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CENTRAL LIMIT THEOREM IN NEGATIVE CURVATURE

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We prove a central limit theorem for the distance of the Brownian point on the universal cover of a compact negatively curved Riemannian manifold. The technical point is a contraction property for the leafwise Brownian motion along the stable foliation.

Let M be a closed Riemannian manifold with negative sectional curvature, and consider the Brownian motion $(\tilde{\omega}_t)_{t \in \mathbb{R}_+}$ on the universal cover $\tilde{M}$ of M. Natural geometric quantities have a linear asymptotic growth along the trajectories. For instance, there are positive numbers l and h such that for a.e. $\tilde{\omega}$

$$\lim_{t \to +\infty} \frac{1}{t} d(x, \tilde{\omega}_t) = l \quad (\text{see } [8]),$$

$$\lim_{t \to +\infty} -\frac{1}{t} \log G(x, \tilde{\omega}_t) = h \quad (\text{see } [13]),$$

where d is the distance on $\tilde{M}$ and G is the Green function on $\tilde{M}$. Geometrically these numbers give some information about the harmonic measure on the boundary of $\tilde{M}$ (see [13] and [14]). In this paper we are interested in the following central limit theorem for the same processes.

THEOREM 1. *There are positive numbers σ_0 and σ_1 such that the distribution of the variables*

$$\frac{1}{\sigma_0 \sqrt{t}} [d(x, \tilde{\omega}_t) - tl]$$

and

$$\frac{1}{\sigma_1 \sqrt{t}} [\log G(x, \tilde{\omega}_t) + th]$$

are asymptotically close to the normal distribution when t goes to infinity.

In a more explicit form, the statement of Theorem 1 is that there exists a positive number σ_0 such that for any real r, any x in $\tilde{M}$,

$$\lim_{t \to \infty} \mathbb{P}_x\{(d(x, \tilde{\omega}_t) - tl) \le \sigma_0 r \sqrt{t}\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{r} \exp\left(-\frac{u^2}{2}\right) du,$$

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where $\mathbb{P}_x$ is the probability measure on the space $C_x(\mathbb{R}_+, \tilde{M})$ of continuous paths $(\tilde{\omega}_t)_{t \geq 0}$ with $\tilde{\omega}_0 = x$ which describes the Brownian motion on $\tilde{M}$ starting from x. There is an analogous statement for $\mathbb{P}_x\{(\log G(x, \tilde{\omega}_t) + th) \leq \sigma_1 r\sqrt{t}\}$.

In the case when $\tilde{M}$ is a symmetric space of negative curvature, direct computations yield $\sigma_0^2 = 2$ and $\sigma_1^2 = 2h$. One of these relations has a geometric meaning.

THEOREM 2. *Consider the number σ_1 obtained in Theorem 1. Then $\sigma_1^2 \geq 2h$, with equality if and only if the manifold M is asymptotically harmonic.*

Recall that we say that M is asymptotically harmonic if the mean curvature of the horospheres in $\tilde{M}$ is constant (see [5] for the properties of asymptotically harmonic manifolds).

The main tool in the proof of Theorems 1 and 2 is to introduce leafwise Brownian motion on the stable foliation associated with geodesic flow. We can replace our processes by semi-Markovian processes driven by this leafwise Brownian motion, and the main point is to show that there is enough contraction in this process. Following [4], [9] and [19], we show that there is contraction in spaces of Hölder continuous functions of sufficiently small exponent (Theorem 3). Our general result below is a central limit theorem for leafwise 1-forms evaluated on leafwise Brownian paths when the codifferential of the form is Hölder continuous (see Section 5 for a more precise statement).

This formulation was introduced by Le Jan [18] and will be investigated further in a companion paper. Here we follow the ideas and the work of Guivarc'h [8].

1. Brownian motion along the stable foliation. Let M be a closed Riemannian manifold with negative sectional curvature, SM be the unit tangent bundle to M, $(\phi_t)_{t \in \mathbb{R}}$ be the geodesic flow on SM and $W^s(v)$ be the stable manifold of the element v in SM:

$$W^s(v) = \left\{w : \exists\, s \in \mathbb{R} \text{ so that } \lim_{t \to \infty} d(\phi_{t+s}v, \phi_t w) = 0\right\}.$$

The sets W^s form a continous foliation of SM, and a neighborhood of v in $W^s(v)$ is canonically diffeomorphic to a neighborhood in M of the footpoint of v. Fix v and the leaf $W^s(v)$. This family of diffeomorphisms defines a metric g_s on the leaf $W^s(v)$. The associated Laplacian Δ_s is defined on C^2 functions on the leaf. We still denote by Δ_s the operator on functions on SM given by Δ_s applied to the restriction of the function to the ambient leaf whenever it makes sense. We shall consider the leafwise Brownian motion $(\omega_t)_{t \in \mathbb{R}_+}$, that is, the Markov process with continuous trajectories on SM and with generator Δ_s (see [6]).

We shall write probability transitions of the leafwise Brownian motion as follows. Recall that geodesics in $\tilde{M}$ are said to be equivalent if they remain a bounded distance apart and that the space of equivalent classes of unit-speed geodesics is the boundary $\partial\tilde{M}$. For each point x in $\tilde{M}$ and each point ξ in $\partial\tilde{M}$, there is a unique unit-speed geodesic starting from x in the class of ξ. We use this property to identify the unit sphere at x with the sphere at infinity $\partial\tilde{M}$. Then if (x, ξ) is the lift of some v in SM, the trajectories of ω_t starting from v can be obtained by projecting on SM the trajectories $(\tilde{\omega}_t, \xi)_{t \in \mathbb{R}_+}$, where $(\tilde{\omega}_t)_{t \in \mathbb{R}_+}$ is the Brownian motion on $\tilde{M}$ starting at x. Choose a fundamental domain M_0 and identify as above SM with $M_0 \times \partial\tilde{M}$. The transition densities of ω_t are given by

$$q_t((x, \xi), d(y, \eta)) = \sum_{\Gamma} \left[p_t(x, y) \mu_t^{x, y}(\gamma)\, dy\, \delta_{\gamma^{-1}\xi}(\eta) \right],$$

where $p_t(x, y)$ is the heat kernel on M_0, δ_ζ is the Dirac measure at ζ and for all t, x, y, $\mu_t^{x, y}$ is the probability measure on the deck transformation group Γ such that

$$\tilde{p}_t(x, \gamma y) := p_t(x, y) \mu_t^{x, y}(\gamma)$$

defines the heat kernel $\tilde{p}_t(x, \cdot)$ on $\tilde{M}$.

Let f be a continuous function on SM. We write $Q_t f$ for the continuous function on SM defined by

$$\begin{aligned} Q_t f(x, \xi) &= \mathbb{E}_{(x, \xi)} f(\omega_t) \\ &= \int f(y, \eta) q_t((x, \xi), d(y, \eta)) \\ &= \int_{M_0} p_t(x, y)\, dy \left[\sum_{\Gamma} \mu_t^{x, y}(\gamma) f(y, \gamma^{-1}\xi) \right]. \end{aligned}$$

By [6] (see also [15]), there is a unique Q-invariant probability measure ω on SM. That is, there is a unique probability measure ω on satisfying, for all continuous f, all positive t,

$$\int f\, d\omega = \int Q_t f\, d\omega.$$

We shall use a slightly stronger result. Define on $M_0 \times \partial\tilde{M} \times \partial\tilde{M}$ the probability transitions $q_t^2((x, \xi_1, \xi_2), d(y, \eta_1, \eta_2))$ by

$$q_t^2((x, \xi_1, \xi_2), d(y, \eta_1, \eta_2)) = \sum_{\Gamma} \left[p_t(x, y) \mu_t^{x, y}(\gamma)\, dy\, \delta_{\gamma^{-1}\xi_1}(\eta_1)\, \delta_{\gamma^{-1}\xi_2}(\eta_2) \right]$$

and the corresponding operator Q_t^2 on continuous functions on $M_0 \times \partial\tilde{M} \times \partial\tilde{M}$:

$$Q_t^2 f(x, \xi_1, \xi_2) = \int f(y, \eta_1, \eta_2) q_t^2((x, \xi_1, \xi_2), d(y, \eta_1, \eta_2)).$$

PROPOSITION 1. *There is a unique probability measure ω^2 on $M_0 \times \partial\tilde{M} \times \partial\tilde{M}$ satisfying*

$$\int Q_t^2 f\, d\omega^2 = \int f\, d\omega^2$$

for all continuous functions f and all positive t. The measure ω^2 *is given by*

$$\int f\,d\omega^2 = \int_{SM} f(x,\xi,\xi)\,d\omega(x,\xi).$$

PROOF. Let μ be a Q_t^2 invariant probability measure and f be a continuous function on $M_0 \times \partial\tilde{M} \times \partial\tilde{M}$.

We write $m_0 = dx/\mathrm{vol}\,M_0$ for the probability invariant under the Brownian motion on M_0 and $(\mu_x)_{x\in M_0}$ for a family of disintegrations for μ associated with the projection on M_0. We have for all continuous functions f on $M_0 \times \partial\tilde{M} \times \partial\tilde{M}$,

$$\int f\,d\mu = \int_{M_0}\left(\int f(v,\xi_1,\xi_2)\,d\mu_x(\xi_1,\xi_2)\right)dm_0(x).$$

Write π for the Γ-covariant projection on $\tilde{M} \times \partial\tilde{M} \times \partial\tilde{M}$ and set $\tilde{f} = f\cdot\pi$.

We write

$$\begin{aligned}
\int f\,d\mu &= \lim_{t\to\infty}\int Q_t^2 f\,d\mu\\
&= \lim_{t\to\infty}\int\left[\int_{M_0} p_t(x,y)\,dy\left(\sum_{\Gamma}\mu_t^{x,y}(\gamma)f(y,\gamma^{-1}\xi_1,\gamma^{-1}\xi_2)\right)\right]d\mu(x,\xi_1,\xi_2)\\
&= \lim_{t\to\infty}\int\left[\int_{M_0} p_t(x,y)\,dy\left(\sum_{\Gamma}\mu_t^{x,y}(\gamma)\tilde{f}(\gamma y,\xi_1,\xi_2)\right)\right]\\
&\quad\times d\mu_x(\xi_1,\xi_2)\,dm_0(x)\\
&= \lim_{t\to\infty}\int_{M_0} dm_0(x)\left[\int \tilde{p}_t(x,\tilde{y})\tilde{f}(\tilde{y},\xi_1,\xi_2)\,d\tilde{y}\,d\mu_x(\xi_1,\xi_2)\right]\\
&= \lim_{t\to\infty}\int_{M_0} dm_0(x)\mathbb{E}_x\left(\int\tilde{f}(\tilde{\omega}_t,\xi_1,\xi_2)\,d\mu_x(\xi_1,\xi_2)\right).
\end{aligned}$$

We obtained the last line by exchanging the order of integration of the variables $\tilde{y}$ and (ξ_1,ξ_2).

For x, y in $\tilde{M}$, denote by V_y^x the unit vector in $S_y\tilde{M}$ pointing toward x. Then $\int\tilde{f}(\tilde{y},\xi_1,\xi_2)\,d\mu_x(\xi_1,\xi_2)$ is close to $\tilde{f}(\tilde{y},V_y^x,V_y^x)$ as soon as $\tilde{y}$ is sufficiently far from x and not too close in direction to ξ_1 or ξ_2. Since as t goes to infinity, $d(x,\tilde{\omega}_t)$ goes to infinity a.e. and the limit distribution of the direction of $\tilde{\omega}_t$ is continuous, $\int\tilde{f}(\tilde{\omega}_t,\xi_1,\xi_2)\,d\mu_x(\xi_1,\xi_2)$ is close to $\tilde{f}(\tilde{\omega}_t,V_{\tilde{\omega}_t}^x,V_{\tilde{\omega}_t}^x)$ with probability close to 1. Therefore, we may write

$$\begin{aligned}
\int f\,d\mu &= \lim_{t\to\infty}\int_{M_0}\mathbb{E}_x\tilde{f}\left(\tilde{\omega}_t,V_{\tilde{\omega}_t}^x,V_{\tilde{\omega}_t}^x\right)dm_0(x)\\
&= \int f(x,\xi,\xi)\,d\omega(x,\xi).
\end{aligned}$$

To obtain the last equality, observe that, setting $F(x, \xi) = f(x, \xi, \xi)$, we have, in the same way,

$$\int f(x, \xi, \xi)\, d\omega(x, \xi) = \lim_{t \to \infty} \int Q_t F(x, \xi)\, d\omega(x, \xi)$$

$$= \lim_{t \to \infty} \int_{M_0} \mathbb{E}_x \tilde{F}\left(\tilde{\omega}_t, V^x_{\tilde{\omega}_t}\right) dm_0(x). \qquad \square$$

Write N for the operator on continuous functions on SM which associates with f the constant $\int f\, d\omega$. From the above argument it also follows that for all continuous f, $\lim_{t \to \infty} Q_t f = Nf$. In the next section, we introduce subspaces of functions where this convergence is exponential. The central limit theorem—and other limit theorems—will follow by standard arguments.

2. Hölder continuous functions and contraction. We shall define Hölder norms on $C(SM)$. We first recall some definitions from hyperbolic geometry (see, e.g., [7]). For p in $\tilde{M}$ and ξ, η in $\partial\tilde{M}$, we denote $(\xi|\eta)_p$ the quantity

$$(\xi|\eta)_p = \lim_{\substack{x \to \xi \\ y \to \eta}} \tfrac{1}{2}(d(p, x) + d(p, y) - d(x, y)).$$

For τ small enough, $d_p(\xi, \eta) := \exp(-\tau(\xi|\eta)_p)$ defines a distance on $\partial\tilde{M}$. If p, q are points in $\tilde{M}$, then the distances d_p and d_q are conformally equivalent. Also as $\xi \to \eta$, the following limit exists and defines the Busemann function $\psi_{p,\eta}(q)$:

$$\psi_{p,\eta}(q) = \lim_{\xi \to \eta}((\xi|\eta)_q - (\xi|\eta)_p).$$

We let $\mathbb{L}_\tau$ be for the space of bounded continuous functions f on SM such that $\|f\|_\tau$ is finite, where

$$\|f\|_\tau = \sup_{x, \xi} |f(x, \xi)| + \sup_{x, \xi_1, \xi_2} |f(x, \xi_1) - f(x, \xi_2)| \exp\left(\tau(\xi_1|\xi_2)_x\right).$$

The main result of this paper is the following theorem:

THEOREM 3. *For every τ small enough, there exist $C > 0$ and $\zeta < 1$ such that, for all $t > 0$,*

$$\|Q_t - N\|_\tau \le C\zeta^t.$$

We shall prove Theorem 3 in Section 3. We first discuss consequences of Theorem 3.

COROLLARY 1. *Let f be a function in $\mathbb{L}_\tau$, $\int f\, d\omega = 0$. Then there exists a unique, up to an additive constant function, u in $\mathbb{L}_\tau$ such that $\Delta_s u = -f$. Moreover, the function u is C^2 along the leaves of the stable foliation.*

PROOF. We set $u = \int_0^\infty Q_t f\,dt$. By Theorem 3 the integral makes sense in $\mathbb{L}_\tau$ and is the uniform limit of $\int_0^T Q_t f\,dt$. We claim that on each stable leaf W^s, the limit u is a weak solution of the elliptic equation $\Delta_s u = -f$. In fact, we have, for any g in $C_K^2(W^s)$,

$$\begin{aligned}\int \Delta_s g \cdot u &= \lim_{T\to\infty} \int \Delta g\left(\int_0^T Q_t f\,dt\right)\\ &= \lim_{T\to\infty} \int g\left(\Delta \int_0^T Q_t f\,dt\right)\\ &= \lim_{T\to\infty} \int g Q_T f - \int gf = -\int gf.\end{aligned}$$

It follows that u is C^2 along the leaves and is a strong solution of $\Delta_s u = -f$. The uniqueness of u follows from Theorem 3, since if $\Delta_s u = 0$, then $Q_t u = u$ and u is constant. Further regularity of the function u will be discussed in Section 4. □

Let α be a section of the bundle $C(TW_s^*)$ of 1-forms on W^s, and assume that on each leaf, α is of class C^1 and closed. We can define $\int_{\omega(0,t)} \alpha$ in the following way:

Choose (p, ξ) in $\tilde{M} \times \partial\tilde{M}$ which projects to ω_0. Consider on $\tilde{M}$ the lifted trajectory $\omega(0,t)$ with $\tilde{\omega}(0) = p$. Consider also the 1-form $\tilde{\alpha}$ on $\tilde{M}$ such that $\pi^*\tilde{\alpha}_z = \alpha_{\pi(z,\xi)}$. The form $\tilde{\alpha}$ is closed and let A be a function on $\tilde{M}$ such that $\tilde{\alpha} = dA$. Define then, for all $t \geq 0$, $\int_{\omega(0,t)} \alpha = A(\tilde{\omega}_t) - A(\tilde{\omega}_0)$.

This process $(\int_{\omega(0,t)} \alpha)_{t\geq 0}$ does not depend on the choices of the lifted trajectory or on the primitive A. Observe also that the function A is of class C^2 on $\tilde{M}$ and that

$$\Delta A = \operatorname{div} \operatorname{grad} A = -\delta\, dA = -\delta\tilde{\alpha}.$$

A direct application of Itô's formula (see [12]) shows that $(M_t)_{t\geq 0}$ is a martingale for the filtration of the Brownian motion, where M_t is given by

$$M_t = \int_{\omega(0,t)} \alpha + \int_0^t \delta_s \alpha(\omega_r)\,dr.$$

Here $\delta_s \alpha$ denotes the codifferential of α associated with the metric along the leaves. That is, $\delta_s \alpha = -\operatorname{div}_s \alpha^\#$, where $\alpha^\#$ is the vector field associated with α by g_s-duality in TW^s and div_s is the divergence along W^s defined by the metric g_s. The increasing process of M_t is given by $2\|\alpha(\omega_t)\|^2\,dt$.

COROLLARY 2. *Let α: $SM \to (TW_s)^*$ be a section of the bundle of closed 1-forms α along the stable leaves, such that α is C^1 along the leaves and such that the function $\delta_s \alpha$ is globally Hölder continuous on SM. Define $\int_{\omega(0,t)} \alpha$ as* *above. Then there exists a Hölder continuous function u, which is C^2 along the leaves, such that*

$$\left(\int_{\omega(0,t)}\alpha + t\int \delta_s \alpha\, d\omega + u(\omega_t) - u(\omega_0)\right)_{t\geq 0}$$

is a real-valued martingale with increasing process $2\|\alpha + du\|^2(\omega_t)\,dt$.

PROOF. The function $-\delta_s\alpha + \int\delta_s\alpha\,d\omega$ is Hölder continuous and has 0 integral. By Corollary 1, there exists a Hölder continuous function u such that $\Delta_s u = \delta_s\alpha - \int\delta_s\alpha\,d\omega$. Consider $\int_{\omega(0,t)}(\alpha + du)$. We get that the process $(M_t)_{t\geq 0}$ is a martingale with increasing process $2\|\alpha + du\|^2(\omega_t)\,dt$, where M_t is given by

$$\begin{aligned} M_t &= \int_{\omega(0,t)}(\alpha + du) + \int_0^t \delta_s(\alpha + du)(\omega_r)\,dr \\ &= \int_{\omega(0,t)}\alpha + \int_{\omega(0,t)} du + \int_0^t\left(\int\delta_s\alpha\,d\omega\right)dr. \end{aligned}$$ □

We can apply Corollary 2 to particular 1-forms. For (x,ξ) in $\tilde{M}$ consider $\psi_{x,\xi}(y)$ the Busemann function on $\tilde{M}$ at ξ and $k_\xi(x,\cdot)$ the Poisson kernel at ξ. Set $\alpha_0 = d\psi_{x,\xi}$ and $\alpha_1 = d\log k_\xi(x,\cdot)$. The 1-forms α_0 and α_1 are closed along the stable leaves, C^∞ along the leaves and such that $\delta_s\alpha_i$ is Hölder continuous, $i = 0, 1$. In fact, for $i = 0$ we have that $\delta_s\alpha_0 = -\Delta_s\psi_{x,\xi}$ is the mean curvature of the stable horosphere, which is Hölder continuous [3, 11]. For $i = 1$, we have $\delta_s\alpha_1 = \|d\log k_\xi(x,\cdot)\|^2$, which is Hölder continuous ([10], Lemma 3.2). We conclude the next corollary.

COROLLARY 3. *There exist Hölder continuous functions u_0, u_1 on SM such that for any ξ, the process $(M_t^0)_{t\geq 0}$ [respectively, $(M_t^1)_{t\geq 0}$],*

$$M_t^0 = \psi_{(\tilde{\omega}_0,\xi)}(\tilde{\omega}_t) - tl + u_0\pi(\tilde{\omega}_t,\xi) - u_0\pi(\tilde{\omega}_0,\xi)$$

[*respectively*,

$$M_t^1 = \log k_\xi(\tilde{\omega}_0,\tilde{\omega}_t) + th + u_1\pi(\tilde{\omega}_t,\xi) - u_1\pi(\tilde{\omega}_0,\xi)\Big]$$

is a martingale with increasing process

$$2\|\xi + \nabla u_0\|^2(\tilde{\omega}_t)\,dt \quad \Big[\textit{respectively}, 2\|\nabla\log k_\xi(x,\cdot) + \nabla u_1\|^2(\tilde{\omega}_t)\,dt\Big].$$

In the statement of Corollary 3, we used that $l = -\int\delta_s\alpha_0\,d\omega$ and $h = \int\|\nabla\log k\|^2\,d\omega$ [13].

We can now achieve the proof of Theorem 1. Fix (x_0,ξ) arbitrarily. Then the martingales $(M_t^0)_{t\geq 0}$ and $(M_t^1)_{t\geq 0}$ are continuous and have moments of all orders. The respective variances $(1/t)\mathbb{E}_{x_0,\xi}\int_0^t 2\|\xi + \nabla u_0\|^2(\tilde{\omega}_r)\,dr$ and

$(1/t)\mathbb{E}_{x_0,\xi}\int_0^t 2\|\nabla \log k_\xi(x,\cdot) + \nabla u_1\|^2(\tilde{\omega}_r)\,dr$ converge to, respectively, σ_0^2 and σ_1^2, where

$$\sigma_0^2 = 2\int \|\xi + \nabla u_0\|^2\, d\omega,$$

$$\sigma_1^2 = 2\int \|\nabla \log k_\xi(x,\cdot) + \nabla u_1\|^2\, d\omega.$$

Observe that $\sigma_i^2 > 0$, $i = 0, 1$, for otherwise we would get $\xi = -\nabla u_0$ and $\psi_{x,\xi}$ bounded on $\tilde{M}$ [respectively, $\nabla \log k_\xi(x_1) = -\nabla u_1$ and $k_\xi(x,\cdot)$ bounded on $\tilde{M}$], which is impossible. Therefore, we can write that the distributions under $\mathbb{P}_{x_0,\xi}$ of $M_t^0/\sigma_0\sqrt{t}$ and of $M_t^1/\sigma_1\sqrt{t}$ converge to the normal distribution as t goes to infinity. We observe now that when t goes to infinity, the process $(\psi_{x,\xi}(\tilde{\omega}_t) - d(x,\tilde{\omega}_t))$ converges $\mathbb{P}_{x_0,\xi}$ a.e. to the a.e. finite number $(\xi|\tilde{\omega}_\infty)_x$. It follows that

$$\frac{1}{\sigma_0\sqrt{t}}\big(\psi_{x,\xi}(\tilde{\omega}_t) - d(x,\tilde{\omega}_t)\big) + \frac{1}{\sigma_0\sqrt{t}}\big(u_0(\tilde{\omega}_t) - u_0(\tilde{\omega}_0)\big)$$

converges a.e. to 0 so that the distribution of $[1/(\sigma_0\sqrt{t})](d(x,\tilde{\omega}_t) - tl)$ is asymptotically normal as well.

Analogously let z_t be the point on the geodesic ray $(\tilde{\omega}_t, \xi)$ closest to x. We have $\mathbb{P}_{x_0,\xi}$ a.e. that $\sup_t d(x, z_t)$ is finite. By the boundary Harnack inequality [1, 2], as soon as $\psi_{z_t,\xi}(\tilde{\omega}_t) \geq 1$, we make a bounded error when replacing $k_\xi(z_t,\tilde{\omega}_t)$ by $G(z_t,\tilde{\omega}_t)$. Altogether we may write that $\mathbb{P}_{x_0,\xi}$ a.e. we have

$$\limsup_{t\to\infty} |\log G(x,\tilde{\omega}_t) - \log k_\xi(x,\tilde{\omega}_t)| < -\infty.$$

We conclude as above that the distribution of $[1/(\sigma_1\sqrt{t})](\log G(x,\tilde{\omega}_t) + th)$ is asymptotically normal.

To prove Theorem 2, we use the expression for σ_1^2 that we obtained above,

$$\sigma_1^2 = 2\int \|\nabla \log k_\xi(x,\cdot) + \nabla u_1\|^2\, d\omega.$$

Recall that $h = \int \|\nabla \log k_\xi(x,\cdot)\|^2\, d\omega$ and that for any continuous function u, which is C^2 along the stable leaves, we have

$$\int \langle \nabla \log k_\xi(x,\cdot), \nabla u\rangle\, d\omega = -\int \Delta u\, d\omega = 0$$

(see [15] and [20]).

Substituting in the above expression, we get

$$\sigma_1^2 = 2h + 2\int \|\nabla u_1\|^2\, d\omega.$$

This proves the inequality $\sigma_1^2 \geq 2h$ and that we have equality only if u_1 is constant, that is, if $\|\nabla \log k_\xi(x,\cdot)\|^2$ is constant and equal to h. This is possible only when M is asymptotically harmonic (see, e.g., [16]).

(The argument in [16] is to consider the measure μ of maximal entropy H for the geodesic flow and to write the inequalities

$$\underset{(1)}{h \le} lH \underset{(2)}{\le} l\int X \log k\, d\mu \underset{(3)}{\le} l\sqrt{h} \underset{(4)}{\le} h.$$

(1) and (4) are in [13], (2) is in [14] for any invariant measure and (3) is Cauchy–Schwarz. Equality in (3) or (4) implies asymptotically harmonic.)

3. Proof of Theorem 3. (Compare with the proofs of [19], Proposition 4, [4], Théorème 3.7, or [17], Proposition 4.28.) The main ingredient is the property of average contraction:

PROPOSITION 2. *For T large enough, we have for all x in M_0, all ξ, η, $\xi \neq \eta$ in $\partial\tilde{M}$,*

$$\frac{1}{T}\mathbb{E}_{x,\xi}\Big(\big(\gamma_T^{-1}\xi|\gamma_T^{-1}\eta\big)_{y_T} - (\xi|\eta)_x\Big) \ge \frac{l}{4},$$

where we write $\tilde{\omega}_t = \gamma_t y_t$, $y_t \in M_0$.

PROOF. Assume not. Then there exist numbers T_n, $T_n \to \infty$, and points x_n, ξ_n, η_n, $\xi_n \neq \eta_n$, such that

$$\frac{1}{T_n}\mathbb{E}_{x_n,\xi_n}\Big(\big(\gamma_{T_n}^{-1}\xi_n|\gamma_{T_n}^{-1}\eta_n\big)_{y_{T_n}} - (\xi_n|\eta_n)_{x_n}\Big) < \frac{l}{4}.$$

Observe that for all $\xi \neq \eta$, $\gamma \in \Gamma$, x, y in M_0, we have the a priori bound

$$\big|(\gamma^{-1}\xi|\gamma^{-1}\eta)_y - (\xi|\eta)_x\big| \le 2d(x, \gamma x) + 2\operatorname{diam} M_0.$$

Hence we can find t_0 so small that

$$(*) \qquad \sup_{0\le t\le t_0}\ \sup_{x,\xi}\ \sup_{\eta\neq\xi}\mathbb{E}_{x,\xi}\big|(\gamma_T^{-1}\xi|\gamma_t^{-1}\eta)y_t - (\xi|\eta)_x\big| \le \frac{l}{4}.$$

By using $(*)$ and suitably relabelling x_n, ξ_n, η_n, we find a sequence of integers $N_j \to \infty$ and points x_j, ξ_j, η_j such that, for all j,

$$\frac{1}{N_j t_0}\mathbb{E}_{x,j,\xi_j}\Big(\big(\gamma_{N_j t_0}^{-1}\eta_j|\gamma_{N_j t_0}^{-1}\xi_j\big)_{y_{N_j t_0}} - (\eta_j|\xi_j)_{x_j}\Big) < \frac{l}{2}.$$

Write now ϕ for the function on $M_0 \times \partial\tilde{M} \times \partial\tilde{M}$ defined for $x, \eta \neq \xi$ by

$$\phi(x, \xi, \eta) = \frac{1}{t_0}\mathbb{E}_{x,\xi}\Big(\big(\gamma_{t_0}^{-1}\xi|\gamma_{t_0}^{-1}\eta\big)_{y_{t_0}} - (\eta|\xi)_x\Big)$$

and observe that ϕ has a continuous extension to the diagonal, which we will still write as ϕ, given by

$$\phi(x, \xi, \xi) = \frac{1}{t_0}\mathbb{E}_{x,\xi}\big(\psi_{x,\xi}(\gamma_{t_0} y_{t_0})\big).$$

Our assumption says that there exists a sequence of integers N_j, $N_j \to \infty$, and points x_j, ξ_j, η_j with the property that, for all j,

$$\frac{1}{N_j}\sum_{k=0}^{N_j-1}\mathbb{E}_{x_j,\xi_j}\Big(\phi\big(y_{kt_0}, \gamma_{kt_0}^{-1}, \xi_j\gamma_{kt_0}^{-1}\eta_j\big)\Big) < \frac{l}{2};$$

in other words, we have

$$\frac{1}{N_j}\sum_{k=0}^{N_j-1} Q_{kt_0}^2\phi(x_j, \xi_j, \eta_j) < \frac{l}{2}.$$

Now take a weak limit μ of a subsequence of the sequence of probability measures μ_j on the compact space $M_0 \times \partial\tilde{M} \times \partial\tilde{M}$ defined by

$$\mu_j = \frac{1}{N_j}\sum_{k=0}^{N_j-1} Q_{kt_0}^2\big((x_j, \xi_j, \eta_j), d(\cdot,\cdot,\cdot)\big).$$

The measure μ is $Q_{t_0}^2$-invariant and satisfies $\int\phi\, d\mu \le l/2$.

Let $\mu' = (1/t_0)\int_0^{t_0}(Q_s^2)\mu\, ds$. The measure μ' is Q^2 invariant. By Proposition 1, μ' coincides with ω^2. Using again $(*)$ we find that

$$\int\phi\, d\omega^2 \le \frac{3l}{4}.$$

On the other hand, we can write

$$\int\phi\, d\omega^2 = \frac{1}{t_0}\int\mathbb{E}_{x,\xi}\big(\psi_{(x,\xi)}(\tilde{\omega}_{t_0})\big)\, d\omega = \lim_{t\to\infty}\frac{1}{t}\int\mathbb{E}_{x,\xi}\big(\psi_{(x,\xi)}(\tilde{\omega}_t)\big)\, d\omega = l,$$

a contradiction. □

PROPOSITION 3. *There is a number $\tau_0 > 0$ such that for any τ, $0 < \tau < \tau_0$, there exists $\zeta_1(\tau) < 1$, such that for t large enough, x in M and all ξ, η, $\xi \ne \eta$, we have*

$$\mathbb{E}_{x,\xi}\left(\frac{\exp\big(-\tau(\gamma_t^{-1}\xi|\gamma_t^{-1}\eta)y_t\big)}{\exp(-\tau(\xi|\eta)_x)}\right) < \zeta_1^t.$$

PROOF. Observe that, if we write $u(x, \xi, \eta, t)$ for

$$u(x,\xi,\eta,t) = \mathbb{E}_{x,\xi}\left(\frac{\exp\big(-\tau(\gamma_t^{-1}\xi|\gamma_t^{-1}\eta)_{y_t}\big)}{\exp(-\tau(\xi|\eta)_x)}\right),$$

we have, using the Markov property,

$$\sup_{x,\xi,\eta} u(x,\xi,\eta,t_1+t_2) \le \sup_{x,\xi,\eta} u(x,\xi,\eta,t_1)\sup_{x,\xi,\eta} u(x,\xi,\eta_2,t_2)$$

so that it is sufficient to prove the statement of Proposition 3 for a fixed time T. More precisely, we prove Proposition 3 if we find for a fixed T and τ sufficiently small numbers $C_0(\tau) > 0$ and $\xi_0(\tau) < 1$ such that (a) and (b) hold:

$$\text{(a)} \qquad \sup_{x,\xi,\eta} \sup_{0 \leq t < T} u(x,\xi,\eta,t) \leq C_0(\tau),$$

$$\text{(b)} \qquad \sup_{x,\xi,\eta} u(x,\xi,\eta,T) \leq \zeta_0(\tau).$$

Choose T given by Proposition 2 and write

$$\frac{\exp\left(-\tau(\gamma_t^{-1}\xi|\gamma_t^{-1}\eta)_{y_t}\right)}{\exp(-\tau(\xi|\eta)_x)} \leq 1 - \tau\left((\gamma_t^{-1}\xi|\gamma_t^{-1}\eta)_{y_t} - (\xi|\eta)_x\right)$$
$$+ 2\tau^2\left[(2d(x,\gamma_t x) + C_1)^2 \exp(2d(x,\gamma_t x) + C_1)\right],$$

where $C_1 = 2 \operatorname{diam} M_0$.

Comparison with a space of constant negative curvature ([12], Theorem VI.5.1) gives that for a fixed T we can find C such that for all t, $0 \leq t \leq T$,

$$\mathbb{E}_x\left((2d(x,\tilde{\omega}_t) + 3C_1)^2 \exp(2d(x,\tilde{\omega}_t) + 3C_1)\right) \leq C.$$

We get, using Proposition 2,

$$u(x,\xi,\eta,T) \leq 1 - \tau\frac{l}{4} + 2\tau^2 C$$

and for $t \leq T$,

$$u(x,\xi,\eta,t) \leq 1 + \tau C + 2\tau^2 C.$$

For τ sufficiently small, we have

$$\zeta_0(\tau) := 1 - \tau\frac{l}{4} + 2\tau^2 C < 1$$

and this proves properties (a) and (b). □

We now prove Theorem 3. Consider f in $\mathbb{L}_\tau$, with τ small enough that Proposition 3 applies. We have to estimate

$$\|Q_t f - \int f\, d\omega\|_\tau.$$

We first have for t large enough, and for all x, ξ_1, ξ_2,

$$\left|(Q_t f(x,\xi_1) - Q_t f(x,\xi_2)) \exp(\tau(\xi_1|\xi_2)_x)\right|$$
$$\leq \int_{M_0} p_t(x,y)\, dy\left(\sum_\Gamma \mu_t^{x,y}(\gamma)\left|f(y,\gamma^{-1}\xi_1) - f(y,\gamma^{-1}\xi_2)\right|\right)\exp(\tau(\xi_1|\xi_2)_x)$$
$$\leq \|f\|_r \int_{M_0} p_t(x,y)\, dy\left(\sum_\Gamma \mu_t^{x,y}(\gamma)\frac{\exp\left(-\tau(\gamma^{-1}\xi_1|\gamma^{-1}\xi_2)_y\right)}{\exp(-\tau(\xi_1|\xi_2)_x)}\right)$$
$$\leq \zeta_1^t \|f\|_\tau \quad \text{by Proposition 3.}$$

In particular, we can write, for t large enough,

$$|Q_t f(x,\xi_1) - \int Q_t(x,\xi_2)\,d\omega_x(\xi_2)| \le \xi_1^t \|f\|_\tau.$$

Set $F_t(x) = \int Q_t f(x,\xi)\,d\omega_x(\xi)$. The function F_t is continuous on M, $|F_t| \le \|f\|_\tau$ and by the Doeblin property of the Brownian motion on M, we can find a number ζ_2, $\zeta_2 < 1$, such that for t large enough, all x in M,

$$\left|\int_M p_t(x,y)F_t(y)\,dy - \int_M F_t(x)\,dm(x)\right|$$
$$\le \|F_t\|_\infty \int \left| p_t(x,y) - \frac{1}{\operatorname{vol} M}\right| dy$$
$$\le \zeta_2^t \|f\|_\tau.$$

Combining the two estimates, we get

$$\left|Q_{2t} f - \int F_t\,dm\right| \le |Q_t(Q_t f - F_t)| + \left|Q_t F_t - \int F_t\,dm\right|$$
$$\le (\zeta_1^t + \zeta_2^t)\|f\|_\tau.$$

Theorem 3 follows if we observe that

$$\int F_t\,dm = \int Q_t f\,d\omega = \int f\,d\omega$$

by the invariance of ω. □

4. Regularity of the potential in Corollary 1. Recall that for a function f in $\mathbb{L}_\tau$ such that $\int f\,d\omega = 0$, we constructed a Hölder continuous function u, which is C^2 along the stable leaves, such that $\Delta_s u = -f$. In this section we study the regularity of u. Since the stable foliation is Hölder continuous, we expect that if f is C^∞ along stable leaves and that all leafwise jets of f are Hölder continuous, then u will have the same regularity. The proof we give below is standard; we have chosen to express it using the Brownian motion on $\tilde{M}$ since we use this construction anyway in the next section. So we first recall the construction of the Brownian motion on $\tilde{M}$ ([12], Section V.4).

We are given an n-dimensional Euclidean Brownian motion $\{w^1(t),\dots,w^n(t)\}_{t\in\mathbb{R}_+}$ starting at $(0,0,\dots,0)$ (in this paper we differ from [12] in that our Euclidean Brownian motion has for infinitesimal generator the Laplacan Δ and not $\frac{1}{2}\Delta$), and we consider on the orthonormal frame bundle $O(\tilde{M})$ the canonical horizontal vector fields $\tilde{L}_1,\dots,\tilde{L}_n$. That is, $\tilde{L}_j(x,e_1,\dots,e_n)$ is the horizontal lift of e_j.

The canonical Brownian motion on the orthonormal bundle $O(\tilde{M})$ is given by the solution $r(w,t)$ of the Stratonovich SDE:

$$dr(t) = \sum_k \tilde{L}_k(r(t))\circ dw^k(t),$$
$$r(0) = r.$$

The Brownian motion $(\tilde{\omega}_t)_{t\in\mathbb{R}_+}$ is defined as the projection on $\tilde{M}$ of $r(w,t)$ for any choice of $r(0)$ which projects in $\tilde{\omega}_0$.

The process $(\tilde{\omega}_t)_{t \in \mathbb{R}_+}$ has continuous trajectories and the strong Markov property.

Let $C_s^\infty(SM)$ denote the space of functions on SM which are C^∞ along the stable leaves and such that the jets of all degrees are Hölder continuous.

PROPOSITION 4. *Let f be a function in C_s^∞ with $\int f \, d\omega = 0$, and let u be such that $\Delta_s u = -f$. Then $u \in C_s^\infty$.*

PROOF. We have

$$u(x, \xi) = \lim_{T \to \infty} \int_0^T \int_{\tilde{M}} \tilde{p}_t(x, \tilde{y}) f\pi(\tilde{y}, \xi) \, d\tilde{y} \, dt$$
$$= \lim_{T \to \infty} \mathbb{E}_x\left(\int_0^T f\pi(\tilde{\omega}_t, \xi) \, dt \right),$$

where π: $S\tilde{M} \to SM$ is the projection. Fix a ball B in $\tilde{M}$ centered at x_0 in M_0 and such that $d(M_0, \partial B) > 0$ and write T_B for the entrance time of the Brownian motion $\tilde{\omega}$ in $\tilde{M} \setminus B$. We have

$$u(x, \xi) = \lim_{T \to \infty} \mathbb{E}_x\left(\int_0^{T \wedge T_B} f\pi(\tilde{\omega}_t, \xi) \, dt \right) + \mathbb{E}_x\left(\int_{T \wedge T_B}^T f\pi(\tilde{\omega}_t, \xi) \, dt \right),$$

where $T_\wedge T_B = \min(T, T_B)$.

The first expectation converges to

$$\int_B g_B(x, \tilde{y}) f\pi(\tilde{y}, \xi) \, d\tilde{y},$$

where g_B is the Green function inside B.

Using the strong Markov property, we may write the second term as

$$\int \left(\mathbb{E}_z \int_0^{T-\tau} f\pi(\tilde{\omega}_t, \xi) \, dt \right) d\varepsilon_x^T(z, \tau),$$

where ε_x^T is the distribution of the variable $(\tilde{\omega}_{T_B \wedge T}, T_B \wedge T)$.

By Theorem 3, we may write, uniformly in z, τ,

$$\mathbb{E}_z \int_0^{T-\tau} f\pi(\tilde{\omega}_t, \xi) \, dt = u\pi(z, \xi) + O(\zeta^{T-\tau}).$$

As T goes to infinity, we have

$$\lim_{T \to \infty} \int u\pi(z, \xi) \, d\varepsilon_x^T(z, \tau) = \int u\pi(z, \xi) \, d\varepsilon_x(z),$$

where ε_x is the distribution of $\tilde{\omega}_{T_B}$, because $u\pi$ is continuous and

$$\lim_{T \to \infty} \int \zeta^{T-\tau} \, d\varepsilon_x^T(z, \tau) = \lim_{T \to \infty} \mathbb{E}_x(\zeta^{T - T_B \wedge T}) = 0$$

by the Lebesgue dominated convergence theorem. Hence we may write

$$u(x, \xi) = \int_B g_B(x, \tilde{y}) f\pi(\tilde{y}, \xi) \, d\tilde{y} + \int_{\partial B} u\pi(z, \xi) \, d\varepsilon_x(z).$$

The regularity of u follows from the regularity of g_B and from the regularity of the density of ε_x with respect to the Lebesgue measure on ∂B. □

5. Central limit theorem for integrals of 1-forms. In this section, we state a more general form of Corollary 2 above. We consider α, a section of the bundle $(TW_s)^*$ of 1-forms which are of class C^4 along the leaves and globally Hölder continuous on SM. We want to define $\int_{\omega(0,t)} \alpha$. By lifting to $S\tilde{M}$, this amounts to defining $\int_{\tilde{\omega}(0,t)} \tilde{\alpha}$, where $\tilde{\alpha}$ is the 1-form on $\tilde{M}$ such that $\pi^* \tilde{\alpha}_z = \alpha_{\pi(z,\xi)}$.

We follow [12], Section VI.6. The Brownian motion being constructed as above, we consider the scalarization $\overline{\alpha}_1, \dots, \overline{\alpha}_n$ of the form $\tilde{\alpha}$ on $O(\tilde{M})$. That is, $\{\overline{\alpha}_i(z,e)\}$ is a system of components of $\tilde{\alpha}_z$ read in the frame e. Since $\tilde{\alpha}$ is of class C^4, the functions $\overline{\alpha}_i(z,e)$ are of class C^3 on $O(\tilde{M})$. We define then $\int_{\tilde{\omega}[0,t]} \tilde{\alpha}$ by the following Stratonovich stochastic integral:

$$\int_{\tilde{\omega}[0,t]} \tilde{\alpha} = \sum_k \int_0^t \overline{\alpha}_k(r(s)) \circ dw^k(s).$$

The process $(\int_{\tilde{\omega}[0,t]} \tilde{\alpha})_{t \in \mathbb{R}_+}$ is a real-valued process with continuous trajectories defined on the same probability space as the Brownian motion $\{w^1(t), \dots, w^n(t)\}_{t \in \mathbb{R}_+}$. From [12], Theorem VI. 6.1, we recall that the process M_t defined by

$$M_t = \int_{\tilde{\omega}(0,t)} \tilde{\alpha} + \int_0^t \delta_s \alpha(\tilde{\omega}_s)\, ds$$

is a real-valued martingale with respect to the natural filtration of $\{w^j\}$, with associated increasing process $2\|\tilde{\alpha}(\tilde{\omega}_t)\|^2\, dt$.

COROLLARY 4. *Let $\alpha\colon SM \to (TW_s)^*$ be a section of the bundle of 1-forms along the stable leaves, which is of class C^4 along the leaves and such that the function $\delta_s \alpha$ is globally Hölder continuous on SM. Define $\int_{\omega(0,t)} \alpha$ as above. Then there exists a Hölder continuous function u such that*

$$\left(\int_{\omega(0,t)} \alpha + t \int \delta_s \alpha \, d\omega + u(\omega_t) - u(\omega_0) \right)$$

is a real-valued martingale with increasing process $2\|\alpha + du\|^2(\omega t)\, dt$.

In particular, there is a number σ^2 such that the variable

$$\frac{1}{\sqrt{t}} \left(\int_{\omega(0,t)} \alpha + t \int \delta_s \alpha \, d\omega \right)$$

is asymptotically distributed like $N(0, \sigma^2)$. We have $\sigma^2 = 0$ if and only if $\alpha = -du$.

The proof is the same as the proof of Corollary 2. By Theorem 3 again, we have $\sigma^2 = 2\int \|\alpha + du\|^2 \, d\omega$ and the last conclusion follows.

We write the conclusion of Corollary 4 more explicitly in the case when $\sigma^2 > 0$: With the above notation, there is a number σ^2 such that for all real r, all (x_0, ξ),

$$\lim_{t \to \infty} \mathbb{P}_{x_0,\xi} \left(\int_{\omega(0,t)} \alpha + t \int \delta_s \alpha \, d\omega \le \sigma r \sqrt{t} \right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{r} \exp\left(-\frac{x^2}{2} \right) dx.$$

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