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UNIFORM LOCAL PROBABILITY APPROXIMATIONS: IMPROVEMENTS ON BERRY–ESSEEN

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Let $X_1, X_2, \ldots$ be independent, mean zero, uniformly bounded random variables with $S_n = X_1 + \cdots + X_n$. Optimal criteria are determined on the length and location of an interval Γ so that $P(S_n \in \Gamma)$ is proportional to $(|\Gamma|/\sqrt{\operatorname{Var} S_n}) \wedge 1$. The proof makes an unusual use of support considerations.

1. Introduction. Let $X, X_1, X_2, \ldots$ be i.i.d., mean zero, bounded random variables. Let n be a fixed natural number, $S_n = \sum_{j=1}^n X_j$ and Γ be an interval. If $S_n/\sqrt{\operatorname{Var} S_n}$ has a continuous density $f(x)$ which is bounded away from 0 and infinity on a bounded region B, then for $\Gamma/\sqrt{\operatorname{Var} S_n} \subset B$,

$$P(S_n \in \Gamma) = P\left(\frac{S_n}{\sqrt{\operatorname{Var} S_n}} \in \frac{\Gamma}{\sqrt{\operatorname{Var} S_n}}\right) \asymp \frac{|\Gamma|}{\sqrt{\operatorname{Var} S_n}}, \tag{1.1}$$

where $|A| \equiv \sup\{|x-y|\colon x, y \in A\}$ denotes the diameter of the set A and $A_1 \asymp A_2$ means there exist $C_1 > 0$ and $C_2 < \infty$ such that $C_1 A_1 \le A_2 \le C_2 A_1$. Even if $S_n/\sqrt{\operatorname{Var} S_n}$ does not have a density (let alone a bounded one), the approximation in (1.1) may still hold. If this is to be the case, intervals Γ of arbitrarily small length are not permitted (otherwise $S_n/\sqrt{\operatorname{Var} S_n}$ would "usually" have a density). Notice that if X took on only the values $-a$ and $1-a$, then the support of S_n would consist of atoms 1 unit apart. Consequently, if (1.1) is to hold for all of the above X distributions and all integers n, the intervals Γ must contain an at least half-closed interval of length at least equal to the diameter of the support of X. Since nonlattice variables can be infinitesimally close to lattice variables, the condition on the minimal length of Γ cannot be eliminated by a nonlattice assumption. Additionally, since $P(S_n \in \Gamma)$ is at most 1, the validity of (1.1) requires that $|\Gamma| = O(\sqrt{\operatorname{Var} S_n})$. Moreover, if such a Γ is located too far into the tail of the distribution, then $P(S_n \in \Gamma) = o(|\Gamma|/\sqrt{\operatorname{Var} S_n})$. [See, e.g., Bikyalis (1966).]

Our main result identifies the family of intervals for which an appropriate analogue of (1.1) holds and provides an extension to independent (but

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not necessarily identically distributed) mean zero, uniformly bounded random variables.

THEOREM 1.1. *Fix any* $0 < \varepsilon \le \frac{1}{2}$. *For any integer* $n \ge 1$, *let* $X_1, \dots, X_n$ *be independent mean zero, uniformly bounded random variables with sum* S_n. *Let*

$$L = \max_{1 \le j \le n} (|\operatorname{supp} X_j|) < \infty. \tag{1.2}$$

Let $y_{n,\varepsilon}^-$ *and* $y_{n,\varepsilon}^+$ *be unique reals such that*

$$y_{n,\varepsilon}^- = \inf\{y\colon\ P(S_n \le y) \ge \varepsilon\} \tag{1.3}$$

and

$$y_{n,\varepsilon}^+ = \sup\{y\colon\ P(S_n \le y) \le 1 - \varepsilon\}. \tag{1.4}$$

Let Γ *be any at least half-closed interval of reals such that*

$$|\Gamma| \ge L \tag{1.5}$$

and

$$\Gamma \cap [y_{n,\varepsilon}^-, y_{n,\varepsilon}^+] \ne \varnothing. \tag{1.6}$$

Then there exists $c_\varepsilon > 0$, *depending only on* ε *(and not otherwise on* $n, \{X_j\}, L$ *or* Γ*), such that*

$$P(S_n \in \Gamma) \ge c_\varepsilon \left(1 \wedge \frac{|\Gamma|}{\sqrt{\operatorname{Var} S_n}}\right). \tag{1.7}$$

Moreover, by direct application of the Berry–Esseen theorem,

$$P(S_n \in \Gamma) \le \left(\left(\frac{|\Gamma|}{\sqrt{2\pi}} + 2c^* L\right)\frac{1}{\sqrt{\operatorname{Var} S_n}} \wedge 1\right), \tag{1.8}$$

where c^* *is the constant determined from the Berry–Esseen theorem.*

Combining (1.7) and (1.8), Theorem 1.1 establishes that whenever the interval Γ has length at least L and some portion of Γ intersects the center of the S_n distribution, namely, between the εth and $(1-\varepsilon)$th quantiles, $P(S_n \in \Gamma)$ is proportional to the length of the interval over the standard deviation, provided this ratio is not larger than 1. However, as the central limit theorem would indicate, this bound is too large as Γ departs from the center of the distribution. A result of Bikyalis (1966) implies that if (1.6) fails, then the upper bound in (1.8) can indeed be reduced by a factor of γ_ε, where $\gamma_\varepsilon \downarrow 0$ as $\varepsilon \downarrow 0$.

Concentration function results identify the order of magnitude of the maximum probability concentrated in an interval of a specified length l. Theorem 1.1 shows that, provided this length is at least L, all intervals of length l located near the center of the distribution have essentially the same probability content.

Although nonasymptotic, Theorem 1.1 also bears some relationship to local limit theorems. Perhaps the most general references in the independent but not identically distributed case are those of McDonald (1979a, b) and Muhkin (1991), which consider lattice random variables, and Maejima (1980), which considers random variables with uniformly bounded, continuous densities.

The remainder of the paper focuses on the proof of Theorem 1.1. A judicious decomposition of partial sums of certain triangular arrays into two components, one of which has uniformly bounded variance, allows Theorem 1.1 to be obtained from the Berry–Esseen theorem plus support considerations. Quite surprisingly, support considerations are the essence of establishing a version of Theorem 1.1, first for sums of infinitely many random variables whose sum of variances is finite (Section 2) and then for weakly convergent partial sums from a triangular array (Section 3). Section 4 then completes the proof of Theorem 1.1.

An easy reduction will simplify the notation. If $L = 0$, the result is trivial. Hence it may be supposed that $0 < L < \infty$. Dividing each random variable by L, it may furthermore be supposed that $L = 1$, which will be assumed throughout the sequel.

2. Support considerations and the concentration of infinite sums of independent random variables. For any closed set F, let diam F denote the diameter of F and let

$$l_F \equiv \inf\{l > 0\colon\ [x, x+l] \cap F \neq \phi\ \forall\ x \in (-l + \inf F, \sup F)\}. \tag{2.1}$$

For closed sets F and G, let $F + G \equiv \{x + y\colon\ x \in F \text{ and } y \in G\}$. Clearly, $l_{F+G} \le \max\{l_F, l_G\}$. In fact, the following lemma holds.

LEMMA 2.1. *Let F and G be nonempty closed subsets of $\mathbf{R}$. Then*

$$l_{F+G} \le \max\{|l_F - l_G|, \min\{l_F, l_G\}\}. \tag{2.2}$$

Moreover, if $l_F = 0$, then

$$l_{F+G} \le \begin{cases} (l_G - \operatorname{diam} F)^+, & \text{if } \operatorname{diam} F < \infty, \\ 0, & \text{if } \operatorname{diam} F = \infty. \end{cases} \tag{2.3}$$

The proof of this lemma involves no probability and is contained in the Appendix. An immediate corollary follows.

COROLLARY 2.2. *Let $F_1, \dots, F_n$ be nonempty closed subsets of $\mathbf{R}$ such that $l_{F_j} \le 1$ for each $1 \le j \le n$. Then $l_{\sum_{j=1}^n F_j} \le 1$. Moreover, if $l_{\sum_{j=1}^n F_j} = 1$, then:*

(i) $l_{F_j} = 1$ *for some* $1 \le j \le n$.
(ii) $l_{F_j} < 1 \Rightarrow l_{F_j} = 0$.
(iii) $l_{F_j} = 0 \Rightarrow F_j$ *consists of a single point.*

Observe that if $W_1, \dots, W_n$ are independent random variables with supports $F_1, \dots, F_n$, then $W_1 + \cdots + W_n$ has support $F_1 + \cdots + F_n$. Moreover, if F is the support of any random variable W, then for any $l > l_F$, $P(W \in [z, z + l]) > 0$ for all $z \in [-l_F + \inf F, \sup F)$.

We now obtain a slight extension of these ideas.

THEOREM 2.3. *Let $W_0, W_1, W_2, \dots$ be independent mean zero random variables such that*

$$\sum_{j=0}^{\infty} \operatorname{Var} W_j < \infty. \tag{2.4}$$

Let F_j denote the support of W_j. Suppose that for each $j \geq 1$ there exists $0 < a_j < 1$ such that

$$F_j \subset [-a_j, 1 - a_j]. \tag{2.5}$$

Suppose also that

$$l_{F_0} \leq 1. \tag{2.6}$$

Let $S_\infty = \sum_{j=0}^{\infty} W_j$ and denote its support by B_∞. Then

$$l_{B_\infty} \leq 1. \tag{2.7}$$

If $l_{B_\infty} = 1$, then (2.8) *holds where*:

$$\begin{aligned} &\text{(i) } l_{F_j} = 1 \ \textit{for some} \ j \geq 0; \\ &\text{(ii) } l_{F_j} < 1 \ \Rightarrow \ l_{F_j} = 0; \\ &\text{(iii) } l_{F_j} = 0 \ \Rightarrow \ P(W_j = 0) = 1. \end{aligned} \tag{2.8}$$

For a partial converse, suppose there exists $x_0 \in \mathbf{R}$ such that

$$\operatorname{supp} W_0 = \{x_0 + j : \ j = 0, \pm 1, \pm 2, \dots\} \cap [\inf W_0, \sup W_0]. \tag{2.9}$$

Then

$$\text{(2.8)(i)–(iii) } \textit{imply } l_{B_\infty} = 1. \tag{2.10}$$

Furthermore, if $z \in (-1 + \inf B_\infty, \sup B_\infty)$, then both

$$P(S_\infty \in (z, z + 1]) > 0 \quad \textit{and} \quad P(S_\infty \in [z, z + 1)) > 0 \tag{2.11}$$

hold if $l_{B_\infty} \leq 1$ and either $l_{B_\infty} < 1$ or (2.9) *holds.*

PROOF. Let B_n denote the support of $S_n = \sum_{j=0}^{n} W_j$. From Corollary 2.2, it follows that $l_{B_n} \leq 1$. Therefore, (2.7) will follow if

$$\liminf_{n \to \infty} l_{B_n} \geq l_{B_\infty}. \tag{2.12}$$

Moreover, if (2.12) holds and $l_{B_\infty} = 1$, then $\lim_{n \to \infty} l_{B_n} = 1$. Applying (2.5) and (2.6) together with an inductive argument based on Lemma 2.1 yields the existence of an integer $n_0 \geq 0$ such that $l_{B_n} = 0$ for $0 \leq n < n_0$ and $l_{B_n} = 1$ for $n \geq n_0$. Corollary 2.2 then implies that (2.8) holds.

To prove (2.12), fix any $\varepsilon > 0$. By (2.4), there exists $n_\varepsilon < \infty$ such that $\sum_{j>n_\varepsilon} \operatorname{Var} W_j < \frac{1}{8}\varepsilon^2$. Hence, using Chebyshev's inequality, for all $n \geq n_\varepsilon$, $P(|S_\infty - S_n| \geq \varepsilon/2) \leq \frac{1}{2}$. Take any $n \geq n_\varepsilon$ and any $z \in B_n$. There exists $\delta > 0$ such that $P(|S_n - z| < \varepsilon/2) \geq \delta$. Hence, by independence of S_n and $S_\infty - S_n$, $P(|S_\infty - z| < \varepsilon) \geq \delta/2$. Consequently, $z \in B_{\infty,\varepsilon}$, where for any set Q define $Q_\varepsilon \equiv \{x: \ d(x,Q) \leq \varepsilon\}$ with $d(x,Q) \equiv \inf\{|x-y|: \ y \in Q\}$. Thus, $B_n \subset B_{\infty,\varepsilon}$ for all $n \geq n_\varepsilon$ and so for any $k > 0$ and all $n \geq n_\varepsilon$,

$$B_n \cap [-k,k] \subseteq B_{\infty,\varepsilon} \cap [-k,k]. \tag{2.13}$$

We will now show that for any $\varepsilon > 0$ and $k > 0$ there exists $m_{\varepsilon,k}$ such that for all $n \geq m_{\varepsilon,k}$,

$$B_\infty \cap [-k,k] \subseteq B_{n,\varepsilon} \cap [-k,k]. \tag{2.14}$$

Fix $\varepsilon > 0$ and $k > 0$. By compactness of $B_\infty \cap [-k,k]$, there exists $\delta_{\varepsilon,k} > 0$ such that for *all* $z \in B_\infty \cap [-k,k]$, $P(|S_\infty - z| < \varepsilon/2) > \delta_{\varepsilon,k}$. There also exists $m_{\varepsilon,k} < \infty$ such that for all $n \geq m_{\varepsilon,k}$, $P(|S_n - S_\infty| < \varepsilon/2) > 1 - \delta_{\varepsilon,k}/2$. Hence, for all such z, $P(|S_n - z| < \varepsilon) > \delta_{\varepsilon,k}/2$. Therefore, for all $n \geq m_{\varepsilon,k}$,

$$B_\infty \cap [-k,k] \subseteq B_{n,\varepsilon},$$

whence (2.14) holds.

There exist $x_k < y_k$ such that

$$[x_k, y_k] \cap B_\infty \cap [-k,k] = \{x_k\} \cup \{y_k\}$$

and $y_k - x_k = l_{B_\infty \cap [-k,k]}$. By (2.13), $(x_k + \varepsilon, y_k - \varepsilon) \subseteq B_n^c$. Hence, for all $n \geq n_\varepsilon$, $l_{B_n} \geq y_k - x_k - 2\varepsilon = l_{B_\infty \cap [-k,k]} - 2\varepsilon$. Since $\varepsilon > 0$ is arbitrary, $\liminf_{n\to\infty} l_{B_n} \geq l_{B_\infty \cap [-k,k]}$. This being valid for each $k > 0$, (2.12) holds.

If (2.9) and (2.8)(i)–(iii) hold, then for some z_n,

$$B_n = \{z_n + j: \quad j = 0, \pm 1, \pm 2, \dots\} \cap [\inf B_n, \sup B_n]$$

and for some $0 \leq n_0 < \infty$, $l_{B_n} = 1$ for all $n \geq n_0$ (so that B_n contains at least two points for $n \geq n_0$). Combining (2.13) and (2.14), it follows that $l_{B_\infty} = 1$. Thus, (2.10) holds.

(2.11) obviously holds if $l_{B_\infty} < 1$. When $l_{B_\infty} = 1$ and (2.9) holds, then since (2.8) also holds, each X_j for $j \geq 1$ is either a point mass at 0 or a two-point distribution living on $\{-a_j, 1 - a_j\}$. Hence, B_∞, the support of S_∞, consists of points exactly one unit apart. Thus, (2.11) follows. □

COROLLARY 2.15. *Let W_0 be a nonconstant infinitely divisible random variable whose Lévy measure ν_0 has support in $[-1,1]$. Let F_0 denote the support of W_0. Then $l_{F_0} \leq 1$ and $l_{F_0} = 1$ if and only if W_0 has no Gaussian component and ν_0 has support in $\{1\} \cup \{-1\}$.*

PROOF. Take any such W_0, ν_0 and F_0. There is no loss of generality in making several assumptions. First, assume W_0 has no Gaussian component, since otherwise $l_{F_0} = 0$. Next assume that $EW_0 = 0$ since $l_{\text{supp }(W_0 - EW_0)} = l_{\text{supp } W_0}$. If ν_0 has support in $\{1\} \cup \{-1\}$, we are done. So suppose $(-1,1) \cap \text{supp } \nu_0 \neq \emptyset$. Replacing W_0 by $-W_0$, if necessary, assume that supp $\nu_0 \cap (0,1) \neq \emptyset$. Finally, assume that $\nu_0(\{0\}) = 0$.

Decompose ν_0 as $\nu_0 = \nu_0^+ + \nu_0^-$, where $\nu_0^+(A) = \nu(A \cap (0,1])$ and $\nu_0^-(A) = \nu_0(A \cap [-1,0))$. Now write $W_0 = W_0^+ + W_0^-$ where W_0^+ and W_0^- are independent, mean zero, infinitely divisible laws with supports F_0^+ and F_0^- and Lévy measures ν_0^+ and ν_0^-, respectively. Since $\nu_0((0,1)) > 0$, Corollary 2.2 can be invoked to conclude that $l_{F_0} < 1$ if $l_{F_0^-} \leq 1$ and $l_{F_0^+} < 1$. We will focus on F_0^+ since F_0^- can be handled in a similar manner.

Before analyzing $l_{F_0^+}$, several observations are needed. Consider any Lévy measure ν with $\nu([a,b]) < \infty$ where $[a,b]$ is the smallest closed interval containing supp ν and $0 < a < b \leq 1$. The probability measure

$$\mu = \text{Pois } \nu \equiv e^{-|\nu|} \sum_{k=q}^{\infty} \frac{\nu^{*k}}{k!} \quad \text{has} \quad \text{supp } \mu = \bigcup_{k=1}^{\infty} G_k^0$$

where $G_1^0 \equiv \text{supp } \nu \cup \{0\}$ and $G_k^0 \equiv G_{k-1}^0 + G_1^0$ for $k \geq 1$. For each n, there exist $x_n, y_n \in \bigcup_{k=1}^{\infty} G_k^0$ with $x_n < y_n$, $(x_n, y_n) \in \left(\bigcup_{k=1}^{\infty} G_k^0\right)^c$, and $y_n - x_n \geq l_{\cup_{k=1}^{\infty} G_k^0} - \frac{1}{n}$. Because $G_1^0 \subseteq G_2^0 \subseteq \cdots$, there exist k_n such that $x_n,\ y_n \in G_{k_n}^0$ and $(x_n, y_n) \in \left(\bigcup_{j=1}^{\infty} G_j^0\right)^c = \left(\bigcup_{j=k_n}^{\infty} G_j^0\right)^c \subseteq (G_{k_n}^0)^c$. Therefore, $l_{G_{k_n}^0} \geq y_n - x_n$. Consequently,

$$l_{\cup_{k=1}^{\infty} G_k^0} \leq \sup l_{G_n^0} \leq \sup\{\max l_{G_{n-1}^0}, l_{G_1^0}\} = l_{G_1^0} \leq \max\{a, b-a\}.$$

Furthermore, centering μ to create a mean zero law will not change $l_{\text{supp } \mu}$.

Write $\nu_0^+ = \sum_{j=1}^{\infty} \nu_i$ where $\nu_n(A) = \nu(A \cap (2^{-n}, 2^{-n+1}])$ for $n \geq 1$. Let Y_n^+ be independent random variables with $\mathscr{L}(Y_n^+) = \delta_{c_n} * \text{Pois } \nu_n$, where c_n is chosen so that $EY_n^+ = 0$. For all $n \geq 1$, $0 \leq l_{Y_n^+} \leq 1$ with both inequalities strict whenever $\nu_n((0,1)) > 0$. Furthermore, $\nu_n((0,1)) > 0$ for some $n \geq 1$ since $\nu_0((0,1)) > 0$. Since ν_0 has support in a compact set, W_0 has finite variance as do W_0^+ and Y_n^+ for all n. Therefore, Theorem 2.3 implies that $l_{F_0^+} = l_{\text{supp } \sum_{j=1}^{\infty} Y_j^+} < 1$. Analogously, it can be shown that $l_{F_0^-} \leq 1$. Hence $l_{F_0} < 1$, thereby completing the proof. □

3. Results for triangular arrays.

We now extend Theorem 2.4 to triangular arrays.

THEOREM 3.1. *For each $n \geq 1$, let $W_{n1}, W_{n2}, \ldots, W_{nk_n}$ be independent mean zero random variables with supports $F_{n1}, \ldots, F_{nk_n}$, respectively. Suppose that for each such F_{nj} there exists $0 < a_{nj} < 1$ such that*

$$F_{nj} \subseteq [-a_{nj}, 1 - a_{nj}]. \tag{3.1}$$

Suppose also that there exists a random variable S such that

$$\mathscr{L}\left(\sum_{j=1}^{k_n} W_{nj}\right) \to \mathscr{L}(S). \tag{3.2}$$

Let F denote the support of S and $\tilde{F}_K \equiv F \cap [-K, K]$. Then for any $K > 0$, any $0 < \varepsilon < 1$ and any sequence $x_n \in [\varepsilon - 1 + \inf \tilde{F}_K, -\varepsilon + \sup \tilde{F}_K]$, both

$$\liminf_{n\to\infty} P\left(\sum_{j=1}^{k_n} W_{nj} \in (x_n, x_n + 1]\right) > 0 \tag{3.3}$$

and

$$\liminf_{n\to\infty} P\left(\sum_{j=1}^{k_n} W_{nj} \in [x_n, x_n + 1)\right) > 0. \tag{3.4}$$

PROOF. If S has a Gaussian component, then (3.3) and (3.4) are trivial. Hence, it may be assumed that S has no Gaussian component. For ease of exposition, replace each W_{nj} by $-W_{nj}$ if necessary so that it may also be assumed that if S has an infinitely divisible part with Lévy measure ν, then $\nu(0, 1] > 0$. Under this assumption, we prove (3.3). The proof of (3.4) is analogous, requiring no further change in W_{nj}.

We intend to proceed as follows: First, we reorder the variables and obtain a limit distribution for the sums as well as the individual variates. Second, we partition the sums into two subsums, the second of which converges to an infinitely divisible law. Third, we show that if (3.3) is to fail, the limit distribution of the individual terms and partial sums must be discrete with atoms located one unit apart. Furthermore, both endpoints of the limit interval $[x^*, x^*+1]$ of $[x_n, x_n+1]$ may be assumed to be atoms of the limit distribution. Fourth, to obtain an explicit means of relating the events $\{S = x^*\}, \{S = x^* + 1\}$ and $\{\sum_{j=1}^{k_n} W_{nj} \in (x_n, x_n + 1]\}$, we introduce another partition of the W_{nj} for $1 \le j \le k_n$ into two groups whose sums T_{n1} and T_{n2} converge in law to independent, discrete-valued, mean zero random variables T_1 and T_2, each having atoms one unit apart, with $-\infty < y_1 \equiv \operatorname{ess\,inf} T_1 < 0$ and T_2 unbounded above if T_1 is unbounded above. [Note $\mathscr{L}(S) = \mathscr{L}(T_1 + T_2)$.] Finally, with these variates, if (3.3) fails, then there exist $\delta_n \to 0$ such that

$$\lim_{n\to\infty} P(T_{n1} + T_{n2} \in (x_n - \delta_n, x_n], |T_{n1} - y_1| \le \delta_n) = P(S = x^*, T_1 = y_1),$$

which will be shown to be positive. Using a certain representation of T_{n1}, we will connect the occurrence of $\{T_{n1} + T_{n2} \in (x_n - \delta_n, x_n], |T_{n1} - y_1| \le \delta_n\}$ with what amounts to the occurrence of $\{T_{n1} + T_{n2} \in (x_n + 1 - 2\delta, x_n + 1], |T_{n1} - 1 - y_1| \le \delta\}$ for any fixed $0 < \delta \le \frac{1}{2}$, thereby establishing that (3.3) must in fact hold.

Having outlined our method of proof, we are ready to begin. By reindexing, if necessary, it may be supposed that for each n,

$$\operatorname{Var} W_{n1} \geq \operatorname{Var} W_{n2} \geq \cdots \geq \operatorname{Var} W_{nk_n}.$$

Moreover, by adding identically zero variables, if necessary, it may be assumed that $k_1 \leq k_2 \leq \cdots$.

Let

$$S_{nk} = \sum_{j=1}^{k} W_{nj}$$

and

$$v_n^2 = \operatorname{Var} S_{nk_n} = \sum_{j=1}^{k_n} EW_{nj}^2.$$

Fix any $K > 0$, any $0 < \varepsilon < 1$ and any $x_n \in [-1+\varepsilon+\inf \tilde{F}_K, -\varepsilon+\sup \tilde{F}_K]$. By passing to subsequences if necessary and using a diagonal argument, it may be assumed that there exist extended reals x^*, $0 \leq a_j \leq 1$ and $0 \leq v_\infty \leq \infty$, and independent random variables $-a_j \leq W_j \leq 1 - a_j$ such that as $n \to \infty$, $x_n \to x^*$, $v_n \to v_\infty$, $a_{nj} \to a_j$ and $\mathscr{L}(W_{nj}) \to \mathscr{L}(W_j)$. By bounded convergence, $EW_j = \lim_{n\to\infty} EW_{nj} = 0$ and $EW_j^2 = \lim_{n\to\infty} EW_{nj}^2$.

If $v_\infty = \infty$, then by the central limit theorem for triangular arrays, S_{nk_n}/v_n converges in law to a standard normal, which contradicts (3.2). Hence, $v_\infty < \infty$. From the "usual algebra" involving fourth moments of sums of independent random variables and the uniform bounds of the summands, it follows that the fourth moments of S_{nk_n} are uniformly bounded. So by uniform integrability,

$$ES = \lim_{n\to\infty} E \sum_{j=1}^{k_n} W_{nj} = 0$$

and

$$ES^2 = \lim_{n\to\infty} S_{nk_n}^2 = v_\infty^2.$$

For any j^*,

$$E\left(\sum_{j=1}^{j^*} W_j^2\right) = \lim_{n\to\infty} E(S_{nj^*})^2 \leq v_\infty^2.$$

Hence, there exists $\sigma_\infty \leq v_\infty$ such that

$$\sum_{j=1}^{\infty} EW_j^2 = \sigma_\infty^2.$$

Moreover, there exist integers $1 \leq j_1 \leq j_2 \leq \cdots$ with $j_n \to \infty$ such that

$$\mathscr{L}(S_{nj_n}) \to \mathscr{L}\left(\sum_{j=1}^{\infty} W_j\right).$$

Let

$$\bar{S}_n = S_{nk_n} - S_{nj_n} = \sum_{j=j_n+1}^{k_n} W_{nj}.$$

Since $E\bar{S}_n^2 \le v_n^2 \to v_\infty^2 < \infty$, $\{\bar{S}_n\}$ is tight. By another subsequence argument, there exists a random variable W_0, independent of $\{W_j, j \ge 1\}$, such that

$$\mathscr{L}(\bar{S}_n) \to \mathscr{L}(W_0).$$

Therefore,

$$\mathscr{L}\left(\sum_{j=0}^{\infty} W_j\right) = \lim_{n\to\infty} \mathscr{L}(S_{nj_n} + \bar{S}_n) = \mathscr{L}(S).$$

Clearly, $EW_0 = 0$ and $EW_0^2 + \sigma_\infty^2 = v_\infty^2$.

We claim that W_0 is infinitely divisible. This follows from the fact that $\{W_{nj}\colon\ j_n < j \le k_n\}$ are u.a.n.: for any $\varepsilon > 0$,

$$\begin{aligned}\lim_{n\to\infty} \sup_{j>j_n} P(|W_{nj}| > \varepsilon) &\le \lim_{n\to\infty} \sup_{j>j_n} \frac{EW_{nj}^2}{\varepsilon^2} \\ &\le \lim_{n\to\infty} \sup_{j>j_n} \varepsilon^{-2} \sum_{i=1}^{j} \frac{EW_{ni}^2}{j} \\ &\le \lim_{n\to\infty} \frac{v_n^2}{\varepsilon^2 j_n} = 0.\end{aligned}$$

Since $|W_{nj}| \le 1$ for each j and n, the Lévy measure ν of W_0 has its support in $[-1, 1]$. Theorem 2.3 now entails that $l_F \le 1$. By weak convergence,

$$\liminf_{n\to\infty} P\left(\sum_{j=1}^{k_n} W_{nj} \in (x_n, x_n + 1]\right) \ge P(S \in (x^*, x^* + 1)). \tag{3.5}$$

If either $l_F < 1$ or both $l_F = 1$ and S has an atom in (x^*, x^*+1), the probability in the right-hand side of (3.5) is positive and we are done. We may therefore assume that $l_F = 1$ and that S has no atom in $(x^*, x^* + 1)$.

Invoking Theorem 2.3:

(i) $l_{F_j} = 1$ for some $j \ge 0$.
(ii) $l_{F_j} < 1 \Rightarrow l_{F_j} = 0$.
(iii) $l_{F_j} = 0 \Rightarrow P(W_j = 0) = 1$.

For $j \ge 1$, $l_{F_j} = 1$ implies that the mean zero variate W_j concentrates on the two-point set $\{-a_j, 1 - a_j\}$. For $j = 0$, Corollary 2.15 implies that W_0 has no Gaussian component and that its Lévy measure ν has its support in $\{-1\} \cup \{1\}$. Hence, S is discrete with its atoms located exactly one unit apart. Since $x^* \in [-1 + \varepsilon + \inf S, -\varepsilon + \sup S]$ and $(x^*, x^* + 1)$ contains no atom of S, both x^* and $x^* + 1$ are atoms of S.

Let $p_0 = \min\{P(S = x^*), P(S = x^* + 1)\} > 0$. We need to write S_{nk_n} as a sum of two groups of variates. If $W_1 \not\equiv 0$, note that $0 < a_1 < 1$ and let $J_{n1} \equiv \{1\}$ and $J_{n2} = \{2, 3, \dots, k_n\}$. Otherwise, $\mathscr{L}(S) = \mathscr{L}(W_0)$ is the difference of two independent Poisson random variables, the first having parameter $\nu(\{1\}) > 0$ and the second $\nu(\{-1\})$, adjusted by a centering constant to have mean zero. In the latter case ($W_1 \equiv 0$), there exists a partition of $\{1, 2, \dots, k_n\}$ into two disjoint subsets J_{n1} and J_{n2} such that

$$\lim_{n\to\infty} \sum_{j\in J_{n1}} EW_{nj}^2 = \lambda^*,$$

where $\lambda^* = \frac{1}{2}\min\{\nu\{1\}, p_0\}$ and

$$\mathscr{L}\left(\sum_{j\in J_{n1}} W_{nj}\right) \to \mathscr{L}(N_{\lambda^*} - \lambda^*),$$

where N_{λ^*} is Poisson with parameter λ^*.

For $i = 1, 2$, let

$$T_{ni} = \sum_{j\in J_{ni}} W_{nj}.$$

There exist independent T_j such that

$$\mathscr{L}(T_{nj}) \to \mathscr{L}(T_j) \quad \text{and} \quad \mathscr{L}(S) = \mathscr{L}(T_1 + T_2).$$

Each T_j is discrete with its adjacent atoms one unit apart.

Let

$$y_1 = \operatorname{ess\,inf} T_1 = \begin{cases} -a_1, & \text{if } W_1 \not\equiv 0, \\ -\lambda^*, & \text{if } W_1 \equiv 0. \end{cases}$$

The support of T_1 is either $\{y_1, y_1 + 1\}$ or else $\{y_1 + k : k = 0, 1, \dots\}$. When T_1 is unbounded above, then T_2 is also unbounded above.

We assert that

$$P(S = x^*, T_1 = y_1) > 0. \tag{3.6}$$

PROOF OF (3.6). Let $k^* = \min\{k \ge 0 : P(S = x^*, T_1 = y_1 + k) > 0\}$. The set defining k^* is nonempty and $P(T_2 = x^* - y_1 - k^*) > 0$. If $k^* = 0$, we are done. It remains to see that k^* cannot be positive. If $k^* > 0$, then $P(S = x^*, T_1 = y_1 + k^* - 1) = 0$, which implies that $P(T_2 = x^* - y_1 - k^* + 1) = 0$. Since T_2 has atoms one unit apart, $P(T_2 = x^* - y_1 - k^*) > 0$ and $P(T_2 = x^* - y_1 - k^* + 1) = 0$, necessarily $P(T_2 \le x^* - y_1 - k^*) = 1$. Therefore, the support of T_1 must be $\{y_1, y_1 + 1\}$, which implies that k^* cannot be greater than 1. Hence, $k^* = 1$. This produces a contradiction as follows:

$$\begin{aligned} 0 &< P(S = x^* + 1) \\ &= P(S = x^* + 1, T_2 \le x^* - y_1 - k^*) \\ &\le P(T_1 \ge y_1 + k^* + 1) = 0. \end{aligned}$$

Hence, $k^* = 0$ and (3.6) holds. □

To obtain a final contradiction, suppose that

$$\lim_{n\to\infty} P(T_{n1} + T_{n2} \in (x_n, x_n + 1]) = 0. \tag{3.7}$$

Then for all $0 < \delta < 1$,

$$\lim_{n\to\infty} P(T_{n1} + T_{n2} \in (x_n - \delta, x_n]) = P(S = x^*). \tag{3.8}$$

Combining (3.6) and (3.8), there exists $\delta_n > 0$, $\delta_n \downarrow 0$ such that

$$\begin{aligned} &\lim_{n\to\infty} P(T_{n1} + T_{n2} \in (x_n - \delta_n, x_n], |T_{n1} - y_1| \le \delta_n) \\ &\quad = P(S = x^*, T_1 = y_1) > 0. \end{aligned} \tag{3.9}$$

We intend to employ (3.9) to contradict (3.7) by means of a coupling argument. The coupling enables us to transfer information concerning $\{T_{n1}+T_{n2} \in (x_n - \delta_n, x_n]\}$ to $\{T_{n1} + T_{n2} \in (x_n + 1 - 2\delta, x_n + 1]\}$ for any fixed $0 < \delta \le \frac{1}{2}$.

Let

$$y_2 = \begin{cases} 1 - a_1, & \text{if } W_1 \not\equiv 0, \\ 1, & \text{if } W_1 \equiv 0 \end{cases}$$

and take any $0 < y^* < y_2$. Then let

$$\mathscr{Q}_n = \sum_{j \in J_{n1}} I(W_{nj} > y^*).$$

Clearly,

$$\lim_{n\to\infty} P(\mathscr{Q}_n = 0) = \begin{cases} 1 - a_1, & \text{if } W_1 \not\equiv 0, \\ e^{-\lambda^*}, & \text{if } W_1 \equiv 0 \end{cases} \tag{3.10}$$

and

$$\lim_{n\to\infty} P(\mathscr{Q}_n = 1) = \begin{cases} a_1, & \text{if } W_1 \not\equiv 0, \\ \lambda^* e^{-\lambda^*}, & \text{if } W_1 \equiv 0. \end{cases} \tag{3.11}$$

Now introduce mutually independent random variables $\{Y_{ni+}, Y_{ni-}, M_n\colon 1 \le i \le k_n\}$ satisfying

$$\begin{aligned} \mathscr{L}(Y_{ni+}) &= \mathscr{L}(W_{ni} \mid W_{ni} > y^*), \\ \mathscr{L}(Y_{ni-}) &= \mathscr{L}(W_{ni} \mid W_{ni} \le y^*), \\ P(M_n = j) &= P(W_{nj} > y^* \mid \mathscr{Q}_n = 1). \end{aligned}$$

Put $T_{n1-} = \sum_{j\in J_{n1}} Y_{nj-}$ and $D_n = Y_{nM_n+} - Y_{nM_n-}$. Note that $D_n \le 1$, $D_n \xrightarrow{P} 1$ and $T_{n1-} \xrightarrow{P} y_1$. Take any $0 < \delta \le \frac{1}{2}$. By (3.7),

$$\begin{aligned}
0 &= \lim_{n\to\infty} P(T_{n1} + T_{n2} \in (x_n + 1 - 2\delta, x_n + 1]) \\
&\ge \lim_{n\to\infty} P(\mathscr{Q}_n = 1) P(D_n + T_{n1-} + T_{n2} \in (x_n + 1 - 2\delta, x_n + 1]) \\
&\ge \lim_{n\to\infty} P(\mathscr{Q}_n = 1) P(T_{n1-} + T_{n2} \in (x_n - \delta, x_n]) \\
&\quad (\text{since } D_n \le 1 \text{ and } D_n \xrightarrow{P} 1) \\
&= \lim_{n\to\infty} P(\mathscr{Q}_n = 1) P(T_{n1} + T_{n2} \in (x_n - \delta, x_n] \mid \mathscr{Q}_n = 0) \\
&\qquad\qquad (\text{by construction}) \\
&\ge \lim_{n\to\infty} \frac{P(\mathscr{Q}_n = 1)}{P(\mathscr{Q}_n = 0)} P(T_{n1} + T_{n2} \in (x_n - \delta, x_n], |T_{n1} - y_1| \le \delta_n) \\
&> 0 \quad [\text{by (3.9)–(3.11)}].
\end{aligned}$$

This contradiction of (3.7) establishes the theorem. □

4. Proof of Theorem 1.1. We only need to show (1.7). Recall that we may assume $L = \max_{1\le j\le n}(|\operatorname{supp} X_j|) = 1$. Suppose that Theorem 1.1 fails to hold. Then there exists $\varepsilon > 0$, a triangular array of rowwise independent mean zero random variables $W_{n1}, W_{n2}, \dots, W_{nk_n}$ for some integers $1 \le k_1 \le k_2 \le \cdots$ such that, for some $0 < a_{nj} < 1$,

$$P(-a_{nj} \le W_{nj} \le 1 - a_{nj}) = 1$$

and intervals Γ_n, at least half-closed, of length at least 1 such that

$$\Gamma_n \cap [\bar{y}^-_{n,\varepsilon}, \bar{y}^+_{n,\varepsilon}] \ne \varnothing, \tag{4.1}$$

where

$$\begin{aligned}
\bar{y}^-_{n,\varepsilon} &= \inf\left\{y\colon P\left(\sum_{j=1}^{k_n} W_{nj} \le y\right) \ge \varepsilon\right\}, \\
\bar{y}^+_{n,\varepsilon} &= \sup\left\{y\colon P\left(\sum_{j=1}^{k_n} W_{nj} \le y\right) \le 1 - \varepsilon\right\}
\end{aligned} \tag{4.2}$$

and

$$\lim_{n\to\infty} \left(1 \vee \frac{\sqrt{\operatorname{Var}(\sum_{j=1}^{k_n} W_{nj})}}{|\Gamma_n|}\right) P\left(\sum_{j=1}^{k_n} W_{nj} \in \Gamma_n\right) = 0. \tag{4.3}$$

There exist $\gamma_{n1} < \gamma_{n2}$ with $\gamma_{n2} - \gamma_{n1} \ge 1$ such that

$$(\gamma_{n1}, \gamma_{n2}) \subset \Gamma_n \subseteq [\gamma_{n1}, \gamma_{n2}].$$

Without loss of generality it may be assumed that $\gamma_{n2} \in \Gamma_n$ whenever $\gamma_{n2} < \infty$.

By passing to subsequences if necessary it may further be supposed that

$$\sigma_n^2 \equiv \mathrm{Var}\left(\sum_{j=1}^{k_n} W_{nj}\right) \to \sigma_\infty^2$$

for some $0 \le \sigma_\infty \le \infty$.

CASE 1. $\sigma_\infty < \infty$. By Chebyshev's inequality, $\{\sum_{j=1}^{k_n} W_{nj}\}_{n\ge1}$ is a tight sequence of random sums. By passing to subsequences if necessary, it may also be assumed that there is some random variable S of variance σ_∞^2 such that

$$\mathscr{L}\left(\sum_{j=1}^{k_n} W_{nj}\right) \to \mathscr{L}(S).$$

There exists x_n such that $(x_n, x_n+1] \in \Gamma_n$ and $(x_n, x_n+1] \cap [\bar{y}_{n,\varepsilon}^-, \bar{y}_{n,\varepsilon}^+] \ne \varnothing$. Since $\sigma_n \to \sigma_\infty < \infty$, $\bar{y}_{n,\varepsilon}^-$ and $\bar{y}_{n,\varepsilon}^+$ are uniformly bounded by Chebyshev's inequality. Hence, so is x_n. As usual, it may therefore be assumed that x_n converges to some x^* (finite). Let $y_0^- = \operatorname{ess\,inf} S$ and $y_0^+ = \operatorname{ess\,sup} S$. If $y_0^- - 1 < x^* < y_0^+$, then Theorem 3.1 contradicts (4.3). We need to consider two further subcases.

Subcase (i). $x^* \ge y_0^+$. Then $x^* + \frac{1}{2}$ is a point of continuity of the S-distribution. Hence

$$\lim_{n\to\infty} P\left(\sum_{j=1}^{k_n} W_{nj} \ge x_n + \tfrac{1}{2}\right) = P\left(S \ge x^* + \tfrac{1}{2}\right) = 0.$$

Therefore,

$$\begin{aligned} 0 &= \lim_{n\to\infty} P\left(\sum_{j=1}^{k_n} W_{nj} \in \Gamma_n\right) \\ &\ge \limsup_{n\to\infty} P\left(\sum_{j=1}^{k_n} W_{nj} \in (x_n, x_n+1]\right) \\ &= \limsup_{n\to\infty} P\left(\sum_{j=1}^{k_n} W_{nj} > x_n\right) \\ &\ge \varepsilon \end{aligned}$$

by (4.2) since $x_n < \bar{y}_{n,\varepsilon}^+$, which gives a contradiction.

Subcase (ii). $x^* \le y_0^- - 1$. Proceeding as in subcase (i),

$$\begin{aligned} 0 &= \lim_{n\to\infty} P\left(\sum_{j=1}^{k_n} W_{nj} \in \Gamma_n\right) \\ &\ge \limsup_{n\to\infty} P\left(\sum_{j=1}^{k_n} W_{nj} \in (x_n, x_n+1]\right) \end{aligned}$$

$$= \limsup_{n\to\infty} P\left(\sum_{j=1}^{k_n} W_{nj} \le x_n + 1 \right)$$

$$\ge \limsup_{n\to\infty} P\left(\sum_{j=1}^{k_n} W_{nj} \le \bar{y}_{n,\varepsilon}^- \right)$$

$$\ge \varepsilon,$$

yielding the third contradiction.

Hence, the case $\sigma^2 < \infty$ of Theorem 1.1 is established. Moreover, because Theorem 1.1 now holds for $\sigma^2 < \infty$, it follows that for every $0 < v < \infty$ and every $0 < \varepsilon << 1$, there exists $c_{\varepsilon,v} > 0$ such that whenever the conditions of Theorem 1.1 hold with the additional constraint that $\limsup \operatorname{Var} \sum_{j=1}^{k_n} W_{nj} \le v^2$, the conclusion (1.7) of Theorem 1.1 holds with c_ε replaced by $c_{\varepsilon,v}$.

CASE 2. $\sigma_\infty = \infty$. If $\limsup_{n\to\infty}(\gamma_{n2} - \gamma_{n1})/\sigma_n > 0$, then by the central limit theorem, $\limsup_{n\to\infty} P(\sum_{j=1}^{k_n} W_{nj} \in \Gamma_n) > 0$, which contradicts (4.3). Hence, it may be supposed that

$$\limsup_{n\to\infty} \frac{\gamma_{n2} - \gamma_{n1}}{\sigma_n} = 0 \tag{4.4}$$

and

$$\lim_{n\to\infty} \frac{\gamma_{n1}}{\sigma_n} = z \text{ for some } |z| < \infty. \tag{4.5}$$

Writing $(\gamma_{n1}, \gamma_{n2}]$ as the union of $\lceil \gamma_{n2} - \gamma_{n1} \rceil$ intervals of length 1, it follows [from (4.3)] that for some $\gamma_{n1} \le \gamma_{n1}^* \le \gamma_{n2} - 1$,

$$\lim_{n\to\infty} \sigma_n P\left(\sum_{j=1}^{k_n} W_{nj} \in (\gamma_{n1}^*, \gamma_{n1}^* + 1] \right) = 0. \tag{4.6}$$

Let $S_n^{(1)} = \sum_{j=1}^{j_n} W_{nj}$ and $S_n^{(2)} = \sum_{j=j_n+1}^{k_n} W_{nj}$, where j_n is chosen to satisfy

$$v^2 \le \sum_{j=1}^{j_n} EW_{nj}^2 < \tfrac{1}{4} + v^2 \tag{4.7}$$

with $v = 1.3\sqrt{2\pi e^{z^2}}$ and z as in (4.5). Let $\sigma_{ni}^2 = \operatorname{Var} S_n^{(i)}$ for $i = 1, 2$. Notice that $v^2 \le \sigma_{n1}^2 \le \frac{1}{4} + v^2$ and put

$$\tilde{c} = c_{1/8,\sqrt{1/4+v^2}}. \tag{4.8}$$

We need to contradict (4.6), which says that the probability $S_n = S_n^{(1)} + S_n^{(2)}$ is in an interval of length 1 is of lower order than $1/\sigma_n$. A two stage procedure will now be used to home in on intervals of length 1 by intervals of larger length. The idea we employ is based on observing that in order for $S_n = S_n^{(1)} + S_n^{(2)}$ to hit the interval $(\gamma_{n1}^*, \gamma_{n1}^* + 1]$, it suffices first that $S_n^{(2)}$ land in a relatively large interval I_n about γ_{n1}^*, say $I_n = (\gamma_{n1}^* - \sigma_{n1}, \gamma_{n1}^* + \sigma_{n1}]$. Second, regardless of which $\gamma_{n1}^* + y \in I_n$ that $S_n^{(2)}$ happens to equal, if $S_n^{(1)} \in (-y, -y+1]$ (an event whose probability is uniformly bounded away from zero for $-\sigma_{n1} \le -y < \sigma_{n1}$ and $\sigma_{n1} \ge 2$), then S_n will be directed into the desired haven, the interval $(\gamma_{n1}^*, \gamma_{n1}^* + 1]$:

$$
\begin{aligned}
0 &= \lim_{n\to\infty} \sigma_n P\left(\sum_{j=1}^{k_n} W_{nj} \in (\gamma_{n1}^*, \gamma_{n1}^* + 1] \right) \quad \text{[by (4.6)]} \\
&\ge \limsup_{n\to\infty} \sigma_n \int_{(\gamma_{n1}^* - \sigma_{n1}, \gamma_{n1}^* + \sigma_{n1}]} P(S_n^{(1)} \in (-y + \gamma_{n1}^*, -y + \gamma_{n1}^* + 1])\, dP(S_n^{(2)} \le y) \\
&\ge \limsup_{n\to\infty} \sigma_n P(S_n^{(2)} \in (\gamma_{n1}^* - \sigma_{n1}, \gamma_{n1}^* + \sigma_{n1}]) \\
&\qquad \times \inf_{y \in (\gamma_{n1}^* - \sigma_{n1}, \gamma_{n1}^* + \sigma_{n1}]} P(S_n^{(1)} \in (-y + \gamma_{n1}^*, -y + \gamma_{n1}^* + 1]) \\
&\ge \limsup_{n\to\infty} \sigma_n P(S_n^{(2)} \in (\gamma_{n1}^* - \sigma_{n1}, \gamma_{n1}^* + \sigma_{n1}]) \inf_{-\sigma_{n1} \le x < \sigma_{n1}} P(S_n^{(1)} \in (x, x+1]) \\
&\ge \limsup_{n\to\infty} \frac{\tilde{c}\sigma_n}{\sigma_{n1}} P(S_n^{(2)} \in (\gamma_{n1}^* - \sigma_{n1}, \gamma_{n1}^* + \sigma_{n1}]) \quad \text{(by Case 1)} \\
&\ge \limsup_{n\to\infty} \frac{\tilde{c}\sigma_n}{\sigma_{n1}} \left(\frac{2\sigma_{n1}}{\sigma_{n2}\sqrt{2\pi}} e^{-z^2/2} - \frac{1.6}{\sigma_{n2}} \right) \quad \text{(by the Berry–Esseen theorem)} \\
&\ge \limsup_{n\to\infty} \frac{\tilde{c}}{\sigma_{n1}} \left(\frac{2\sigma_{n1}}{\sqrt{2\pi e^{z^2}}} - 1.6 \right) \quad \left(\text{since } \frac{\sigma_n}{\sigma_{n2}} \to 1 \right) \\
&\ge \frac{\tilde{c}}{\sqrt{\frac{1}{4} + v^2}} \quad \left(\text{since } \frac{2\sigma_{n1}}{\sqrt{2\pi e^{z^2}}} \ge \frac{2v}{\sqrt{2\pi e^{z^2}}} \ge 2.6 \text{ and } \sigma_{n1}^2 \le \frac{1}{4} + v^2 \right) \\
&> 0.
\end{aligned}
$$

This contradiction completes the proof of Theorem 1.1. □

REMARK. The first case of the proof of Theorem 1.1 establishes that whenever the conditions of Theorem 1.1 hold and $\limsup \sum_{j=1}^{k_n} W_{nj} \le v^2 < \infty$, then the conclusion (1.7) of Theorem 1.1 holds with c_ε replaced by $c_{\varepsilon,v}$. As v increases, $c_{\varepsilon,v}$ decreases to $c_\varepsilon = c_{\varepsilon,\infty}$ which, by the proof of the second case, is found to be strictly positive. Consequently, for fixed ε, the $c_{\varepsilon,v}$ are uniformly bounded away from 0.

APPENDIX

PROOF OF LEMMA 2.1. We prove (2.2) first. It clearly holds if l_F or $l_G = \infty$ or if l_F or $l_G = 0$. Without loss of generality we may assume $0 < l_F \le l_G < \infty$. For the time being, assume F and G are compact. Let $m = \max\{l_G - l_F, l_F\}$ and take any $z \in [-m + \inf(F+G), \sup(F+G)]$. Clearly if $z \in [-m + \inf(F+$

$G), \inf(F+G)]$ or if $z = \sup(F+G)$, $[z, z+m] \cap (F+G) \neq \phi$, so it may be assumed that $\inf(F+G) < z < \sup(F+G)$. Let

$$z_* = \sup\{x+y\colon\ x \in F, y \in G, x+y \le z\},$$
$$z^* = \inf\{x+y\colon\ x \in F, y \in G \text{ and } x+y \ge z\}.$$

There exist $x_*, x^* \in F$ and $y_*, y^* \in G$ such that $x_* + y_* = z_*$ and $x^* + y^* = z^*$. If $z = z^*$, then $[z, z] \in F+G$. So suppose $z \neq z^*$. Then $z_* < z < z^*$.

CASE 1. $x_* < \sup F$. In this case we find an element of $(F+G) \cap [z, z+l_F]$ by incrementing x_*: There exists $x^{**} \in (x_*, x_*+l_F] \cap F$. Since $x^{**} > x_*$, necessarily $x^{**} + y_* > z$. Therefore, $[z, z+l_F] \cap (F+G) \neq \phi$.

CASE 2. $\inf F < x^* \le x_* = \sup F$. In this case we show $[z, z+l_F] \cap (F+G) \neq \phi$ by shrinking x^*: There exists $x_{**} \in F \cap [x^* - l_F, x^*)$. It follows that

$$x_{**} + y^* < x^* + y^* = z^*$$

and so in fact $x_{**} + y^* < z$. Therefore,

$$z^* - z < x^* - x_{**} \le l_F,$$

whence $[z, z+l_F] \cap (F+G) \neq \phi$.

CASE 3. $\inf F = x^* < x_* = \sup F$. By the definition of l_F, there exist $x', x'' \in F$ such that $x'' = x' + l_F$. Let $F_1 = \{x'\} \cup \{x''\}$. Let

$$z^{**} = \inf\{x+y \ge z\colon\ x \in F_1 \text{ and } y \in G\},$$
$$z_{**} = \sup\{x+y \le z\colon\ x \in F_1 \text{ and } y \in G\}.$$

Note that the sets defining z^{**} and z_{**} are nonempty under the assumptions of Case 3. Also $z_{**} \le z_* < z < z^* \le z^{**}$. There exist $x^{\#}, x_{\#} \in F_1$ and $y^{\#}, y_{\#} \in G$ such that $x_{\#} + y_{\#} = z_{**}$ and $x^{\#} + y^{\#} = z^{**}$. If $x_{\#} \le x^{\#}$, then by Cases 1 and 2 applied to F_1 and G, $z^* - z \le z^{**} - z \le l_F$ and so $[z, z+l_F] \cap (F+G) \neq \phi$. It may therefore be assumed that $x^{\#} = x' < x_{\#} = x''$. Consequently, $y_{\#} < y^{\#}$. Now if $y^{\#} - y_{\#} \le l_G$, then

$$\begin{aligned} z^* - z &< z^{**} - z_{**} \\ &= x^{\#} + y^{\#} - (x_{\#} + y_{\#}) \\ &\le -l_F + l_G \end{aligned}$$

so that $[z, z + (l_G - l_F)] \cap (F+G) \neq \phi$. Finally, if $y^{\#} - y_{\#} > l_G$, let $y' = \inf\{y \in G\colon\ y > y_{\#}\}$ and $y'' = \sup\{y \in G\colon\ y < y^{\#}\}$. Since $y' \le y_{\#} + l_G < y^{\#}$, it follows from the definition of y'' that $y' \leq y''$. Note also that for $y \in G$ with $y > y_{\#}, x_{\#} + y \geq z^{**}$ so $x_{\#} + y' \geq z^{**}$. Similarly, $x^{\#} + y'' \leq z_{**}$. Therefore,

$$\begin{aligned} z^* - z < z^{**} - z_{**} \\ &\leq x_{\#} + y' - (x^{\#} + y'') \\ &\leq x_{\#} - x^{\#} \\ &= l_F \end{aligned}$$

and so again $[z, z + l_F] \cap (F + G) \neq \phi$. This completes the proof of (2.2) if F and G are compact.

If F and G are not both compact, let $F_n = F \cap [-n, n]$ and $G_n = G \cap [-n, n]$. Then

$$\begin{aligned} l_{F+G} &\leq \lim_{n\to\infty} l_{F_n+G_n} \\ &\leq \lim_{n\to\infty} \max\{|l_{F_n} - l_{G_n}|, \min\{l_{F_n}, l_{G_n}\}\} \\ &= \max\{|l_F - l_G|, \min\{l_F, l_G\}\}. \end{aligned}$$

To prove (2.3), assume that $l_F = 0$ and $\operatorname{diam} F = \infty$. Then $F = (-\infty, b]$ or $[a, \infty)$ or $(-\infty, \infty)$. Without loss of generality, it may be assumed that $F = [a, \infty)$. Let $c = \inf G$. Since $F + G = \bigcup_{y \in G}[y + a, \infty)$, it is clear that

$$F + G = \begin{cases} (-\infty, \infty), & \text{if } c = -\infty, \\ [a + c, \infty), & \text{if } c > -\infty \end{cases}$$

and so $l_{F+G} = 0$.

Now suppose $\operatorname{diam} F < \infty$. Then $F = [a, b]$ for some $-\infty < a \leq b < \infty$. Without loss of generality, it may also be assumed that $a < b$ and $l_G < \infty$. Take any $z \in (\inf(F + G), \sup(F + G))$. Without loss of generality, it may also be assumed that $z \notin F + G$. Let $b^* = \sup\{x \leq b\colon x + y = z$ for some $x \in F$ and $y \in [\inf G, \sup G]\}$. Let $d^* = z - b^*$. There exists $y_1 \in G$ such that $0 < y_1 - d^* \leq l_G$. Clearly $a = \inf F \leq b^*$. The interval $[a + y_1, b^* + y_1] \in F + G$, so necessarily $a + y_1 > z$. Hence $a + y_1 - z = b^* + y_1 - z - (b^* - a) \leq l_G - (b^* - a)$. Therefore, if $b^* = b$ we are done. If $b^* < b$, then $d^* = \inf G$ and so, $z = b^* + d^* \in F + G$, a contradiction. Therefore, $[z, z + (l_G - \operatorname{diam} F)^+] \cap (F + G) \neq \phi$. □

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