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ASYMPTOTIC SHAPES FOR STATIONARY FIRST PASSAGE PERCOLATION

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This paper deals with first passage percolation where the usual i.i.d. condition is weakened to stationarity (and ergodicity). The well known asymptotic shape result is known to extend to this case. It is easy to give necessary conditions for a compact set $B \subset \mathbb{R}^d$ to arise as the asymptotic shape for some stationary measure on the passage times. Our main result says that these conditions are also sufficient.

1. Introduction and main result. First passage percolation was introduced by Hammersley and Welsh (1965) and can be viewed as a model for the spread of a fluid through a porous medium. Let $\mathbb{Z}^d$ be the set of all d-tuples $x = (x(1), \dots, x(d))$ such that $x(1), \dots, x(d)$ are integers. Let $\mathbf{0}$ denote the origin. Consider the nearest neighbor graph on $\mathbb{Z}^d$, $d \geq 2$, that is, the graph whose vertex set is $\mathbb{Z}^d$ and where there is an (undirected) edge connecting x and y for every pair (x, y) such that $|x - y| = 1$ (here and throughout $|\cdot|$ denotes the Euclidean norm). The edge joining x and y is denoted $e(x, y)$. Each edge e is associated with a nonnegative random variable $T(e)$ which is interpreted as the time it takes for the fluid to pass through e. A connected sequence r of edges $(e(x_0, x_1), e(x_1, x_2), \dots, e(x_{n-1}, x_n))$ is called a path, and

$$D(r) = x_n - x_0$$

is the total displacement as one moves along the path. If $r_1, \dots, r_k$ are paths such that for $i = 1, \dots, k-1$, r_i ends where r_{i+1} starts, then we define the concatenation $r = (r_1, \dots, r_k)$ in the obvious way. The passage time $T(r)$ of r is defined

$$T(r) = \sum_{i=1}^{n} T(e(x_{i-1}, x_i)).$$

The travel time from x to y is

$$T(x, y) = \inf\{T(r)\colon\ r \text{ is a path from } x \text{ to } y\}.$$

Clearly,

$$T(x, y) = T(y, x), \qquad \forall\, x, y \in \mathbb{Z}^d. \tag{1}$$

We also have that travel times are subadditive, that is,

$$T(x, y) \leq T(x, z) + T(z, y), \qquad \forall\, x, y, z \in \mathbb{Z}^d. \tag{2}$$

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Suppose fluid enters the medium at the origin at time zero. The random set

$$\tilde{B}(t) = \{x \in \mathbb{Z}^d\colon\ T(\mathbf{0}, x) \le t\}$$

is interpreted as the set of sites reached by the fluid by time t. One of the main objects of interest in first passage percolation is the limiting behavior of $\tilde{B}(t)$ as $t \to \infty$. To describe this behavior, it is convenient to replace $\tilde{B}(t)$ by

$$\bar{B}(t) = \{x + y\colon\ x \in \tilde{B}(t),\ y \in \bar{U}\},$$

where

$$\bar{U} = \{y = (y(1), \dots, y(d))\colon\ |y(i)| \le \tfrac{1}{2},\ 1 \le i \le d\}$$

is the closed unit cube centered at the origin. Thereby, the lattice structure of $\tilde{B}$ is "smoothed out." For $x \notin \mathbb{Z}^d$, $T(\mathbf{0}, x)$ will be interpreted as $\inf\{t\colon\ \bar{B}(t) \ni x\}$.

In the case when the passage times $T(e)$ are stationary [i.e., for any n the distribution of $(T(e(x_1 + z, y_1 + z)), \dots, T(e(x_n + z, y_n + z)))$ does not depend on $z \in \mathbb{Z}^d$] and ergodic (i.e., any event that is invariant under all translations has probability 0 or 1) with finite first moment, the subadditive relation (2) implies that

$$\mu(x) = \lim_{r \to \infty} \frac{E[T(\mathbf{0}, rx)]}{r}$$

exists for any $x \in \mathbb{R}^d$. It then follows from Kingman's subadditive ergodic theorem [Kingman (1973)] that

$$\lim_{r \to \infty} \frac{T(\mathbf{0}, rx)}{r} = \mu(x) \quad \text{a.s.} \tag{3}$$

This is referred to as the existence of an asymptotic speed.

The majority of attention has been focused upon the case when the passage times are i.i.d., the main result being that $\bar{B}(t)$, under suitable moment conditions, has an asymptotic shape. This is made precise in the following theorem, proved in the case $d = 2$ by Cox and Durrett (1981) and for higher dimensions by Kesten (1986). See also Durrett (1988).

THEOREM 1.1. *Let F be a nonnegative probability distribution such that if $T_1, \dots, T_{2d}$ are i.i.d. with distribution F, we have*

$$E(\min\{T_1^d, \dots, T_{2d}^d\}) < \infty.$$

Suppose the passage times $T(e)$ are i.i.d. with distribution F. Then there exists a nonrandom convex set $B_0 \subseteq \mathbb{R}^d$ with nonempty interior such that either (a) *or* (b) *holds*:

(a) *B_0 is compact and for all $\varepsilon > 0$ we a.s. have*

$$(1 - \varepsilon)B_0 \subseteq \frac{\bar{B}(t)}{t} \subseteq (1 + \varepsilon)B_0 \quad \textit{eventually};$$

(b) *for all* $M > 0$ *we a.s. have*

$$\frac{\bar{B}(t)}{t} \supseteq \{x \in \mathbb{R}^d,\ |x| \leq M\} \quad \textit{eventually}$$

(*this corresponds to* $B_0 = \mathbb{R}^d$).

This means that the a.s. convergence in (3) holds in all directions simultaneously. Due to the subadditivity of the travel times, we have that B_0 must be convex. By symmetry, B_0 must also be invariant under permutations of the coordinate axes as well as under reflections in the coordinate hyperplanes. A precise condition on F distinguishing the cases (a) and (b) is known (see Section 2). In case (a), the exact shape of B_0 seems extremely difficult to compute; it is only known in the trivial case when F is a one-point distribution. Consult Kesten (1987) for a nice review of i.i.d. first passage percolation.

Recently some attention has been drawn to non-i.i.d. cases as well. Fontes and Newman (1993) consider certain dependent models of possible relevance to physics. Boivin (1990) considers the general stationary and ergodic case, and proves the following theorem.

THEOREM 1.2. *Let* $F_1, \ldots, F_d$ *be nonnegative probability distributions such that for* $i = 1, \ldots, d$ *there is an* $\varepsilon > 0$ *such that* F_i *has finite moment of order* $d + \varepsilon$. *Suppose the passage times are stationary and ergodic, and that an edge oriented in the ith coordinate direction has distribution* F_i. *Then there exists a deterministic, continuous and nonnegative function* μ *on* $\{x \in \mathbb{R}^d\colon |x| = 1\}$ *such that*

$$\lim_{\substack{|x|\to\infty \\ x \in Z^d}} \left(\frac{T(\mathbf{0}, x)}{|x|} - \mu\left(\frac{x}{|x|}\right)\right) = 0 \quad a.s.$$

This is the stationary case analogue of Theorem 1.1. In the special case where the $T(e)$'s are also bounded, this was first noted by Derriennic [see Kesten (1986), page 259]. The quantity $\mu(x/|x|)$ is the inverse asymptotic speed in the direction $x/|x|$. To phrase the result in terms of asymptotic shapes, let

$$B_0 = \left\{ x \in \mathbb{R}^d\colon\ |x|\mu\left(\frac{x}{|x|}\right) \leq 1 \right\}.$$

If $\mu(x/|x|) > 0$ everywhere, then B_0 is compact with nonempty interior, and conclusion (a) of Theorem 1.1 holds. Since isotropy is not assumed, B_0 does not have the invariance properties of the i.i.d. case, except for the fact that

$$x \in B_0 \quad \Leftrightarrow \quad -x \in B_0, \tag{4}$$

which can equivalently be stated $\mu(x/|x|) = \mu(-x/|x|)$ and which follows easily from (1). We will use the word symmetric to indicate that (4) holds. Convexity follows as in the i.i.d. case. Hence, symmetry, convexity and nonempty interior are necessary conditions for a set $B \subseteq \mathbb{R}^d$ to arise as limiting shape in stationary first passage percolation.

If $\mu(x/|x|) \equiv 0$, we have infinite asymptotic speed in all directions, and situation (b) in Theorem 1.1 applies.

There is a third case, however: when μ takes on both zero and strictly positive values. Then neither of the two scenarios of Theorem 1.1 apply. A trivial example in two dimensions is when all horizontal edges have passage time 0 and all vertical edges have passage time 1. For more interesting examples, see Section 2.

Our main theorem concerns the first of these three cases. It says that the conditions of symmetry and convexity are not only necessary but also sufficient conditions for a compact set with nonempty interior to arise as a limiting shape for some stationary measure on the passage times.

THEOREM 1.3. *Let $\mathscr{C}$ be the set of all subsets of $\mathbb{R}^d$ that are compact, convex and symmetric with nonempty interior. Let $\mathscr{C}^*$ be the set of all compact subsets of $\mathbb{R}^d$ with nonempty interior that can arise as limiting shapes for stationary first passage percolation. Then $\mathscr{C}^* = \mathscr{C}$.*

If the edges are thought of as microscopic, we thus have a precise condition telling us which kinds of macroscopic linear spread can be modelled by stationary first passage percolation. The problem of determining a similar condition for i.i.d. first passage percolation seems to be difficult.

From the above discussions $\mathscr{C}^* \subseteq \mathscr{C}$ is clear so we only need to prove $\mathscr{C} \subseteq \mathscr{C}^*$. The proof will be by construction. That is, for any $B \in \mathscr{C}$ we construct a stationary measure μ_B which will yield B as a limiting shape. We remark that μ_B has the following very nice properties. First, passage times are bounded. Second, μ_B is Bernoulli, which means isomorphic (in the ergodic-theoretical sense) to an i.i.d. measure and implies mixing. Our measure also has trivial tail σ-fields.

The rest of this paper is organized as follows. In Section 2 we discuss the possible finiteness of the asymptotic speed and the related issue of compactness of the asymptotic shape, while in Section 3 we prove Theorem 1.3.

2. On the compactness of B_0. For i.i.d. first passage percolation, a precise condition for compactness of the asymptotic shape B_0 was given in Kesten (1986): B_0 is compact if and only if

$$F(0) < p_c(\mathbb{Z}^d, \text{ bond}),$$

where $p_c(\mathbb{Z}^d, \text{ bond})$ is the critical value for standard d-dimensional bond percolation [see Grimmett (1989) or Kesten (1987)]. This is to say that $\bar{B}(t)$ grows faster than linearly if and only if the expected distance which can be travelled from the origin along edges with zero passage time only is infinite.

Moving on to the stationary case, one might hope to find some similar condition guaranteeing compactness of B_0. However, as we shall see, it is possible to get superlinear growth of $\bar{B}(t)$ even if a.s. no edges have zero passage time.

In fact, it is not possible to give any condition on the marginal distributions $F_1, \dots, F_d$ of the passage times which is sufficient for the compactness of B_0, except for the rather obvious fact that B_0 is compact if there is a $\delta > 0$ such that $F_i(\delta) = 0$ for $i = 1, \dots, d$.

EXAMPLE 2.1. Let $F_1, \dots, F_d$ have finite moments of order $d + \varepsilon$ for some $\varepsilon > 0$, so that Theorem 1.2 applies. Suppose furthermore that at least one of the F_i's (F_1, say) supports the interval $[0, \delta]$ for all $\delta > 0$ (note that this does not imply that F_1 has a point mass at 0). We construct a stationary measure on the passage times, with marginals $F_1, \dots, F_d$, as follows. Let $\{X_i\}_{i \in Z}$ be i.i.d. random variables with distribution F_1. Let each edge e oriented in the first coordinate direction have passage time X_i, where i is the second coordinate of the vertices joined by e. Finally, let all other edges have passage times which are independent (and distributed according to their respective marginals). The measure obtained in this way yields infinite asymptotic speed in the first coordinate direction. To see this, note that for any $\varepsilon > 0$ it is possible to travel at asymptotic speed at least $1/\varepsilon$ by first moving along the second coordinate axis until a vertex incident to some edge e oriented in the first coordinate direction with $T(e) \le \varepsilon$ is encountered, and then turning and moving in the first coordinate direction.

EXAMPLE 2.2. Let the passage times be as in the previous example, with the extra assumption that the distributions $F_2, \dots, F_d$ are bounded away from 0. Then the asymptotic speed is clearly finite in all directions except in the first coordinate direction. By an obvious modification we can get, for any $i \in \{1, \dots, d\}$, infinite speed in exactly i of the d coordinate directions.

The next example shows that finite asymptotic speed in each of the d coordinate directions is not a sufficient condition for compactness of the asymptotic shape.

EXAMPLE 2.3. Let $X = 0$ or 1, each with probability $\frac{1}{2}$. Let $d = 2$ and let the passage times be given by

$$T\big(e\big((x(1), x(2)), (x(1)+1, x(2))\big)\big) = \begin{cases} X, & \text{if } x(1) + x(2) \text{ is even,} \\ 1 - X, & \text{if } x(1) + x(2) \text{ is odd,} \end{cases}$$

$$T\big(e\big((x(1), x(2)), (x(1), x(2)+1)\big)\big) = \begin{cases} 1 - X, & \text{if } x(1) + x(2) \text{ is even,} \\ X, & \text{if } x(1) + x(2) \text{ is odd;} \end{cases}$$

see Figure 1. It is easy to check that the measure on the passage times thus obtained is stationary and ergodic and that the asymptotic speed is infinite in the directions

$$\frac{x}{|x|} = \pm \frac{(1,1)}{\sqrt{2}}$$

and finite in all others.

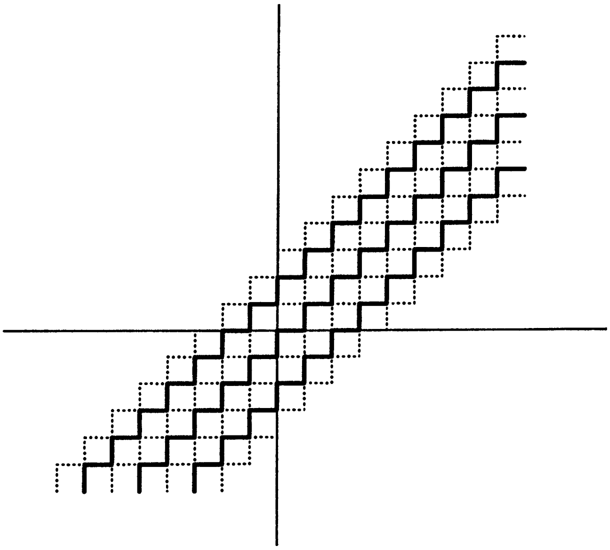

FIG. 1. *All bold edges have the same passage time X; all dotted edges have the same passage time* $1 - X$.

None of these examples is mixing, but measures with similar properties and which, in addition, are mixing (even Bernoulli) can be constructed using, for example, the technique of Section 3.

3. Proof of Theorem 1.3. We begin with some preliminaries.

LEMMA 3.1. *Let* $B \subseteq \mathbb{R}^d$ *be convex, let* $x_1, \ldots, x_n \in \mathbb{R}^d$ *and let* $\alpha_1, \ldots, \alpha_n$ *be positive numbers such that for* $i = 1, \ldots, n$ *we have* $\alpha_i^{-1} x_i \in B$. *Then*

$$\frac{x_1 + \cdots + x_n}{\alpha_1 + \cdots + \alpha_n} \in B.$$

PROOF. We have

$$\frac{x_1 + \cdots + x_n}{\alpha_1 + \cdots + \alpha_n} = \frac{\alpha_1}{\alpha_1 + \cdots + \alpha_n} \frac{x_1}{\alpha_1} + \cdots + \frac{\alpha_n}{\alpha_1 + \cdots + \alpha_n} \frac{x_n}{\alpha_n} \in B$$

by convexity. □

For a set

$$E = \{e(x_1, y_1), \ldots, e(x_n, y_n)\}$$

of edges and $z \in \mathbb{Z}^d$, let

$$T_z E = \{e(x_1 + z, y_1 + z), \ldots, e(x_n + z, y_n + z)\}.$$

In words, T_zE is E shifted over the vector z. If η: $E \to \mathbb{R}^+$ is a configuration assigning passage times to E, we write $T_z\eta$ for the configuration on T_zE given by $T_z\eta(e(x_i + z, y_i + z)) = \eta(e(x_i, y_i))$.

We now turn to the construction which will prove Theorem 1.3. Let $B \in \mathscr{C}$ be the desired asymptotic shape. Let

$$h = \inf\{|x|: x \in \mathbb{R}^d \setminus B\},$$

and note that

$$h > 0$$

due to the fact that B is symmetric with nonempty interior whence it has $\mathbf{0}$ in its interior.

Let $(x_1, x_2, \dots)$ be a dense sequence of points on the boundary of B. What we will construct is a stationary measure which yields something which at first sight looks like a complete mess of finite chains of fast edges in an environment of slow edges, but whose asymptotic shape is surprisingly tractable. The chains will be fairly straight (more like sticks, perhaps) and oriented in directions $x_1/|x_1|, x_2/|x_2|, \dots$.

For each i, let z_i denote the integer point closest to $2^i(x_i/|x_i|)$, in the Euclidean norm and with some arbitrary convention in case of ties. Let r_i be a path from $\mathbf{0}$ to z_i such that the following hold:

(i) The path r_i is as short as possible, in that it consists of exactly $|z_i(1)| + \cdots + |z_i(d)|$ edges.

(ii) No vertex in r_i is further away than $\sqrt{d}$ units from a straight line through $\mathbf{0}$ and z_i.

The reader may easily check that such a path exists. Let E_i be the set of edges given by

$$E_i = \{e\text{: } e \text{ is an edge either in } r_i \text{ or incident to some vertex in } r_i\};$$

see Figure 2.

Define for $i = 1, 2, \dots$, the configuration η_i: $E_i \to \mathbb{R}^+$ of passage times by

$$\eta_i(e) = \begin{cases} |x_i(1)|/|x_i|^2, & \text{if } e \text{ is in } r_i \text{ and oriented in the} \\ & \quad \text{first coordinate direction,} \\ \vdots & \\ |x_i(d)|/|x_i|^2, & \text{if } e \text{ is in } r_i \text{ and oriented in the} \\ & \quad d\text{th coordinate direction,} \\ h^{-1}(1 + 2\sqrt{d}), & \text{if } e \text{ is not in } r_i. \end{cases}$$

The reason for these choices will hopefully become clear later, but a few words can be said immediately. The reason for the choice of passage times for the edges in r_i is that $|x_i(j)|/|x_i|^2$ equals $1/|x_i|$ times the length of the projection of the jth unit vector onto x_i. These are designed for yielding fast asymptotic speed (close to $|x_i|$) in direction $x_i/|x_i|$. The other edges are designed for preventing higher speed than what is allowed by B. An edge in itself is of course

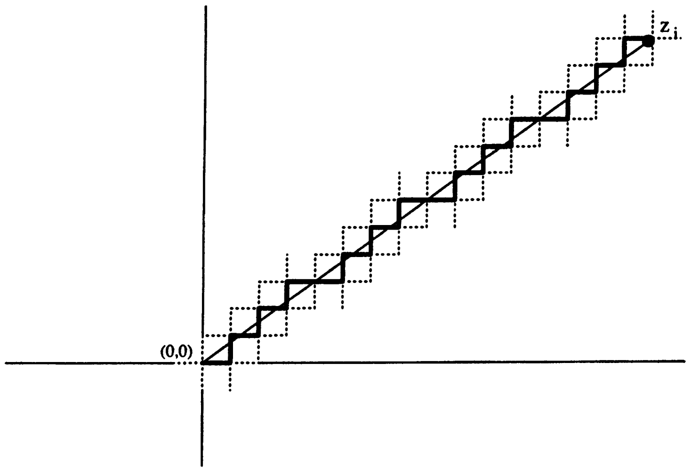

FIG. 2. *The path r_i is formed by the bold edges and the dotted edges are the slow edges surrounding r_i. Together the bold and the dotted edges form the set E_i.*

within the "speed limits" of B if it has passage time at least h^{-1}. The term $h^{-1}2\sqrt{d}$ is for compensating for sideways displacements (perpendicular to $|x_i|$) in r_i; these displacements can be of length up to $2\sqrt{d}$.

Now, let $\{X_x\}_{x\in Z^d}$ and $\{Y_x\}_{x\in Z^d}$ be arrays of random variables such that

$$\{X_x\}_{x\in Z^d} \text{ are i.i.d. with } P(X_x = j) = \begin{cases} \frac{1}{2}, & \text{for } j = 0, \\ 3^{-j}, & \text{for } j = 1, 2, \dots, \end{cases}$$

$$\{Y_x\}_{x\in Z^d} \text{ are i.i.d. with } Y_x \text{ uniformly distributed on } [0,1]$$

and

$$\{X_x\}_{x\in Z^d} \text{ is independent of } \{Y_x\}_{x\in Z^d}.$$

The passage times will be determined from $\{X_x\}_{x\in Z^d}$ and $\{Y_x\}_{x\in Z^d}$ in the following way. For every $x \in \mathbb{Z}^d$ and whenever $X_x \neq 0$, the edges in $T_x E_i$ will be assigned passage times according to $T_x \eta_i$, where $i = X_x$. This will of course lead to contradictions, because the different $T_x E_i$'s will sometimes overlap. This is resolved in the following way. Whenever two or more $T_x E_i$'s try to determine the passage time of the same edge, the one with the largest value of X_x (i.e., of i) wins. In case of a tie here, the one with the largest value of Y_x wins. This works because a.s. only finitely many $T_x E_i$'s will compete for each edge: the expected number of competitors is less than or equal to

$$\sum_{i=1}^{\infty} 3^{-i} d 2^{i+1} = 2d \sum_{i=1}^{\infty} \left(\tfrac{2}{3}\right)^i < \infty$$

since each E_i contains at most $d2^{i+1}$ edges in each coordinate direction by (i) above. Finally, let all remaining edges have passage time $h^{-1}(1 + 2\sqrt{d})$. This is our construction.

The passage times are bounded by $h^{-1}(1+2\sqrt{d})$, and they are stationary and Bernoulli since they are obtained in a translation invariant way from an i.i.d. process [see Ornstein and Weiss (1987)]. Given a realization of the process, we call an edge e slow if $T(e) = h^{-1}(1+2\sqrt{d})$, and fast otherwise.

Let B° denote the interior of B. What we need in order to show that B is the asymptotic shape for our process is the following:

(a) $(\bar{B}(t))/t \not\ni y$ eventually a.s. for any $y \in \mathbb{R}^d \setminus B$;
(b) $(\bar{B}(t))/t \ni y$ eventually a.s. for any $y \in B^\circ$.

Statement (b) is equivalent to

$$\text{(b}'\text{)} \qquad \lim_{t\to\infty} \frac{E[T(\mathbf{0}, ty)]}{t} < 1 \quad \text{for any } y \in B^\circ.$$

We have existence of the limit in (b′) and equivalence with (b) due to the a.s. convergence in (3), the boundedness of the passage times and Theorem 1.2. We will prove (a) and (b′).

We first prove (a). For $y \in \mathbb{Z}^d$, let r be a path from $\mathbf{0}$ to y. The path r can be written as the concatenation $(r_1^f, r_1^s, r_2^f, \dots, r_n^f, r_n^s)$, where $r_1^f, r_2^f, \dots, r_n^f$ consist of fast edges only and $r_1^s, r_2^s, \dots, r_n^s$ consist of slow edges only (r_1^f and r_n^s may be empty). We call $r_1^f, r_2^f, \dots, r_n^f$ fast subpaths and $r_1^s, r_2^s, \dots, r_n^s$ slow subpaths. It is clear from the construction that each fast subpath gets its values from one $T_x\eta_i$ only (this is due to the "layer" of slow edges that surround the fast ones in each η_i). The displacement $D(r_j^f)$ travelled by a fast subpath (whose edges are in T_xE_i) can be written

$$D(r_j^f) = D_\parallel(r_j^f) + D_\perp(r_j^f),$$

where $D_\parallel(r_j^f)$ is the component of $D(r_j^f)$ obtained by projecting it on the straight line associated with $T_x\eta_i$, and $D_\perp(r_j^f)$ is the component perpendicular to this line. We have, by the choice of passage times for the fast edges,

$$\frac{D_\parallel(r_j^f)}{T(r_j^f)} = \pm x_i \in B.$$

We also have

$$|D_\perp(r_j^f)| \le 2\sqrt{d}.$$

Hence,

$$\frac{D_\perp(r_j^f)}{2h^{-1}\sqrt{d}} \in B$$

and Lemma 3.1 implies

$$\frac{D(r_j^f)}{T(r_j^f) + 2h^{-1}\sqrt{d}} \in B.$$

For each slow edge e we have

$$\frac{D(e)}{T(e) - 2h^{-1}\sqrt{d}} \in B.$$

Since the number of fast subpaths of r exceeds the number of slow edges in r by at most 1, we have, again using Lemma 3.1,

$$\frac{D(r)}{T(r) + 2h^{-1}\sqrt{d}} \in B.$$

Pick $y \in \mathbb{R}^d$ and $t \in \mathbb{R}$. Taking infimum of the passage times of all paths from $\mathbf{0}$ to ty, we get (using the fact that B is closed)

$$\frac{ty}{T(\mathbf{0}, ty) + 2h^{-1}\sqrt{d}} \in B$$

and letting $t \to \infty$, the term $2h^{-1}\sqrt{d}$ can be neglected, so that if $y \notin B$,

$$\limsup_{t\to\infty} \frac{t}{T(\mathbf{0}, ty)} < 1$$

(again using closedness of B). By (3) the lim sup is in fact a limit, so we have

$$\lim_{t\to\infty} \frac{T(\mathbf{0}, ty)}{t} > 1$$

and (a) follows.

We proceed to prove (b′). For any $x \in B^\circ$ and any i, we can find $\varepsilon > 0$, k and $i_1, \dots, i_k$ such that $i \le i_1 < \cdots < i_k$ and x belongs to the convex hull of $(1+\varepsilon)^{-1}x_{i_1}, \dots, (1+\varepsilon)^{-1}x_{i_k}$. Hence it suffices to show that, for any $\varepsilon > 0$,

$$\lim_{t\to\infty} \frac{E[T(\mathbf{0}, tx_i)]}{t} < 1 + \varepsilon \tag{5}$$

for all sufficiently large i. For fixed i, (5) follows by subadditivity once we can show that

$$\frac{|x_i| E[T(\mathbf{0}, z_i)]}{2^i} < 1 + \varepsilon. \tag{6}$$

The idea now is to give an upper bound for the left-hand side of (6) by suggesting a specific choice of path from $\mathbf{0}$ to z_i. Due to the choice of passage times for the fast edges in $T_x E_i$, the left-hand side would equal 1 if (a) a $T_x r_i$ path would start at $\mathbf{0}$ (and thus end at z_i) and (b) this path were not interrupted by other configurations. Both (a) and (b) fail (with high probability). However, the time lost is reasonably small if we adopt the following strategy:

First find the $T_x r_i$ path whose starting point x is closest to the origin (in the L^1-norm with some arbitrary convention in case of a tie) and move to x using as few edges as possible. Then move along $T_x r_i$ (regardless of whether the edges are covered by configurations other than $T_x \eta_i$) until it ends, at $x + z_i$, and finally go from $x + z_i$ to z_i using as few edges as possible.

We have that

$$E[T(\mathbf{0}, z_i)] \leq \frac{2^i}{|x_i|} + 2ah^{-1}(1+2\sqrt{d}) + bh^{-1}(1+2\sqrt{d}), \tag{7}$$

where a is the expected L^1-distance from $\mathbf{0}$ to the nearest starting point of a $T_x r_i$-path (the 2 in front of a comes from the movement back to z_i at the end of the path) and b is the expected number of edges in $T_x r_i$ which are covered by another configuration of higher priority.

Since each integer point is the starting point of a $T_x r_i$ path with probability 3^{-i}, independently of all other points, it is easy to see that

$$a \leq C3^{i/d}$$

for some constant $C = C(d)$. To get an upper bound for b, note that $T_x r_i$ has at most $d2^i$ edges and that each edge is covered by a configuration of higher priority with probability at most

$$\sum_{j=i}^{\infty} 3^{-j} d 2^{j+1} = 2d \sum_{j=i}^{\infty} (\tfrac{2}{3})^j = 6d(\tfrac{2}{3})^i$$

so that

$$b \leq 6d^2(\tfrac{4}{3})^i.$$

Substituting a and b in (7) we get

$$E[T(\mathbf{0}, z_i)] \leq \frac{2^i}{|x_i|} + \left(2C3^{i/d} + 6d^2\left(\frac{4}{3}\right)^i\right)h^{-1}(1+2\sqrt{d}),$$

so that

$$\begin{aligned}\frac{|x_i| E[T(\mathbf{0}, z_i)]}{2^i} &\leq 1 + \left(2C\left(\frac{3^{1/d}}{2}\right)^i + 6d^2\left(\frac{2}{3}\right)^i\right)h^{-1}(1+2\sqrt{d}) \\ &\leq 1 + (2C + 6d^2)\left(\frac{\sqrt{3}}{2}\right)^i h^{-1}(1+2\sqrt{d}),\end{aligned}$$

which is smaller than $1+\varepsilon$ whenever

$$(2C + 6d^2)\left(\frac{\sqrt{3}}{2}\right)^i h^{-1}(1+2\sqrt{d}) < \varepsilon,$$

that is, whenever

$$i > \frac{\log(((2C + 6d^2)(1+2\sqrt{d}))/h\varepsilon)}{\log(2/\sqrt{3})}.$$

So (6) holds for all such i, and the proof of Theorem 1.3 is complete. □

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