how the invariance and symmetry algebras of the diffusion (ξ) are related to the geometry of the Riemannian manifold (M, g) in terms of the following Lie algebras (see Definitions 2.4 and 2.5).

in terms of the following Lie algebras (see Definitions 2.4 and 2.5).

DEFINITION 5.6. We denote by $\mathscr{P}_g(M)$ the Lie algebra of *parallel vector fields* of (M, g), namely,

$$\begin{aligned} \mathscr{P}_g(M) &= \left\{X \in \mathscr{X}(M) | L_X g = 0 \text{ and } dX^\flat = 0\right\} \\ &= \{X \in \mathscr{X}(M) | A_X = 0\}, \quad \text{by (131) and (132)}. \end{aligned} \tag{38}$$

We denote by $\mathscr{I}_g(M)$ the Lie algebra of *infinitesimal isometries* of (M, g), namely,

$$\mathscr{I}_g(M) = \{X \in \mathscr{X}(M) | L_X g = 0\}. \tag{39}$$

We denote by $\mathscr{H}_g(M)$ the Lie algebra of *infinitesimal homothetic transformations* of (M, g), namely,

$$\mathscr{H}_g(M) = \{X \in \mathscr{X}(M) | \exists\, \lambda \in \mathbb{R},\, L_X g = \lambda g\}. \tag{40}$$

We denote by $\mathscr{C}_g(M)$ the Lie algebra of *infinitesimal conformal transformations* of (M, g), namely,

$$\mathscr{C}_g(M) = \{X \in \mathscr{X}(M) | \exists\, \rho \in C^\infty(M),\, L_X g = \rho g\}. \tag{41}$$

If $L_X g = \rho g$, we write $\rho = \eta_g(X)$.

REMARK 5.3. These Lie algebras are precisely those of the Lie groups introduced in Definition 4.3.

The following lemma is a crucial tool to relate symmetries of $\mathscr{L}$ with geometric properties of (M, g). (It is proven in the Appendix.)

LEMMA 5.7. *For any smooth function ρ, the linear partial differential operator of order less than 2, $[\Delta_g, X] - \rho\Delta_g$, is in fact a first-order operator (identified with a vector field) if and only if $X \in \mathscr{C}_g(M)$ and $\eta_g(X) = \rho$ (see Definition 5.6). Then, necessarily, we have*

$$\left[\Delta_g, X\right] - \rho\Delta_g = -\frac{n-2}{2}\nabla_g \rho = -\frac{n-2}{2}\nabla_g \eta_g(X). \tag{42}$$

PROPOSITION 5.8. *The following equalities between algebras hold*:

$$\mathscr{I}nv(\mathscr{L}) = \left\{X \in \mathscr{I}_g(M) | [B, X] = 0 \textit{ and } L_X c = 0\right\},$$

$$\mathscr{S}ym(\mathscr{L}) = \left\{X \in \mathscr{C}_g(M) | [B, X] - \frac{n-2}{4}\nabla_g \eta_g(X) - \eta_g(X) B = 0 \textit{ and } L_X c = -\rho c\right\}. \tag{43}$$

PROOF. By (32) and (3), we have

$$X \in \mathscr{I}nv(\mathscr{L}) \quad \Leftrightarrow \quad \tfrac{1}{2}\left[\Delta_g, x\right] + [B, X] + [c, X] = 0.$$

We know from Lemma 5.7 that $[\Delta_g, X]$ is a vector field if and only if $X \in \mathscr{I}_g(M)$ (see Definition 5.6) and that necessarily $[\Delta_g, X] = 0$. Therefore, since $[B, X]$ is a vector field and $[c, X] = -L_X c$ is a function, we have

$$\begin{aligned} X \in \mathscr{Inv}(\mathscr{L}) \quad &\Leftrightarrow \quad [\Delta_g, X] \in \mathscr{X}(M) \text{ and} \\ &\qquad \tfrac{1}{2}[\Delta_g, X] = -[B, X] \text{ and } L_X c = 0 \\ &\Leftrightarrow \quad X \in \mathscr{I}_g(M) \text{ and } 0 = \tfrac{1}{2}[\Delta_g, X] = -[B, X] \text{ and } L_X c = 0 \\ &\Leftrightarrow \quad X \in \mathscr{I}_g(M) \text{ and } [B, X] = 0 \text{ and } L_X c = 0. \end{aligned}$$

By (36) and (3), we have

$$\begin{aligned} X \in \mathscr{Sym}(\mathscr{L}) \quad \Leftrightarrow \quad &\exists\, \rho \in C^\infty(M), \tfrac{1}{2}[\Delta_g, X] + [B, X] + [c, X] \\ &= \tfrac{1}{2}\rho\Delta_g + \rho B + \rho c. \end{aligned}$$

By Lemma 5.7, $[\Delta_g, X] - \rho\Delta_g$ is a vector field if and only if $X \in \mathscr{C}_g(M)$ and $\eta_g(X) = \rho$. Then necessarily $[\Delta_g, X] - \rho\Delta_g = -\frac{1}{2}(n-2)\nabla_g \rho$ and, therefore,

$$\begin{aligned} X \in \mathscr{Sym}(\mathscr{L}) \quad \Leftrightarrow \quad &\exists\, \rho \in C^\infty(M), [\Delta_g, X] - \rho\Delta_g \in \mathscr{X}(M), \\ &\tfrac{1}{2}\big([\Delta_g, X] - \rho\Delta_g\big) = -[B, X] + \rho B \text{ and } L_X c = -\rho c \\ \Leftrightarrow \quad &X \in \mathscr{C}_g(M), -\frac{n-2}{4}\nabla_g \eta_g(X) = -[B, X] + \eta_g(X) B \\ &\text{and } L_X c = -\rho c. \end{aligned}$$ □

This last proposition makes it possible to describe $\mathscr{Inv}(\mathscr{L})$ and $\mathscr{Sym}(\mathscr{L})$ more precisely in the following cases.

EXAMPLE 5.1. If $\mathscr{L} = \frac{1}{2}\Delta_g$, that is, if $\mathscr{L}$ is an intrinsic Laplacian (see Definition 2.3), then

$$\begin{aligned} \mathscr{Inv}(\mathscr{L}) &= \mathscr{I}_g(M), \\ \mathscr{Sym}(\mathscr{L}) &= \mathscr{C}_g(M) \quad (\text{if } n = 2) \\ &= \mathscr{H}_g(M) \quad (\text{if } n \neq 2). \end{aligned} \tag{44}$$

When M is compact or (M, g) is complete and nonflat, then $\mathscr{H}_g(M) = \mathscr{I}_g(M)$ (see [13]). In any of these cases, we can conclude that $\mathscr{Sym}(\mathscr{L}) = \mathscr{Inv}(\mathscr{L})$ when $n \neq 2$.

EXAMPLE 5.2. If there exists $\varphi \in C^\infty(M)$ such that $\mathscr{L} = \frac{1}{2}\Delta_g + \nabla_g \varphi$, then

$$\begin{aligned} \mathscr{Inv}(\mathscr{L}) &= \{X \in \mathscr{I}_g(M) | L_X \varphi = \text{constant}\}, \\ \mathscr{Sym}(\mathscr{L}) &= \{X \in \mathscr{C}_g(M) | L_x \varphi = -\tfrac{1}{4}(n-2)\eta_g(X) + \text{constant}\}. \end{aligned} \tag{45}$$

Indeed, by Proposition 5.8, this is a simple consequence of formula (135) in Lemma A.3.

EXAMPLE 5.3. If $\mathscr{L} = \frac{1}{2}\Delta_g + B$, where $B \in \mathscr{I}_g(M)$, then $B \in \mathscr{Inv}(\mathscr{L})$.

5.2. *The time-dependent case.* We denote by $\mathscr{X}_{\mathrm{td}}(M)$ the algebra of time-dependent vector fields on M.

DEFINITION 5.9. We identify a time-dependent vector field $Z \in \mathscr{X}_{\mathrm{td}}(M)$ with a vector field $\overline{Z}$ on $\mathbb{R} \times M$ so that $\mathscr{X}_{\mathrm{td}}(M)$ can be seen as a subalgebra of $\mathscr{X}(\mathbb{R} \times M)$. The vector field $[\partial/\partial t, \overline{Z}]$ on $\mathbb{R} \times M$ can be identified with a time-dependent vector field that we denote $\partial_t Z \in \mathscr{X}_{\mathrm{td}}(M)$.

REMARK 5.3. In a given coordinate system, where $Z = \sum_{i=1}^{n} Z^i(t, x)\, \partial/\partial x_i$, we have

$$\partial_t Z = \sum_{i=1}^{n} \frac{\partial Z^i(t, x)}{\partial t} \frac{\partial}{\partial x_i}.$$

DEFINITION 5.10. The time-dependent vector field Z on M is said to be a *td-invariance infinitesimal transformation* of (ξ) if:

(i) Z is a proper vector field;
(ii) the global flow $\Phi_s^{\overline{Z}}$ that $\overline{Z}$ generates on $\mathbb{R} \times M$ is such that its projection on M is a td-invariance transformation of (ξ) for all s.

The *td-invariance algebra* of $\mathscr{L}$ is the following algebra of time-dependent vector fields on M:

$$\mathscr{I}\!nv_{\mathrm{td}}(\mathscr{L}) = \{Z \in \mathscr{X}_{\mathrm{td}}(M) | \partial_t Z + [\mathscr{L}, Z] = 0\}. \tag{46}$$

DEFINITION 5.11. The time-dependent vector field Z on M is said to be a *td-symmetry infinitesimal transformation* of (ξ) if:

(i) Z is a proper vector field;
(ii) the global flow $\Phi_s^{\overline{Z}}$ that $\overline{Z}$ generates on $\mathbb{R} \times M$ is such that its projection on M is a td-invariance transformation of (ξ) for all s.

The *td-symmetry algebra* of $\mathscr{L}$ is the following set of time-dependent vector fields on M:

$$\mathscr{S}\!ym_{\mathrm{td}}(\mathscr{L}) = \{Z \in \mathscr{X}_{\mathrm{td}}(M) | \exists\, \rho \in C^\infty(\mathbb{R} \times M),\ \partial_t Z + [\mathscr{L}, Z] = \rho \mathscr{L}\}. \tag{47}$$

REMARK 5.4. As subsets of vector fields on $\mathbb{R} \times M$, $\mathscr{I}\!nv_{\mathrm{td}}(\mathscr{L})$ and $\mathscr{S}\!ym_{\mathrm{td}}(\mathscr{L})$ are Lie algebras (by the same proof as in Lemma 5.2).

The following lemmas are proved as in the time-independent case by replacing $\mathscr{L}$ by $\partial/\partial t + \mathscr{L}$ and considering equalities between time-dependent operators *on* M (and not on $\mathbb{R} \times M$).

LEMMA 5.12. *Every td-invariance infinitesimal transformation of* (ξ) *belongs to* $\mathscr{I}\!nv_{\mathrm{td}}(\mathscr{L})$. *Every proper vector field of* $\mathscr{I}\!nv_{\mathrm{td}}(\mathscr{L})$ *is a td-invariance infinitesimal transformation of* (ξ).

LEMMA 5.13. *Every td-symmetry infinitesimal transformation of* (ξ) *belongs to* $\mathcal{S}ym_{\mathrm{td}}(\mathcal{L})$. *Every proper vector field of* $\mathcal{S}ym_{\mathrm{td}}(\mathcal{L})$ *is a td-symmetry infinitesimal transformation of* (ξ).

The sets $\mathcal{I}nv_{\mathrm{td}}(\mathcal{L})$ and $\mathcal{S}ym_{\mathrm{td}}(\mathcal{L})$ can be characterized by way of the following definition.

DEFINITION 5.14. For a time-dependent vector field $Z \in \mathscr{X}_{\mathrm{td}}(M)$ and $t \in \mathbb{R}$, we define the time-independent vector field $Z_t \in \mathscr{X}(M)$ by freezing t.

After noticing that, for all t, the time-independent vector field $(\partial_t Z)_t$ coincides with $\partial_t Z_t$, obtained by differentiating Z_t (see Remark 5.3), the following proposition can be proved as in the time-independent case.

PROPOSITION 5.15. *The following equalities between algebras hold*:

$$\mathcal{I}nv_{\mathrm{td}}(\mathcal{L}) = \{Z \in \mathscr{X}_{\mathrm{td}}(M) | \forall\, t,\, Z_t \in \mathscr{I}_g(M) \text{ and } \partial_t Z + [B, Z_t] = 0 \text{ and } L_{Z_t} c = 0\},$$

$$(48)\quad \mathcal{S}ym_{\mathrm{td}}(\mathcal{L}) = \left\{Z \in \mathscr{X}_{\mathrm{td}}(M) | \forall\, t,\, Z_t \in \mathscr{C}_g(M) \text{ and } L_{Z_t} c = -\eta_g(Z_t) c \right.$$
$$\left. \partial_t Z + [B, Z_t] - \frac{n-2}{4} \nabla_g \eta_g(Z_t) - \eta_g(Z_t) B = 0 \right\}.$$

Here, the description of $\mathcal{I}nv_{\mathrm{td}}(\mathcal{L})$ and $\mathcal{S}ym_{\mathrm{td}}(\mathcal{L})$ can be carried on. We shall show that $Z \in \mathcal{I}nv_{\mathrm{td}}(\mathcal{L})$ [or $\mathcal{S}ym_{\mathrm{td}}(\mathcal{L})$] is determined by its "initial value" Z_0 and that this latter must belong to one of the following Lie algebras.

DEFINITION 5.16. Let us introduce the algebras of vector fields on M given by

$$(49)\quad \begin{aligned} \mathscr{I}_{\mathcal{L}} &= \{X \in \mathscr{X}(M) | \forall\, k \in \mathbb{N},\, \mathrm{ad}^k_{\mathcal{L}}(X) \in \mathscr{X}(M)\}, \\ \mathscr{C}_{\mathcal{L}} &= \{X \in \mathscr{X}(M) | \forall\, k \in \mathbb{N},\, \mathrm{ad}^k_{\mathcal{L}}(X) \in C^\infty(M) \oplus \mathscr{X}(M)\}, \end{aligned}$$

where $\mathrm{ad}^k_{\mathcal{L}}$ is defined in (33).

Note that $\mathrm{ad}^k_{\mathcal{L}}(X)$ is a linear differential operator, the order of which is increasing with k, and $\mathscr{C}_{\mathcal{L}}$ consists of vector fields X for which this order remains bounded by 1. Both $\mathscr{I}_{\mathcal{L}}$ and $\mathscr{C}_{\mathcal{L}}$ are Lie algebras by the same proof as in Lemma 5.2.

PROPOSITION 5.17. $\mathscr{I}_{\mathcal{L}}$ *and* $\mathscr{C}_{\mathcal{L}}$ *are finite-dimensional Lie algebras given by*

$$(50)\quad \begin{aligned} \mathscr{I}_{\mathcal{L}} &= \{X \in \mathscr{I}_g(M) | \forall\, k \in \mathbb{N},\, \mathrm{ad}^k_B(X) \in \mathscr{I}_g(M)\}, \\ \mathscr{C}_{\mathcal{L}} &= \left\{X \in \mathscr{C}_g(M) | \forall\, k \in \mathbb{N},\, \mathrm{ad}^k_{(-B + \frac{1}{4}(n-2)\nabla_g \eta_g(\cdot) - \eta_g(\cdot) B)}(X) \in \mathscr{C}_g(M)\right\}. \end{aligned}$$

Moreover,

$$
\begin{aligned}
&Z \in \mathscr{I}nv_{\mathrm{td}}(\mathscr{L}) &&\Leftrightarrow\quad Z_0 \in \mathscr{I}_{\mathscr{L}} \textit{ and } Z_t = \exp(-t\,\mathrm{ad}_B)(Z_0),\\
(51)\quad &Z \in \mathscr{S}ym_{\mathrm{td}}(\mathscr{L}) &&\Leftrightarrow\quad Z_0 \in \mathscr{C}_{\mathscr{L}} \textit{ and}\\
& && \qquad Z_t = \exp\bigl(t\,\mathrm{ad}_{(-B+\frac{1}{4}(n-2)\nabla_g\eta_g(\cdot)-\eta_g(\cdot)B)}\bigr)(Z_0).
\end{aligned}
$$

PROOF. To begin with, we prove the following implications: For $Z \in \mathscr{X}_{\mathrm{td}}(M)$, we have

$$
(52)\qquad
\begin{aligned}
Z \in \mathscr{I}nv_{\mathrm{td}}(\mathscr{L}) &\Rightarrow \forall\, k \in \mathbb{N}, \forall\, t,\ \partial_t^k Z \in \mathscr{I}_{\mathscr{L}},\\
Z \in \mathscr{S}ym_{\mathrm{td}}(\mathscr{L}) &\Rightarrow \forall\, k \in \mathbb{N}, \forall\, t,\ \partial_t^k Z \in \mathscr{C}_{\mathscr{L}}.
\end{aligned}
$$

By (46), if $Z \in \mathscr{I}nv_{\mathrm{td}}(\mathscr{L})$, we have

$$\mathrm{ad}_{\mathscr{L}}(Z_t) = [\mathscr{L}, Z_t] = -\partial_t Z \in \mathscr{X}(M).$$

Now, successive derivations of the previous formula provide

$$
\begin{aligned}
\partial_t Z_t &= -[\mathscr{L}, Z_t] = -\mathrm{ad}_{\mathscr{L}}(Z_t),\\
\partial_t^2 Z_t &= -\partial_t[\mathscr{L}, Z_t] = -\partial_t(\mathscr{L}Z_t - Z_t\mathscr{L}) = -(\mathscr{L}\partial_t Z_t - \partial_t Z_t\,\mathscr{L})\\
&= -\mathrm{ad}_{\mathscr{L}}(\partial_t Z_t) = \mathrm{ad}_{\mathscr{L}}^2(Z_t)\\
&\ \ \vdots\\
\partial_t^k Z_t &= (-1)^k\,\mathrm{ad}_{\mathscr{L}}^k(Z_t), \qquad \forall\, k \in \mathbb{N}.
\end{aligned}
$$

Since $\partial_t^k Z_t \in \mathscr{X}(M)$, the first implication is proved. With similar arguments, the second implication can be proved by induction.

This done, we can carry on. If $X \in \mathscr{I}_{\mathscr{L}}$, then in particular $\mathrm{ad}_{\mathscr{L}}(X) \in \mathscr{X}(M)$ and, by Lemma 5.7, we can conclude that $X \in \mathscr{I}_g(M)$. However, $\mathscr{I}_g(M)$ is at most of dimension $\frac{1}{2}(n(n+1))$ [13, page 238] so that $\mathscr{I}_{\mathscr{L}}$ is a finite-dimensional Lie algebra. Moreover, by Lemma 5.7, since $X \in \mathscr{I}_g(M)$, we have

$$\mathrm{ad}_{\mathscr{L}}(X) = [\mathscr{L}, X] = [B, X] = \mathrm{ad}_B(X) \in \mathscr{X}(M).$$

Therefore, the vector field $\mathrm{ad}_B(X)$ belongs to $\mathscr{I}_{\mathscr{L}}$ since it coincides with $\mathrm{ad}_{\mathscr{L}}(X)$. Now, by definition of $\mathscr{I}_{\mathscr{L}}$ in (49), $\mathscr{I}_{\mathscr{L}}$ is stable under the action of $\mathrm{ad}_{\mathscr{L}}$. Then $\mathrm{ad}_B(X) \in \mathscr{I}_g(M)$ and we have

$$\mathrm{ad}_{\mathscr{L}}^2(X) = \mathrm{ad}_{\mathscr{L}}(\mathrm{ad}_B(X)) = \mathrm{ad}_B(\mathrm{ad}_B(X)) = \mathrm{ad}_B^2(X).$$

By induction, we thus prove that if $X \in \mathscr{I}_{\mathscr{L}}$, then for all k, $\mathrm{ad}_B^k(X) \in \mathscr{I}_g(M)$. The reverse inclusion is then straightforward.

Now, since $\mathscr{I}_{\mathscr{L}}$ is finite dimensional, we have, by (48),

$$
\begin{aligned}
Z \in \mathscr{I}nv_{\mathrm{td}}(\mathscr{L}) &\Rightarrow Z_0 \in \mathscr{I}_{\mathscr{L}} \text{ and } \partial_t Z = -\mathrm{ad}_B(Z_t)\\
&\Rightarrow Z_0 \in \mathscr{I}_{\mathscr{L}} \text{ and } \frac{dZ_t}{dt} = -\mathrm{ad}_B(Z_t)\\
&\Rightarrow Z_0 \in \mathscr{I}_{\mathscr{L}} \text{ and } Z_t = \exp(-t\,\mathrm{ad}_B)(Z_0).
\end{aligned}
$$

By (46), the reverse implication is clear.

The case of $\mathcal{S}ym_{\mathrm{td}}(\mathcal{L})$ can be treated with similar arguments. □

This last proposition makes it possible to describe $\mathcal{I}nv_{\mathrm{td}}(\mathcal{L})$ and $\mathcal{S}ym_{\mathrm{td}}(\mathcal{L})$ more precisely in the following cases.

EXAMPLE 5.4. If $\mathcal{L} = \frac{1}{2}\Delta_g$, that is, if $\mathcal{L}$ is an intrinsic Laplacian (see Definition 2.3), then

$$\begin{aligned} \mathcal{I}nv_{\mathrm{td}}(\mathcal{L}) &= \{Z \in \mathcal{X}_{\mathrm{td}}(M) | \forall\, t, Z_t \in \mathcal{I}_g(M)\}, \\ \mathcal{S}ym_{\mathrm{td}}(\mathcal{L}) &= \{Z \in \mathcal{X}_{\mathrm{td}}(M) | \forall\, t, Z_t \in \mathcal{C}_g(M)\} \quad \text{if } n = 2. \end{aligned} \tag{53}$$

EXAMPLE 5.5. If $\mathcal{L} = \frac{1}{2}\Delta_g + B$, where $B \in \mathcal{I}_g(M)$, then

$$\mathcal{I}nv_{\mathrm{td}}(\mathcal{L}) = \{Z \in \mathcal{X}_{\mathrm{td}}(M) | \forall\, t, Z_t \in \mathcal{I}_g(M)\}.$$

6. Geometry and symmetry properties of partial differential equations associated with $\mathcal{L}$. Up to now, we have focused on symmetries of the diffusion process (ξ) and, for this, we have considered transformations of the manifold M (or $\mathbb{R} \times M$ in the time-dependent case). In this section, we are no longer interested in the diffusion (ξ) for itself but rather in certain PDE's associated with $\mathcal{L}$ as illustrated below.

1. A function f, a solution of $\mathcal{L}f = 0$ (or of $\partial_t f + \mathcal{L}f = 0$), is such that $f(\xi(t))$ [or $f(t, \xi(t))$] is a local martingale.
2. A positive function f, a solution of $\mathcal{L}^* f = 0$, defines an invariant measure for the diffusion (ξ).
3. The density f of $\xi(t)$, if it exists, satisfies $\partial_t f - \mathcal{L}^* f = 0$.

The symmetries of these different PDE's, when they exist, are an ingredient to calculate special solutions of these PDE's, namely, functions with a certain degree of symmetry (i.e., invariant under some group action). The reader will find in [16], page 192, the basic computational procedure. In the same spirit, the existence of symmetries is related to specific decompositions of the generator $\mathcal{L}$ and thus to functions of the diffusion process (ξ) which still are diffusion processes (see [10]). Moreover, it is explained by Rosencrans [19] how to use the symmetry properties of the parabolic equation $u_t = \mathcal{L}u$ to solve certain perturbed equations $w_t = (\mathcal{L} + P)w$ by a process of quadrature from the former. As an application, if $(z(t))$ is a diffusion with generator $\mathcal{L} + P$, then in such cases there exists (x', t') computed by quadrature from (x, t) such that $\mathbb{E}_x(\varphi(\xi)) = \mathbb{E}_{x'}(\varphi(z(t')))$.

The mathematical framework of this section differs from previous sections by the fact that the object under study is no longer a stochastic process on M (or rather on M'), but a smooth function, from M to $\mathbb{R}$ or $\mathbb{R} \times M$ to $\mathbb{R}$, which is a solution of one of the PDE's $\mathcal{L}f = 0$, $\mathcal{L}^* f = 0$, $\partial_t f - \mathcal{L}f = 0$, $\partial_t f - \mathcal{L}^* f = 0$, $\partial_t f - \mathcal{L}f = 0$ or $\partial_t f + \mathcal{L}^* f = 0$. Therefore, the infinitesimal symmetries that we shall focus on here will be vector fields on $M \times \mathbb{R}$ or $\mathbb{R} \times M \times \mathbb{R}$ as will be shown subsequently.

The PDE's under scope in this section are all second-order linear equations, dealt with by Ovsjannikov [17]. We shall draw upon his results for the elliptic equation $\mathscr{L}f = 0$ and upon our own results [3] for the parabolic equation $\partial_t f - \mathscr{L}f = 0$. However, our approach here consists in pointing out and developing common intrinsic tools for the study of these PDE's.

We start by preliminary remarks and definitions.

6.1. *Preliminary remarks and definitions.* In this section, and for technical reasons, not only smoothness, but also analyticity is required for $\mathscr{L}$.

ASSUMPTION 6.1. The operator $\mathscr{L}$ is assumed to be analytic [the functions $a^{ij}(\cdot)$, $b^i(\cdot)$ and $c(\cdot)$ in (1) are analytic] and nondegenerate elliptic.

In the above-mentioned equations $\mathscr{L}^* f = 0$, $\partial_t f - \mathscr{L}^* f = 0$ and $\partial_t f + \mathscr{L}^* f = 0$, the dual operator $\mathscr{L}^*$ is defined with respect to the Riemannian measure Ω_g as follows.

DEFINITION 6.2. For any smooth differential operator P on M, P^* is defined by duality by

$$\int_M (Pf_1) f_2 \Omega_g = \int_M f_1 (P^* f_2) \Omega_g, \qquad \forall f_1, f_2 \in C_K^\infty(M). \tag{54}$$

The following properties are well known.

LEMMA 6.3. Let $X_0, X_1, \dots, X_m$ be smooth vector fields on M. Then

$$X_0^* = -X_0 - \operatorname{div}_g X_0, \tag{55}$$

$$\Delta_g^* = \Delta_g, \tag{56}$$

$$\mathscr{L}^* = \tfrac{1}{2}\Delta_g - B - \operatorname{div}_g B + c. \tag{57}$$

The following lemma is easy to prove [17, 3] and will be useful in the sequel.

LEMMA 6.4. *For any smooth function θ and differential operator P, one defines the differential operator P_θ by*

$$P_\theta f \overset{\text{def}}{=} e^{-\theta} P(e^\theta f), \qquad \forall f \in C^\infty(M).$$

We have

$$\begin{aligned} \mathscr{L}_\theta f &= \tfrac{1}{2}\Delta_g f + L_{(B+\nabla_g \theta)} f - \left(\tfrac{1}{2}\Delta_g \theta + L_B \theta\right) f \\ &\quad + \tfrac{1}{2} g(\nabla_g \theta, \nabla_g \theta) f + cf, \\ \mathscr{L}_\theta^* f &= \tfrac{1}{2}\Delta_g f + L_{(-B+\nabla_g \theta)} f - \left(\tfrac{1}{2}\Delta_g \theta - L_B \theta\right) f \\ &\quad + \tfrac{1}{2} g(\nabla_g \theta, \nabla_g \theta) f + cf - (\operatorname{div}_g B) f. \end{aligned} \tag{58}$$

Two tensors associated with $\mathscr{L}$ will play a central role throughout this section. Their definition requires the geometric tools introduced in Definition 2.5.

DEFINITION 6.5. If $\mathscr{L}$ is given by (3), the skew-symmetric $(1,1)$ tensor field $K_{\mathscr{L}}$ and the function $H_{\mathscr{L}}$ are defined by

$$\begin{aligned} K_{\mathscr{L}} &= A_B - A_B^*, \\ H_{\mathscr{L}} &= \operatorname{div}_g B + g(B,B) - 2c. \end{aligned} \tag{59}$$

By Lemma A.2, $K_{\mathscr{L}}$ is also characterized by

$$g(K_{\mathscr{L}}X,Y) = -dB^\flat(X,Y) \quad \text{or} \quad (K_{\mathscr{L}}X)^\flat = -i_X(dB^\flat), \qquad \forall\, X,Y \in \mathscr{X}(M). \tag{60}$$

In the sequel, it will prove useful to have various expressions of these tensors. The proof of the following lemma can be found in the Appendix.

LEMMA 6.6. *Let $\phi \in D(M)$ and let $\alpha\colon M \hookrightarrow)0,+\infty($ be a smooth function. We have*

$$\begin{aligned} &\operatorname{met}(\mathscr{L}^*) = \operatorname{met}(\mathscr{L}), \qquad K_{\mathscr{L}^*} = -K_{\mathscr{L}}, \qquad H_{\mathscr{L}^*} = H_{\mathscr{L}}, \\ &\operatorname{met}(\mathscr{L}^\phi) = \phi_*(\operatorname{met}(\mathscr{L})), \qquad K_{\mathscr{L}^*} = \phi_*(K_{\mathscr{L}}), \qquad H_{\mathscr{L}^*} = \phi_*(H_{\mathscr{L}}), \\ &\operatorname{met}\left(\frac{1}{\alpha}\mathscr{L}\right) = \alpha\operatorname{met}(\mathscr{L}), \qquad K_{(1/\alpha)\mathscr{L}} = K_{\mathscr{L}}, \\ &H_{(1/\alpha)\mathscr{L}} = \frac{1}{\alpha}\big(H_{\mathscr{L}} - \Delta_g\mu - g(\nabla_g\mu,\nabla_g\mu)\big) \quad \textit{with } \mu = \frac{n-2}{4}\log\alpha. \end{aligned} \tag{61}$$

If $X \in \mathscr{C}_g(M)$, we have

$$K_{\mathscr{L}}X = -\eta_g(X)B + [B,X] + \nabla_g(g(X,B)), \tag{62}$$

and if there exists $\varphi \in C(M)$ such that B in (3) *can be written*

$$B = B_0 + \nabla_g\varphi \quad \textit{with } \operatorname{div}_g B_0 = 0, \tag{63}$$

then

$$\begin{aligned} K_{\mathscr{L}}X &= -\eta_g(X)B_0 + [B_0,X] + \nabla_g(g(X,B_0)), \\ H_{\mathscr{L}} &= 2e^{-\varphi}\big(\tfrac{1}{2}\Delta_g + B_0 - c\big)e^{\varphi} + g(B_0,B_0). \end{aligned} \tag{64}$$

We end this section with the definition of a local group action on functions, classical in symmetry groups of PDE's (for which our references will be [16] and [17]).

We consider a general manifold N, which will be either M or $M \times \mathbb{R}$ in the sequel, and define an action on functions from N to $\mathbb{R}$ by an action on their graphs as follows (see Figure 1).

PROPOSITION 6.7 [16, 17]. *Let N be a manifold. Let $\zeta \in \mathscr{X}(N \times \mathbb{R})$, and let f be a smooth function on an open domain* $\mathrm{Dom}(f) \subset N$ *containing the point z_0. If the domain of f is sufficiently shrunk around z_0, then, for all ε small enough, the image of the graph of f under Φ_ε^ζ is well defined and is the graph of a function $\tilde{\Phi}_\varepsilon^\zeta \cdot f$.*

This local group action induces an infinitesimal action. In the following lemma, we see a general formula (see [20] for instance).

LEMMA 6.8. *Let $\mathfrak{U}(N \times \mathbb{R})$ be the subalgebra of $\mathscr{X}(N \times \mathbb{R})$ of vector fields ζ of the form*

$$\zeta = X + m \times u \frac{\partial}{\partial u}, \quad \text{where } X \in \mathscr{X}(N) \text{ and } m \in C^\infty(N). \tag{65}$$

For any $\zeta \in \mathfrak{U}(N \times \mathbb{R})$, let $\tilde{\zeta} \in \mathscr{X}(N) \oplus C^\infty(N)$ be the linear partial differential operator of order less than 1 on N given by

$$\tilde{\zeta} = -X + m. \tag{66}$$

We have, for all $z \in N$ and $f \in C^\infty(N)$,

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0}\Big(\big(\tilde{\Phi}_\varepsilon^\zeta \cdot f\big)(x)\Big) = -(Xf)(x) + m(x)f(x) = \big(\tilde{\zeta}f\big)(x). \tag{67}$$

With these tools, we can define infinitesimal symmetries of a linear PDE.

DEFINITION 6.9. Let P be a linear differential operator on N. An infinitesimal symmetry of the PDE

$$Pf = 0 \tag{68}$$

is a vector field $\zeta \in \mathscr{X}(N \times \mathbb{R})$ such that, for all functions f satisfying (68) and all ε small enough, $\tilde{\Phi}_\varepsilon^\zeta \cdot f$ satisfies also (68). We denote by $\mathfrak{S}_P \subset \mathscr{X}(N \times \mathbb{R})$ the set consisting of all these infinitesimal symmetries.

REMARK 6.1. Since (68) is linear, $\mathfrak{S}_P$ contains the subalgebra $\mathfrak{S}_P^\infty$ of $\mathscr{X}(N \times \mathbb{R})$ consisting of vector fields ζ of the form $\zeta = f_0(z)\, \partial/\partial u$, where f_0 satisfies (68).

6.2. *Geometry and symmetries of the PDE $\mathscr{L}f = 0$.* The first two theorems of this section are part of Ovsjannikov's study [17] (we, however, give intrinsic proofs in the Appendix). Then we develop different approaches to the analysis of the set $\mathfrak{S}_{\mathscr{L}}$ of infinitesimal symmetries of the elliptic equation $\mathscr{L}f = 0$ and give examples.

THEOREM 6.10 [17]. *The set $\mathfrak{S}_{\mathscr{L}}$ of infinitesimal symmetries of the elliptic equation $\mathscr{L}f = 0$ satisfies the following properties*:

(i) *$\mathfrak{S}_{\mathscr{L}}$ is a Lie algebra which can be written as a direct sum*

$$\mathfrak{S}_{\mathscr{L}} = \big(\mathfrak{S}_{\mathscr{L}} \cap \mathfrak{U}(M \times \mathbb{R})\big) \oplus \mathfrak{S}_{\mathscr{L}}^\infty. \tag{69}$$

(ii) *If* $\zeta \in \mathfrak{U}(M \times \mathbb{R})$, *the following equivalence is satisfied*:

$$\zeta \in \mathfrak{S}_{\mathscr{L}} \quad \Leftrightarrow \quad \exists\, \rho \in C^\infty(M), \left[\mathscr{L}, \tilde{\zeta}\right] = \rho\mathscr{L}. \tag{70}$$

This last theorem enables us to characterize $\mathfrak{S}_{\mathscr{L}}$ in terms of the following algebra $\mathscr{Sym}(\mathscr{L})$ of linear partial differential operators of order less than 1 on M. This algebra extends the definition (36) in Definition 5.4, which was limited to first-order operators (vector fields).

DEFINITION 6.11. The symmetry algebra of $\mathscr{L}$ is the extension of Definition 5.4 to linear partial differential operators of order less than 1 on M, namely,

$$\begin{aligned}\mathscr{Sym}(\mathscr{L}) = \{X + m \in \mathscr{X}(M) \oplus C^\infty(M) | \exists\, \rho \in C^\infty(M),\\ [\mathscr{L}, X + m] = \rho\mathscr{L}\}.\end{aligned} \tag{71}$$

We keep the same notation for both.

The symmetry algebra of $\mathscr{L}$ can be studied by geometric means.

THEOREM 6.12 [17]. *A linear partial differential operator* $X + m \in \mathscr{X}(M) \oplus C^\infty(M)$ *of order less than* 1 *on* M *belongs to* $X + m \in \mathscr{Sym}(\mathscr{L})$ *if and only if* $X \in \mathscr{C}_g(M)$ *and*

$$K_{\mathscr{L}}X + \nabla_g\left(m - \frac{n-2}{4}\eta_g(X) - g(X,B)\right) = 0, \tag{72}$$

$$\left(L_X + \eta_g(X)\right)\left(H_{\mathscr{L}} + \frac{n-2}{4(n-1)}R_g\right) = 0, \tag{73}$$

where R_g *is the scalar curvature of the Riemannian manifold* (M, g). *Equation* (72) *can be replaced by*

$$-\eta_g(X)B + [B, X] + \nabla_g\left(m - \tfrac{1}{4}(n-2)\eta_g(X)\right) = 0 \tag{74}$$

and (73) *by any of the equivalent forms*

$$\begin{aligned}&\tfrac{1}{4}(n-2)\Delta_g\eta_g(X) + \left(L_X + \eta_g(X)\right)H_{\mathscr{L}} = 0 \quad \textit{or}\\ &\tfrac{1}{2}\Delta_g m + L_B m = \left(L_X + \eta_g(X)\right)c.\end{aligned} \tag{75}$$

REMARK 6.2. In a theorem [17, page 356], Ovsjannikov asserts that if $H = H_{\mathscr{L}} + ((n-2)/(4(n-1)))R_g \neq 0$, then $X + m \in \mathscr{Sym}(\mathscr{L})$ implies that X is an isometry for the metric g. In fact, if $H > 0$, this implies that X is an isometry for the metric Hg since $L_X(Hg) = H\eta_g(X)g + (L_X H)g = (L_X H + \eta_g(X)H)g = 0$ by (73).

EXAMPLE 6.1. As an example, if $\mathscr{L} = \frac{1}{2}\Delta_g$, that is, if $\mathscr{L}$ is an intrinsic Laplacian (see Definition 2.3), then:

(i) If $n = 2, \mathcal{S}ym(\mathcal{L}) = \mathcal{C}_g(M) \oplus \mathbb{R}$.

(ii) If $n \neq 2$, $X + m \in \mathcal{S}ym(\mathcal{L}) \Leftrightarrow X \in \mathcal{C}_g(M)$ and $\Delta_g \eta_g(X) = 0$ and $m = \frac{1}{4}(n-2)\eta_g(X) +$ constant.

Theorem 6.12 may take a simpler form when M is compact and two dimensional thanks to the following decomposition found in [1], page 557.

LEMMA 6.13. *When M is compact, any vector field X can be written in a unique way as*

$$X = X_0 + \nabla_g \varphi \quad \textit{with } \operatorname{div}_g X_0 = 0 \textit{ and } \varphi \in C^\infty(M). \tag{76}$$

PROPOSITION 6.14. *Let $n = 2$. Assume that M is compact, and let B in* (3) *be written as*

$$B = B_0 + \nabla_g \varphi \quad \textit{with } \operatorname{div}_g B_0 = 0 \textit{ and } \varphi \in C^\infty(M). \tag{77}$$

Let $X + m \in \mathcal{X}(M) \oplus C^\infty(M)$ be a linear partial differential operator of order less than 1 *on M. We have*

$$X + m \in \mathcal{S}ym(\mathcal{L}) \Leftrightarrow \begin{cases} X \in \mathcal{C}_g(M), & m = L_X\varphi + \textit{constant}, \\ [B_0, X] = \eta_g(X)B_0, & (L_X + \eta_g(X))H_{\mathcal{L}} = 0. \end{cases} \tag{78}$$

PROOF. By (64), (74) and (73), $X + m$ belongs to $\mathcal{S}ym(\mathcal{L})$ if and only if

$$X \in \mathcal{C}_g(M), \qquad -\eta_g(X)B_0 + [B_0, X] + \nabla_g(m - L_X\varphi) = 0$$

and

$$(L_X + \eta_g(X))H_{\mathcal{L}} = 0,$$

where, since $\operatorname{div} B_0 = 0$ and $\operatorname{div} X = \eta_g(X)$ when $n = 2$ by (134), we have, by (129),

$$\operatorname{div}(-\eta_g(X)B_0 + [B_0, X]) = -L_{B_0}\eta_g(X) + L_{B_0}\operatorname{div}_g X = 0.$$

By uniqueness of the decomposition (76), this ends the proof. □

6.3. *Geometry and symmetries of the PDE $\mathcal{L}^* f = 0$.* In the first proposition of this section, we show how the symmetries of the PDE $\mathcal{L}^* f = 0$ are related to those of the PDE $\mathcal{L} f = 0$. Then we point out how the geometry of $\mathcal{L}$ may help in computing invariant measures for the associated diffusion process (when $\mathcal{L}1 = 0$), thus extending a result of Ikeda and Watanabe [12].

PROPOSITION 6.15. *Let $X + m \in \mathcal{X}(M) \oplus C^\infty(M)$ be a linear partial differential operator of order less than* 1 *on M. We have*

$$X + m \in \mathcal{S}ym(\mathcal{L}^*) \quad \Leftrightarrow \quad -X + m - \frac{n-2}{2}\eta_g(X) \in \mathcal{S}ym(\mathcal{L}). \tag{79}$$

PROOF. By (61) in Lemma 6.6, we have $\mathrm{met}(\mathscr{L}^*) = \mathrm{met}(\mathscr{L})$, $K_{\mathscr{L}^*} = -K_{\mathscr{L}}$, $H_{\mathscr{L}^*} = H_{\mathscr{L}}$ and thus

$$
\begin{aligned}
&X + m \in \mathscr{Sym}(\mathscr{L}^*)\\
&\quad\Leftrightarrow\quad X \in \mathscr{C}_g(M) \text{ and}\\
&\qquad\begin{cases}
-K_{\mathscr{L}}X + \nabla_g\left(m - \dfrac{n-2}{4}\eta_g(X) - g(X, -B)\right) = 0,\\
\dfrac{n-2}{4}\Delta_g \eta_g(X) + \big(L_X + \eta_g(X)\big)H_{\mathscr{L}} = 0,
\end{cases}\\
&\quad\Leftrightarrow\quad X \in \mathscr{C}_g(M) \text{ and}\\
&\qquad\begin{cases}
K_{\mathscr{L}}(-X) + \nabla_g\left(\left(m - \dfrac{n-2}{2}\eta_g(X)\right)\right.\\
\qquad\qquad\left. - \dfrac{n-2}{4}\eta_g(-X) - g(-X, B)\right) = 0,\\
\dfrac{n-2}{4}\Delta_g \eta_g(X) + \big(L_X + \eta_g(X)\big)H_{\mathscr{L}} = 0,
\end{cases}\\
&\quad\Leftrightarrow\quad -X + m - \frac{n-2}{2}\eta_g(X) \in \mathscr{Sym}(\mathscr{L}).
\end{aligned}
$$
□

The following proposition is an extension of Theorem 4.6 in [12].

PROPOSITION 6.16. *Assume that, in* (3), $c = 0$ *and* B *can be written as*

(80) $$B = B_0 + \nabla_g \varphi \quad \textit{with } \mathrm{div}_g B_0 = 0 \textit{ and } \varphi \in C^\infty(M) \textit{ such that } L_{B_0}\varphi = 0.$$

Then $\mathscr{L}^* e^{2\varphi} = 0$. *In particular, if* $\int_M e^{2\varphi}\Omega_g < +\infty$, *a diffusion process with generator* $\mathscr{L}$ *and infinite lifetime has an invariant probability with density* (*with respect to the Riemannian measure*) *proportional to* $e^{2\varphi}$.

PROOF. By (58), we have, for any smooth function θ,

$$
\begin{aligned}
e^{-\theta}\mathscr{L}^* e^{\theta} = \mathscr{L}_\theta^* 1 &= \tfrac{1}{2}\Delta_g\theta - L_B\theta + \tfrac{1}{2}g(\nabla_g\theta, \nabla_g\theta) - \mathrm{div}_g B\\
&= \tfrac{1}{2}\Delta_g(\theta - 2\varphi) - L_{B_0}\theta + \tfrac{1}{2}g\big(\nabla_g(\theta - 2\varphi), \nabla_g\theta\big)
\end{aligned}
$$

so that, if we choose $\theta = 2\varphi$, we obtain $\mathscr{L}^* e^{2\varphi} = 0$. □

6.4. *Symmetries of the equations* $\partial_t f + \mathscr{L}f = 0$ *and* $\partial_t f - \mathscr{L}f = 0$. Here, the base space for solutions of these PDE's is the manifold $N = \mathbb{R} \times M$. We refer to Section 6.1 for the basic definitions.

The following characterization of the set $\mathfrak{S}_{\partial_t - \mathscr{L}}$ is adapted from [16], [17] and [19] and can be found in [3].

DEFINITION 6.17. Let $\mathfrak{U}(\mathbb{R} \times M \times \mathbb{R})$ be the subalgebra of $\mathscr{X}(\mathbb{R} \times M \times \mathbb{R})$ of vector fields ζ of the form (independent of the coordinate system on M)

$$\zeta = \zeta^0(t)\partial_t + \sum_{i=1}^{n} \zeta^i(t,x)\partial_{x_i} + \zeta^{n+1}(t,x)u\frac{\partial}{\partial u}. \tag{81}$$

For any $\zeta \in \mathfrak{U}(\mathbb{R} \times M \times \mathbb{R})$ and $t \in \mathbb{R}$, let $\hat{\zeta}_t$ and $\partial_t\hat{\zeta}_t$ be the linear partial differential operators of order less than 2 in $\mathbb{R}\mathscr{L} \oplus \mathscr{X}(M) \oplus C^\infty(M)$ given by

$$\begin{aligned} \hat{\zeta}_t &= -\zeta^0(t)\mathscr{L} - \sum_{i=1}^{n} \zeta^i(t,x)\frac{\partial}{\partial x_i} + \zeta^{n+1}(t,x), \\ \partial_t\hat{\zeta}_t &= -\partial_t\zeta^0(t)\mathscr{L} - \sum_{i=1}^{n} \partial_t\zeta^i(t,x)\partial_{x_i} + \partial_t\zeta^{n+1}(t,x). \end{aligned} \tag{82}$$

As in Section 5.2, we shall show that elements of $\mathfrak{S}_{\partial_t - \mathscr{L}}$ are determined by their "initial values" $\hat{\zeta}_0$ and that this latter must belong to the following Lie algebra $\mathscr{F}_{\mathscr{L}}$ of second-order differential operators on M.

THEOREM 6.18 [3]. *The set $\mathfrak{S}_{\partial_t - \mathscr{L}}$ of infinitesimal symmetries of the parabolic equation $\partial_t f - \mathscr{L}f = 0$ satisfies the following properties*:

(i) *$\mathfrak{S}_{\partial_t - \mathscr{L}}$ is a Lie algebra which can be written as a direct sum*

$$\mathfrak{S}_{\partial_t - \mathscr{L}} = \left(\mathfrak{S}_{\partial_t - \mathscr{L}} \cap \mathfrak{U}(\mathbb{R} \times M \times \mathbb{R})\right) \oplus \mathfrak{S}^\infty_{\partial_t - \mathscr{L}}. \tag{83}$$

(ii) *The following set $\mathscr{F}_{\mathscr{L}}$ of second-order differential operators on M,*

$$\begin{aligned} \mathscr{F}_{\mathscr{L}} = \{P \in \mathbb{R}\mathscr{L} \oplus \mathscr{X}(M) \oplus C^\infty(M) | \forall\, k \in \mathbb{N}, \\ \operatorname{ad}^k_{\mathscr{L}}(P) \in \mathbb{R}\mathscr{L} \oplus \mathscr{X}(M) \oplus C^\infty(M)\}, \end{aligned} \tag{84}$$

is a finite-dimensional Lie algebra.

(iii) *If $\zeta \in \mathfrak{U}(\mathbb{R} \times M \times \mathbb{R})$, the following equivalence is satisfied*:

$$\zeta \in \mathfrak{S}_{\partial_t - \mathscr{L}} \quad \Leftrightarrow \quad \hat{\zeta}_0 \in \mathscr{F}_{\mathscr{L}} \quad \text{and} \quad \forall\, t \in \mathbb{R}, \qquad \hat{\zeta}_t = \exp(t\, \operatorname{ad}_{\mathscr{L}})(\hat{\zeta}_0). \tag{85}$$

(iv) *$\zeta \mapsto \hat{\zeta}_0$ is an antiisomorphism from $\mathfrak{S}_{\partial_t - \mathscr{L}} \cap \mathfrak{U}(\mathbb{R} \times M \times \mathbb{R})$ to $\mathscr{F}_{\mathscr{L}}$.*

Let us notice that in (85) the time t can be reversed so that $\mathfrak{S}^f_{\partial_t + \mathscr{L}}$ can be characterized by replacing $\mathscr{L}$ by $-\mathscr{L}$ in the following study.

With this last theorem, the analysis of infinitesimal symmetries of $\partial_t - \mathscr{L}$ (or $\partial_t + \mathscr{L}$) draws heavily upon that of the Lie algebra $\mathscr{F}_{\mathscr{L}}$ defined in (84). Rosencrans [19] introduced the *perturbation algebra* of $\mathscr{L}$. It is the algebra $\mathscr{P}_{\mathscr{L}}$ of linear partial differential operators of order less than 1 on M such that $\mathscr{F}_{\mathscr{L}} = \mathbb{R}\mathscr{L} \oplus \mathscr{P}_{\mathscr{L}}$.

THEOREM 6.19. *The operator $X + m \in \mathscr{X}(M) \oplus C^\infty(M)$ belongs to $\mathscr{P}_{\mathscr{L}}$ if and only if there exists a sequence $(X_i)_{i \in N}$ in $\mathscr{K}_g(M)$ which satisfies one of the* *following equivalent inductions*:

$$
\begin{aligned}
&X_0 = X,\\
&X_1 = K_{\mathscr{L}} X_0 + \nabla_g(m - g(X_0, B))\\
&\qquad\qquad\qquad [or\ = -\eta_g(X_0)B + [B, X_0] + \nabla_g m],\\
&X_{i+2} = K_{\mathscr{L}} X_{i+1} + \tfrac{1}{2}\nabla_g(L_X, H_{\mathscr{L}} + \eta_g(X_i)H_{\mathscr{L}})\\
&\qquad [or\ = -\eta_g(X_{i+1})B + [B, X_{i+1}]\\
&\qquad\qquad + \nabla_g(g(X_{i+1}, B) + \tfrac{1}{2}L_{X_i} H_{\mathscr{L}} + \tfrac{1}{2}\eta_g(X_i)H_{\mathscr{L}})]
\end{aligned}
\tag{86}
$$

or

$$
\begin{aligned}
&X_0 = X,\\
&X_1^\flat = -i_{X_0}(dB^\flat) + d(m - g(X_0, B)),\\
&X_{i+2}^\flat = -i_{X_{i+1}}(dB^\flat) + \tfrac{1}{2}d(L_{X_i} H_{\mathscr{L}} + \eta_g(X_i)H_{\mathscr{L}}),
\end{aligned}
\tag{87}
$$

where the skew-symmetric (1,1) *tensor field* $K_{\mathscr{L}}$ *and the function* $H_{\mathscr{L}}$ *are defined in* (59). *Moreover, we have*

$$
\mathrm{ad}_{\mathscr{L}}^{k+1}(X + m) = \eta_g(X_k)\mathscr{L} + X_{k+1} + \textit{function}. \tag{88}
$$

PROOF. Induction (86) follows Proposition 3.2 in [3], where it may be noted that we used the metric $a = 2g$. The equation giving X_{i+2} differs from that in [3], which was

$$
X_{i+2} = K_{\mathscr{L}} X_{i+1} + \tfrac{1}{2}[X_i, \nabla_g H_{\mathscr{L}}] + 2\eta_g(X_i)\nabla_g H_{\mathscr{L}}. \tag{89}
$$

However, they are equivalent by formula (135) in Lemma A.3 of the Appendix. Moreover, the alternative forms in brackets are a straightforward consequence of (62) in Lemma 6.6.

The equivalent form (87) is obtained by duality, thanks to the identity (60). Equation (88) comes from the proof of Proposition 3.2 in [3]. □

It may be noted that the infinitesimal symmetries of the elliptic equation $\mathscr{L}f = 0$ are related to the infinitesimal *conformal* transformations of (M, g), while those of the parabolic equation $\partial_t f - \mathscr{L}f = 0$ are related to the infinitesimal *homothetic* transformations of (M, g).

REMARK 6.3. By Theorem 6.12, if $X + m \in \mathscr{S}ym(\mathscr{L})$ with $X \in \mathscr{K}_g(M)$, then $X + m \in \mathscr{P}_{\mathscr{L}}$ and $X_i = 0$ for $i \geq 1$.

The previous theorem may take simpler forms thanks to the decomposition (76).

PROPOSITION 6.20. *Assume that* M *is compact and let* B *in* (3) *be written as*

$$
B = B_0 + \nabla_g \varphi \quad \textit{with } \mathrm{div}_g B_0 = 0 \textit{ and } \varphi \in C^\infty(M). \tag{90}
$$

Then the linear partial differential operator of order less than 1, $X + m \in \mathscr{X}(M) \oplus C^{\infty}(M)$, *belongs to* $\mathscr{P}_{\mathscr{L}}$ *if and only if*:

(i) $m = L_X \varphi + constant$.
(ii) *The sequence* $(X_i)_{i \in \mathbb{N}}$ *defined by*

$$X_0 = X \quad and \quad X_{i+1} = [B_0, X_i] \tag{91}$$

satisfies

$$\forall i \in \mathbb{N}, \qquad X_i \in \mathscr{I}_g(M) \quad and \quad L_{X_i}\left(\frac{\left(\frac{1}{2}\Delta_g + B_0 - c\right)e^{\varphi}}{e^{\varphi}} \right) = constant. \tag{92}$$

Proof. We recall that when M is compact, then $\mathscr{K}_g(M) = \mathscr{I}_g(M)$ (see [13]). By (64) and (86), $X + m$ belongs to $\mathscr{P}_{\mathscr{L}}$ if and only if there exists a sequence $(X_i)_{i \in \mathbb{N}}$ in $\mathscr{I}_g(M)$ which satisfies

$$X_0 = X,$$
$$X_1 = [B_0, X_0] + \nabla_g(m - L_{X_0}\varphi),$$
$$X_{i+2} = [B_0, X_{i+1}] + \nabla_g\left(g(X_{i+1}, B_0) + \tfrac{1}{2}L_{X_i}H_{\mathscr{L}}\right).$$

Now, if $X_i \in \mathscr{I}_g(M)$, we have both $\operatorname{div}_g X_i = 0$ and $\operatorname{div}_g[B_0, X_i] = 0$ by (134) and (129). Therefore, by uniqueness of the decomposition (76), $X + m$ belongs to $\mathscr{P}_{\mathscr{L}}$ if and only if there exists a sequence $(X_i)_{i \in \mathbb{N}}$ in $\mathscr{I}_g(M)$ which satisfies

$$X_0 = X,$$
$$X_i = [B_0, X_0] \text{ and } m - L_{X_0}\varphi = \text{constant},$$
$$X_{i+2} = [B_0, X_{i+1}] \text{ and } g(X_{i+1}, B_0) + \tfrac{1}{2}L_{X_i}H_{\mathscr{L}} = \text{constant},$$

where

$$\begin{aligned} g(X_{i+1}, B_0) &= g([B_0, X_i], B_0) \\ &= -g\left(L_{X_i}(B_0), B_0\right) \quad [\text{by } (4)] \\ &= -\tfrac{1}{2}L_{X_i}(g(B_0, B_0)) \quad \left[\text{by } (4) \text{ since } L_{X_i}g = 0\right]. \end{aligned}$$

The expression of $H_{\mathscr{L}}$ is given by (64) and we get

$$g(X_{i+1}, B_0) + \frac{1}{2}L_{X_i}H_{\mathscr{L}} = L_{X_i}\left(\frac{\left(\frac{1}{2}\Delta_g + B_0 - c\right)e^{\varphi}}{e^{\varphi}} \right). \qquad \square$$

The following proposition can be proved in the same way.

Proposition 6.21. *Assume that* $\mathscr{K}_g(M) = \mathscr{I}_g(M)$ *and that* (M, g) *has no parallel vector fields* [*see* (38)]. *If* B *in* (3) *can be written as*

$$B = B_0 + \nabla_g \varphi \quad with\ B_0 \in \mathscr{I}_g(M) \text{ and } \varphi \in C^{\infty}(M), \tag{93}$$

then the linear partial differential operator of order less than 1, $X + m \in$

$\mathscr{X}(M) \oplus C^\infty(M)$, *belongs to* $\mathscr{P}_{\mathscr{L}}$ *if and only if*:

(i) $m = L_X \varphi + constant$

(ii) *The sequence* $(X_i)_{i \in \mathbb{N}}$ *defined by*

$$X_0 = X \quad and \quad X_{i+1} = [B_0, X_i] \tag{94}$$

satisfies

$$\forall\, i \in \mathbb{N}, \qquad L_{X_i}\left(\frac{\left(\frac{1}{2}\Delta_g + B_0 - c\right)e^{\varphi}}{e^{\varphi}} \right) = constant. \tag{95}$$

6.5. *Symmetries of the equations* $\partial_t f + \mathscr{L}^* f = 0$ *and* $\partial_t f - \mathscr{L}^* f = 0$. The symmetries of the PDE $\partial_t f - \mathscr{L}^* f = 0$ (resp., $\partial_t f + \mathscr{L}^* f = 0$) are related to those of the PDE $\partial_t f - \mathscr{L} f = 0$ (resp., $\partial_t f + \mathscr{L} f = 0$) as shown in the following proposition.

PROPOSITION 6.22. *We have* $\mathscr{P}_{\mathscr{L}^*} = \mathscr{P}^*_{\mathscr{L}}$.

PROOF. By (61) in Lemma 6.6, $\operatorname{met}(\mathscr{L}^*) = \operatorname{met}(\mathscr{L})$, $K_{\mathscr{L}^*} = -K_{\mathscr{L}}$ and $H_{\mathscr{L}^*} = H_{\mathscr{L}}$. Thus, the linear differential operator of order less than 1, $X + m \in \mathscr{X}(M) \oplus C^\infty(M)$, belongs to $\mathscr{P}_{\mathscr{L}^*}$ if and only if there exists a sequence $(X_i)_{i \in \mathbb{N}}$ in $\mathscr{K}_g(M)$ which satisfies the induction

$$\begin{aligned} X_0 &= X, \\ X_1 &= -K_{\mathscr{L}} X_0 + \nabla_g\big(m - g(X_0, -B)\big), \\ X_{i+2} &= -K_{\mathscr{L}} X_{i+1} + \tfrac{1}{2}\nabla_g\big(L_{X_i} H_{\mathscr{L}} + \eta_g(X_i) H_{\mathscr{L}}\big) \end{aligned}$$

if and only if $-X + m$ belongs to $\mathscr{P}_{\mathscr{L}}$ [with corresponding sequence $((-1)^i X_i)_{i \in \mathbb{N}}$ in $\mathscr{K}_g(M)$ for induction (86)]. However, $(X + m)^* = -X - \operatorname{div}_g X + m$, where $\operatorname{div}_g X = \text{constant}$ by (134). □

6.6. *The two-dimensional case*. We assume that $n = 2$. Some of the previous results may take a simpler form thanks to the following definition.

DEFINITION 6.23. For any vector field Z, the scalar $\operatorname{rot}_g Z$ is defined by $dZ^\flat = (\operatorname{rot}_g Z)\Omega_g$.

The following lemma is straightforward application of Theorem 6.19.

LEMMA 6.24. *The linear partial differential operator of order less than* 1, $X + m \in \mathscr{X}(M) \oplus C^\infty(M)$, *belongs to* $\mathscr{P}_{\mathscr{L}}$ *if and only if there exists a sequence* $(X_i)_{i \in \mathbb{N}}$ *in* $\mathscr{K}_g(M)$ *which satisfies the induction*

$$\begin{aligned} X_0 &= X, \\ X_1^\flat &= -\operatorname{rot}_g B\big(i_{X_0}\Omega_g\big) + d\big(m - g(X_0, B)\big), \\ X_{i+2}^\flat &= -\operatorname{rot}_g B\big(i_{X_{i+1}}\Omega_g\big) + \tfrac{1}{2} d\big(L_{X_i} H_{\mathscr{L}} + \eta_g(X_i) H_{\mathscr{L}}\big). \end{aligned} \tag{96}$$

PROPOSITION 6.25. *Assume that* $\mathscr{H}_g(M) = \mathscr{I}_g(M)$ *and* $\mathscr{P}_g(M) = \{0\}$. *Then, if* B *in* (3) *is such that* $\operatorname{rot}_g B$ = *constant, we have*

$$(97) \qquad X + m \in \mathscr{P}_{\mathscr{L}} \quad \Leftrightarrow \quad \begin{cases} X \in \mathscr{I}_g(M) \text{ and } L_X H_{\mathscr{L}} = \text{constant} \\ \text{and } \operatorname{rot}_g B(i_X \Omega_g) = d(m - g(X, B)). \end{cases}$$

Moreover, if the manifold M *is simply connected, there always exists a function* m *satisfying* $\operatorname{rot}_g B(i_X \Omega_g) = d(m - g(X, B))$.

PROOF. The sequence $(X_i)_{i \in \mathbb{N}}$ in $\mathscr{I}_g(M)$ of (96) is such that $X_i = 0$ for $i \geq 0$. Indeed, we have

$$dX_1^\flat = -\operatorname{rot}_g B\, d(i_{X_0} \Omega_g) + d^2(m - g(X_0, B)) = -\operatorname{rot}_g B \operatorname{div}_g X_0 \Omega_g,$$

$$dX_{i+2}^\flat = -\operatorname{rot}_g B \operatorname{div}_g X_{i+1} \Omega_g,$$

since, by (6), we have

$$d(i_X \Omega_g) = L_X \Omega_g - i_X d\Omega_g = \operatorname{div}_g X \Omega_g.$$

However, $\operatorname{div}_g X_0 = \operatorname{div}_g X_{i+1} = 0$ so that necessarily $dX_i^\flat = 0$ for $i \geq 0$. Since $\mathscr{P}_g(M) = \{0\}$, we conclude that $X_i = 0$ for $i \geq 0$. With this property, this completes the proof of the first equivalence.

On the other hand, the 1-form $\operatorname{rot}_g B(i_X \Omega_g)$ is closed since $\operatorname{rot}_g B$ = constant and $d(i_X \Omega_g) = \operatorname{div}_g X \Omega_g = 0$. Thus, if the manifold M is simply connected, there exists $\psi \in C^\infty(M)$ such that $\operatorname{rot}_g B(i_X \Omega_g) = d\psi$ and it suffices to take $m = g(X, B) + \psi$. □

7. Questions of modelling. Examples. In this section, we give various applications of the techniques developed above. We are particularly interested in outlining how symmetries can be destroyed or created when passing from deterministic to stochastic systems.

7.1. *Review of symmetries of deterministic systems.* To illustrate the concepts of the Lie theory, let us start with an example. S. Lie has shown that a one-parameter group of planar transformations Φ_t^Y with infinitesimal generator Y leaves the differential equation $du/dx = U(x, u)/X(x, u)$ invariant if and only if the vector field $Z = X\, \partial/\partial x + U\, \partial/\partial u$ satisfies $[Y, Z] = \gamma Z$, where γ is a function [11, page 144]. What is more, if $Y = \xi\, \partial/\partial x + v\, \partial/\partial u$, then $(X\xi - Uv)^{-1}$ is an integrating factor of the 1-form $X du - u dx$. As an example, calculation shows that the rotation group generated by the vector field $Y = -u\, \partial/\partial x + x\, \partial/\partial u$ leaves the differential equation

$$\frac{du}{dx} = \frac{u + x(x^2 + u^2)}{x - u(x^2 + u^2)}$$

invariant. The term $-1/(u^2+x^2)$ indeed is an integrating factor since

$$(u^2+x^2)^{-1}\big((x-u(x^2+u^2))\,du-(u+x(x^2+u^2))\,dx\big)$$
$$=d\left(\arctan\left(\frac{u}{x}\right)-\frac{1}{2}(x^2+u^2)\right)$$

and the solutions are thus part of the curves $\arctan(u/x)-\frac{1}{2}(x^2+u^2)=$ constant.

Now, let us turn to the general case of autonomous systems. Let U be any vector field on M and consider the ordinary differential equation

$$\dot{x}=U(x). \tag{98}$$

Here, the object under study is no longer a stochastic process on M, but a smooth function from $\mathbb{R}$ to M, which is a solution of the ODE (98). Therefore, the infinitesimal symmetries upon which we focus are vector fields on $\mathbb{R}\times M$.

DEFINITION 7.1. An infinitesimal symmetry of the ODE (98) is a vector field $Z\in\mathscr{X}(\mathbb{R}\times M)$ such that, for all functions f $[=(t\mapsto x(t))]$ satisfying (98) and for all ε small enough, $\widetilde{\Phi^Z_\varepsilon}\cdot f$ also satisfies (98).

By the specific role played by the variable t in (98), the characterization of the infinitesimal symmetries of the ODE (98) takes the following form.

LEMMA 7.2. *Let $Z\in\mathscr{X}(\mathbb{R}\times M)$ be written in the form*

$$Z=Z^0(t,x)\frac{\partial}{\partial t}+Z_t, \tag{99}$$

where, for all t, $Z_t\in\mathscr{X}(M)$ (see Section 5.2). Then Z is an infinitesimal symmetry of the ODE (98) if and only if

$$\partial_t Z_t=[U,Z_t]-\frac{\partial Z^0}{\partial t}U. \tag{100}$$

In particular, if $Z\in\mathscr{X}(M)$ (i.e., if Z is time-independent), this reduces to $[U,Z]=0$.

PROOF. We know by Exercise 2.19 in [16] that the vector field Z is an infinitesimal symmetry of (98) if and only if there exists a function γ such that

$$\left[\frac{\partial}{\partial t}-U,Z\right]=\gamma\left(\frac{\partial}{\partial t}-U\right).$$

Using (99), this can be rewritten as

$$\frac{\partial Z^0}{\partial t}\frac{\partial}{\partial t}+\partial_t Z_t-[U,Z_t]=\gamma\left(\frac{\partial}{\partial t}-U\right)$$

and we get

$$\frac{\partial Z^0}{\partial t} = \gamma \quad \text{and} \quad \partial_t Z_t - [U, Z_t] = -\gamma U. \qquad \square$$

7.2. *Symmetries of deterministic systems and diffusion processes.* One way to extend the deterministic smooth system (98) to a stochastic one is by considering a diffusion process (ξ) with generator $\mathscr{L}_\varepsilon = U + \varepsilon \times$ elliptic operator, for a "small" noise parameter ε. If we fix a Riemannian metric g_0 on M, it may be natural to choose

$$\mathscr{L}_\varepsilon = \frac{\varepsilon}{2}\Delta_{g_0} + U \tag{101}$$

or, more generally, $\mathscr{L}_t = \varepsilon(\frac{1}{2}\Delta_{g_0} + W) + U$, where $W \in \mathscr{X}(M)$. We address the following problem: What properties of (98) are preserved in the diffusion (ξ) of generator $\mathscr{L}_\varepsilon$ given by (101)? The following proposition yields a partial answer to our question.

PROPOSITION 7.3. *Let $\mathscr{L}_\varepsilon$ be given by* (101), *and let (ξ) be a diffusion with generator $\mathscr{L}_\varepsilon$. Let $X \in \mathscr{X}(M)$.*

(i) *Assume that there exists $\varepsilon > 0$ such that the vector field X on M is an infinitesimal symmetry of the diffusion process (ξ) with generator $\mathscr{L}_\varepsilon$.*

(a) *If $X \in \mathscr{H}_{g_0}(M)$, then the vector field $Z = \eta_{g_0}(X)t(\partial/\partial t) + X$ on $\mathbb{R} \times M$ is an infinitesimal symmetry of the ODE* (98).

(b) *If $n = 2$, then the vector field $Z = \eta_{g_0}(X)t(\partial/\partial t) + X$ on $\mathbb{R} \times M$ is an infinitesimal symmetry of the ODE* (98).

(ii) *Assume that there exists $\gamma \in C^\infty(M)$ such that $[U, X] = \gamma U$ [so that, in particular, the vector field $\gamma t(\partial/\partial t) + X$ on $\mathbb{R} \times M$ is an infinitesimal symmetry of the ODE* (98) *by Lemma* 7.2].

(a) *If $\gamma =$ constant and $X \in \mathscr{H}_{g_0}(M)$ with $\eta_{g_0}(X) = \gamma$, then, for all $\varepsilon \geq 0$, the vector field X on $\mathbb{R} \times M$ is an infinitesimal symmetry of the diffusion process (ξ) with generator $\mathscr{L}_\varepsilon$.*

(b) *If $n = 2$ and $X \in \mathscr{C}_{g_0}(M)$ with $\eta_{g_0}(X) = \gamma$, then, for all $\varepsilon > 0$, the vector field X on $\mathbb{R} \times M$ is an infinitesimal symmetry of the diffusion process (ξ) with generator $\mathscr{L}_\varepsilon$.*

PROOF. By Proposition 5.8, we know that the vector field X is an infinitesimal symmetry of the diffusion process (ξ) with generator $\mathscr{L}_\varepsilon$ if and only if

$$X \in \mathscr{C}_{g_0}(M) \quad \text{and} \quad [U, X] - \eta_{g_0}(X)U = \varepsilon\frac{n-2}{4}\nabla_{g_0}\eta_{g_0}(X) \tag{102}$$

since $\mathrm{met}(\mathscr{L}_\varepsilon) = g_0/\varepsilon$.

(i) Assume that X is an infinitesimal symmetry of the diffusion process (ξ) with generator $\mathscr{L}_\varepsilon$, for a given $\varepsilon > 0$.

(a) In this case where $X \in \mathscr{H}_{g_0}(M)$, we have $\nabla_{g_0}\eta_{g_0}(X) = 0$ and by (102) we get $[U, X] - \eta_{g_0}(X)U = 0$. Thus $Z = \eta_{g_0}(X)t(\partial/\partial t) + X$ satisfies (100) and we conclude with Lemma 7.2.

(b) In the case $n = 2$, we have $\frac{1}{4}(n-2)\nabla_{g_0}\eta_{g_0}(X) = 0$ and we conclude as previously.

(ii) This is straightforward by (102). □

Infinitesimal symmetries of $\mathscr{L}_\varepsilon$ are, in full generality, vector fields on $\mathbb{R} \times M$ (and not only on M as in the previous proposition). This remark motivates the following result, which will be useful for the forthcoming applications.

PROPOSITION 7.4. *Let* $\mathscr{L}_\varepsilon$ *be given by* (101). *Let also* $X \in \mathscr{X}(M)$ *and* $m \in C^\infty(M)$ *be given.*

(i) *Assume that* $n \neq 2$. *Then, for all* $\varepsilon > 0$, $X + m/\varepsilon \in \mathrm{Sym}(\mathscr{L}_\varepsilon)$ *if and only if*

$$X \in \mathscr{H}_{g_0}(M), \qquad -\eta_{g_0}(X)U + [U, X] + \nabla_{g_0}m = 0, \qquad \Delta_{g_0}m = 0 \quad \textit{and} \quad L_U m = 0. \tag{103}$$

(ii) *Assume that* $n = 2$. *Then, for all* $\varepsilon > 0$, $X + m/\varepsilon \in \mathrm{Sym}(\mathscr{L}_\varepsilon)$ *if and only if*

$$X \in \mathscr{C}_{g_0}(M), \qquad -\eta_{g_0}(X)U + [U, X] + \nabla_{g_0}m = 0, \qquad \Delta_{g_0}m = 0 \quad \textit{and} \quad L_U m = 0. \tag{104}$$

PROOF. Since $\mathrm{met}(\mathscr{L}_\varepsilon) = g_0/\varepsilon$, we know by Theorem 6.12 that $X + m/\varepsilon \in \mathrm{Sym}(\mathscr{L}_\varepsilon)$ if and only if

$$X \in \mathscr{C}_{g_0}(M), \qquad -\eta_{g_0}(X)U + [U, X] + \varepsilon\nabla_{g_0}\left(\frac{m}{\varepsilon} - \frac{n-2}{4}\eta_{g_0}(X)\right) = 0$$

and

$$\frac{\varepsilon}{2}\Delta_{g_0}\left(\frac{m}{\varepsilon}\right) + L_U\left(\frac{m}{\varepsilon}\right) = 0.$$

If this holds for all $\varepsilon > 0$, we get

$$X \in \mathscr{C}_{g_0}(M), \qquad -\eta_{g_0}(X)U + [U, X] + \nabla_{g_0}m = 0,$$
$$\frac{n-2}{4}\eta_{g_0}(X) = \text{constant},$$
$$\Delta_{g_0}m = 0 \quad \text{and} \quad L_U m = 0.$$

This completes the proof. □

7.3. *Gradient dynamics with noise.* Let (M, g_0) be a Riemannian manifold, and consider the deterministic system

$$\dot{x} = -\nabla_{g_0} V(x), \quad \text{where } V \geq 0. \tag{105}$$

Here again we extend this deterministic smooth system to a stochastic one by considering a diffusion process (ξ) with generator

$$\mathscr{L}_\varepsilon = \varepsilon\left(\tfrac{1}{2}\Delta_{g_0} + W\right) - \nabla_{g_0} V, \quad \text{where } W \in \mathscr{X}(M). \tag{106}$$

Certain properties of the deterministic system (105) still remain in the diffusion with generator $\mathscr{L}_\varepsilon$ given by (106).

PROPOSITION 7.5. *Let $\mathscr{L}_\varepsilon$ be given by* (106) *and let (ξ) be a diffusion with generator $\mathscr{L}_\varepsilon$ and having infinite lifetime. The following properties hold.*

(i) *If $L_W V = 0$, $\operatorname{div}_{g_0} W = 0$ and $\int e^{-2V/\varepsilon}\Omega_{g_0} < +\infty$, then (ξ) has an invariant probability proportional to $e^{-2V/\varepsilon}$ (with respect to the Riemannian measure). In particular, the "most probable" points are those for which $e^{-2V/\varepsilon}$ is maximum, that is, the minima of V which are the stationary points of the deterministic system* (105).

(ii) *If $V \geq 0$ and $(\frac{1}{2}\Delta_{g_0} + W)V \leq 0$, then the process $(V(\xi))$ is a nonnegative supermartingale [note that V is a Lyapunov function for* (105) *since $(d/dt)V(x_t) = -\|\nabla_{g_0} V(x_t)\|^2 \leq 0$].*

PROOF. We just outline the proofs.

(i) It is clear from Proposition 6.16.

(ii) By (106), we have $\mathscr{L}_\varepsilon V = \varepsilon(\frac{1}{2}\Delta_{g_0} + W)V - g_0(\nabla_{g_0} V, \nabla_{g_0} V) \leq 0$. □

It can be noted that we have a certain latitude with the metric. Indeed, if $f\colon \mathbb{R} \to \mathbb{R}$ is such that $f' > 0$, then we have, by (124),

$$\nabla_{g_0} V = \nabla_{f'(V) g_0} f(V). \tag{107}$$

Therefore, another possibility for extending the deterministic smooth system (105) to a stochastic one is to consider a diffusion process (ξ) with generator

$$\mathscr{L}_\varepsilon = \varepsilon\left(\tfrac{1}{2}\Delta_{f'(V) g_0} + W\right) - \nabla_{f'(V) g_0} f(V), \quad \text{where } W \in \mathscr{X}(M). \tag{108}$$

If $L_W V = 0$, $\operatorname{div}_{g_0} W = 0$, an invariant measure may now exist as soon as f is such that $\int e^{-f(V)/\varepsilon} f'(V)^{n/2}\Omega_{g_0} \leq +\infty$ (to be compared with the condition $\int e^{-2V/\varepsilon}\Omega_{g_0} < +\infty$ in the previous proposition).

7.4. *Modelling noises in systems for invariance purposes.* In this section, we raise the following question: Given a vector field describing the dynamics of a deterministic system, is it possible to model this latter as a diffusion process with specified symmetries? We illustrate this problem in a practical situation.

Consider a target for which we assume that the speed vector is fixed and known (see Figure 2) and the associated tracking problem. The modelling consists of taking linear coordinates in the plane containing both the trajectory of the target and the observation point (see [14]). In the absence of noise, the evolution equation of the target is given by

$$\dot{x}_1 = -V, \qquad \dot{x}_2 = 0. \tag{109}$$

For filtering purposes, the evolution of the state vector x_t is usually modelled as a stochastic process solution of the stochastic differential equation

$$\begin{aligned} dx_1(t) &= -V\,dt + \sqrt{\varepsilon}\, dv_1(t), \\ dx_2(t) &= \sqrt{\varepsilon}\, dv_2(t), \end{aligned} \tag{110}$$

where $\sqrt{\varepsilon}$ is the standard deviation of the noise process. The related observation process y_t is modelled as

$$dy_t = h(x_t)\, dt + dw_t. \tag{111}$$

Here v_t and w_t are assumed to be independent Brownian motions. The conditional density p_t of the state x_t given the past observations $(y_s,\ s \le t)$

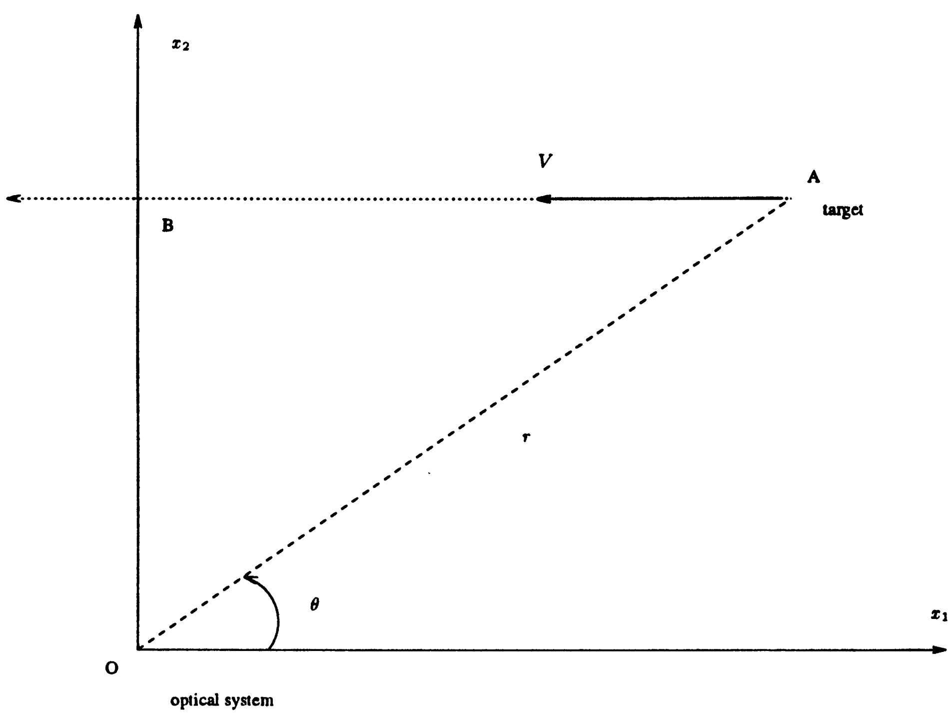

FIG. 2. *Two-dimensional tracking example.*

satisfies the so-called Zakai equation:

$$(112)\quad dp_t = \left(\frac{\varepsilon}{2} \frac{\partial^2 p_t}{\partial x_1^2} + \frac{\varepsilon}{2} \frac{\partial^2 p_t}{\partial x_2^2} + V \frac{\partial p_t}{\partial x_1} - \frac{1}{2} \|h\|^2 p_t \right) dt + \sum_{i=1}^{p} h_i p_t \circ dy_t^i.$$

When the output function h is a function of the distance to the target, it can be shown by direct computation that there exists a particular solution with symmetry of the Zakai equation (112) as follows.

PROPOSITION 7.6. *If $h = (h_1, \dots, h_p)'$ depends only on $\|x\|$, there exists a particular solution of the Zakai equation* (112) *of the form*

$$p_t(x_1, x_2) = \exp\left(\frac{v}{\varepsilon} x_1 \right) q_t\left(\sqrt{x_1^2 + x_2^2} \right) = \exp\left(\frac{v}{\varepsilon} r \cos \theta \right) q_t(r),$$

where q_t satisfies the following stochastic PDE on $]0, +\infty[$:

$$(113)\quad dp_t = \left(\frac{\varepsilon}{2} \frac{\partial^2 q_t}{\partial r^2} + \frac{\varepsilon}{2r} \frac{\partial q_t}{\partial r} - \frac{V^2}{2\varepsilon} q_t - \frac{\|h\|^2}{2} q_t \right) dt + \sum_{i=1}^{p} h_i q_t \circ dy_t^i.$$

This fact can be explained by the existence of symmetries for the filtering problem and will be developed in a forthcoming paper (see [4]). However, we simply want to stress here that the existence of such a symmetry results from a fit between the state noise and the output function. Another choice of diffusion to model the deterministic system (110) would not necessarily be adapted to another type of output function. This is suggested by the following proposition.

PROPOSITION 7.7. *Let $U \in \mathscr{X}(M)$ and $h_1, \dots, h_p \in C^\infty(M)$ be given. For $\varepsilon > 0$ and $y = (y_1, \dots, y_p) \in \mathbb{R}^p$, let us define*

$$(114)\quad \mathscr{L}_\varepsilon = \frac{\varepsilon}{2} \Delta_{g_0} + U \quad \text{and} \quad \mathscr{L}_\varepsilon^y = \frac{\varepsilon}{2} \Delta_{g_0} + U - \frac{\|h\|^2}{2} + \sum_{i=1}^{p} y_i h_i.$$

Let $X \in \mathscr{X}(M)$ and $m \in C^\infty(M)$ be such that

$$(115)\quad \begin{aligned} &X \in \mathscr{I}_{g_0}(M), \qquad L_X h_1 = \cdots = L_X h_p = 0, \qquad [U, X] + \Delta_{g_0} m = 0, \\ &\qquad\qquad \Delta_{g_0} m = 0, \qquad L_U m = 0. \end{aligned}$$

Then, for all $\varepsilon > 0$, $-X + (m/\varepsilon) u(\partial/\partial u)$ is an infinitesimal symmetry of $\partial_t f - \mathscr{L}_\varepsilon^y f$, for all $y = (y_1, \dots, y_p) \in \mathbb{R}^p$.

PROOF. Let $\varepsilon > 0$ and $y = (y_1, \dots, y_p) \in \mathbb{R}^p$ be fixed. By Theorem 6.19, $-X + (m/\varepsilon) u(\partial/\partial u)$ is an infinitesimal symmetry of $\partial_t f - \mathscr{L}_\varepsilon^y f$ if and only if there exists a sequence $(X_i)_{i \in \mathbb{N}}$ in $\mathscr{H}_g(M)$ which satisfies the induction

$$\begin{aligned} X_0 &= X, \\ X_1 &= -\eta_{g_0}(X_0) B + [B, X_0] + \nabla_{g_0} m, \\ X_{i+2} &= -\eta_{g_0}(X_{i+1}) B \\ &\quad + [B, X_{i+1}] + \nabla_{g_0}\left(g_0(X_{i+1}, B) + \tfrac{1}{2} L_{X_i} H_{\mathscr{L}_\varepsilon^y} + \tfrac{1}{2} \eta_{g_0}(X_i) H_{\mathscr{L}_\varepsilon^y} \right), \end{aligned}$$

where

$$H_{\mathscr{L}_\varepsilon^y} = H_{\mathscr{L}_\varepsilon} + \|h\|^2 - 2\sum_{i=1}^{p} y_i h_i.$$

Under our assumptions, we see by Proposition 7.4 that $X + m/\varepsilon \in \mathrm{Sym}(\mathscr{L}_\varepsilon)$ and, thanks to Remark 6.3, it can be seen that the sequence $(X_i)_{i\in\mathbb{N}} = (X, 0, \dots, 0, \dots)$ satisfies the previous induction. □

COROLLARY 7.8. *If $U = -V(\partial/\partial x_1)$, $h = \tilde{h}(r)$ and $g_0 = g_{\mathbb{R}^2}$ so that*

$$\mathscr{L}_\varepsilon^y = \frac{\varepsilon}{2}\Delta - V\frac{\partial}{\partial x_1} - \frac{\|h\|^2}{2} + yh,$$

then, for all $y \in \mathbb{R}$ and $\varepsilon > 0$,

$$\zeta = -\frac{\partial}{\partial\theta} - \frac{V}{\varepsilon} r\sin\theta u\frac{\partial}{\partial u}$$

is an infinitesimal symmetry of the parabolic equation $\partial_t f - \mathscr{L}_\varepsilon^y f = 0$.

PROOF. With the notation of Proposition 7.7, we have $X = \partial/\partial\theta$ and $m = -Vr\sin\theta = -Vx_2$. It can easily be checked that (115) is satisfied. □

We now apply these results to the case of bearing only measurements [2], which is known to be a difficult problem in tracking. If the measurements are available at discrete instants of a uniform partition $0 = t_0 < \cdots < t_k < \cdots$, with time step $\delta t = t_{k+1} - t_k$, we can note

$$z_k = \theta(x(t_k)) = \arctan\left(\frac{x_2(t_k)}{x_1(t_k)}\right).$$

Suppose that, after having measured z_k, we transform it into $H(z_k)$, where $H(\theta) = \log\tan(\theta/2)$, and process the new observations $H(z_0), \dots, H(z_k), \dots$. With our choice of continuous representation for the observations, we have $y(t_k) = H(z_k)$ and this leads to a continuous time output function h in (111) given by

$$\begin{aligned} dy(t) &\simeq H(z_{k+1}) - H(z_k) = (H\circ\theta)(x(t_{k+1})) - (H\circ\theta)(x(t_k)) \\ &= -V\frac{\partial(H\circ\theta)}{\partial x_1}(x(t_k))\,\delta t + \cdots, \quad \text{by (109)}, \end{aligned}$$

that is, after calculation,

$$h(x) = -\delta t\, V\frac{\partial(H\circ\theta)}{\partial x_1}(x) \propto \frac{1}{r}.$$

Thus, Proposition 7.6 applies and the Zakai equation (112) may be "reduced" to a lower dimensional state (this is developed in a forthcoming paper [4]).

7.5. *Examples.* We give examples of diffusions with symmetries when the metric associated with the generator is that of the hyperbolic plane. Let H^2 denote the hyperbolic plane $M = \mathbb{R} \times]0, +\infty[$ with metric

$$g_{H^2} = \frac{1}{x_2^2}\left(dx_1^2 + dx_2^2\right). \tag{116}$$

If Δ, div and ∇ denote the Laplacian, the divergence and the gradient for the useful flat metric on M, we have, by Lemma A.1,

$$\begin{aligned} \nabla_{H^2} &= x_2^2\, \nabla, \\ \operatorname{div}_{H^2} &= \operatorname{div} - \frac{2}{x_2}\, dx_2, \\ \Delta_{H^2} &= x_2^2 \Delta. \end{aligned} \tag{117}$$

The infinitesimal isometries of H^2 are

$$\begin{aligned} \mathscr{I}_{H^2}(M) = \mathbb{R} - \operatorname{Span}\Bigg\{ &\frac{\partial}{\partial x_1}, x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2}, \\ &\left(x_1^2 - x_2^2\right)\frac{\partial}{\partial x_1} + 2x_1x_2\frac{\partial}{\partial x_2}\Bigg\}, \end{aligned} \tag{118}$$

and it can be seen that $\mathscr{K}_{H^2}(M) = \mathscr{I}_{H^2}(M)$ and that $\mathscr{P}_{H^2}(M) = \{0\}$.

EXAMPLE 7.1. Let

$$\mathscr{L} = \frac{1}{2}x_2^2\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right) + \lambda x_2 \frac{\partial}{\partial x_2}.$$

Then $\operatorname{met}(\mathscr{L}) = g_{H^2}$ and $\mathscr{L}$ can be written

$$\mathscr{L} = \tfrac{1}{2}\Delta_{H^2} + \lambda \Delta_{H^2}(\log x_2).$$

By Proposition 6.6, we find

$$K_{\mathscr{L}} = 0 \quad \text{and} \quad H_{\mathscr{L}} = x_2^{-\lambda}\Delta_{H^2} x_2^{\lambda} = x_2^{-\lambda} x_2^2 \Delta x_2^{\lambda} = \text{constant}.$$

Therefore, by Theorem 6.12, we find that

$$\mathscr{S}\!\mathit{ym}(\mathscr{L}) = \left\{X + \lambda L_X \log x_2 + \text{constant, with } X \in \mathscr{I}_{H^2}(M)\right\}.$$

EXAMPLE 7.2. Let

$$\mathscr{L} = \tfrac{1}{2}\Delta_{H^2} + B_0 + \lambda\, \nabla_{H^2} \log x_2,$$

where

$$B_0 = \kappa_1 \frac{\partial}{\partial x_1} + \kappa_2\left(x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2}\right).$$

By Proposition 6.21, we find that $B_0 + \lambda L_{B_0} \log x_2 + \text{constant} \in \mathscr{P}_{\mathscr{L}}$.

EXAMPLE 7.3. Let

$$\mathscr{L} = \tfrac{1}{2}\Delta_{H^2} + \lambda_1 \nabla_{H^2} \log x_1 + \lambda_2 \nabla_{H^2} \log x_2.$$

If $x_1 = r\cos\theta$ and $x_2 = r\sin\theta$, we find by Proposition 6.6 that

$$K_{\mathscr{L}} = 0 \quad \text{and} \quad H_{\mathscr{L}} = \text{constant} \times \tan^2\theta.$$

Therefore, it is a consequence of Theorem 6.12 that $r(\partial/\partial r)$ + constant belongs to $\mathscr{Sym}(\mathscr{L})$ and also to $\mathscr{P}_{\mathscr{L}}$ by Remark 6.3.

EXAMPLE 7.4. Let

$$\mathscr{L} = \tfrac{1}{2}\Delta_{H^2} + \lambda \nabla_{H^2} \log r + \lambda_2 \nabla_{H^2} \psi(\theta).$$

By Proposition 6.6, we find

$$K_{\mathscr{L}} = 0 \quad \text{and} \quad H_{\mathscr{L}} = \text{function}(\theta).$$

We conclude as in Example 7.3.

APPENDIX

A.1. Some geometric identities. For all geometric tools introduced in this Appendix, our reference is [13]. In this section, M is a manifold of dimension n equipped with a Riemannian metric g.

LEMMA A.1. *Let $\phi \in D(M)$, $X, Y \in \mathscr{X}(M)$, $f \in C^\infty(M)$ and $\alpha\colon M \hookrightarrow]0, +\infty[$ be a smooth function. The following identities hold*:

$$\phi_*(\alpha g) = \phi_*(\alpha)\phi_*(g) = \alpha \circ \phi^{-1}\phi_*(g), \tag{119}$$

$$\nabla_{\phi_*(g)} f = \phi_*\big(\nabla_g(\phi^*(f))\big), \tag{120}$$

$$\Omega_{\phi_*(g)} = \phi_*(\Omega_g), \tag{121}$$

$$\operatorname{div}_{\phi_*(g)} X = \phi_*\big(\operatorname{div}_g(\phi_*(X))\big), \tag{122}$$

$$\Delta_{\phi_*(g)} f = \phi_*\big(\Delta_g(\phi^*(f))\big) = \Delta_g^{\phi} f, \tag{123}$$

$$\nabla_{\alpha g} f = \frac{1}{\alpha}\nabla_g f, \tag{124}$$

$$\Omega_{\alpha g} = \alpha^{n/2}\Omega_g, \tag{125}$$

$$\begin{aligned} \operatorname{div}_{\alpha g} X &= \operatorname{div}_g X + \frac{n}{2} L_X(\log\alpha) \\ &= \operatorname{div}_g X + \frac{n}{2} g\big(X, \nabla_g(\log\alpha)\big), \end{aligned} \tag{126}$$

$$\begin{aligned} \Delta_{\alpha g} &= \frac{1}{\alpha}\Delta_g + \frac{n-2}{2\alpha}\nabla_g(\log\alpha) \\ &= \frac{1}{\alpha}\Delta_g + \frac{n-2}{2}\nabla_{\alpha g}(\log\alpha). \end{aligned} \tag{127}$$

The following properties are easily proved.

LEMMA A.2. *Let X, Y and Z be vector fields on M. Then*

$$\operatorname{div}_g Z = -\operatorname{trace}(A_Z), \tag{128}$$

$$\operatorname{div}_g([Y,Z]) = Y\cdot(\operatorname{div}_g Z) - Z\cdot(\operatorname{div}_g Y), \tag{129}$$

$$A_X Y - A_Y X = [X,Y], \tag{130}$$

$$g((A_Z + A_Z^*)X, Y) = -(L_Z g)(X,Y), \tag{131}$$

$$g((A_Z - A_Z^*)X, Y) = -dZ^\flat(X,Y), \tag{132}$$

$$\nabla_g(g(X,Y)) = -A_X^* Y - A_Y^* X. \tag{133}$$

LEMMA A.3. *For any $X \in \mathscr{C}_g(M)$ and smooth function φ, we have*

$$\operatorname{div}_g X = \frac{n}{2}\eta_g(X) \tag{134}$$

and

$$\left[X, \nabla_g \varphi\right] = \nabla_g(L_X\varphi) - \eta_g(X)\nabla_g\varphi. \tag{135}$$

PROOF. By (128) and (131) in Lemma A.2, we have

$$\begin{aligned}\operatorname{div}_g X &= -\operatorname{trace}(A_X) = -\operatorname{trace}(A_X^*) = -\frac{1}{2}\operatorname{trace}(A_X + A_X^*)\\ &= -\frac{1}{2}\operatorname{trace}\big(-\eta_g(X)\big) = \frac{n}{2}\eta_g(X).\end{aligned}$$

Let Y and Z be vector fields. We have, by (4),

$$L_X g(Y,Z) = (L_X g)(Y,Z) + g(L_X Y, Z) + g(Y, L_X Z).$$

Now, since $X \in \mathscr{C}_g(M)$ and $Z = \nabla_g\varphi$, this gives

$$L_X L_Y \varphi = \eta_g(X) L_Y\varphi + L_{[X,Y]}\varphi + g\big(Y, \left[X, \nabla_g\varphi\right]\big)$$

and, therefore,

$$g\big(Y, \left[X,\nabla_g\varphi\right]\big) = L_Y L_X\varphi - \eta_g(X)L_Y\varphi = g\big(Y, \nabla_g(L_X\varphi)\big) - \eta_g(X) g(Y, \nabla_g\varphi),$$

which gives the result. □

LEMMA A.4. *Let X be a vector field on M with local flow Φ_s^X. Then, for every $x \in M$ and $f \in C^\infty(M)$, the function*

$$\mathscr{L}^{\Phi_s^X} f = \mathscr{L}\big(f\circ\Phi_s^X\big)\circ\Phi_{-s}^X \tag{136}$$

is well defined in a neighborhood of x for s small enough and we have

$$\left.\frac{d}{ds}\right|_{s=0}\big(\mathscr{L}^{\Phi_s^X} f\big)(x) = [\mathscr{L}, X]f(x). \tag{137}$$

PROOF. Formula (136) is a local version of (10). For y in a neighborhood of x, we have, for any smooth function φ,

$$\begin{aligned}\big(\varphi\circ\Phi_s^X\big)(y) &= \varphi(y) + s\epsilon_0(\varphi, y, s)\\ &= \rho(y) + s(X\varphi)(y) + s^2\epsilon_1(\rho, y, s),\end{aligned} \tag{138}$$

where, for $k = 0, 1$,

$$\epsilon_k(\varphi, y, s) = \int_0^1 \frac{(1-h)^k}{k!} \left. \frac{d^{k+1}\varphi(\Phi_u^X(y))}{du^{k+1}} \right|_{u=sh} dh. \tag{139}$$

Therefore,

$$\begin{aligned}
\mathscr{L}^{\Phi_s^X} f(x) &= \mathscr{L}(f \circ \Phi_s^X)(\Phi_{-s}^X(x)) \\
&= (\mathscr{L}f)(\Phi_{-s}^X(x)) + s\mathscr{L}(Xf)(\Phi_{-s}^X(x)) + s^2(\mathscr{L}\epsilon_1(f,\cdot,s))(\Phi_{-s}^X(x)) \\
&= (\mathscr{L}f)(x) - s(X(\mathscr{L}f))(x) + s^2\epsilon_1(\mathscr{L}f, x, -s) + s(\mathscr{L}(Xf))(x) \\
&\quad - s^2\epsilon_0(\mathscr{L}(Xf), x, -s) + s^2(\mathscr{L}\epsilon(f,\cdot,s))(\phi_{-s}^X(x))
\end{aligned}$$

so that

$$\left. \frac{d}{ds}(\mathscr{L}^{\Phi_s^X} f)(x) \right|_{s=0} = -(X(\mathscr{L}f))(x) + (\mathscr{L}(Xf))(x) = [\mathscr{L}, X] f(x). \quad \square$$

A.2. Proofs.

PROOF OF LEMMA 5.7. Thanks to the identities of Lemma A.2, we have

$$\begin{aligned}
[\Delta_g, X] f &= \operatorname{div}_g(\nabla_g(Xf)) - X \cdot \operatorname{div}_g(\nabla_g f) \\
&= \operatorname{div}_g(\nabla_g g(X, \nabla_g f)) - X \cdot \operatorname{div}_g(\nabla_g f) \\
&= \operatorname{div}_g(-A_X^* \nabla_g f - A_{\nabla_g f}^* X) - X \cdot \operatorname{div}_g(\nabla_g f) \quad [\text{by } (133)] \\
&= \operatorname{div}_g(-A_X^* \nabla_g f - A_X \nabla_g f + A_X \nabla_g f - A_{\nabla_g f} X) - X \cdot \operatorname{div}_g(\nabla_g f) \\
&= -\operatorname{div}_g((A_X^* + A_X)\nabla_g f) + \operatorname{div}_g(A_X \nabla_g f - A_{\nabla_g f} X) - X \cdot \operatorname{div}_g(\nabla f) \\
&= -\operatorname{div}_g((A_X^* + A_X)\nabla_g f) + \operatorname{div}_g([X, \nabla_g f]) - X \cdot \operatorname{div}_g(\nabla_g f) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad [\text{by } (130)] \\
&= -\operatorname{div}_g((A_X + A_X^*)\nabla_g f) - \nabla_g(\operatorname{div}_g X) \cdot f \quad [\text{by } (129)].
\end{aligned}$$

On the other hand, we have, for any smooth function ρ,

$$\rho\Delta_g f = \rho \operatorname{div}_g(\nabla_g f) = \operatorname{div}_g(\rho \nabla_g f) - \nabla_g \rho \cdot f,$$

so that

$$[\Delta_g, X] f - \rho\Delta_g f = -\operatorname{div}_g((A_X + A_X^* + \rho)\nabla_g f) + \nabla_g(\rho - \operatorname{div}_g X) \cdot f. \tag{140}$$

Therefore, $[\Delta_g, X] - \rho\Delta_g$ is a vector field if and only if

$$Zf = \operatorname{div}_g((A_X + A_X^* + \rho)\nabla_g f)$$

defines a first-order operator, that is, if it satisfies $Zf^2 = 2fZf$. This leads to $A_X + A_X^* + \rho I = 0$ or $L_X g = \rho g$ by Lemma A.2, that is, $X \in \mathscr{C}_g(M)$ and $\rho = \eta_g(X)$ (see Definition 5.6). Then, by (140) and (134), we have

$$[\Delta_g, X] - \rho\Delta_g = \nabla_g(\rho - \operatorname{div}_g X) = -\frac{n-2}{2} \nabla_g \rho. \quad \square$$

PROOF OF LEMMA 6.6. In (61), we just prove the last equation and leave the rest to the reader. Let $\mathscr{L}' = (1/\alpha)\mathscr{L}$. With $g' = \alpha g$, we have $g'(K_{\mathscr{L}'}X, Y) = -d(B')^{\flat'}(X, Y)$, where

$$g'(B', X) = \alpha g(B', X) = \alpha g\left(\frac{1}{\alpha}B - \frac{n-2}{4}\nabla_{g'}(\log\alpha), X\right),$$

so that $(B')^{\flat'} = B^\flat - \frac{1}{4}(n-2)d(\log\alpha)$ and therefore $d(B')^{\flat'} = dB^\flat$ and $K_{\mathscr{L}'} = K_{\mathscr{L}}$ since $d^2(\log\alpha) = 0$. Moreover,

$$\begin{aligned}
H_{\mathscr{L}'} &= \operatorname{div}_{g'} B' + g'(B', B') - \frac{2c}{\alpha}\\
&= \operatorname{div}_g B' + \frac{n}{2}g(B', \nabla_g(\log\alpha)) + \alpha g(B', B') - \frac{2c}{\alpha} \quad [\text{by } (126)]\\
&= \operatorname{div}_g\left(\frac{1}{\alpha}\left(B - \frac{n-2}{4}\nabla_g \log\alpha\right)\right)\\
&\quad + \frac{n}{2\alpha}g\left(B - \frac{n-2}{4}\nabla_g(\log\alpha), \nabla_g(\log\alpha)\right) - \frac{2c}{\alpha}\\
&\quad + \frac{1}{\alpha}g\left(B - \frac{n-2}{4}\nabla_g(\log\alpha), B - \frac{n-2}{4}\nabla_g(\log\alpha)\right) - \frac{2c}{\alpha}\\
&= \frac{1}{\alpha}\operatorname{div}_g B - \frac{n-2}{4\alpha}\Delta_g(\log\alpha)\\
&\quad - \frac{1}{\alpha}g\left(B - \frac{n-2}{4}\nabla_g(\log\alpha), \nabla_g(\log\alpha)\right) - \frac{2c}{\alpha}\\
&\quad + \frac{1}{\alpha}g(B, \nabla_g(\log\alpha)) + \frac{1}{\alpha}g(B, B)\\
&\quad + \frac{4-n^2}{16\alpha}g(\nabla_g(\log\alpha), \nabla_g(\log\alpha)) - \frac{2c}{\alpha}\\
&= \frac{1}{\alpha}\operatorname{div}_g B - \frac{n-2}{4\alpha}\Delta_g(\log\alpha)\\
&\quad - \frac{(n-2)^2}{16\alpha}g(\nabla_g(\log\alpha), \nabla_g(\log\alpha)) + \frac{1}{\alpha}g(B, B) - \frac{2c}{\alpha}\\
&= \frac{1}{\alpha}H_{\mathscr{L}} - \frac{n-2}{4\alpha}\Delta_g(\log\alpha) - \frac{(n-2)^2}{16\alpha}g(\nabla_g(\log\alpha), \nabla_g(\log\alpha))\\
&= \frac{1}{\alpha}\left(H_{\mathscr{L}} - \Delta_g\mu - g(\nabla_g\mu, \nabla_g\mu)\right),
\end{aligned}$$

where $\mu = \frac{1}{4}(n-2)\log\alpha$.

The identity (62) is proved as follows:

$$-\eta_g(X)B + [B, X] = -\eta_g(X)B - A_X B + A_B X \quad [\text{by } (130)]$$

$$= A_X^* B + A_B^* X + (A_B - A_B^*) X$$
$$\big[\text{since } A_X + A_X^* + \eta_g(X) = 0\big]$$
$$= (A_B - A_B^*) X - \nabla_g(g(X, B)) \quad [\text{by (133)}].$$

As for (64), we have $B^\flat = B_0^\flat + d\varphi$, so that $dB^\flat = dB_0^\flat + d^2\varphi = dB_0^\flat$. For $X \in \mathscr{C}_g(M)$, we have

$$K_{\mathscr{L}} X = i_X(dB^\flat) = i_X\big(dB_0^\flat\big) = (A_{B_0} - A_{B_0}^*) X$$
$$= -\eta_g(X) B_0 + [B_0, X] + \nabla_g(g(X, B_0)).$$

A simple calculation gives the expression of $H_{\mathscr{L}}$, with the help of (58). □

Proof of Theorem 6.10. The proof follows the lines of that of Proposition 2.2 in [3].

For the introduction of the formalism of jet spaces, we follow [16]. If $U^{(2)}$ is the jet space of order 2 associated with smooth functions from M to $\mathbb{R}$, a generic point in $U^{(2)}$ is denoted by $u^{(2)} = (u, (u_{x_i}, 1 \le i \le n), (u_{x_i x_j}, 1 \le i \le j \le n))$ for a given coordinate system $x_1, \ldots, x_n$. If f is a smooth function on M, the 2-jet of f (or second prolongation of f) at x_0 is the collection $\mathrm{pr}^{(2)}[f](x_0)$ of partial derivatives of f at x up to order 2. For $\zeta \in \mathfrak{X}(M \times \mathbb{R})$, the second prolongation of ζ is a vector field $\mathrm{pr}^{(2)}[\zeta]$ of $M \times U^{(2)}$ whose action on a smooth function F on $M \times U^{(2)}$ is given by

$$(\mathrm{pr}^{(2)}[\zeta] \cdot F)(x, u^{(2)}) = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} F\Big(x_\varepsilon, \mathrm{pr}^{(2)}\big[\tilde{\Phi}_\varepsilon^\zeta \cdot f\big](x_\varepsilon)\Big),$$

where f is any smooth function such that $\mathrm{pr}^{(2)}[f](x) = u^{(2)}$ and where $(x_\varepsilon, u_\varepsilon) = \Phi_\varepsilon^\zeta(x, f(x))$.

If Γ is the smooth (even analytic) map on $M \times U^{(2)}$ given by

$$\Gamma(x, u^{(2)}) = \sum_{i,j=1}^n a^{ij}(x) u_{x_i x_j} + \sum_{i=1}^n b^i(x) u_{x_i} + c(x) u,$$

it is proved in [16], Theorem 2.71, that $\mathfrak{S}_{\mathscr{L}}$ coincides with the Lie algebra of infinitesimal symmetries of Γ when Γ is nondegenerate in the sense of Definition 2.70 in [16]. This happens to be the case, in particular, since $\mathscr{L}$ is analytic and equation $\mathscr{L}f = 0$ can be written

$$\partial^2_{x_1 x_1} f = \frac{1}{a^{11}(x)} \left(- \sum_{(i,j) \ne (1,1)}^n a^{ij}(x)\, \partial^2_{x_i x_j} f - \sum_{i=1}^n b^i(x)\, \partial_{x_i} f - c(x) f \right),$$

namely, in Kovalevskaya form [16, page 166]. Therefore, $\mathfrak{S}_{\mathscr{L}}$ is a Lie algebra.

It follows from the study on second-order linear equations in [17] that, since equation $\mathscr{L}f = 0$ is not *very degenerate* in the sense of [17], page 345, then the decomposition (i) is satisfied (see [17], page 347) and we have, for any $\xi \in \mathfrak{U}(M \times \mathbb{R})$,

$$(141) \qquad \zeta \in \mathfrak{S}_{\mathscr{L}} \quad \Leftrightarrow \quad \begin{cases} \exists\, \nu \in C^\infty(M \times \mathbb{R}), \forall\, (x, u^{(2)}) \in M \times U^{(2)}, \\ \mathrm{pr}^{(2)}[\zeta] \Gamma(x, u^{(2)}) - \nu(x, u) \Gamma(x, u^{(2)}) = 0. \end{cases}$$

At last, to prove equivalence (70), we notice that when $\zeta \in \mathfrak{U}(M \times \mathbb{R})$ is given by (65) we have

$$\mathrm{pr}^{(2)}[\zeta]\Gamma(x, \mathrm{pr}^{(2)}[f](x)) = \left(\left[\mathscr{L}, \tilde{\zeta}\right] f + m\mathscr{L} f\right)(x)$$

by (67). Therefore, if we prove that the function ν in (141) does not depend on u, then equivalence (70) derives from equivalence (141) with $u^{(2)} = \mathrm{pr}^{(2)}[f](t, x)$, for any f, and $\rho = \nu - m$. For this purpose, we notice that $\mathrm{pr}^{(2)}[\zeta]\Gamma(x, u^{(2)})$ is a linear form in $u^{(2)}$ since $[\mathscr{L}, \tilde{\zeta}] + m\mathscr{L}$ is a linear differential operator. By (141), this linear form is zero as soon as the linear form $\Gamma(x, \cdot)$ is zero and therefore these linear forms are proportional, that is, $\nu(x, u) = \nu(x)$. □

PROOF OF THEOREM 6.12. By (3), we have

$$\begin{aligned}
&[\mathscr{L}, X + m] = \rho\mathscr{L} \\
&\Leftrightarrow \quad \frac{1}{2}[\Delta_g, X] - \frac{1}{2}\rho\Delta_g = -\left[\frac{1}{2}\Delta_g + B + c, m\right] - [B + c, X] \\
&\qquad\qquad + \rho(B + c) \\
&\Leftrightarrow \quad \frac{1}{2}[\Delta_g, X] - \frac{1}{2}\rho\Delta_g = -\nabla_g m' - [B, X] + \rho B \\
&\qquad\qquad + L_X c + \rho c - \frac{1}{2}\Delta_g m - L_B m \\
&\qquad\qquad \Big[\text{since } [\Delta_g, m]\varphi = \mathrm{div}_g(m\nabla_g\varphi + \varphi\nabla_g m) - m\Delta_g\varphi \\
&\qquad\qquad\qquad = 2g(\nabla_g\varphi, \nabla_g m) + \Delta_g m\varphi\Big] \\
&\Leftrightarrow \quad \frac{1}{2}[\Delta_g, X] - \frac{1}{2}\rho\Delta_g = -\nabla_g m - [B, X] + \rho B \\
&\qquad \text{and } \frac{1}{2}\Delta_g m + L_B m = L_X c + \rho c \\
&\qquad\qquad \left(\text{since } [\Delta_g, X]1 - \rho\Delta_g 1 = 0\right) \\
&\Leftrightarrow \quad \begin{cases} X \in \mathscr{C}_g(M) \text{ and } \rho = \eta_g(X), \\ -\dfrac{n-2}{4}\nabla_g\eta_g(X) = -\nabla_g m - [B, X] + \eta_g(X)B, \\ \dfrac{1}{2}\Delta_g m + L_B m = L_X c + \eta_g(X)c \end{cases} \\
&\qquad\qquad (\text{by Lemma 5.7}).
\end{aligned}$$

Now, if $X \in \mathscr{C}_g(M)$ and $\nabla_g m = \frac{1}{4}(n-2)\nabla_g\eta_g(X) + [X, B] + \eta_g(X)B$, we have

$$\frac{1}{2}\Delta_g m + L_B m = L_X c + \eta_g(X)c$$

$$\Leftrightarrow \quad \frac{1}{2}\mathrm{div}_g(\nabla_g m) + g(B, \nabla_g m) = L_X c + \eta_g(X)c$$

$$\Leftrightarrow \quad \frac{n-2}{4}\left(\frac{1}{2}\Delta_g + B\right)\eta_g(X)$$
$$+ \frac{1}{2}\mathrm{div}_g\big([X,B] + \eta_g(X)B\big) + g(B,[X,B]) + \eta_g(X)g(B,B)$$
$$= L_X c + \eta_g(X)c$$

$$\Leftrightarrow \quad \frac{n-2}{8}\Delta_g\eta_g(X) + \frac{1}{2}\big(L_X + \eta_g(X)\big)H_{\mathscr{L}} = 0,$$

where we used the identities

$$\begin{aligned}\mathrm{div}_g\big([X,B] + \eta_g(X)B\big) &\\ = X\cdot(\mathrm{div}_g B) - B\cdot(\mathrm{div}_g X) + \eta_g(X)\mathrm{div}_g B + B\cdot\eta_g(X) &\quad \text{[by (129)]}\\ = \big(X + \eta_g(X)\big)\cdot(\mathrm{div}_g B) - \frac{n-2}{2}B\cdot\eta_g(X) &\quad \text{[by (134)]}\end{aligned}$$

and

$$\begin{aligned} g(B,[X,B]) &= \tfrac{1}{2}\big(g(L_X B, B) + g(B, L_X B)\big)\\ &= \tfrac{1}{2}\big(L_X g(B,B) - (L_X g)(B,B)\big)\\ &= \tfrac{1}{2}\big(L_X g(B,B) - \eta_g(X)g(B,B)\big).\end{aligned}$$

Therefore, at this stage we have proven that

$$[\mathscr{L}, X+m] = \rho\mathscr{L} \quad \Leftrightarrow \quad \begin{cases} X \in \mathscr{C}_g(M) \text{ and } \rho = \eta_g(X),\\ -\dfrac{n-2}{4}\nabla_g\eta_g(X) = -\nabla_g m - [B,X] + \eta_g(X)B,\\ \dfrac{n-2}{4}\Delta_g\eta_g(X) + \big(L_X + \eta_g(X)\big)H_{\mathscr{L}} = 0.\end{cases}$$

This yields (74) and (75). The equality (72) is a straightforward application of (62). To finish, it is proven in [17], page 351, that

$$\Delta_g\eta_g(X) = \frac{1}{n-1}\big(L_X + \eta_g(X)\big)R_g, \tag{142}$$

where R_g is the scalar curvature of the Riemannian manifold (M, g). This explains (73). □

REFERENCES

[1] ABRAHAM, R., MARSDEN, J. E. and RATIU, T. (1988). *Manifolds, Tensor Analysis, and Applications*. Springer, New York.

[2] BAR-SHALOM, Y. and FORTMANN, T. E. (1988). *Tracking and Data Association*. Academic Press, New York.

[3] COHEN DE LARA, M. (1991). A note on the symmetry group and perturbation algebra of a parabolic partial differential operator. *J. Math. Phys.* **32** 1444–1449.

[4] COHEN DE LARA, M. (1993). Reduction of the Zakai equation by invariance group techniques with an application to tracking. Preprint.

[5] DOUBROVINE, B., NOVIKOV, S. and FOMENKO, A. (1982). *Géométrie Contemporaine, Méthodes et Applications* **1**. Mir, Moscow.

[6] EMERY, M. (1989). *Stochastic Calculus in Manifolds*. Springer, Berlin.

[7] GALLOT, S., HULIN, D. and LAFONTAINE, J. (1987). *Riemannian Geometry*. Springer, New York.

[8] GLOVER, J. (1991a). Symmetry groups and translation invariant representations of Markov processes. *Ann. Probab.* **19** 562–586.

[9] GLOVER, J. (1991b). Symmetry groups of Markov processes and the diagonal principle. *J. Theoret. Probab.* **14** 417–440.

[10] GLOVER, J. and MITRO, J. (1990). Symmetries and functions of Markov processes. *Ann. Probab.* **18** 655–668.

[11] HELGASON, S. (1978). *Differential Geometry, Lie Groups and Symmetric Spaces*. Academic Press, New York.

[12] IKEDA, N. and WATANABE, S. (1989). *Stochastic Differential Equations and Diffusion Processes*. North-Holland, Amsterdam.

[13] KOBAYASHI, S. and NOMIZU, K. (1963). *Foundations of Differential Geometry*. Wiley, New York.

[14] LÉVINE, J. and PIGNIÉ, G. (1986). Exact finite dimensional filters for a class of nonlinear discrete-time systems. *Stochastics* **10** 97–132.

[15] LIAO, M. (1992). Symmetry groups of Markov processes. *Ann. Probab.* **20** 563–578.

[16] OLVER, P. J. (1986). *Applications of Lie Groups to Differential Equations*. Springer, New York.

[17] OVSJANNIKOV, L. V. (1982). *Group Analysis of Differential Equations*. Academic Press, New York.

[18] ROGERS, L. C. G. and WILLIAMS, D. (1987). *Diffusions, Markov Processes, and Martingales* **2**. Wiley, New York.

[19] ROSENCRANS, S. I. (1976). Perturbation algebra of an elliptic operator. *J. Math. Anal. Appl.* **56** 317–329.

[20] ROSENCRANS, S. I. (1977). Linearity of the group of a parabolic equation. *J. Math. Anal. Appl.* **61** 537–551.

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