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EXACT ASYMPTOTICS FOR THE PROBABILITY OF EXIT FROM A DOMAIN AND APPLICATIONS TO SIMULATION

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We study the asymptotics of the exit probability $\mathbb{P}^{\varepsilon}_{x,s}\{\tau \le T\}$, where τ is the exit time from an open set and $\mathbb{P}^{\varepsilon}_{x,s}$ is the law of a diffusion process with a small parameter ε multiplying the diffusion coefficient. We consider the case of the Brownian bridge in many dimensions, this choice being motivated by applications to numerical simulation. The method uses recent results reducing the problem to the solution of a system of linear first-order PDE's.

1. Introduction. Let $D \subset \mathbb{R}^n$ be an open set, and let us consider the diffusion process X^{ε} which is associated with the stochastic differential equation

$$
\begin{aligned}
dX_t^{\varepsilon} &= b(X_t^{\varepsilon}, t)\,dt + \sqrt{\varepsilon}\,\sigma(X_t^{\varepsilon}, t)\,dB_t, \\
X_s^{\varepsilon} &= x \in D.
\end{aligned} \tag{1.1}
$$

Let τ denote the exit time of X^{ε} from D. In this paper we are concerned with the problem of finding the exact asymptotics of

$$
\mathbb{P}^{\varepsilon}_{x,s}\{\tau \le T\}, \tag{1.2}
$$

where $T > 0$ and $\mathbb{P}^{\varepsilon}_{x,s}$ is the law of X^{ε}. Of course this quantity tends to 1 if the solution of the deterministic ODE

$$
\begin{aligned}
\dot{x}_t &= b(x_t, t), \\
x_s &= x
\end{aligned}
$$

exits from D before ($<$) time T. In any case the Ventsel–Freidlin theory of large deviations states that

$$
\log \mathbb{P}^{\varepsilon}_{x,s}\{\tau \le T\} \sim -\frac{1}{\varepsilon} u(x,s), \tag{1.3}
$$

where the function u is defined in the following way. Define the action functional I on the set of all continuous paths γ by

$$
I_s(\gamma) = \tfrac{1}{2}\int_s^T \left\langle a(\gamma_t, t)^{-1}(\gamma_t' - b(\gamma_t, t)), \gamma_t' - b(\gamma_t, t)\right\rangle dt \tag{1.4}
$$

if γ is absolutely continuous and $I_s(\gamma) = +\infty$ otherwise (here $a = \sigma\sigma^*$). Then $u(x, s)$ is the infimum of I_s taken on the set of all paths starting at x at time s and such that $\gamma(t) \in D^C$ for some $t \le T$.

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The theory of large deviations, more precisely, states that if the minimizing path γ for I_s is unique, then the asymptotics of the quantity in (1.2) are the same as

$$\mathbb{P}^{\varepsilon}_{x,s}\{\tau \le T,\ X^{\varepsilon} \in B_{\delta}(\gamma)\}, \tag{1.5}$$

where by $B_{\delta}(\gamma)$ we denote the neighborhood (a *tube*) of radius δ of γ. This suggests that the requested asymptotics should depend only on the behavior (curvature?) of the boundary near the point ϕ at which the minimizer γ reaches ∂D.

The goal of this paper is to give an exact equivalent for $\mathbb{P}^{\varepsilon}_{x,s}\{\tau \le T\}$, that is, to remove the log in (1.3). More precisely, we deal with the situation in which X^{ε} is a Brownian bridge with a small parameter multiplying the diffusion coefficient.

The interest for this particular situation, besides the natural question of determining how the boundary near the exit point affects the asymptotics, is motivated by the following application. Let W be an n-dimensional Brownian motion, and let $f\colon D \to \mathbb{R}$ be a function on the open set D. Suppose we want to evaluate numerically the quantity

$$\mathbb{E}^{x}\left[\int_0^{\tau} f(W_s)\,ds\right]. \tag{1.6}$$

Such quantities appear naturally in the stochastic representation of the solutions of second-order linear PDE's. One way to do that is to fix a step ε and to simulate the subsequent positions of a path $W_0 = x, W_{\varepsilon}, W_{2\varepsilon}, \dots, W_{n\varepsilon}, \dots$. If N is the first index such that $W_{N\varepsilon} \notin D$, one then approximates the functional

$$I(W) = \int_0^{\tau} f(W_s)\,ds$$

by the sum

$$I_{\varepsilon}(W) = \varepsilon \sum_{k=0}^{N-1} f(W_{k\varepsilon}).$$

Then, averaging the functional I_{ε} over many independently simulated paths, one has the desired approximation of the expectation in (1.6).

The drawback to this procedure is that one implicitly assumes that $N\varepsilon$ is a good approximation of the exit time τ. Indeed this is not the case, since, if $k < N$, we know that $W_{(k-1)\varepsilon}$ and $W_{k\varepsilon}$ are still in D, but there is a strictly positive probability that the path has performed an excursion out of D in the time interval $[(k-1)\varepsilon, k\varepsilon]$. If f is nonnegative, this means that in the above described procedure I_{ε} systematically overestimates the functional I.

One way to handle the problem is as follows: at each step (k, say), first simulate $W_{(k+1)\varepsilon}$. If $W_{(k+1)\varepsilon} \in D$, then compute the probability p that the process has gone out of D in the time interval $[k\varepsilon, (k+1)\varepsilon]$, given the positions $W_{k\varepsilon}$ and $W_{(k+1)\varepsilon}$, and with probability p kill the process and choose $(k+1)\varepsilon$ as an approximation of the exit time τ, and with probability $1-p$

continue the simulation of the path. The law of a Brownian motion on a given time interval $[k\varepsilon, (k+1)\varepsilon]$ conditioned on its positions $W_{k\varepsilon} = x$ and $W_{(k+1)\varepsilon} = y$ is, up to a translation in the time scale, a Brownian bridge conditioned to be in y at time ε and starting at x. Thus the killing probability p is the exit probability from D of such a process.

Unfortunately this is not an easy quantity to compute, so we suggest replacing it with *its asymptotics as* $\varepsilon \to 0$. In the last section we produce results of simulations showing that the overestimation above can actually produce inaccurate results, which can be considerably improved by the suggested correction with an acceptable price in terms of CPU time.

Of course it might be important to obtain the exact asymptotics also for other conditioned processes, in order to apply the above described technique to processes other than Brownian motion. We think that the argument developed here, which relies on recent results of Fleming and James, might be able to handle more general diffusion processes, but we do not know whether the results would be explicit enough to be effectively implemented in a computer program. Possibly these more general situations should be handled by approximating the conditioned diffusion with the bridge arising from the conditioning of the diffusion obtained by freezing its drift and diffusion coefficients, a situation that should be reduced to the Brownian bridge.

The above described procedure might be useful in other problems in numerical simulation. In particular, it should improve existing techniques of simulation of diffusion processes with reflection, or other conditions, at the boundary.

The table of contents is as follows. In Section 2 we recall the results of Fleming and James, reducing the computation of the requested asymptotics to the solution of a linear partial differential problem of the first order. In Section 3 we study the geometry of the characteristics of this system for the (multidimensional) Brownian bridge and show that the results of Fleming and James are applicable in this situation. We compute the asymptotics in Section 4 (Theorem 4.4), and its dependence on the geometry of ∂D is made explicit; also, some examples are given. Section 5 gives a geometric interpretation of the asymptotics, whereas Section 6 contains the results of the simulations and some related questions.

2. A general result. Let X^ε be the solution of (1.1). We shall assume $\sigma \equiv I$ (the identity matrix) and that the vector field b: $D \times [0,T] \to \mathbb{R}^n$ is Lipschitz continuous in x uniformly in t.

In this section we recall a result giving the asymptotics for the exit probability (1.2) under suitable assumptions. Let us define the function u: $D \times [0,T] \to \mathbb{R}$ by

$$u(x,s) = \inf I_s(\gamma),$$

where I_s is as in (1.4), the infimum being taken on the set of all paths starting at x at time s and such that $\gamma(t) \in D^C$ for some $t \le T$. It can be shown that u solves the Hamilton–Jacobi equation

$$(2.1)\qquad \begin{aligned} \frac{\partial u}{\partial s} + b\cdot Du - \frac{1}{2}|Du|^2 &= 0 && \text{in } D\times]0,T[,\\ u(x,s) &= 0 && \text{on } \partial D\times[0,T],\\ u(x,s) &\to +\infty && \text{as } t\nearrow T,\ x\in D. \end{aligned}$$

The above described equation should be considered in the sense of viscosity solutions [Fleming and Souganidis (1986)] and is intended in the classical sense at each point at which u is differentiable.

The next result, due to Fleming and James (1992), gives an expansion for the exit probability (1.2), at least for (x, s) in a set $N \subset D \times [0,T]$ satisfying the following assumptions.

ASSUMPTION (A). (a) $N\subset D\times[0,T']$, $T' < T$ and N is an open set.
(b) $u\in\mathscr{C}^\infty(\overline{N})$.
(c) Let us write

$$\beta(x,s) = b(x,s) - Du(x,s), \qquad (x,s)\in\overline{N}.$$

Let $\gamma_{x,s}$ be the solution of

$$(2.2)\qquad \begin{aligned} \dot\gamma_{x,s}(t) &= \beta(t,\gamma_{x,s}(t)),\\ \gamma_{x,s}(s) &= x, \end{aligned}$$

and let $z_{x,s}$ be the first point at which $\gamma_{x,s}$ first reaches ∂N. If

$$\Gamma_1 = \{z_{x,s}, (x,s)\in N\},$$

then $\Gamma_1\subset\partial D$. Moreover Γ_1 is a $\mathscr{C}^\infty$-manifold which is relatively open in ∂N, and $\gamma_{x,s}$ crosses ∂N nontangentially.

Then one has [Fleming and James (1992)] the following theorem.

THEOREM 2.1. *Let $N\subset D\times[0,T[$ be such that Assumption* (A) *is satisfied. Then for $(x,s)\in N$ the following expansion holds*:

$$\begin{aligned}\mathbb{P}^\varepsilon_{x,s}\{\tau\le T\} = \exp\left(-\frac{u(x,s)}{\varepsilon}\right)&\exp[-w(x,s)]\\ &\times(1+\psi_1(x,s)\varepsilon+\cdots+\psi_m(x,s)\varepsilon^m+o(\varepsilon^m)),\end{aligned}$$

where $w\colon N\to\mathbb{R}^+$ is the solution of

$$\begin{aligned}\frac{\partial w}{\partial s} + (b-Du)Dw &= -\frac{1}{2}\Delta u \quad \text{in } N,\\ w &= 0 \quad \text{on } \partial D\times[0,T[\,\cap\,\overline{N},\end{aligned}$$

whereas the coefficients ψ_i *are defined by iteration by* $\psi_0 = 1$ *and*

$$\frac{\partial \psi_k}{\partial s} + (b - Du) D\psi_k = -\left[\frac{1}{2}(|Dw|^2 - \Delta w)\psi_{k-1} - \langle Dw, D\psi_{k-1}\rangle + \frac{1}{2}\Delta \psi_{k-1}\right] \quad in\ N,$$

$$\psi_k = 0 \quad on\ \partial D \times [0, T[\ \cap \overline{N}.$$

REMARK 2.2. The original result in Fleming and James (1992) deals with a more general situation (b also depending on ε, σ nonconstant etc.).

Assumption (A) also insures that the differential systems or w and ψ_k can actually be solved by characteristics (the characteristics of both systems are, moreover, the same). Thus the solution of the differential system of Theorem 2.1 is simple: one has to solve the ordinary equation

$$\dot{\gamma}_t = \beta(t, \gamma_t),$$
$$\gamma_s = x,$$

where $\beta = b - Du$, and then

$$w(x, s) = \exp\left(\int_s^\tau \frac{1}{2}\Delta u(\gamma_t, t)\, dt\right), \tag{2.3}$$

τ being the time at which γ reaches ∂N.

REMARK 2.3. One might ask whether the expansion of Theorem 2.1 still holds if Assumption (A) is not satisfied. To make this point clear, let us consider the example of Brownian motion (i.e., $b \equiv 0$). The exact asymptotics for the exit time in this situation can be found in Baldi (1991) or in Bellaïche [(1981), page 185], and they are of the form

$$\mathbb{P}^{\varepsilon}_{x,s}\{\tau \le T\} \sim c_1 \varepsilon \exp\left(-\frac{u(x,s)}{\varepsilon}\right),$$

for some constant $c_1 > 0$ (at least for "most" points $x \in D$). Thus the first term before the exponential is *of order* 1 in ε, whereas in the asymptotics of Theorem 2.1 the first term is a constant, that is, of order 0 in ε.

These two situations show another difference: as pointed out in Fleming and James (1992), under Assumption (A) the minimizing path joining a point $(x, s) \in N$ to the boundary reaches ∂D at a time $T' < T$. In the case of the Brownian motion, the boundary is always reached exactly at time T. It seems that for the asymptotics of the exit time two typical situations appear, depending on whether the expansion begins with a term of order 1 or not, and it might be interesting to classify these two situations. The time at which the minimizers reach the boundary seems to make a difference.

A related result worth mentioning is given in Azencott (1985). He deals with expansions of quantities of the type

$$\mathbb{P}^{\varepsilon}_{x,s}\{X^{\varepsilon} \in \Gamma\},$$

Γ being a set of paths, for which certain hypotheses of regularity of the boundary (as an infinite-dimensional manifold) hold. In principle these results might be applicable to our situation with Γ = {set of paths which reach D^C before time T}. Although in general it is difficult to check Azencott's assumptions, they are certainly not satisfied in the range of application of Theorem 2.1 because also Azencott's expansion begins with a term of order 1 in ε.

3. The geometry of the Brownian bridge. In this section we make some remarks concerning the geometry of the minimizers of the action functional I_s for the Brownian bridge, which will allow us to determine for which points (x, s) Theorem 2.1 can be applied. Most of this section consists of the usual remarks connecting the uniqueness of the minimizers and the regularity of the solution u of the Hamilton–Jacobi equation.

We shall consider the following family of diffusion processes (Brownian bridge with a small parameter):

$$dX^{\varepsilon}(t) = -\frac{X^{\varepsilon}(t) - y}{1-t}\,dt + \sqrt{\varepsilon}\,dB_t, \qquad X^{\varepsilon}(s) = x, \tag{3.1}$$

where y is a fixed point in the open set D. If by $\Gamma_{x,s}$ we denote the set of all absolutely continuous paths γ such that $\gamma(s) = x$ and $\gamma(t) \in D^C$ for some $0 < t < 1$, and we write

$$J_{s,t}(\gamma) = \frac{1}{2}\int_s^t \left| \dot{\gamma}(r) + \frac{\gamma(r) - y}{1-r} \right|^2 dr,$$

then

$$u(x,s) = \inf_{\gamma \in \Gamma_{x,s}} J_{s,1}(\gamma).$$

LEMMA 3.1. *We have*

$$u(x,s) = \inf_{\phi \in \partial D} \inf_{0<t<1} \frac{1}{2}\left\{ \frac{|x-\phi|^2}{t-s} + \frac{|y-\phi|^2}{1-t} - \frac{|x-y|^2}{1-s} \right\}. \tag{3.2}$$

PROOF. If $\gamma \in \Gamma_{x,s}$, let $0 < t < 1$ and $\phi \in \partial D$ be such that $\gamma(t) = \phi$. The infimum of $J_{s,1}$ over such paths is a typical problem of calculus of variations with fixed endpoints. The Lagrange equation is

$$\ddot{\gamma}(r) = 0,$$

meaning that the minimum is attained by the line segment joining x to ϕ at uniform speed. In order to minimize $J_{s,1}$, γ will be chosen to be a solution of

$$\dot{\gamma}(r) = -\frac{\gamma(r) - y}{1-r}, \qquad \gamma(t) = \phi$$

on the time interval $[t, 1]$ (by the way, this gives the line segment from ϕ to y driven at uniform speed). On this path the functional $J_{s,1}$ takes the value

$$J_{s,1}(\gamma) = \frac{1}{2}\int_s^t \left| \frac{\phi - x}{t - s} + \frac{1}{1 - r}\left[x - y + \frac{r - s}{t - s}(\phi - x) \right] \right|^2 dr$$

(the integrand vanishes on $[t, 1]$). Integration gives the value

$$\frac{1}{2}\left\{ \frac{1}{1-t} - \frac{1}{1-s} \right\} \left| (\phi - x)\frac{1-s}{t-s} + x - y \right|^2$$
$$= \frac{1}{2}\left\{ \frac{|x-\phi|^2}{t-s} + \frac{|y-\phi|^2}{1-t} - \frac{|x-y|^2}{1-s} \right\}.$$

Of course the minimum of $J_{s,1}$ over all paths starting at x at time s and reaching the boundary before time 1 is given by the minimum of this quantity in $\phi \in \partial D$ and $t, s \le t \le 1$. □

It is easy to compute the infimum in t of the left-hand side of (3.2) for fixed ϕ. The minimum indeed is attained for

(3.3) $$t = t(x, s) := s + (1 - s)\frac{|x - \phi|}{|x - \phi| + |y - \phi|},$$

which gives

(3.4) $$u(x, s) = \inf_{\phi \in \partial D} \frac{1}{2(1-s)}\left\{ (|x - \phi| + |y - \phi|)^2 - |x - y|^2 \right\}.$$

REMARK 3.2. The quantity $t(x, s)$ is the time at which the path $\gamma_{x,s}$, which minimizes the action functional starting at x at time s, reaches the boundary. It is important to remark that $t(x, s)$ is constant along $\gamma_{x,s}$; more precisely, $t(\gamma_{x,s}(r), r) = t(x, s)$ for every $s \le r \le t(x, s)$. It is also clear that $t(x, s) < 1$ for every $s < 1$.

REMARK 3.3. The drift for the Brownian bridge is

$$b(x, t) = -\frac{x - y}{1 - t}$$

and has a singularity at $t = 1$. This does not present a real difficulty in applying Theorem 2.1, where the drift is supposed to be Lipschitz continuous. Indeed, as remarked in the Introduction, one has

$$\mathbb{P}^{\varepsilon}_{x,s}\{\tau \le T\} \sim \mathbb{P}^{\varepsilon}_{x,s}\{\tau \le T, X^{\varepsilon} \in B_{\delta}(\gamma)\} \sim \mathbb{P}^{\varepsilon}_{x,s}\{\tau \le \eta\},$$

where η is the supremum of the exit times from D of the paths in $B_{\delta}(\gamma)$. If δ is chosen smaller than $d(y, \partial D)/2$, then η is strictly smaller than 1 and only the behavior of the process in the time interval $[0, \eta]$ is relevant, where the drift coefficient is regular.

In the following we shall write

$$I(x,\phi)=\frac{1}{2}\big((|x-\phi|+|y-\phi|)^2-|x-y|^2\big), \tag{3.5}$$

$$t_0=t_0(x)=\frac{|x-\phi|}{|x-\phi|+|y-\phi|}. \tag{3.6}$$

One should remark that the level sets of I are ellipsoids of revolution whose foci are located at x and y.

Also, from (3.6) it follows that

$$|x-\phi(x)|=\frac{t_0}{1-t_0}|y-\phi(x)|. \tag{3.7}$$

If $\tilde{D}$ is the set of all points $x\in D$ such that the infimum in (3.4) is attained at only one point $\phi\in\partial D$, then one can define a function $\phi\colon\tilde{D}\to\partial D$ by

$$\phi(x)=\arg\min_{\xi\in\partial D} I(x,\xi)=\arg\min_{\xi\in\partial D}[|x-\xi|+|y-\xi|] \tag{3.8}$$

so that, for $x\in\tilde{D}$,

$$u(x,s)=\frac{1}{(1-s)}I(x,\phi(x)). \tag{3.9}$$

Let $\Psi\colon\mathbb{R}^n\to\mathbb{R}$ be a smooth function, and let $\phi_0\in\partial D$ be a local minimum of Ψ on ∂D; ϕ_0 is said to be *nondegenerate* if for some local system of coordinates $G\colon\mathbb{R}^{n-1}\supset U\to\partial D$ the Hessian of $\Psi\circ G$ is positive definite at $G^{-1}(\phi_0)$.

DEFINITION 3.4. A point $x\in D$ is said to be *regular* if the following hold:

(a) The line segment joining x to y is entirely contained in D.

(b) The minimum of $\phi\to I(x,\phi)$ on ∂D is attained at a unique point $\phi_0=\phi(x)$. Moreover, ∂D is $\mathscr{C}^\infty$ in a neighborhood of ϕ_0, and ϕ_0 is a nondegenerate minimum for $\phi\to I(x,\phi)$.

PROPOSITION 3.5. *Let $x\in D$ be a regular point. Then there exists a neighborhood $\mathscr{U}$ of x such that if $p\in\mathscr{U}$, then p is a regular point. Moreover, the mapping $p\to\phi(p)$ is $\mathscr{C}^\infty$ on $\mathscr{U}$ and u is $\mathscr{C}^\infty(\mathscr{U}\times[0,T[)$.*

PROOF. Because of (3.9) we only need to prove that there exists a neighborhood $\mathscr{U}$ of x such that if $p\in\mathscr{U}$, then p is also a regular point and the mapping $p\to\phi(p)$ is $\mathscr{C}^\infty(\mathscr{U})$.

Let (U,G) be a local system of coordinates of ∂D near $\phi_0=\phi(x)$, and let us consider the function $F\colon D\times U\to\mathbb{R}$ defined by $F(x,z)=I(x,G(z))$. One has

$$D_zF(x,G^{-1}(\phi_0))=0$$

because $G^{-1}(\phi_0)$ is a local minimum of $z\to F(x,z)$. Moreover, the assumption of nondegeneracy of the Hessian of $z \to F(x,z)$ at $z = G^{-1}(\phi_0)$ allows us to apply the implicit function theorem, obtaining that for p in a neighborhood U_1 of x there exists a unique point $z(p)$ in a neighborhood V_1 of $G^{-1}(\phi_0)$ such that

$$D_z F(p, z(p)) = 0.$$

This means that $z(p)$ is a critical point of $z \to F(p,z)$.

Now the assumption that ϕ_0 is the unique minimum of $\phi \to I(x,\phi)$ allows us easily to deduce that for any neighborhood W of ϕ_0 there exists a neighborhood U_2 of x such that for every $p \in U_2$ the minimum of $\phi \to I(p,\phi)$ is attained in W.

Thus for $p \in U_1 \cap U_2$ the minimum of $z \to F(x,z) = I(x, G(z))$ is attained in $V_1 \cap G^{-1}(W)$. However, in this set $z(p)$ is the unique critical point of $z \to F(p,z)$, which is thus the minimum. The mapping $p \to z(p)$ is $\mathscr{C}^\infty$ because of the implicit function theorem, and the same is true for $p \to \phi(p) = G(z(p))$. □

In the next section we shall compute the asymptotics of the exit probability for the Brownian bridge starting at a regular point. We see now that every $x \in D$ "in general position" is regular.

Let Γ_1 and Γ_2 be two smooth hypersurfaces mutually tangent at a point ϕ, and let us denote by T_ϕ the tangent hyperplane at the two surfaces at ϕ. If $w \in T_\phi$, let γ_1 and γ_2 be two paths on Γ_1 and Γ_2, respectively, such that

$$\gamma_1(0) = \gamma_2(0) = \phi, \qquad \dot{\gamma}_1(0) = \dot{\gamma}_2(0) = w.$$

Then

$$|\gamma_1(t) - \gamma_2(t)| \le \int_0^t |\dot{\gamma}_1(s) - \dot{\gamma}_2(s)|\, ds = O(t^2).$$

We say that at ϕ there is a contact of order greater than 1 between Γ_1 and Γ_2 if for some $w \in T_\phi$ one has

$$|\gamma_1(t) - \gamma_2(t)| = o(t^2)$$

as $t \to 0$.

As we shall see in Section 5, Γ_1 and Γ_2 have a contact of order greater than 1 if and only if the Weingarten maps L_1 and L_2 at ϕ of Γ_1 and Γ_2, respectively, are such that $L_1 - L_2$ is not invertible.

PROPOSITION 3.6. *Let Ψ be a smooth function whose gradient never vanishes on ∂D, and let ϕ_0 be a local minimum of Ψ on ∂D. Then ϕ_0 is not a nondegenerate minimum of Ψ on ∂D if and only if at ϕ_0 the two surfaces ∂D and $\{\Psi = \Psi(\phi_0)\}$ have order of contact greater than* 1.

PROOF. Let (U, G) be a local system of coordinates at ϕ_0. If $H = \operatorname{Hess}\Psi(\phi_0)$ was not invertible, there would be a vector $v \in \mathbb{R}^{n-1}$ such that

$$0 = \langle Hv, v\rangle = \frac{\partial^2}{\partial v^2} \Psi \circ G\big(G^{-1}(\phi_0)\big).$$

Let $\partial G/\partial v \in T_{\phi_0}$ be the corresponding tangent vector, and let γ_1 and γ_2 be the paths on ∂D and $\Psi = \Psi(\phi_0)$, respectively, such that

$$\gamma_1(0) = \gamma_2(0) = \phi_0, \qquad \dot{\gamma}_1(0) = \dot{\gamma}_2(0) = \frac{\partial G}{\partial v}.$$

Let us assume that there exists a neighborhood U_0 of 0 such that $\gamma_1(t) \neq \gamma_2(t)$ for $t \in U_0$; otherwise there is nothing to prove. Then, for some $0 \le \tau \le 1$,

$$\begin{aligned} &\Psi(\gamma_1(t)) - \Psi(\gamma_2(t)) \\ &\quad = \langle \operatorname{grad} \Psi(\gamma_1(t) + \tau(\gamma_2(t) - \gamma_1(t))), \gamma_1(t) - \gamma_2(t) \rangle \\ &\quad = \langle \operatorname{grad} \Psi(\phi_0), \gamma_1(t) - \gamma_2(t) \rangle + o(t^2). \end{aligned} \tag{3.10}$$

Moreover,

$$\frac{d}{dt}(\Psi(\gamma_1(t)) - \Psi(\gamma_2(t)))\bigg|_{t=0} = \frac{d}{dt}\Psi(\gamma_1(t))\bigg|_{t=0} = 0$$

since ϕ_0 is a minimum of Ψ on D; but also

$$\frac{d^2}{dt^2}(\Psi(\gamma_1(t)) - \Psi(\gamma_2(t)))\bigg|_{t=0} = \frac{d^2}{dt^2}\Psi(\gamma_1(t))\bigg|_{t=0} = \langle Hv, v \rangle = 0,$$

so that $\Psi(\gamma_1(t)) - \Psi(\gamma_2(t)) = o(t^2)$. Putting this together with (3.10) we have

$$\begin{aligned} o(t^2) &= \Psi(\gamma_1(t)) - \Psi(\gamma_2(t)) \\ &= |\gamma_1(t) - \gamma_2(t)| \left\langle \operatorname{grad} \Psi(\phi_0), \frac{\gamma_1(t) - \gamma_2(t)}{|\gamma_1(t) - \gamma_2(t)|} \right\rangle + o(t^2) \\ &= |\gamma_1(t) - \gamma_2(t)| \langle \operatorname{grad} \Psi(\phi_0), n \rangle + o(t^2), \end{aligned}$$

where n denotes the normal at both surfaces at ϕ_0. Since $\operatorname{grad} \Psi(\phi_0)$ does not vanish and points in the same direction as n, this implies that $|\gamma_1(t) - \gamma_2(t)| = o(t^2)$. □

REMARK 3.7. Because of (3.8), $D_\phi I(x, \phi)$ is a vector which is orthogonal to ∂D at $\phi = \phi(x)$. If we denote by $n(\phi)$ the *inner* normal to ∂D at ϕ, computing the derivative $D_\phi I(x, \phi)$ implies

$$\frac{x - \phi}{|x - \phi|} + \frac{y - \phi}{|y - \phi|} = \lambda n(\phi),$$

for some $\lambda \in \mathbb{R}$. It is easy to see that

$$\lambda = 2\left\langle \frac{y - \phi}{|y - \phi|}, n(\phi) \right\rangle,$$

from which we derive

$$\frac{x - \phi}{|x - \phi|} = 2\left\langle \frac{y - \phi}{|y - \phi|}, n(\phi) \right\rangle n(\phi) - \frac{y - \phi}{|y - \phi|}. \tag{3.11}$$

In particular, (3.11) implies that if x is a regular point and $\gamma_{x,s}$ is the solution of (2.2), then $\gamma_{x,s}$ reaches the boundary nontangentially. Indeed in the time interval $[s, t(x,s)]$ the path $\gamma_{x,s}$ follows the line segment from x to $\phi(x)$ at uniform speed. From (3.11) if $\langle x-\phi, n(\phi)\rangle = 0$, then also $\langle y - \phi, n(\phi)\rangle = 0$, and this implies that ϕ lies on the line segment joining x to y, which is not possible since x is a regular point and the line segment joining x to y must lie entirely in D.

4. The asymptotics for the Brownian bridge. In this section we compute the Laplacian of u at a point (x, s), where x is assumed to be regular, and solve the differential system of Theorem 2.1 by computing w by (2.3).

We start with the first derivatives of u (we can assume $s = 0$). From (3.9),

$$\frac{\partial u}{\partial x_i}(x,0) = \frac{\partial I}{\partial x_i}(x,\phi(x)) + \sum_{j=1}^{n} \frac{\partial I}{\partial \phi_j}(x,\phi(x)) \frac{\partial \phi_j}{\partial x_i}(x),$$

However, it is easy to recognize that in the previous formula the last term is equal to 0, because $[\partial\phi/\partial x_i](x)$ is a tangent vector to ∂D at $\phi = \phi(x)$ for $i = 1,\dots,n$, whereas $D_\phi I(x,\phi)$ is orthogonal to ∂D. Thus

$$Du(x,0) = D_x I(x,\phi(x)) = \frac{x-\phi}{|x-\phi|}(|x-\phi| + |y-\phi|) - (x-y) \tag{4.1}$$

and

$$\begin{aligned}\frac{\partial^2 u}{\partial x_i^2}(x,0) &= \frac{\partial}{\partial x_i}\left[-\phi_i + y_i + (x_i - \phi_i)\frac{|y-\phi|}{|x-\phi|}\right]\\ &= -\frac{\partial \phi_i}{\partial x_i} + \left(1 - \frac{\partial \phi_i}{\partial x_i}\right)\frac{|y-\phi|}{|x-\phi|}\\ &\quad + \frac{x_i - \phi_i}{|x-\phi|^2}\left[-\frac{|x-\phi|}{|y-\phi|}\sum_{j=1}^{n}(y_j - \phi_j)\frac{\partial \phi_j}{\partial x_i}\right.\\ &\quad \left. - \frac{|y-\phi|}{|x-\phi|}\sum_{j=1}^{n}(x_j-\phi_j)\left(\delta_{ij} - \frac{\partial \phi_j}{\partial x_i}\right)\right].\end{aligned}$$

In the calculus of the Laplacian we make use of the following lemma, which implies that, after summation on i, the contribution of the term in the last two lines is equal to

$$-\frac{|y-\phi|}{|x-\phi|},$$

giving

$$\frac{1}{2}\Delta u(x,s) = \frac{1}{2(1-s)}\left\{-\left(1 + \frac{|y-\phi|}{|x-\phi|}\right)\sum_{i=1}^{n}\frac{\partial \phi_i}{\partial x_i} + (n-1)\frac{|y-\phi|}{|x-\phi|}\right\}. \tag{4.2}$$

LEMMA 4.1. *If x is a regular point,*

$$\sum_{i=1}^{n} (x_i - \phi_i)\frac{\partial \phi_j}{\partial x_i} = 0, \tag{4.3}$$

for every $j = 1, \dots, n$.

PROOF. It is easy to see that if q is any point in the segment joining x to ϕ, then also $q \in \tilde{D}$ and $\phi(q) = \phi(x)$. It suffices now to remark that the quantity in (4.3) is just, up to a multiplicative constant, the derivative of ϕ at x in the direction of $\phi(x)$. □

In order to conclude the computation of Δu, we want to obtain an expression for the quantity

$$\sum_{i=1}^{n} \frac{\partial \phi_i}{\partial x_i}.$$

This will be done by the implicit function theorem, as hinted in the proof of Proposition 3.6. The fact that x is regular justifies all the differentiation we are going to do. Our notation is as before, with (G, U) a local system of coordinates of ∂D at $\phi(x)$. Let $z \in U$ be the point such that $\phi(x) = G(z)$. Thus $z = z(x)$ is defined as

$$z = \operatorname*{arg\,min}_{w \in U} \left(|x - G(w)| + |y - G(w)|\right) := \operatorname*{arg\,min}_{w \in U} F(x, w).$$

Since $z(x)$ is in the interior of U, it is necessary that all derivatives of H vanish at $z(x)$:

$$\frac{\partial F}{\partial z_k}(x, z) := H_k(x, z) = -\left\langle \frac{x - G(z)}{|x - G(z)|} + \frac{y - G(z)}{|y - G(z)|}, \frac{\partial G}{\partial z_k} \right\rangle = 0, \tag{4.4}$$

for $k = 1, \dots, n-1$. Relation (4.4) allows us to state that the function $x \to z(x)$ is defined implicitly by

$$H(x, z) = 0,$$

where $H(x, z) = (H_1(x, z), \dots, H_{n-1}(x, z))$. The derivatives of z are thus given by the implicit function theorem

$$\frac{\partial z}{\partial x} = -\left(\frac{\partial H}{\partial z}\right)^{-1} \frac{\partial H}{\partial x},$$

where

$$\frac{\partial z}{\partial x} = \left(\frac{\partial z_k}{\partial x_i}\right)_{ki},$$

$$\frac{\partial H}{\partial z} = \left(\frac{\partial H_j}{\partial z_k}\right)_{jk},$$

$$\frac{\partial H}{\partial x} = \left(\frac{\partial H_j}{\partial x_i}\right)_{ji}.$$

Thus we have

$$\sum_{i=1}^{n} \frac{\partial \phi_i}{\partial x_i} = \sum_{i=1}^{n} \frac{\partial G_i(z(x))}{\partial x_i} = -\operatorname{tr}\left\{ \frac{\partial G}{\partial z} \left(\frac{\partial H}{\partial z} \right)^{-1} \frac{\partial H}{\partial x} \right\}$$
$$= -\operatorname{tr}\left\{ \left(\frac{\partial H}{\partial z} \right)^{-1} \frac{\partial H}{\partial x} \frac{\partial G}{\partial z} \right\}.$$

LEMMA 4.2. *Let x be a point in $\tilde{D}$, and let us define the matrices $A = (a_{ij})_{ij}$ and $B = (b_{ij})_{ij}$ by*

$$a_{ij} = \frac{1}{|y - G(z)|} \left\{ \left\langle \frac{\partial G}{\partial z_i}, \frac{\partial G}{\partial z_j} \right\rangle - \left\langle \frac{y - G(z)}{|y - G(z)|}, \frac{\partial G}{\partial z_i} \right\rangle \left\langle \frac{y - G(z)}{|y - G(z)|}, \frac{\partial G}{\partial z_j} \right\rangle \right\},$$

$$b_{ij} = 2 \left\langle \frac{y - G(z)}{|y - G(z)|}, n \right\rangle \left\langle n, \frac{\partial^2 G}{\partial z_i \, \partial z_j} \right\rangle.$$

Then

$$\frac{\partial H}{\partial z} = \frac{1}{t_0} A - B, \qquad \frac{\partial H}{\partial x} \frac{\partial G}{\partial z} = -\frac{1 - t_0}{t_0} A.$$

PROOF. By a plain computation,

$$\frac{\partial H_j}{\partial z_k} = -\left\langle -\frac{\partial G / \partial z_k}{|x - G(z)|} + \frac{x - G(z)}{|x - G(z)|^3} \left\langle x - G(z), \frac{\partial G}{\partial z_k} \right\rangle \right.$$
$$\left. - \frac{\partial G / \partial z_k}{|y - G(z)|} + \frac{y - G(z)}{|y - G(z)|^3} \left\langle y - G(z), \frac{\partial G}{\partial z_k} \right\rangle, \frac{\partial G}{\partial z_j} \right\rangle$$
$$- \left\langle \frac{x - G(z)}{|x - G(z)|} + \frac{y - G(z)}{|y - G(z)|}, \frac{\partial^2 G}{\partial z_k \, \partial z_j} \right\rangle$$

and, collecting the terms together,

$$\frac{\partial H_j}{\partial z_k} = \left\langle \frac{\partial G}{\partial z_k}, \frac{\partial G}{\partial z_j} \right\rangle \left\{ \frac{1}{|x - G(z)|} + \frac{1}{|y - G(z)|} \right\}$$
$$- \frac{1}{|x - G(z)|^3} \left\langle x - G(z), \frac{\partial G}{\partial z_k} \right\rangle \left\langle x - G(z), \frac{\partial G}{\partial z_j} \right\rangle$$
$$- \frac{1}{|y - G(z)|^3} \left\langle y - G(z), \frac{\partial G}{\partial z_k} \right\rangle \left\langle y - G(z), \frac{\partial G}{\partial z_j} \right\rangle$$
$$- \left\langle \frac{x - G(z)}{|x - G(z)|}, \frac{\partial^2 G}{\partial z_k \, \partial z_j} \right\rangle + \left\langle \frac{y - G(z)}{|y - G(z)|}, \frac{\partial^2 G}{\partial z_k \, \partial z_j} \right\rangle.$$

Now we replace everywhere the quantity $x - G(z)$ with a function of $y - G(z)$ and t_0. More precisely, we replace $|x - \phi|$ by the left-hand side of (3.7), and $(x - \phi)/|x - \phi|$ by the left-hand side of (3.11). Using the fact that, for all $k = 1, \dots, n - 1$, n is orthogonal to $\partial G/\partial z_k$, for all $k = 1, \dots, n - 1$ one gets, finally,

$$\begin{aligned}\frac{\partial H_j}{\partial z_k} = \frac{1}{t_0}\Bigg\{ & \frac{\langle \partial G/\partial z_k, \partial G/\partial z_j \rangle}{|y - G(z)|} - \frac{1}{|y - G(z)|}\left\langle \frac{y - G(z)}{|y - G(z)|}, \frac{\partial G}{\partial z_k} \right\rangle \\ & \times \left\langle \frac{y - G(z)}{|y - G(z)|}, \frac{\partial G}{\partial z_j} \right\rangle \Bigg\} \\ & - 2\left\langle \frac{y - G(z)}{|y - G(z)|}, n \right\rangle \left\langle n, \frac{\partial^2 G}{\partial z_k\, \partial z_j} \right\rangle \\ = {} & \frac{1}{t_0} A - B.\end{aligned}$$

Similarly,

$$\begin{aligned}\frac{\partial F_k}{\partial x_i} &= \sum_l - \left\{ \frac{\delta_{il}}{|x - G(z)|} - \frac{(x_l - G_l(z))(x_i - G_i(z))}{|x - G(z)|^3} \right\} \frac{\partial G_l}{\partial z_k} \\ &= -\frac{1}{|x - G(z)|} \left\{ \frac{\partial G_i}{\partial z_k} - \frac{(x_i - G_i(z))}{|x - G(z)|} \left\langle \frac{x - G(z)}{|x - G(z)|}, \frac{\partial G}{\partial z_k} \right\rangle \right\}.\end{aligned}$$

Thus

$$\begin{aligned}\left(\frac{\partial H}{\partial x} \frac{\partial G}{\partial z} \right)_{jk} &= -\frac{1}{|x - G(z)|} \\ & \quad \times \left\{ \left\langle \frac{\partial G}{\partial z_k}, \frac{\partial G}{\partial z_j} \right\rangle - \left\langle \frac{x - G(z)}{|x - G(z)|}, \frac{\partial G}{\partial z_k} \right\rangle \left\langle \frac{x - G(z)}{|x - G(z)|}, \frac{\partial G}{\partial z_j} \right\rangle \right\} \\ &= -\frac{1 - t_0}{t_0} \frac{1}{|y - G(z)|} \\ & \quad \times \left\{ \left\langle \frac{\partial G}{\partial z_k}, \frac{\partial G}{\partial z_j} \right\rangle - \left\langle \frac{y - G(z)}{|y - G(z)|}, \frac{\partial G}{\partial z_k} \right\rangle \left\langle \frac{y - G(z)}{|y - G(z)|}, \frac{\partial G}{\partial z_j} \right\rangle \right\} \\ &= -\frac{1 - t_0}{t_0} A.\end{aligned}$$

□

Let x be a regular point and let q be a point in the line segment joining x to $\phi(x)$. It is easy to check that q is regular.

As we already remarked, $q \in \tilde{D}$ and $\phi(q) = \phi(x)$; moreover, the minimizing path of the action functional I_s starting at q is the polygonal path formed by the line segments joining q to $\phi(q) = \phi(x)$ and $\phi(q)$ to y. This is a consequence of the fact that for the points in $\tilde{D}$ the minimizer joining x to ∂D is unique. Moreover, by Lemma 4.2 the Hessian of $F(q, z)$ at $z = G^{-1}(\phi(q)) = G^{-1}(\phi(x))$ is given by

$$\begin{aligned}\frac{1}{t_0(q)}A - B &= \frac{1}{t_0(x)}A - B + \left(\frac{1}{t_0(q)} - \frac{1}{t_0(x)}\right)A\\ &= \text{Hess}\, F(x, G^{-1}(\phi(x))) + \left(\frac{1}{t_0(q)} - \frac{1}{t_0(x)}\right)A\end{aligned}$$

Now A is a positive definite matrix and $t_0(x) \ge t_0(q)$, because of (3.6). This implies that also the Hessian of $F(q, z)$ at $z = G^{-1}(\phi(q))$ is positive definite, so that, by Remark 3.7, q is also regular.

If x is regular, it is now easy to show a set $N \subset D \times [0, T]$ containing x and satisfying Assumption (A): if $\mathscr{U}$ is the neighborhood of x given by Proposition 3.5 and $V \subset D$ is the set formed by the union of all line segments joining p to $\phi(p)$ for $p \in \mathscr{U}$, then set $N = V \times [0, T'[$, where $T' = \sup_{p \in \mathscr{U}} t_0(p)$.

Lemma 4.2 implies that

$$\sum_{i=1}^{n} \frac{\partial \phi_i}{\partial x_i} = \text{tr}\left[(A - t_0 B)^{-1}(1 - t_0)A\right]$$

and, substituting in (4.2),

$$\begin{aligned}\frac{1}{2}\Delta u = \frac{1}{2(1-s)}\Bigg\{ &-\left(1 + \frac{|y - \phi|}{|x - \phi|}\right)\text{tr}\left[(A - t_0 B)^{-1}(1 - t_0)A\right]\\ &+ (n-1)\frac{|y - \phi|}{|x - \phi|}\Bigg\},\end{aligned}$$

so that, by (3.7),

$$\frac{1}{2}\,\Delta u = \frac{1}{2(1-s)}\left\{-\frac{1}{t_0}\text{tr}\left[(A - t_0 B)^{-1}(1 - t_0)A\right] + (n-1)\frac{1 - t_0}{t_0}\right\}.$$

From (3.3) we have $t_0(1 - s) = (t(x, s) - s)$, so that, expressing t_0 in terms of t and s,

$$\begin{aligned}\frac{1}{2}\,\Delta u(x, s) = \frac{1}{2}\Bigg[&\frac{(n-1)}{t - s} - \frac{(n-1)}{1 - s}\\ &- \frac{1}{t - s}\text{tr}\left[((1 - s)A - (t - s)B)^{-1}(1 - t)A\right]\Bigg].\end{aligned} \tag{4.5}$$

From now on we shall write t instead of $t(x, s)$. It is convenient to simplify the expression above by remarking that

$$\begin{aligned}\frac{(n-1)}{t-s} &- \frac{1}{t-s}\operatorname{tr}\left[((1-s)A-(t-s)B)^{-1}(1-t)A\right]\\ &= \frac{1}{t-s}\operatorname{tr}\left[I-((1-s)A-(t-s)B)^{-1}(1-t)A\right]\\ &= \frac{1}{t-s}\operatorname{tr}\left[((1-s)A-(t-s)B)^{-1}\{(1-s)A-(t-s)B-(1-t)A\}\right]\\ &= \operatorname{tr}\left[((1-s)A-(t-s)B)^{-1}(A-B)\right]\\ &= \operatorname{tr}\left[((1-s)I-(t-s)A^{-1}B)^{-1}(I-A^{-1}B)\right],\end{aligned}$$

so that, finally,

$$\begin{aligned}\frac{1}{2}\Delta u(x,s) &= \frac{1}{2}\left[-\frac{(n-1)}{1-s} + \operatorname{tr}\left(((1-s)I-(t-s)A^{-1}B)^{-1}(I-A^{-1}B)\right)\right]\\ &= \frac{1}{2}\left[-\frac{(n-1)}{1-s} + \operatorname{tr}\left(((I-tA^{-1}B)-s(I-A^{-1}B))^{-1}\right.\right.\\ &\qquad\qquad\left.\left.\times(I-A^{-1}B)\right)\right].\end{aligned}$$

We recall that our goal is to compute

$$\exp[-w(x,s)] = \exp\left(-\int_s^t \frac{1}{2}\Delta u(\gamma_{x,s}(r), r)\,dr\right).$$

This is now easy since

$$\begin{aligned}&\frac{1}{2}\Delta u(\gamma_{x,s}(r), r)\\ &\quad= \frac{1}{2}\left[-\frac{(n-1)}{1-r} + \operatorname{tr}\left(((I-tA^{-1}B)-r(I-A^{-1}B))^{-1}(I-A^{-1}B)\right)\right]\end{aligned}$$

[as remarked before, $t(\gamma_{x,s}(r), r) \equiv t(x, s)$ for $s \le r \le t(x, s)$].

REMARK 4.3. The matrix $(1/t_0)A - B$ is positive definite, being the Hessian of the function $F(x, z) = |x - G(z)| + |y - G(z)|$ at $z = G^{-1}(\phi(x))$, since x is regular. However, A is also positive definite so that for $s \le r \le t$ the same is true for $(1 - tA^{-1}B) - r(I - A^{-1}B)$, which can be written

$$(1-r)I-(t-r)A^{-1}B = (1-r)A^{-1}(A-t_0B).$$

This justifies the existence of the inverse of this matrix and the raising to the power $\frac{1}{2}$ in the proof of the following theorem, which is our main result.

THEOREM 4.4. *Let x be a regular point* (*Definition* 3.4). *Then*

$$\exp[-w(x,s)] = (1-s)^{(n-1)/2}\det\big((1-s)I - (t(x,s)-s)A^{-1}B\big)^{-1/2},$$

so that, for $s = 0$,

$$\mathbb{P}^{\varepsilon}_{x,0}\{\tau \le T\} \sim \exp\left(-\frac{u(x,0)}{\varepsilon}\right)\det\big(I - t_0 A^{-1}B\big)^{-1/2}.$$

PROOF. Let us set

$$\Gamma(\lambda) = \exp\left(-\frac{1}{2}\int_s^\lambda \Big(\big((I - tA^{-1}B) - r(I - A^{-1}B)\big)^{-1}(I - A^{-1}B)\Big)\,dr\right).$$

Then

$$\Gamma(\lambda) = \big((I - tA^{-1}B) - \lambda(I - A^{-1}B)\big)^{1/2}\big((I - tA^{-1}B) - s(I - A^{-1}B)\big)^{-1/2}.$$

Indeed, the two left-hand sides agree for $\lambda = s$, and both satisfy the differential equation

$$\Gamma'(\lambda) = -\frac{1}{2}\big((I - tA^{-1}B) - \lambda(I - A^{-1}B)\big)^{-1}(I - A^{-1}B)\Gamma(\lambda)$$

[in the computations we make use of the fact that all the matrices $((1 - tA^{-1}B) - r(I - A^{-1}B))^{-1}$ and $(I - A^{-1}B)$ commute as r varies being, up to scalar multiplicative constants, the resolvents of $A^{-1}B$]. Thus, recalling that $\det(\exp A) = \exp(\operatorname{tr} A)$,

$$\begin{aligned}
\exp[-w(x,s)] &= \left(\frac{1-s}{1-t}\right)^{(n-1)/2} \\
&\quad\times \exp\left(-\frac{1}{2}\int_s^t \operatorname{tr}\big((I - tA^{-1}B) - r(I - A^{-1}B)\big)^{-1}(I - A^{-1}B)\,dr\right) \\
&= \left(\frac{1-s}{1-t}\right)^{(n-1)/2} \\
&\quad\times \det\exp\left(-\frac{1}{2}\int_s^t \big((I - tA^{-1}B) - r(I - A^{-1}B)\big)^{-1}(I - A^{-1}B)\,dr\right) \\
&= \left(\frac{1-s}{1-t}\right)^{(n-1)/2} \det\Big(\Big(\big((I - tA^{-1}B) - t(I - A^{-1}B)\big)^{-1}\Big)^{1/2} \\
&\qquad\qquad\times\big((I - tA^{-1}B) - s(I - A^{-1}B)\big)^{-1/2}\Big) \\
&= \left(\frac{1-s}{1-t}\right)^{(n-1)/2}\sqrt{\frac{\det\big((I - tA^{-1}B) - t(I - A^{-1}B)\big)}{\det\big((I - tA^{-1}B) - s(I - A^{-1}B)\big)}} \\
&= \left(\frac{1-s}{1-t}\right)^{(n-1)/2}\sqrt{\frac{\det\big((1-t)I\big)}{\det\big((1-s)I - (t-s)A^{-1}B\big)}}
\end{aligned}$$

$$= (1-s)^{(n-1)/2} \det((1-s) - (t-s)A^{-1}B)^{-1/2}. \qquad \square$$

One should remark that, as x approaches the boundary and y remains fixed, $t_0(x) \to 0$, because of (3.5), so that for x near ∂D the quantity $\det(I - t_0 A^{-1}B)^{-1/2}$ is approximately 1.

EXAMPLE 4.5 (The case of the half-space). If D is a half-space, then it is easy to see that all its points are regular: indeed an ellipsoid of revolution cannot have order of contact greater than 1 with a hyperplane. The computation of the first coefficient $\exp[-w(x,s)]$ is immediate in this case, because G can be chosen linear, which gives $B \equiv 0$; thus, from Theorem 4.4,

$$\exp[-w(x,0)] \equiv 1,$$

or, equivalently, $w \equiv 0$. It is immediate to see that in this case also all the other coefficients ψ_k in Theorem 2.1 vanish. Thus

$$\mathbb{P}^{\varepsilon}_{x,s}\{\tau < 1\} := q^{\varepsilon}(x,s) = \exp\left(-\frac{u(x,s)}{\varepsilon}\right)(1 + o(\varepsilon^m)),$$

for every $m > 0$. Indeed it is not difficult to check that the above equality holds *exactly*, since the function $(x,s) \to \exp[-u(x,s)/\varepsilon]$ solves the equation for q^{ε}, namely,

$$\begin{aligned} \frac{\partial q^{\varepsilon}}{\partial s} + \frac{\sqrt{\varepsilon}}{2}\Delta q^{\varepsilon} &= 0 \quad \text{in } D \times [0,1[,\\ q^{\varepsilon} &= 1 \quad \text{on } \partial D \times [0,1[,\\ q^{\varepsilon}(x,1) &= 0 \quad \text{if } x \in D. \end{aligned}$$

It suffices to remark that u satisfies the Hamilton–Jacobi equation and that, from (4.5), $\Delta u \equiv 0$.

In this situation it is also easy to compute u explicitly: if D is the hyperplane given by the equation

$$D = \{x, a - \langle x, v\rangle > 0\},$$

where v is a vector of length 1, then

$$u(x,s) = \frac{2}{1-s}(a - \langle y, v\rangle)(a - \langle x, v\rangle)$$

(such a function satisfies the Hamilton–Jacobi equation and its boundary condition). For this computation and the following, it is sometimes useful to use Lemma 6.1 in order to compute the quasipotential u.

For the sake of completeness we write here the asymptotics for the exit probability of a one-dimensional Brownian bridge. If $D =]a,b[$ and $y \in D$, then (3.4) easily gives

$$u(x,s)=\begin{cases}\dfrac{1}{2(1-s)}\Big(\big((x-a)+(y-a)\big)^2-(x-y)^2\Big), & \text{if } x\le a+b-y,\\ \dfrac{1}{2(1-s)}\Big(\big((b-x)+(b-y)\big)^2-(x-y)^2\Big), & \text{if } x\ge a+b-y,\end{cases}$$

or, after simplification,

$$u(x,s)=\begin{cases}\dfrac{2}{(1-s)}(x-a)(y-a), & \text{if } x\le a+b-y,\\ \dfrac{2}{(1-s)}(b-x)(b-y), & \text{if } x\ge a+b-y.\end{cases}$$

Every point except $x=a+b-y$ is regular, and at each regular point u is linear, which gives $w\equiv 0$ and

$$\mathbb{P}^{\varepsilon}_{x,s}\{\tau\le 1\}=\exp\left(\frac{u(x,s)}{\varepsilon}\right)(1+o(\varepsilon^m)),$$

for every $m>0$. By repeating the arguments of the previous example if $x=a+b-y$, then from x there are two minimizers leading to the boundary and, summing the contribution of each, we have

$$\mathbb{P}^{\varepsilon}_{x,s}\{\tau\le 1\}=2\exp\left(\frac{u(x,s)}{\varepsilon}\right)(1+o(\varepsilon^m)),$$

for every $m>0$.

It is fair to remark that the asymptotics for the situation considered in this example should be considered already known. Indeed the multidimensional case can be easily reduced to one dimension by projecting the process in the direction of v and the one-dimensional case is easily treated directly, since for a Brownian motion B the joint distribution of $\sup_{t\le T}B_t$ and B_T is known.

In the following example we are able to handle situations in which the assumption of uniqueness of the minimizing path for the action functional is not satisfied (but there is still a finite number of such minimizers).

EXAMPLE 4.6. Suppose D is the square in Figure 1 with y located at the center. We suppose that $y=0$ and that the sides of the square have length equal to 2.

Suppose that x lies in the upper triangle (excluding the diagonals). Then by an elementary computation the minimizer of the action functional is the line segment joining $x=(x_1,x_2)$ to $(x_1/(2-x_2),1)$. Since the asymptotics of the exit time in (1.2) is the same as for the tube in (1.5) and the latter is

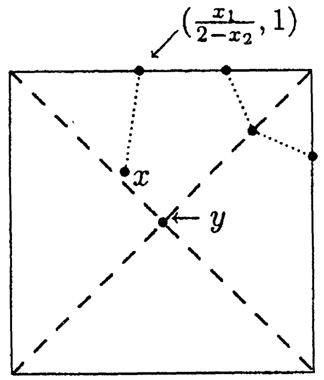

FIG. 1.

unchanged if we replace D with the half-plane $\{x_2 < 1\}$, Example 4.5 gives

$$\mathbb{P}^{\varepsilon}_{x,s}\{\tau \le 1\} \sim \exp\left(-\frac{2(1-x_2)}{\varepsilon(1-s)}\right).$$

Conversely, if $x = (t,t)$ lies on the upper right diagonal (but not in the center), then there are two paths minimizing the action functional, one, γ_1, given by the line segment joining x to $(t/(2-t), 1)$ and a second one symmetric to the first, γ_2, reaching the boundary at $(1, t/(2-t))$. Again large deviations theory states that the asymptotics for the exit time is the same as the asymptotics of the sum of the two probabilities

$$\mathbb{P}^{\varepsilon}_{x,s}\{\tau^{\varepsilon} \le 1, X^{\varepsilon} \in B_{\delta}(\gamma_1)\} + \mathbb{P}^{\varepsilon}_{x,s}\{\tau^{\varepsilon} \le 1, X^{\varepsilon} \in B_{\delta}(\gamma_2)\}.$$

The first probability has the same asymptotics as the exit probability from $\{x, x_2 < 1\}$, whereas the second one has the same asymptotics as the exit probabilities from $\{x, x_1 < 1\}$. These were calculated in the previous example and are both equal to

$$\exp\left(-\frac{2(1-t)}{\varepsilon(1-s)}\right).$$

This argument obviously holds whenever x lies on the diagonals (except the point at the center), so that if $x = (t,t)$, $t \ne 0$, the asymptotic expression for the exit probability is

$$\mathbb{P}^{\varepsilon}_{x,s}\{\tau \le 1\} \sim 2\exp\left(-\frac{2(1-|t|)}{\varepsilon(1-s)}\right).$$

Finally, if $x = 0$, there are four minimizers and the exit probability can be estimated by summing the contribution of the four tubes, which gives

$$\mathbb{P}^{\varepsilon}_{x,s}\{\tau \le 1\} \sim 4\exp\left(-\frac{2}{\varepsilon(1-s)}\right).$$

EXAMPLE 4.7. Let D be the ball of radius R in $\mathbb{R}^n$, with the conditioning point y at the origin; ∂D can be written locally as the graph of a function defined on the ball of radius R of $\mathbb{R}^{n-1}$. For instance, the upper half of S^{n-1} is the graph of

$$\psi(z_1, \dots, z_{n-1}) = \sqrt{R^2 - z_1^2 - \cdots - z_{n-1}^2},$$

$z = (z_1, \dots, z_{n-1})$ being a point in the ball of radius R of $\mathbb{R}^{n-1}$, so that a local system of coordinates is

$$G(z) = G(z_1, \dots, z_{n-1}) = \left(z_1, \dots, z_{n-1}, \sqrt{R^2 - z_1^2 - \cdots - z_{n-1}^2}\right).$$

Since $y = 0$, then of course $|y - \phi(x)| \equiv R$ and $[y - \phi(x)]/|y - \phi(x)|$ is the inner normal of ∂D at $\phi(x)$. In particular, $\langle (y - \phi)/|y - \phi(x)|, n\rangle = 1$ and $\langle y - \phi, \partial G/\partial z_j\rangle = 0$ for every $j = 1, \dots, n-1$. A straightforward computation then gives

$$n = \frac{1}{R}\left(-z_1, \dots, -z_{n-1}, -\sqrt{R^2 - z_1^2 - \cdots - z_{n-1}^2}\right),$$

$$\frac{1}{|y - \phi(x)|}\left\langle \frac{\partial G}{\partial z_k}, \frac{\partial G}{\partial z_j}\right\rangle = \frac{1}{R}\left(\delta_{kj} + \frac{z_k z_j}{R^2 - z_1^2 - \cdots - z_{n-1}^2}\right) = \left\langle n, \frac{\partial^2 G}{\partial z_k\, \partial z_j}\right\rangle.$$

Thus

$$B = 2A,$$

so that

$$\exp[-w(x,0)] = \det(I - t_0 A^{-1}B)^{-1/2} = (1 - 2t_0)^{-(n-1)/2}.$$

It is now easy to compute $t_0(x)$: the minimizing path is the one which goes first from x to the boundary following the radius at point $x/|x|$ and then back to $y = 0$ (again Lemma 6.1 is useful). Thus

$$t_0(x) = \frac{R - |x|}{2R - |x|},$$

$$\exp[-w(x,0)] = \left(\frac{2R - |x|}{|x|}\right)^{(n-1)/2}.$$

For a general domain D it might be impossible to do exact computations, the difficulty being the computation of $\phi(x)$. However, the formulas are explicit enough to be used in simulations: the program should find numerically $\phi(x)$ by local search, which enables a subsequent analytic computation of A and B.

5. Geometric interpretation. We are now going to interpret the meaning of the coefficient $\det(I - t_0 A^{-1}B)^{-1/2}$ which was derived in the previous section.

It is clear from the discussion of the previous sections that if $x \in D$ is regular, then the point $\phi(x) \in \partial D$ of Definition 3.4 is also the point of ∂D at which the smallest ellipsoid E of the form $\{z; |x - z| + |y - z| = \text{const}\}$ and ∂D are mutually tangent. We shall see that $\det(I - t_0 A^{-1}B)^{-1/2}$ is a measure of the contact between ∂D and such an ellipsoid at $\phi(x)$.

It is always possible by a rotation and a translation to assume that $\phi = 0$ and that the common normal to both surfaces points upward in the direction of, say, the nth axis. Thus both the surfaces E and ∂D can be described locally as the graph of two functions, f and g, respectively, of the variables $z_1, \dots, z_{n-1}$. By the previous assumptions the differentials of f and g vanish at $z = 0$.

A system of coordinates on ∂D at $z = 0$ is given by

$$G(z_1, \dots, z_{n-1}) = (z_1, \dots, z_{n-1}, g(z_1, \dots, z_{n-1})), \tag{5.1}$$

so that

$$\frac{\partial G}{\partial z_i} = \Big(0, \dots, 0, \underset{\substack{\uparrow \\ i\text{th coordinate}}}{1}, 0, \dots, 0, \frac{\partial g}{\partial z_i}\Big),$$

$$\frac{\partial G}{\partial z_i}(0) = (0, \dots, 0, \underset{\substack{\uparrow \\ i\text{th coordinate}}}{1}, 0, \dots, 0), \tag{5.2}$$

$$\frac{\partial^2 G}{\partial z_i \, \partial z_j} = \Big(0, \dots, 0, \frac{\partial^2 g}{\partial z_i \, \partial z_j}\Big).$$

Moreover (remember that $\phi = 0$),

$$\left\langle \frac{y - \phi}{|y - \phi|}, n \right\rangle = \frac{y_n}{|y|}$$

and the matrices A and B of Lemma 4.2 take the form

$$A = \left(\frac{1}{|y|}\left(\delta_{ij} - \frac{y_i y_j}{|y|^2}\right)\right)_{ij}, \qquad B = \left(2\frac{y_n}{|y|} \frac{\partial^2 g}{\partial z_i \, \partial z_j}\right)_{ij}. \tag{5.3}$$

A natural way to compare the contact between E and ∂D is to compare the Hessians of f and g at 0. More precisely, we have the following lemma.

LEMMA 5.1. *We have*

$$\begin{aligned} I - t_0 A^{-1}B &= I - \operatorname{Hess} f(0)^{-1} \operatorname{Hess} g(0) \\ &= \operatorname{Hess} f(0)^{-1}(\operatorname{Hess} f(0) - \operatorname{Hess} g(0)). \end{aligned}$$

PROOF. We only need to compute the Hessian of f at 0. Now f is defined implicitly by

$$F(z_1, \dots, z_{n-1}, f(z_1, \dots, z_{n-1})) = \text{const},$$

where $F(z) = |x - z| + |y - z|$. We already know that

$$DF(z) = -\left(\frac{x - z}{|x - z|} + \frac{y - z}{|y - z|}\right)$$

and

(5.4) $$DF(0) = -\lambda n,$$

where

$$\lambda = 2\left\langle \frac{y}{|y|}, n \right\rangle = 2\frac{y_n}{|y|}.$$

By the implicit function theorem,

$$\frac{\partial f}{\partial z_i} = -\frac{\partial F/\partial z_i}{\partial F/\partial z_n},$$

so that

$$\frac{\partial^2 f}{\partial z_i\, \partial z_j} = -\frac{(\partial F/\partial z_n)(\partial^2 F/\partial z_i\, \partial z_j) - (\partial F/\partial z_i)(\partial^2 F/\partial z_n\, \partial z_j)}{(\partial F/\partial z_n)^2}.$$

At $z = 0$ one has $\partial F/\partial z_i = 0$ for $i = 1, \dots, n - 1$, because of (5.4) (recall that the normal n lies along the nth axis) and $\partial F/\partial z_n = -\lambda$. Thus, for $i, j = 1, \dots, n - 1$,

$$\frac{\partial^2 f}{\partial z_i\, \partial z_j}(0) = \frac{1}{\lambda}\frac{\partial^2 F}{\partial z_i\, \partial z_j}(0).$$

Moreover,

$$\begin{aligned}\frac{\partial^2 F}{\partial z_i\, \partial z_j}(0) &= \frac{1}{|x|}\left(\delta_{ij} - \frac{x_i x_j}{|x|^2}\right) + \frac{1}{|y|}\left(\delta_{ij} - \frac{y_i y_j}{|x|^2}\right)\\ &= \frac{1}{t_0|y|}\left(\delta_{ij} - \frac{y_i y_j}{|x|^2}\right),\end{aligned}$$

so that, finally,

$$\frac{\partial^2 f}{\partial z_i\, \partial z_j} = \frac{1}{\lambda t_0|y|}\left(\delta_{ij} - \frac{y_i y_j}{|x|^2}\right),$$

and from (5.3) the statement follows. □

We are now going to give a more intrinsic version of the previous lemma.

Let M be a hypersurface of $\mathbb{R}^n$, let ϕ be a point on M and let M_ϕ be the tangent space to M at ϕ. Let $n(x)$ be a $\mathscr{C}^\infty$ unit normal field around ϕ. If

$X = \sum_{i=1}^{n} a_i \, \partial/\partial x_i$ is a vector in M_ϕ, consider the transformation

$$X \to Xn, \tag{5.5}$$

where Xn is the vector whose jth component is $\sum_{i=1}^{n} a_i (\partial n_j / \partial x_i)(\phi)$. Then Xn is still a vector in M_ϕ, so that (5.5) defines a transformation of M_ϕ into itself which is called the Weingarten map [see Hicks (1965), page 21, e.g.]. This map is closely related to the curvature of M at ϕ [Hicks (1965), page 24].

In our situation the two hypersurfaces E and D have the same tangent hyperplane at ϕ on which the Weingarten maps L_1 of E and L_2 of ∂D both operate. Thus a measure of the contact between E and ∂D can be obtained by looking at the difference $L_1 - L_2$. The following statement gives an intrinsic expression for the quantity $\det(I - t_0 A^{-1} B)^{-1/2}$.

Proposition 5.2. *We have*

$$I - t_0 A^{-1} B = L_1^{-1}(L_1 - L_2) = I - L_1^{-1} L_2.$$

Proof. Let us choose the system of coordinates introduced before Lemma 5.1, and let $n(z)$ be the $\mathscr{C}^\infty$ unit normal field pointing in the direction of the positive nth axis. It is then an elementary exercise in differential geometry to check that, in these coordinates,

$$L_1 = -\text{Hess}\, f(0), \qquad L_2 = -\text{Hess}\, g(0).$$

Indeed it suffices to remark that

$$\left\langle \frac{\partial G}{\partial z_i}(z), n(z) \right\rangle \equiv 0,$$

for $i = 1, \dots, n-1$, since $\partial G / \partial z_i$ is always a tangent vector. Thus, if $j = 1, \dots, n-1$, by derivation

$$0 = \frac{\partial}{\partial z_j} \left\langle \frac{\partial G}{\partial z_i}(z), n(z) \right\rangle = \left\langle \frac{\partial^2 G}{\partial z_i \, \partial z_j}(z), n(z) \right\rangle = \left\langle \frac{\partial G}{\partial z_i}(z), \frac{\partial n}{\partial z_j}(z) \right\rangle.$$

However, at $z = 0$, $n(z) = (0, \dots, 0, 1)$ so that, by (5.2), at $z = 0$ the previous identity becomes

$$\frac{\partial^2 g}{\partial z_i \, \partial z_j}(0) + \frac{\partial n_i}{\partial z_j}(0) = 0,$$

which expresses the fact that $-(\partial^2 g / (\partial z_i \, \partial z_j))(0)$ is the ijth entry of the representative matrix of the Weingarten map in the given coordinate system. □

One should remark that whereas the Weingarten map depends on the choice of the normal field (outer or inner), which makes it defined up to a multiplicative factor of -1, the quantity $L_1^{-1}(L_1 - L_2)$ is intrinsic and does not depend on the choice of the normal field.

6. Computing the mean exit time by simulation. We have performed a set of simulations in order to obtain numerically the mean exit time from a ball $D \subset \mathbb{R}^2$ of radius 1 of a Brownian motion starting at the origin. The well-known exact value is $\frac{1}{2}$.

As explained in the Introduction, a raw simulation scheme is biased by a systematic error. In order to get a correction of this error, we compute the asymptotics in ε for the probability that a Brownian motion B goes out of D in the time interval $[k\varepsilon, (k+1)\varepsilon]$, given $W_{k\varepsilon} = x$ and $W_{(k+1)\varepsilon} = y$, $x, y \in D$. The conditioned Brownian motion, up to a translation in the time scale, has the same law as

$$\mathring{W}_s = x + \frac{s}{\varepsilon}(y - x) + B_s - \frac{s}{\varepsilon}B_\varepsilon, \qquad 0 \le s \le \varepsilon.$$

Performing the time change $t = s/\varepsilon$,

$$\begin{aligned} Y_t = \mathring{W}_{t\varepsilon} &= x + t(y - x) + B_{t\varepsilon} - tB_\varepsilon \\ &= x + t(y - x) + \sqrt{\varepsilon}(B_t - tB_1) \quad \text{in law} \end{aligned}$$

for $0 \le t \le 1$. Thus the process Y has the same law as the solution of

$$\begin{aligned} dX^\varepsilon(t) &= -\frac{X^\varepsilon(t) - y}{1 - t}\,dt + \sqrt{\varepsilon}\,dB_t, \\ X^\varepsilon(s) &= x. \end{aligned}$$

Since the probability that $\mathring{W}$ exits from D before time ε is the same as the probability that X^ε exits D before time 1, its asymptotics are given by Theorem 4.4.

Table 1 reports the results of three sets of numerical simulations, each based on 10,000 simulated paths. In the first set (crude 1) the paths were obtained by simple simulation with a step $\varepsilon = 0.01$. In the second (corrected) the paths, still with step $\varepsilon = 0.01$, were killed with the procedure described in the Introduction, using the asymptotics derived in the previous sections. The third set was again obtained by crude simulation but with the step reduced to 0.002, to increase precision.

It can be remarked that the enhancement of the performances provided by the correction procedure is considerable. The increase in CPU time might be

TABLE 1

Type	Step	Estimated mean time	Error	CPU time
Crude 1	0.01	0.557	11.4%	1.00
Corrected	0.01	0.502	0.4%	2.90
Crude 2	0.002	0.527	5.4%	4.85

taken into account, but is considerably less than that observed in the third simulation, which denotes a less dramatic improvement in the results. It should be added that the second set of simulations, in which the correction procedure described in the Introduction was applied, is the only one for which the true value ($\frac{1}{2}$) was in a confidence interval, at any reasonable level. For instance, at level 95% the confidence interval is [0.495, 0.509].

The increase of the CPU time required by the correction procedure is less considerable than might be expected because one may ask the program to compute the killing probability p (a demanding task in terms of computation time) only when the process is near the boundary. Otherwise p is too small to influence the simulation significantly.

It is fair, however, to point out that by far the most time-consuming part in the computation of the killing probability is the determination of $\phi(x)$, which depends on the form of the boundary ∂D. One may expect that if D is more complicated than a ball, the increase in CPU time might be heavier.

The point $\phi(x)$ can be obtained by local search on the boundary; a different procedure to the same goal can be derived easily from the following lemma.

LEMMA 6.1. *Let $\gamma(t) = x + t(y - x)$ be the uniform motion on the line segment joining x to y. Then*

$$(|x - z| + |y - z|)^2 - |x - y|^2 = \min_{0 \le t \le 1} \min_{\xi \in \partial D} \frac{|\xi - \gamma(t)|^2}{t(1 - t)}.$$

Moreover, if x is regular and $\bar{t}$ and ξ are, respectively, the time and the point of ∂D at which the minimum above is attained, then $\bar{t} = t_0$ and $\xi = \phi(x)$.

The lemma says that the cost function is also given by the minimum of

$$t \to \frac{d(\gamma(t), \partial D)^2}{t(1 - t)}, \qquad 0 \le t \le 1,$$

which might be easier to obtain numerically if the boundary ∂D is such that the distance $d(\gamma(t), \partial D)$ is easy to compute (e.g., if D is a ball).

Lemma 6.1 may be restated in terms of elementary geometry by saying that if for each $0 \le t \le 1$ we consider a ball of radius const $\cdot \sqrt{t(1 - t)}$ and centered at $\gamma(t)$, then the union of all such balls is an ellipsoid of revolution having its foci in x and y.

PROOF OF LEMMA 6.1. If

$$H(t) = \frac{|z - (x + t(y - x))|^2}{t(1 - t)},$$

then

$$H'(t) = \frac{1}{t^2(1 - t)^2}\Big[-2t(1 - t)\langle z - (x + t(y - x)), y - x\rangle - (1 - 2t)|z - (x + t(y - x))|^2\Big].$$

Developing the quantity between brackets, the critical points of ϕ are the solutions of

$$t^2\big[|y-x|^2 - 2\langle z-x, y-x\rangle\big] + 2t|z-x|^2 - |z-x|^2 = 0.$$

This second-order equation is easily solved and, having discarded the negative solution and after some simplifications, one finds that the critical point is

$$\bar{t} = \frac{|z-x|}{|z-x| + |z-y|},$$

which is of course also a minimizer. Substituting back in order to compute the minimum and performing some more tedious simplifications one gets, finally,

$$\begin{aligned}\frac{|z-(x+\bar{t}(y-x))|^2}{\bar{t}(1-\bar{t})} &= 2(|z-x|\,|z-y| + \langle z-x, z-y\rangle)\\ &= (|x-z| + |y-z|)^2 - |x-y|^2. \qquad \square\end{aligned}$$

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