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HAUSDORFF MEASURE OF TRAJECTORIES OF MULTIPARAMETER FRACTIONAL BROWNIAN MOTION

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Consider $0 < \alpha < 1$ and the Gaussian process $Y(t)$ on $\mathbb{R}^N$ with covariance $E(Y(t)Y(s)) = |t|^{2\alpha} + |s|^{2\alpha} - |t-s|^{2\alpha}$, where $|t|$ is the Euclidean norm of t. Consider independent copies $X^1, \dots, X^d$ of Y and the process $X(t) = (X^1(t), \dots, X^d(t))$ valued in $\mathbb{R}^d$. In the transient case $(N < \alpha d)$ we show that a.s. for each compact set L of $\mathbb{R}^N$ with nonempty interior, we have $0 < \mu_\varphi(X(L)) < \infty$, where μ_φ denotes the Hausdorff measure associated with the function $\varphi(\varepsilon) = \varepsilon^{N/\alpha} \log\log(1/\varepsilon)$. This result extends work of A. Goldman in the case $\alpha = 1/2$; the proofs are considerably simpler.

1. Introduction. Classical results [1], [5] indicate that for $d \geq 3$, a.s. any portion R of the trajectory of $\mathbb{R}^d$-valued Brownian motion satisfies $0 < \mu_\varphi(R) < \infty$, where φ is the function given by $\varphi(x) = x^2 \log\log(1/x)$ for $0 < x < 1/3$. [The case $d = 2$ ([7], [9]) is also known—the correct function is now $\varphi(x) = x^2 \log 1/x \log\log\log 1/x$—but lies deeper, due to the recurrence properties of Brownian motion in that case, and will not be generalized in the present paper.] This result has been extended by Goldman [2] to the case of Levy's multiparameter Brownian motion from $\mathbb{R}^N$ to $\mathbb{R}^d$. Goldman's achievement is impressive, as he succeeded even though he lacked a key estimate (Corollary 2.3 below). The original motivation for the present work was to provide a simple proof of Goldman's result. It turned out that our approach works in a more general setting.

The basic process considered in this paper is the Gaussian process $X(t)$ from $\mathbb{R}^N$ to $\mathbb{R}^d$ such that

$$E\big(|X(t) - X(s)|^2\big) = d|t-s|^{2\alpha}, \tag{1.1}$$

where $0 < \alpha < 1$ and where $|\cdot|$ denotes the Euclidean distance. Following [3], we will call this process the (N, d, α) Gaussian process. The process considered by Goldman [2] is the $(N, d, 1/2)$ Gaussian process; Brownian motion is the $(1, d, 1/2)$ Gaussian process and fractional Brownian motion is the $(1, d, \alpha)$ Gaussian process. It should be clear that the components of X, which are processes from $\mathbb{R}^N$ to $\mathbb{R}$, are independent copies of the $(N, 1, \alpha)$ Gaussian process.

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We denote by φ the function (defined for $\varepsilon \leq 1/3$)

$$\varphi(\varepsilon) = \varepsilon^{N/\alpha} \log\log\left(\frac{1}{\varepsilon}\right)$$

and we denote by μ_φ the corresponding Hausdorff measure.

THEOREM 1.1. *Assume $N < \alpha d$. Consider $a > 0$. Then a.s. we have*

$$0 < \mu_\varphi\{X(t); |t| \leq a\} < \infty. \tag{1.2}$$

When $N > \alpha d$, a.s. the image of a ball has nonempty interior [6]. What happens in the "critical case" $N = \alpha d$ is unknown (unless $N = 1$, $\alpha = 1/2$, $d = 2$) and is certainly a deeper question. At this point we have succeeded only to prove the right-hand side inequality of (1.2) for the natural choice of φ.

We now comment on the methods and the organisation of the paper. As one moves away from Brownian motion, fewer tools are available, so one must rely on general principles. In Section 2 we recall those principles we will use. In Section 3, we make a few observations about the $(N, 1, \alpha)$ process and prepare the ground to solve dependence-related problems. In Section 4 we prove the basic estimate and in Section 5 we conclude the proof.

2. General facts. Consider a set S and a Gaussian process $(Z(t))_{t \in S}$. We provide S with the distance $d(x, t) = (E(Z(t) - Z(s))^2)^{1/2}$. We denote by $N_d(S, \varepsilon)$ the smallest number of (open) d-balls of radius ε needed to cover S and we denote by D the diameter of S, that is, $D = \sup\{d(x, t);\ s, t \in S\}$.

LEMMA 2.1. *Given $u > 0$, we have*

$$\begin{aligned} P\left(\sup_{s,t\in S} |Z(t) - Z(s)| \geq K\left(u + \int_0^D \sqrt{\log N_d(S, \varepsilon)}\, d\varepsilon\right)\right) \\ \leq \exp\left(-\frac{u^2}{D}\right). \end{aligned} \tag{2.1}$$

PROOF. The fact that

$$E \sup_{s,t\in S} |Z(t) - Z(0)| \leq K\int_0^D \sqrt{\log N_d(S, \varepsilon)}\, d\varepsilon \tag{2.2}$$

is Dudley's bound. A more careful walk through the same steps yields (2.1) ([4], Theorem 11.1). An alternative route is to use (2.2) and the Gaussian isoperimetric inequality ([4], Section 3.1). □

LEMMA 2.2. *Consider a function Ψ such that $N(S, \varepsilon) \le \Psi(\varepsilon)$ for all $\varepsilon > 0$. Assume that for some constant C and all $\varepsilon > 0$ we have $\Psi(\varepsilon)/C \le \Psi(\varepsilon/2) \le C\Psi(\varepsilon)$. Then*

$$P\Big(\sup_{s,t\in S} |Z(t) - Z(s)| \le u\Big) \ge \exp\left(-\frac{\Psi(u)}{K}\right),$$

where K depends on C only.

This is proved in [8] and is the main new ingredient since [2].

COROLLARY 2.3. *Consider the $(N, 1, \alpha)$ Gaussian process $Y(t)$. Then, for a constant K depending on α, N only we have, for $\varepsilon \le 1$,*

$$\exp\left(-\frac{K}{\varepsilon^{N/\alpha}}\right) \le P\Big(\sup_{|t|\le 1}|Y(t)| \le \varepsilon\Big) \le \exp\left(-\frac{1}{K\varepsilon^{N/\alpha}}\right). \tag{2.3}$$

PROOF. The right-hand side is proved essentially in [6]. For the left-hand side, letting $S = \{t \in \mathbb{R}^N;\ |t| \le 1\}$, we see that $N(S, \varepsilon) \le K\varepsilon^{-N/\alpha}$ [since $d(s,t) = |t - s|^{\alpha}$] so that this follows from Lemma 2.2. □

3. Specific facts. The very existence of the $(N, 1, \alpha)$ Gaussian processes $Y(t)$ relies upon the fact that $\mathbb{R}^N$, provided with the distance $d(s,t) = |t - s|^{\alpha}$, is isometric to a subset of a Hilbert space. Such an isometry is provided by Schönberg's formulae

$$\begin{aligned} E(Y(t)Y(u)) &= |t|^{2\alpha} + |u|^{2\alpha} - |t-u|^{2\alpha} \\ &= c\,\mathscr{Re}\int_{\mathbb{R}^N}(1 - e^{i\langle t,x\rangle})(1 - e^{-i\langle u,x\rangle})\frac{dx}{|x|^{N+2\alpha}}, \end{aligned} \tag{3.1}$$

where the constant c depends on α, N only. Equivalently this means that if dm is a random Gaussian scattered measure on $\mathbb{R}^N$, with $E(m(A))^2 = \lambda(A)$ (λ Lebesgue measure), then the process

$$Y(t) = c\int_{\mathbb{R}^N}(1 - \cos\langle t, x\rangle)\frac{dm(x)}{|x|^{\alpha+N/2}} \tag{3.2}$$

is (a version of) the $(N, 1, \alpha)$ process.

In order to solve some dependence problems that are a major obstacle, given $0 < a < b < \infty$, we consider the process

$$Y(a,b,t) = c\int_{a\le|x|\le b}(1 - \cos\langle t, x\rangle)\frac{dm(x)}{|x|^{\alpha+N/2}}.$$

An essential fact is that if $a < b < a' < b'$, the processes $Y(a,b,t)$ and $Y(a',b',t)$ are independent. Also, the next lemma expresses how well $Y(a,b,t)$ approximates $Y(t)$. To simplify notations, for a random variable Z, we write

$\|Z\|_2$ rather than $(EZ^2)^{1/2}$ and we denote by K a constant depending only on N, α, d and that may vary at each occurrence. (Specific constants will be denoted by $K_1, K_2, \dots$.)

LEMMA 3.1. $\|Y(a,b,t) - Y(t)\|_2 \le K_1[|t|^2 a^{2-2\alpha} + b^{-2\alpha}]^{1/2}$.

PROOF. We have

$$E\big((Y(a,b,t) - Y(t))^2\big)$$
$$= \int_{|x|<a} (1 - \cos\langle t, x\rangle)^2 \frac{dx}{|x|^{N+2\alpha}} + \int_{|x|>b} (1 - \cos\langle t, x\rangle)^2 \frac{dx}{|x|^{N+2\alpha}}.$$

In the first integral we bound $(1 - \cos\langle t, x\rangle)^2$ by $|t|^2|x|^2$, and in the second one by 2 to get the required bound. □

LEMMA 3.2. *Consider $b > a > 1$, $1 > r > 0$ and set $A = r^2 a^{2-2\alpha} + b^{-2\alpha}$. Then if $A \le \frac{1}{2}2^{2\alpha}$ and*

$$u \ge K\left(A \log \frac{Kr^{2\alpha}}{A}\right)^{1/2},$$

we have

$$P\Big(\sup_{|t|\le r} |Y(t) - Y(a,b,t)| \ge u\Big) \le \exp\left(-\frac{u^2}{KA}\right).$$

PROOF. Consider $S = \{|t| \le r\}$ and on S the distance

$$d(s,t) = \|(Y(t) - Y(a,b,t)) - (Y(s) - Y(a,b,s))\|_2.$$

Then $d(s,t) \le |t - s|^\alpha$ and thus

$$N_d(s, \varepsilon) \le \frac{Kr^N}{\varepsilon^{N/\alpha}}.$$

Also, by Lemma 3.1, we have $d(0,t) \le K\sqrt{A}$, so that the diameter D of S is $\le K\sqrt{A}$. Thus, by simple estimates,

$$\int_0^D \sqrt{\log N_d(s,\varepsilon)\, d\varepsilon} \le K\int_0^D \sqrt{\log \frac{Kr}{\varepsilon^{1/\alpha}}}\, d\varepsilon$$
$$\le Kr^\alpha \int_0^{Dr^{-\alpha}} \sqrt{\log \frac{K}{x}}\, dx$$
$$\le KD\sqrt{\log \frac{Kr^\alpha}{D}} \le K\left(A \log \frac{Kr^{2\alpha}}{A}\right)^{1/2}.$$

Since $u + B \ge u/2$ for $u \ge B$, the conclusion then follows from (2.1) □

We now denote by $X(a, b, t)$ the Gaussian process from $\mathbb{R}^N$ to $\mathbb{R}^d$ such that its components are independent copies of $Y(a, b, t)$.

COROLLARY 3.3. *With the notations of Lemma* 3.2, *if*

$$u \geq K\left(A \log\frac{Kr^{2\alpha}}{A}\right)^{1/2},$$

we have

$$P\left(\sup_{|t|\leq r}|X(t) - X(a,b,t)| \geq u\right) \leq \exp\left(-\frac{u^2}{KA}\right). \tag{3.3}$$

Let us spell out another basic fact.

LEMMA 3.4. *If $r > 0$, $u \geq Kr^{\alpha}$, then*

$$P\left(\sup_{|t|\leq r}|X(t)| \geq u\right) \leq \exp\left(-\frac{u^2}{Kr^{2\alpha}}\right).$$

PROOF. Reducing to the components, this is a straight forward consequence of (2.1). □

LEMMA 3.5. *If $\varepsilon \leq r^{\alpha}$, we have, for all $0 < a < b$, that*

$$P\left(\sup_{|t|\leq r}|X(a,b,t)| \leq \varepsilon\right) \geq \exp\left(-K\left(\frac{r}{\varepsilon^{1/\alpha}}\right)^N\right).$$

PROOF. It suffices to prove this for $Y(a, b, t)$ rather than $X(a, b, t)$. This is proved as the left-hand side of (2.3), or can be deduced from it. □

4. The main estimate. The main estimate is given in the following proposition.

PROPOSITION 4.1. *There exists a constant $\delta > 0$ with the following property. Given $r_0 \leq \delta$, we have*

$$P\left(\exists\, r, r_0^2 \leq r \leq r_0, \sup_{|t|\leq r}|X(t)| \leq Kr^{\alpha}\left(\log\log\frac{1}{r}\right)^{-\alpha/N}\right)$$
$$\geq 1 - \exp\left(-\left(\log\frac{1}{r_0}\right)^{1/2}\right).$$

PROOF. Consider a number $U > 1$, to be determined later, and for $k \geq 0$, let $r_k = r_0 U^{-2k}$. Consider the largest integer k_0 such that

$$k_0 \leq \frac{\log(1/r_0)}{2\log U}.$$

Thus, for $k \le k_0$ we have $r_k \ge r_0^2$. It thereby suffices to prove that

$$(4.1)\qquad \begin{aligned} &P\left(\exists\, k \le k_0,\ \sup_{|t|\le r_k} |X(t)| \le K r_k^{\alpha}\left(\log\log\frac{1}{r_k}\right)^{-\alpha/N}\right) \\ &\qquad \ge 1 - \exp\left(-\left(\log\frac{1}{r_0}\right)^{1/2}\right). \end{aligned}$$

One certainly wishes to appeal to Lemma 3.5. In order to create independence, let us consider the sequence $a_k = r_0^{-1}U^{2k-1}$ and the processes $X_k(t) := X(a_k, a_{k+1}, t)$. As k varies, these processes are independent. Moreover, it follows from Lemma 3.5 that one can find a constant K_2 such that, if r_0 is small enough, we have for each $k \ge 0$ that

$$\begin{aligned} &P\left(\sup_{|t|\le r_k} |X_k(t)| \le K_2 r_k^{\alpha}\left(\log\log\frac{1}{r_k}\right)^{-\alpha/N}\right) \\ &\qquad \ge \exp\left(-\frac{1}{4}\log\log\frac{1}{r_k}\right) = \frac{1}{(\log(1/r_k))^{1/4}}. \end{aligned}$$

Thus, by independence,

$$(4.2)\qquad \begin{aligned} &P\left(\exists\, k \le k_0,\ \sup_{|t|\le r_k} |X_k(t)| \le K_2 r_k^{\alpha}\left(\log\log\frac{1}{r_k}\right)^{-\alpha/N}\right) \\ &\qquad \ge 1 - \left(1 - \frac{1}{(\log(1/r_0))^{1/4}}\right)^{k_0} \\ &\qquad \ge 1 - \exp\left(-\frac{k_0}{(\log 1/r_0)^{1/4}}\right). \end{aligned}$$

Let now $A_k = r_k^2 a_k^{2-2\alpha} + a_{k+1}^{-2\alpha}$. Thus

$$A_k r_k^{-2\alpha} = (r_k a_k)^{2-2\alpha} + (r_k a_{k+1})^{-2\alpha} = U^{-(2-2\alpha)} + U^{-2\alpha} \le 2U^{-\beta},$$

where $\beta = 2\min(\alpha, 1-\alpha)$. We appeal to Corollary 3.3 (with A_k rather than A) to see that for

$$u \ge K r_k^{\alpha} U^{-\beta/2}(\log U)^{1/2}$$

we have

$$P\left(\sup_{|t|\le r_k} |X(t) - X_k(t)| \ge u\right) \le \exp\left(-\frac{u^2 U^{\beta}}{K r_k^{2\alpha}}\right).$$

Thus we see that provided

$$U^{\beta/2}(\log U)^{-1/2} \geq \left(\log\log\frac{1}{r_0}\right)^{\alpha/N}, \tag{4.3}$$

we can take $u = K_2 r_k^{\alpha}(\log\log(1/r_0))^{-\alpha/N}$ to get

$$P\left(\sup_{|t|\leq r_k}|X(t) - X_k(t)| \geq K_2 r_k^{\alpha}\left(\log\log\frac{1}{r_0}\right)^{-\alpha/N}\right)$$
$$\leq \exp\left(-\frac{U^{\beta}}{K(\log\log(1/r_0))^{2\alpha/N}}\right).$$

Combining with (4.2) we get

$$\begin{aligned}&\left(\exists\, k \leq k_0, \sup_{|t|\leq r_k}|X(t)| \leq 2K_2 r_k^{\alpha}\left(\log\log\frac{1}{r_k}\right)^{-\alpha/N}\right)\\ &\qquad\geq 1 - \exp\left(-\frac{k_0}{(\log(1/r_0))^{1/4}}\right)\\ &\qquad\quad - k_0\exp\left(-\frac{U^{\beta}}{K(\log\log(1/r_0))^{2\alpha/N}}\right).\end{aligned} \tag{4.4}$$

We recall that $k_0 \leq \log(1/r_0)$ and that $k_0 \geq (\log(1/r_0))/4\log U$ (if this number is greater than or equal to 1). Thus we see that one can take (among many other possible choices) $U = (\log(1/r_0))^{1/\beta}$. In that case, (4.3) holds and the right-hand side of (4.4) is at least $1 - \exp(-(\log(1/r_0))^{1/2})$ when $\log 1/r_0$ is large enough. □

COMMENT. Proposition 4.1 is actually much more precise than what we really need, but no extra work is required to get this precision.

5. Proof of Theorem 1.1. The easiest part is the left-hand side of (1.2). The proof is the obvious adaptation of the (simple) argument of [2]. In the proof of Theorem 2.3 of [2], it suffices to replace the use of Theorem 21 of [2] by that of Lemma 7.1 of [6] (of which Theorem 21 is actually a special case). Thus, we turn to the proof of the right-hand side inequality. It suffices to assume $a = 1$. We set $B = \{t \in \mathbb{R}^N;\ \sup_{i\leq N}|t_i| \leq 1\}$. For $k \geq 1$, consider the set

$$R_k = \left\{t \in B;\ \exists\, r, 2^{-2k} \leq r \leq 2^{-k};\right.$$
$$\left.\sup_{|s-t|\leq r}|X(s) - X(t)| \leq K_2 r^{\alpha}\left(\log\log\frac{1}{r}\right)^{-\alpha/N}\right\}, \tag{5.1}$$

where K_3 is the constant of Proposition 4.1. It follows from that proposition [and the fact that $(X(s) - X(t))$ is distributed like $X(s)$] that $P(t \in R_k) \geq 1 - \exp(-\sqrt{k/2})$. Denoting by λ Lebesgue measure on $\mathbb{R}^N$, it follows from the Fubini theorem that the event

$$\Omega_0 = \left\{ \frac{\lambda(R_k)}{\lambda(B)} \geq 1 - \exp\left(-\frac{\sqrt{k}}{4} \right) \text{ infinitely often} \right\}$$

occurs with probability 1. Let us recall that a dyadic cube of order l is a product of intervals of the type $[p2^{-l}, (p+1)2^{-l}[$ $(l \in \mathbb{N})$. It is well known and follows in particular from Lemma 3.1 that the event Ω_1, defined as "for each l large enough, and each dyadic cube C of $\mathbb{R}^N$ of order l that meets B, we have $\sup_{t,s \in C} |X(t) - X(s)| \leq K 2^{-l\alpha}\sqrt{l}$" also has probability 1. Given $\omega \in \Omega_0 \cap \Omega_1$, we show that (with obvious notations) $\mu_\varphi(X_\omega(B)) < \infty$. Consider k such that $\lambda(R_k) \geq \lambda(B)(1 - \exp(-\sqrt{k}/4))$, where the random set R_k is given by (5.1).

We denote by $C_l(x)$ the dyadic cube of order l that contains x. If $x \in R_k$, we can find l with $2k + k_0 \geq l \geq k$ (k_0 depending on d only) such that

$$\sup_{s,t \in C_l(x)} |X(s) - X(t)| \leq K 2^{-l\alpha} (\log\log 2^l)^{-\alpha/N}.$$

[If r witnesses that $x \in R_k$, l is simply the smallest integer for which each point of $C_l(x)$ is within distance r of x.] Thus we can cover R_k by a union V of sets $V_l (k \leq l \leq 2k + k_0)$ such that each set V_l is a union of dyadic cubes C of order l for which

$$\sup_{s,t \in C} |X(s) - X(t)| \leq K 2^{-l\alpha} (\log\log 2^l)^{-\alpha/N}.$$

In particular, $X(C)$ is contained in a ball of radius

$$\varrho_l = K 2^{-\alpha l} (\log\log 2^l)^{-\alpha/N}.$$

Simple estimates show that

$$\varrho_l^{1/\alpha} (\log\log \varrho_l)^{1/N} \leq K 2^{-l}$$

so that $\varphi(\varrho_l) \leq K 2^{-lN} = K\lambda(C)$. Since

$$\sum_l \sum_{C \subset V_l} \lambda(C) = \lambda(V) < K,$$

we see that $X_\omega(V)$ is contained in the union of a family of balls B_i of radius ϱ_i that satisfy $\sum_i \varphi(\varrho_i) \leq K$. Now $B \setminus V$ is contained in a union of dyadic cubes of order $q = 2k + k_0$, none of which meets R_k. There can be at most

$$2^{Nq} \lambda(B \setminus R_k) \leq K 2^{Nq} \exp\left(-\frac{\sqrt{k}}{4} \right)$$

such cubes if k (hence q) have been taken large enough, and since we assume that $\omega \in \Omega_1$, for each such cube C, $X(C)$ is contained in a ball of radius $\varrho = K2^{-q\alpha}\sqrt{q}$. Now

$$\varphi(\varrho) \leq K2^{-Nq}q^{N/2\alpha}\log q$$

so that $X(B \setminus R_k)$ can be covered by a family B_i of balls of radius $\varrho_i (= \varrho)$ such that

$$\sum \varphi(\varrho_i) \leq (K2^{-Nq}q^{N/2\alpha}\log q)\left(K2^{Nq}\exp\left(-\frac{\sqrt{k}}{4}\right)\right) \leq 1$$

for k large enough. Because k was arbitrarily large, the proof is complete. □

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REFERENCES

[1] Ciesielski, Z. and Taylor, S. J. (1962). First passage times and sojorn times for Brownian motion in space and the exact Hausdorff measure of the sample path. *Trans. Amer. Math. Soc.* **10** 434–450.

[2] Goldman, A. (1988). Movement brownien à plusieurs paramètres: mesure de Hausdorff des trajectories. *Astérisque* **167**.

[3] Kahane, J. P. (1985). *Some Random Series of Functions*, 2nd ed. Cambridge Univ. Press.

[4] Ledoux, M. and Talagrand, M. (1991). *Probabilities in Banach Space*. Springer, Berlin.

[5] Levy, P. (1953). La mesure de Hausdorff de la courbe du mouvement brownien. *Giormale dell Instituto Ital. Altuari* **16** 1–37.

[6] Pitt, L. D. (1978). Local times for Gaussian vector fields. *Indiana Univ. Math. J.* **27** 309–330.

[7] Ray, D. (1963). Sojorn times and the exact Hausdorff measure of the sample path for planar Brownian motion. *Trans. Amer. Math. Soc.* **106** 436–444.

[8] Talagrand, M. (1993). New Gaussian estimates for enlarged balls. *Geometric Funct. Anal.* **3** 502–520.

[9] Taylor, S. J. (1964). The exact Hausdorff measure of the sample path for planar Brownian motion. *Proc. Cambridge Phil. Soc.* **60** 253–258.

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