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A BORDERLINE RANDOM FOURIER SERIES[1]

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Consider a mean zero random variable X, and an independent sequence (X_n) distributed like X. We show that the random Fourier series $\sum_{n \geq 1} n^{-1} X_n \exp(2i\pi nt)$ converges uniformly almost surely if and only if $E(|X| \log\log(\max(e^e, |X|))) < \infty$.

1. Introduction. The question considered in this paper was motivated by the problem of exactly which integrability condition on a function f is sufficient to insure the convergence a.e. of its (nonrandom) Fourier series. An example going back to Kolmogorov shows that it does not suffice that $E(|f|LL(f)) < \infty$ [where $LL(x) = \log\log(\max(e^e, x))$]. On the other hand, the work of Carelson and Sjölin [4] shows that the condition $E(|f|\log^+|f|LL|f|) < \infty$ suffices. Lacey [1] observed connections between this question and the question of the uniform convergence of the random Fourier series

$$\sum_{n \geq 1} \frac{X_n}{n} \exp(2i\pi nt), \tag{1.1}$$

where (X_n) is an i.i.d. sequence. He showed in particular that if, for an Orlicz function φ, one could find a sequence (X_n) such that $E\varphi(|X_1|) < \infty$, and such that the series (1.1) does not converge a.s., one could find a function f with $\varphi(f) < \infty$ and such that the Fourier series of f does not converge a.e. As will be demonstrated in the present paper, this approach fails to bring new information, but it turns out that the study of the series (1.1) was in itself a somewhat challenging question.

The modern theory of random Fourier series started with the work of Marcus and Pisier [3] and was later developed further by Marcus [2] and by the author [5], [6]. The point of view of Marcus and Pisier is, given an independent identically distributed sequence (X_n), to characterize the sequences of coefficients (a_n) for which $\sum_{n \geq 1} a_n X_n \exp(2i\pi nt)$ converges. Much greater generality is reached in [6], where series $\sum_{n \geq 1} Y_n \exp(2i\pi nt)$ are studied, for independent sequences (Y_n). (When specialized to the setting of Marcus and Pisier, the results of [6] recover theirs.) Roughly speaking, it can be said that the a.s. convergence of the series $\sum_{n \geq 1} Y_n \exp(2i\pi nt)$ is completely understood in the case where the r.v. Y_n are "away from L^1." This is not, however, the case in situations where the variables Y_n resemble functions in L^1, and in that case there is a gap between the necessary and the

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sufficient conditions given in [6]. (There is indeed an intrinsically inefficient step in the proof of the sufficiency conditions of [6].) In the present paper, in the case where the variables Y_n are specialized to $Y_n = X_n/n$, where X_n are identically distributed, we show how to close this gap.

THEOREM 1.1. *Given an independent identically distributed mean zero sequence* (X_n), *the random Fourier series*

$$\sum_{n=1}^{\infty} \frac{X_n}{n} \exp(2i\pi nt)$$

converges uniformly almost surely if and only if $E(|X|LL(X)) < \infty$.

Previous work on random Fourier series is based on a few powerful and simple ideas. These are unfortunately hidden under a mountain of technical points which largely arise from the generality of the setting, and there is a very real danger that these ideas would be lost forever. One of the motivations in writing the present paper is certainly that in the present case the technicalities are fewer and thereby that the main ideas are easier to grasp. For this reason, the presentation has been made essentially self-contained, hoping that the present work could serve as an introduction to the more complicated work of [6].

The paper is organized as follows. In Section 2, we recall some general facts and we show how to deal with the integrability condition $E(|X|LL(X)) < \infty$. In Section 3, we prove the necessity of this integrability condition. (While this could be deduced from [6], Theorem 1.4, it is as simple, and much clearer to reproduce the argument in our special case.) In Section 4, we prove the sufficiency of the integrability condition. The new ingredients in our approach are better discussed there after the preliminaries have been set up. It is natural to hope that these new ingredients would allow one to improve the general sufficient conditions presented in [6], Theorems 1.3 and 1.6. This line of research remains to be investigated. Another natural line of investigation is the study of the uniform convergence of the series $\sum_{n\geq 1} a_n X_n \exp(2i\pi n)$, where a_n is a decreasing sequence and (X_n) is an i.i.d. sequence. We believe however that the techniques presented here make this study little more than a routine computation.

2. Preliminaries. The framework of our approach is standard. Consider an independent sequence $(\varepsilon_i)_{i\geq 1}$ of Bernoulli random variables [$P(\varepsilon_i = 1) = P(\varepsilon_i = -1) = \frac{1}{2}$] that is independent of the sequence (X_n). It follows from general principles that the convergence of $\sum_{n\geq 1} n^{-1}X_n \exp(2i\pi nt)$ is equivalent to the convergence of $\sum_{n\geq 1} n^{-1}\varepsilon_n X_n \exp(2i\pi nt)$, where convergence means uniform convergence, almost surely. Since the sequences $(\varepsilon_n X_n)$ and $(\varepsilon_n|X_n|)$ have the same distribution, we can and do assume $X_1 \geq 0$. It will be convenient to assume that the sequences X_n, ε_n are defined on a product space $\Omega \times \Omega'$ (provided with a product probability $Q = P \otimes P'$) and that for ω, $\omega') \in \Omega \times \Omega'$, we have $X_n(\omega, \omega') = X_n(\omega)$, $\varepsilon_n(\omega, \omega') = \varepsilon_n(\omega')$. Also, it will be notationally convenient to write $\gamma_n(t) = \exp(2i\pi tn)$ and to work on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ rather than $\mathbb{R}$.

Let us observe that

$$|\gamma_n(t) - \gamma_n(s)| \le \min(2\pi, 2\pi|s - t|n), \tag{2.1}$$

where $|s - t|$ is the distance of s and t in $\mathbb{T}$. Numerical values such as 2π are distracting, so we prefer to write

$$|\gamma_n(t) - \gamma_n(s)| \le K \min(1, n|s - t|). \tag{2.2}$$

There, as well as in the rest of the paper, K denotes a universal constant, not necessarily the same at each occurrence. The uniform convergence a.e. of series

$$\sum_{n \ge 1} a_n \varepsilon_n \gamma_n(t) \tag{2.3}$$

is understood through the work of Marcus and Pisier, and the problem is to decide when the sequence $a_n = a_n(\omega) = n^{-1}X_n(\omega)$ is such that the series (2.3) converges for almost all ω.

After these generalities, we start the technical work. One should certainly expect that Theorem 1.1 relies on a rather precise understanding of the condition $E(XLL(X)) < \infty$. The work required for this is not connected to Fourier series and is of a classical probabilistic spirit. For simplicity, throughout the paper we set $e_p = 2^{2^p}$.

LEMMA 2.1. *Consider a r.v. $X \ge 0$ such that $E(X) < \infty$, $X \ne 0$. Then for each p we can find a number λ_p such that*

$$\sum_{k \ge e_p} E\left(\frac{X^2}{\lambda_p^2 k^2} \wedge 1\right) = 2^p \tag{2.4}$$

and we have

$$\sum_{p \ge 1} \lambda_p 2^p < \infty \quad \Leftrightarrow \quad E(XLL(X)) < \infty.$$

PROOF. It is standard to check that when $E(X) < \infty$, we have $\sum_{k \ge 1} E(k^{-2}X^2 \wedge 1) < \infty$. The function

$$f_k(x) = E\left(\frac{X^2}{k^2x^2} \wedge 1\right)$$

is continuous, decreasing and $\lim_{x \to 0} f_k(x) = P(X \ne 0) >$. The existence of λ_p follows easily.

We now compute the left-hand side of (2.4) as

$$\sum_{k \ge e_p} \int_0^1 2tP(X \ge tk\lambda_p)\, dt = \sum_{k \ge e_p} \int_0^{k\lambda_p} \frac{1}{\lambda_p^2 k^2} 2tP(X \ge t)\, dt$$

$$= \frac{1}{\lambda_p^2} \int_0^\infty \Bigg(\sum_{\substack{k \ge e_p \\ k\lambda_p \ge t}} \frac{1}{k^2} \Bigg) 2tP(X \ge t)\, dt.$$

Using the lower bound $\sum_{k \ge n_0} k^{-2} \ge 1/2n_0$, we get that

$$\frac{2t}{\lambda_p^2} \sum_{\substack{k \ge e_p \\ k\lambda_p \ge t}} \frac{1}{k^2} \ge \frac{1}{\lambda_p} \min\left(\frac{t}{\lambda_p e_p}, 1 \right)$$

and we get from (2.4) that

$$\sum_{k \ge e_p} \int_0^\infty \min\left(\frac{t}{\lambda_p e_p}, 1 \right) P(X \ge t)\, dt \le \lambda_p 2^p \tag{2.5}$$

and thus

$$\int_0^\infty \operatorname{card}\{p; t \ge \lambda_p e_p\} P(X \ge t)\, dt \le \sum_p \lambda_p 2^p. \tag{2.6}$$

If we have $\lambda_p \le 1$ eventually, then for t large we have

$$\operatorname{card}\{p; t \ge \lambda_p e_p\} \ge \frac{1}{K} LL(t).$$

Since the condition $E(XLL(X)) = \infty$ implies

$$\int_0^\infty LL(t) P(X \ge t)\, dt = \infty,$$

we have shown that when $E(XLL(X)) = \infty$, the series $\sum_p \lambda_p 2^p$ diverges.

Using now the bound $\sum_{k \ge n_0} k^{-2} \le K/n_0$, we obtain

$$\int_0^\infty \min\left(\frac{t}{\lambda_p e_p}, 1 \right) P(X \ge t)\, dt \ge \frac{1}{K} \lambda_p 2^p.$$

Consider the set $J = \{p;\ \lambda_p \ge 2^{-2p}\}$. Then

$$\int_0^\infty \operatorname{card}\{p \in J; t \ge \lambda_p e_p\} P(X \ge t)\, dt \ge \frac{1}{K} \sum_{p \in J} \lambda_p 2^p.$$

Now we have

$$\operatorname{card}\{p \in J; t \geq \lambda_p e_p\} \leq KLL(t),$$

so that

$$\int_0^\infty LL(t) P(X \geq t)\, dt \geq \frac{1}{K} \sum_{p \in J} \lambda_p 2^p$$

and the divergence of the series $\Sigma \lambda_p 2^p$ implies the divergence of the series $\Sigma_{p \in J} \lambda_p 2^p$ and that

$$E(XLL(X)) = \infty. \qquad \square$$

COROLLARY 2.2. *Assume that $E(XLL(X)) < \infty$. Then we can find a sequence μ_p such that*:

(2.7) $$\mu_p \geq 2^{-2p} \quad \text{and} \quad \sum_p 2^p \mu_p < \infty;$$

(2.8) *The sequence (μ_p) is decreasing.*

(2.9) $$\text{For each } p, \quad \sum_{k \geq e_p} E\left(\frac{X^2}{\mu_p^2 k^2} \wedge 1 \right) \leq 2^p;$$

(2.10) $$\sum_k E\left(\frac{X}{k} 1_{\{k \mu_{m(k)} \leq X \leq k\}} \right) < \infty;$$

where $m(k)$ is the largest integer such that $2^{-2^{m(k)}} \geq k^{-2}$.

PROOF. To obtain (2.7)–(2.9), we set

$$\mu_p = 2^{-2p} + \sup\{\lambda_q, q \geq p\},$$

where (λ_q) is the sequence constructed in Lemma 2.1.

To prove (2.10), we observe that $2^{m(k)} \leq K \log k$ so that by (2.7) we have $\mu_{m(k)} \geq (K \log k)^{-2}$ and we have

(2.11) $$\sum_k \frac{1}{k} 1_{\{k \mu_{m(k)} \leq X \leq k\}} \leq K \log \frac{k_2}{k_1},$$

where $k_2 = \sup\{k; k \mu_{m(k)} \leq X\}$ and $k_1 = \inf\{k; k \geq X\}$. Now, for $k \mu_{m(k)} \leq X$, we have $X \geq k(K \log k)^{-2}$, so that $k \leq X(K \log X)^2$ and thus $k_2/k_1 \leq (K \log X)^2$. Thus the right-hand side of (2.11) is bounded by $K(1 + LL(X))$ and this proves (2.10). $\square$

3. Necessity. In this section we assume the convergence of the random Fourier series. Consider the random distance d_ω on $\mathbb{T}$ given by

$$d_\omega^2(s,t) = \sum_{k \geq 1} \frac{X_k^2(\omega)}{k^2} |\gamma_k(s) - \gamma_k(t)|^2.$$

As initiated by Marcus and Pisier, the approach is that the space $(\mathbb{T}, d_\omega)$ must satisfy the condition

$$(3.1) \qquad \int_0^\infty (\log N(\mathbb{T}, d_\omega, \varepsilon))^{1/2}\, d\varepsilon < \infty$$

for the series $\Sigma k^{-1} X_k^2(\omega)\varepsilon_k \gamma_k$ to converge uniformly P' a.s. There $N(\mathbb{T}, d_\omega, \varepsilon)$ denotes the smallest number of balls of radius ε for the distance d_ω needed to cover $\mathbb{T}$.

Consider the set $D_p = \{n2^{-2^p}; 0 \le n < 2^{2^p}\}$ and consider the sequence λ_p of Lemma 2.1. We will show that with probability 1, for p large enough, we have

$$(3.2) \qquad \log N\left(D_p, d_\omega, \frac{1}{K}\lambda_p 2^{p/2}\right) \ge \frac{2^p}{K}.$$

Thus (3.1) implies the convergence for the series $\Sigma \lambda_p 2^p$, and thereby that $E(XLL(X)) < \infty$.

To prove (3.2), consider $s, t \in D_p$, $s \ne t$. The key point is the inequality

$$(3.3) \qquad P\left(d_\omega(s,t) \le \frac{1}{K}\lambda_p 2^{p/2}\right) \le \exp\left(\frac{-2^p}{K_1}\right).$$

Indeed, if A is a subset of D_p of cardinality N, we have

$$P\left(\exists\, s, t \in A,\, d_\omega(s,t) \le \frac{1}{K}\lambda_p 2^{p/2}\right) \le N^2 \exp\left(\frac{-2^p}{K_1}\right),$$

so that taking N of order $\exp(2^p/3K_1)$, we get

$$P\left(N\left(D_p, d_\omega, \frac{1}{2K}\lambda_p 2^{p/2}\right) \ge \exp\left(\frac{2^p}{3K_1}\right)\right) \ge 1 - \exp\left(\frac{-2^p}{3K_1}\right),$$

and this implies (3.2). We turn to the proof of (3.3). We claim that

$$(3.4) \qquad \sum_{k \ge e_p} E\left(\frac{X_k^2}{\lambda_p^2 k^2}|\gamma_k(s) - \gamma_k(t)|^2 \wedge 1\right) \ge \frac{2^p}{K}.$$

First, we observe that

$$|\gamma_k(s) - \gamma_k(t)| \ge 1 \quad \Rightarrow \quad E\left(\frac{X^2}{\lambda_p^2 k^2}|\gamma_k(s) - \gamma_k(t)|^2 \wedge 1\right) \ge E\left(\frac{X^2}{\lambda_p^2 k^2} \wedge 1\right).$$

Thus, to prove (3.4) it suffices by (2.4) to prove that for $m \ge 2^p$,

$$(3.5) \qquad \operatorname{card}\{k; 2^m \le k \le 2^{m+1}, |\gamma_k(s) - \gamma_k(t)| \ge 1\} \ge 2^m/K.$$

Now $|\gamma_k(s) - \gamma_k(t)| = |\exp 2\pi i k(s-t) - 1|$ and thus $|\gamma_k(s) - \gamma_k(t)| \ge 1$ whenever $k(s-t) \notin [-\frac{1}{6}, \frac{1}{6}] \bmod 1$.

Now $|s-t| \geq 2^{-2^p}$, $m \geq 2^p$, so that (3.5) and hence (3.4) follows from elementary considerations. Consider now the r.v.

$$Z_k = \frac{X_k^2}{\lambda_p^2 k^2} |\gamma_k(s) - \gamma_k(t)|^2 \wedge 1.$$

To prove (3.3), it suffices by (3.4) to know that

$$P\left(\sum Z_k \leq \tfrac{1}{4} \sum EZ_k\right) \leq \exp\left(-\tfrac{1}{4} \sum EZ_k\right). \tag{3.6}$$

To prove this, one observes that $e^{-x} \leq 1 - x/2$ for $0 \leq x \leq 1$, so that

$$E\exp(-Z_k) \leq E\left(1 - \frac{Z_k}{2}\right) \leq 1 - \frac{1}{2}EZ_k \leq \exp\left(-\frac{1}{2}EZ_k\right)$$

and thus $E\exp(-\Sigma Z_k) \leq \exp(-\frac{1}{2}\Sigma EZ_k)$, from which (3.6) follows by Chebyshev inequality.

4. Sufficiency. Under the condition $E(XLL(X)) < \infty$, we will construct a subset Ω_0 of Ω, with $P(\Omega_0) = 1$, such that if $\omega \in \Omega_0$, the series $R'(t) = \Sigma_{k \geq 1} k^{-1} X_k(\omega)\varepsilon_k(\omega')\gamma_k(t)$ converges uniformly P' a.s. Since $E(X) < \infty$, only infinitely many of the events $\{X_k > k\}$ occur, so setting $Y_k = X_k 1_{\{X_k \leq k\}}$, we can instead consider the series $R(t) = \Sigma_{k \geq 1} k^{-1} Y_k(\omega)\varepsilon_k(\omega')\gamma_k(t)$.

According to Theorem 1.1 of [3], a series $\Sigma_{k \geq 1} a_k \varepsilon_k \gamma_k(t)$ converges uniformly a.s. as soon as it is a.s. bounded. Thus it suffices to prove that

$$\sup_N \sup_{t \in D_N} |R(t)| < \infty \quad \text{a.s.} \tag{4.1}$$

The proof relies on a chaining argument. Given $m \geq 1$ and a point u in D_m, we pick once and for all a point $\varphi_m(u)$ in D_{m-1} such that $|u - \varphi_m(u)| \leq 2^{-2^{n-1}}$. Given t in D_N, we define the sequence $(t_n)_{n \leq N}$ by $t_N = t$ and $t_{n-1} = \varphi_n(t_n)$ for $n \leq N$. Thus $|t_n - t_{n+1}| \leq 2^{-2^n}$.

Consider now a number $n_0 = n_0(\omega, \omega')$ that depends on ω, ω' and that will be constructed later. Since $t_{n_0} \in D_{n_0}$ can take only finitely many values, it suffices to show that for $\omega \in \Omega_0$, we have P' a.s. that

$$\sup_N \sup_{t \in D_N} |R(t) - R(t_{n_0})| < \infty. \tag{4.2}$$

We write

$$R(t) - R(t_0) = \sum_{n_0 \leq n < N} [R(t_{n+1}) - R(t_n)]. \tag{4.3}$$

So far, this is routine. Now each term in the right of (4.3) is the sum of a series. In this series we wish to separate the contributions of the "large" and "small" terms. In the way this was done in previous work (and in [6] in particular), the contributions of some of the "large" terms were implicitly counted many times. While in many situations this does not matter, in the present situation we cannot afford to be crude and we have to use a more precise decomposition (that is the new ingredient of the present paper).

We recall the sequence (μ_n) of Corollary 2.2 and for each k, we define a random sequence $n(k) = n(k, \omega)$ as the smallest integer $n \geq n_0$ for which $Y_k(\omega) > k\mu_n$. For $1 < n \leq N$, we write

$$U_n = U_n(t) = \sum_{k \geq 1} \frac{Y_k(\omega)}{k} \varepsilon_k 1_{\{Y_k(\omega) > k\mu_n\}}(\gamma_k(t_{n+1}) - \gamma_k(t_n)),$$

$$V_n = V_n(t) = \sum_{k \geq 1} \frac{Y_k(\omega)}{k} \varepsilon_k 1_{\{Y_k(\omega) \leq k\mu_n\}}(\gamma_k(t_{n+1}) - \gamma_k(t_n)).$$

(For the simplicity of notation, the dependence on t will be kept implicit.) Thus (4.3) implies

$$R(t) - R(t_{n_0}) = \sum_{n_0 \leq n < N} (U_n + V_n) = \sum_{n_0 \leq n < N} U_n + \sum_{n_0 \leq n < N} V_n$$

and thus

$$|R(t) - R(t_{n_0})| \leq \left| \sum_{n_0 \leq n < N} U_n \right| + \sum_{n_0 \leq n < N} |V_n|. \tag{4.4}$$

Now comes the essential point. Since the sequence (μ_n) decreases for $n \geq n_0$, we have $1_{\{Y_k(\omega) > k\mu_n\}} = 1$ if and only if $n \geq n(k)$, so that

$$\sum_{n_0 \leq n < N} 1_{\{Y_k(\omega) > k\mu_n\}}(\gamma_k(t_{n+1}) - \gamma_k(t_n)) = \gamma_k(t_N) - \gamma_k(t_{n(k)}) \tag{4.5}$$

with the convention that the right-hand side is zero if $n(k) \geq N$. Thus, with the same convention,

$$\sum_{n_0 \leq n < N} U_n = \sum_{k \geq 1} \frac{Y_k(\omega)}{k} \varepsilon_k \big(\gamma_k(t_N) - \gamma_k(t_{n(k)})\big)$$

and thus

$$\left| \sum_{n_0 \leq n < N} U_n \right| \leq \sum_{k \geq 1} \frac{Y_k(\omega)}{k} |\gamma_k(t_N) - \gamma_k(t_{n(k)})|. \tag{4.6}$$

This bound is far superior to what we would get by using first the triangle inequality $|\Sigma U_n| \leq \Sigma |U_n|$, which prevents the cancellation effects of (4.5).

Since $|t_{n+1} - t_n| \leq 2^{-2^n}$, we see that $|t_N - t_{n(k)}| \leq 2 \cdot 2^{-2^{n(k)}}$ and thus by (2.2) we have

$$\left| \sum_{n_0 \leq n < N} U_n \right| \leq K \sum_{k \geq 1} \frac{Y_k(\omega)}{k} \min(1, k 2^{-2^{n(k)}}).$$

We now recall the integer $m(k)$ of Corollary 2.2. Distinguishing whether $n(k) > m(k)$, in which case $k 2^{-2^{n(k)}} \leq k^{-1}$, or whether $n(k) \leq m(k)$ and observing that in that case $Y_k(\omega) \geq k\mu_{m(k)}$, we get

$$\left| \sum_{n_0 \leq n < N} U_n \right| \leq K \sum_{k \geq 1} \frac{Y_k(\omega)}{k^2} + \sum_{k \geq 1} \frac{1}{k} Y_k(\omega) 1_{\{Y_k(\omega) \geq k\mu_{m(k)}\}}.$$

Now since $E(X) < \infty$, we have $\sum_{k \geq 1} E(Y_k/k^2) < \infty$, and since by (2.10) we have

$$E\left(\sum_{k \geq 1} \frac{1}{k} Y_k 1_{\{Y_k \geq k\mu_{m(k)}\}}\right) < \infty,$$

we can assume that for each $\omega \in \Omega_0$, we have

$$\sum_{k \geq 1} \frac{Y_k(\omega)}{k^2} < \infty, \qquad \sum_{k \geq 1} \frac{1}{k} Y_k(\omega) 1_{\{Y_k(\omega) \geq k\mu_{m(k)}\}} < \infty.$$

Thus, by (4.4), we see that

$$|R(t) - R(t_{n_0})| \leq S(\omega) + \sum_{n_0 \leq n < N} |V_n|, \tag{4.7}$$

where $S(\omega)$ is finite on Ω_0. Thus we are reduced to the control of $\sum_{n_0 \leq n < N} |V_n|$ and back to routine arguments. Given $\mu \in D_{n+1}$, consider the r.v.

$$T_n(u, \omega) = \sum_{k \geq 1} \frac{Y_k^2(\omega)}{k^2} 1_{\{Y_k(\omega) \leq k\mu_n\}} |\gamma_k(u) - \gamma_k(\varphi_{n+1}(u))|^2.$$

Using (2.2) we get

$$T_n(u, \omega) \leq T_n(\omega) = K \sum_{k \geq 1} \frac{Y_k^2(\omega)}{k^2} 1_{\{Y_k(\omega) \leq k\mu_n\}} \min(1, k2^{-2^n})^2.$$

Consider the r.v.

$$W_p = \sum_{k \geq e_p} \frac{Y_k^2}{k^2} 1_{\{Y_k \leq k\mu_p\}} \leq \sum_{k \geq e_p} \frac{X_k^2}{k^2} 1_{\{X_k \leq k\mu_p\}}.$$

Since $\mu_p \geq \mu_n$ for $p \leq n$ and since $k2^{-2^n} \leq 2^{2^p - 2^n}$ for $e_{p-1} < k \leq e_p$, we see that

$$T_n \leq \sum_{1 \leq p \leq n} 2^{2^{p+1} - 2^{n+1}} W_{p-1}$$

and thus

$$\sqrt{T_n} \leq \sum_{1 \leq p \leq n} 2^{2^p - 2^n} \sqrt{W_{p-1}}. \tag{4.8}$$

Now by (2.9), we have $EW_p \leq 2^p \mu_p^2$, so that by (2.7),

$$E\left(\sum_{p \geq 1} 2^{p/2} \sqrt{W_p}\right) \leq \sum_{p \geq 1} 2^{p/2} \sqrt{EW_p} < \infty.$$

Thus we can assume that for $\omega \in \Omega_0$ we have $\sum_{p \geq 1} 2^{p/2} \sqrt{W_p(\omega)} < \infty$. It then follows from (4.8) and a routine computation that, for $\omega \in \Omega_0$,

$$S'(\omega) = \sum_{n \geq 1} 2^{n/2} \sqrt{T_n(\omega)} < \infty. \tag{4.9}$$

We now fix $\omega \in \Omega_0$, and we prove that (4.2) holds P' a.s. We recall the subgaussian inequality

$$P'\left(\left|\sum a_k \varepsilon_k(\omega')\right| \geq v\right) \leq \exp\left(-\frac{v^2}{8\sum|a_k|^2}\right)$$

for all complex numbers (a_k). Thus, for each n and each u in D_{n+1}, we get

$$P'\left(\left\{\omega';\ \sum_{k\geq 1}\varepsilon_k(\omega')\frac{Y_k(\omega)}{k}1_{\{Y_k\leq k\mu_n\}}(\gamma_k(\mu)-\gamma_k(\varphi_{n+1}(u)))\geq v\sqrt{T_k(\omega)}\right\}\right)$$
$$\leq \exp\left(\frac{-v^2}{8}\right).$$

We now define the event $\Omega'_q \subset \Omega'$ by

$\forall\, n \geq q, \forall\, u \in D_{n+1}$,

$$\left|\sum_{k\geq 1}\varepsilon_k(\omega')\frac{Y_k(\omega)}{k}1_{\{Y_k\leq k\mu_n\}}(\gamma_k(u)-\gamma_k(\varphi_n(u)))\right| \leq 2^{2+n/2}\sqrt{T_n(\omega)}\,.$$

Thus $\Omega' \setminus \Omega_q$ has probability at most $\sum_{n\geq q} 2^{-2^n}$. Thus, if we define $\Omega'_0 = \bigcup_q \Omega'_q$, we have $P'(\Omega'_0) = 1$. Moreover, for $\omega' \in \Omega'_1$ let us define n_0 as the smallest integer r for which $\omega' \in \Omega'_r$. Then, for $n \geq n_0$, we have $|V_n(\omega')| \leq 2^{2+n/2}\sqrt{T_n(\omega)}$ so that

$$\sum_{n\geq n_0}|V_n(\omega')| \leq 4S_2(\omega).$$

The proof is complete. □

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