SINGULARITY OF SUPER-BROWNIAN LOCAL TIME AT A POINT CATALYST

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In a one-dimensional single point-catalytic continuous super-Brownian motion studied by Dawson and Fleischmann, the occupation density measure λ^c at the catalyst's position $\mathfrak{c}$ is shown to be a singular (diffuse) random measure. The source of this qualitative new effect is the irregularity of the varying medium $\delta_{\mathfrak{c}}$ describing the point catalyst. The proof is based on a probabilistic characterization of the law of the Palm canonical clusters χ appearing in the Lévy–Khintchine representation of λ^c in a historical process setting and the fact that these χ have infinite left upper density (with respect to Lebesgue measure) at the Palm time point.

1. Introduction.

1.1. Motivation and main result.
A *one-dimensional point-catalytic continuous super-Brownian motion* $\mathfrak{X} = \{\mathfrak{X}_t; t \geq 0\}$, in which branching is allowed only at a *single point catalyst*, was discussed in some detail in Dawson and Fleischmann (1994) and Fleischmann (1994). This critical measure-valued branching process $\mathfrak{X}$ is associated with mild solutions of the nonlinear equation

$$(1.1.1) \qquad \frac{\partial}{\partial t} u(t,z) = \kappa \Delta u(t,z) - \delta_c(z) u^2(t,z), \qquad t > 0, \ z \in \mathbb{R}.$$

Here $\kappa > 0$ is a diffusion constant, the (one-dimensional) Laplacian Δ acts on the space variable $z \in \mathbb{R}$ and the *branching rate* degenerates to the Dirac δ-function δ_c describing a *point catalyst* situated at $\mathfrak{c} \in \mathbb{R}$. In fact, the *Laplace transition functional*, which determines this Markov process $\mathfrak{X}$, has the form

$$(1.1.2) \qquad \mathbb{E}\{\exp(\mathfrak{X}_t, -\varphi) \mid \mathfrak{X}_s = \mu\} = \exp(\mu, -u_\varphi(t-s)),$$

$0 \leq s \leq t$, $\varphi \in \Phi_+$, $\mu \in \mathcal{M}_f$. Here u_φ is the unique mild solution of (1.1.1) with initial condition $u|_{t=0+} = \varphi$, where φ lies in some set Φ_+ of nonnegative test functions, $\mathcal{M}_f$ is the space of all finite measures on $\mathbb{R}$ and (m, f) is an abbreviation for the integral $\int dm\, f$ of the function f with respect to the measure m.

It was demonstrated that there exists a version of $\mathfrak{X}$ for which the associated *point-catalytic occupation time process*

$$(1.1.3) \qquad\qquad \mathfrak{Y}_t := \int_0^t dr\, \mathfrak{X}_r(\cdot), \qquad t \geq 0,$$

possesses a *jointly continuous occupation density field* $\mathfrak{y} = \{\mathfrak{y}_t(z); \; t \geq 0, z \in \mathbb{R}\}$ with probability 1. Consequently $\mathfrak{Y}_t(dz) = \mathfrak{y}_t(z)\,dz$, $t \geq 0$, $z \in \mathbb{R}$, in spite of the irregularity of the branching rate δ_c [see Theorem 1.2.4 in Dawson and Fleischmann (1994); hereafter, DF]. The *occupation density measures* $\lambda^z(dr) =: d\mathfrak{y}_r(z), z \in \mathbb{R}$, on $\mathbb{R}_+$ (*super-Brownian local time measures*) were shown to be absolutely continuous a.s. provided that z is different from c [see (1.2.5) there]. Moreover, the super-Brownian local time measure λ^c at the catalyst's location has *carrying Hausdorff–Besicovitch dimension* 1 (cf. Theorem 1.2.5).

On the other hand, DF, Theorem 1.2.3, says, roughly speaking, that, at a fixed time, the density of mass *vanishes* stochastically as the catalyst's position is approached. Together these facts suggest that λ^c is a *singular* diffuse random measure in contrast to the well-known absolute continuity in the case of one-dimensional regular branching [cf. Konno and Shiga (1988) or Reimers (1989)].

The heuristic picture is that density of mass arriving at c normally dies instantaneously due to the infinite branching rate at c, but that additionally bursts of creation of absolutely continuous mass occur on an "exceptional" set of times.

The main purpose of this note is to prove that λ^c is in fact singular a.s. Here we restrict our consideration to the case in which $\mathfrak{X}$ starts off at time 0 with a unit mass δ_c at the catalyst's position c:

THEOREM 1.1.4 (Singularity of λ^c). *Assume that $\mathfrak{X}_0 = \delta_c$. The occupation density measure λ^c at the catalyst's position is with probability 1 a singular (diffuse) random measure on $\mathbb{R}_+$.*

1.2. *Methodology.* Our approach to this requires the development of some tools that may be of some independent interest. First, we consider an enriched version of $\mathfrak{X}$, namely, the *historical point-catalytic super-Brownian motion* $\tilde{\mathfrak{X}} := \{\tilde{\mathfrak{X}}_t; \; t \geq 0\}$, which is necessary for our argument. Here the state $\tilde{\mathfrak{X}}_t$ at time t keeps track of the entire history of the population masses alive at t and their "family relationships" and arises as the diffusion limit of the reduced branching tree structure associated with the approximating branching particle system [cf., for instance, Dynkin (1991b) or Dawson and Perkins (1991)].

In this setting, $\lambda^c(dr)$ is replaced by $\tilde{\lambda}^c(d[r, w])$, where $\tilde{\lambda}^c([r_1, r_2] \times B)$ is the contribution to the "occupation density increment" $\lambda^c([r_1, r_2])$ due to paths in the subset B of c-*Brownian bridge paths* w on $[0, r]$, which start at time 0 at c and also end up in c at time r, $r_1 \leq r \leq r_2$. These c-Brownian bridge paths

w on $[0, r]$ can be interpreted as the particles' trajectories that contributed to the occupation density increment $\lambda^c([r_1, r_2])$.

Next, the infinite divisibility of the law of the random measure $\tilde{\lambda}^c$ allows us to use the framework of *Lévy–Khintchine representation*. Moreover, we can disintegrate the corresponding Lévy–Khintchine measure Q with respect to its intensity measure to obtain its *Palm distributions* $Q^{r,w}(d\chi)$. Roughly speaking, $Q^{r,w}(d\chi)$ is the law of a *canonical cluster* χ but "given that it contains the pair" $[r, w]$. Given such an r and a fixed c-Brownian bridge path w on $[0, r]$ we derive a *probabilistic representation* (see Theorem 3.3.9) for the Laplace functional of the Palm distribution $Q^{r,w}$ in terms of the (deterministic) *Brownian local time measure* $L^c(w, dt)$ at c of this given bridge path w and solutions of an historical version of the singular equation

$$(1.2.1) \qquad -\frac{\partial}{\partial r} u = \kappa \Delta u + f \delta_c - \delta_c u^2, \qquad r > 0$$

[see (3.1.4)].

Exploiting this probabilistic representation of $Q^{r,w}$, the key step of the proof of Theorem 1.1.4 is then to demonstrate that the random measure $\chi(d[r', w'])$ distributed according to $Q^{r,w}$ has with probability 1 at the Palm point r an *infinite left upper density* with respect to the Lebesgue measure dr' (Theorem 4.2.2). This is then shown to imply that the original super-Brownian local time measure λ^c at c cannot have an absolutely continuous component.

A different approach to the singularity result is given in Fleischmann and LeGall (1994).

1.3. *Outline.* In Section 2 the point-catalytic super-Brownian motion $\mathfrak{X}$, its occupation time process $\mathfrak{Y}$ and the occupation density measures λ^z are adapted to the framework of historical processes. Based on this, the Palm representation formula is derived in the following section. The singularity proof then follows in Section 4.

Our general reference for standard facts on random measures is Kallenberg (1983), Chapters 6 and 10, and Kerstan, Matthes and Mecke (1982), Chapters 1, 3 and 11; for Brownian motion and in particular for Brownian local time, we refer to Revuz and Yor (1991), Chapters 6 and 8.

2. Occupation Density Measures in the Setting of Historical Processes.
The purpose of this section is to sketch how the point-catalytic super-Brownian motion $\mathfrak{X}$ and related objects of interest in the present paper can be fitted into the general framework of historical processes. For the latter, we refer, for instance, to Dynkin (1991b) and Dawson and Perkins (1991).

2.1. *Preliminaries: Terminology.* We start by introducing some terminology we will use. If A, B are sets and $a \mapsto B^a$ is a map of A into the set of all

subsets of B, then we often write

$$(2.1.1) \qquad A \mathbin{\hat{\times}} B^{\bullet} := \{[a,b]; \ a \in A, b \in B^a\} = \bigcup_{a \in A} \{a\} \times B^a$$

for the *graph* of this map. Note that $A \mathbin{\hat{\times}} B^{\bullet} \subseteq A \times B$.

We adopt the following conventions. If E is a topological space, then subsets of E will always be equipped with the subset topology. Products of topological spaces will be endowed with the product topology. Measures on a topological space E will be defined on the Borel σ-algebra $\mathscr{B}(E)$ (generated by the open subsets of E). A measure m on E with $m(E \setminus E') = 0$ for some $E' \in \mathscr{B}(E)$, that is, if m is concentrated on E', will also be regarded as a measure on E' (and conversely).

If E_1 is a topological space and E_2 a normed space with norm $\| \cdot \|$, let $\mathbf{B}[E_1, E_2]$ denote the space of all *bounded measurable* maps $f \colon E_1 \mapsto E_2$ equipped with the supremum norm $\|f\|_\infty := \sup\{\|f(e_1)\|; \ e_1 \in E_1\}$, $f \in \mathbf{C}[E_1, E_2]$, of *uniform convergence*. Note that $\mathbf{B}[E_1, E_2]$ is a Banach space if E_2 is. By $\mathbf{C}[E_1, E_2]$ we denote the subspace of all *continuous* (bounded) f in $\mathbf{B}[E_1, E_2]$.

Once and for all we *fix a closed finite time interval* $I := [0, T]$, $0 < T < \infty$. We write $\mathbb{C}$ for the Banach space $\mathbf{C}[I, \mathbb{R}]$ of all real-valued continuous paths on I. The lower index $+$ on the symbol of a space will always refer to the subset of all of its nonnegative members.

2.2. Brownian paths W and Brownian path processes $\widetilde{W}$.

In this subsection we provide some preliminaries on historical processes starting at the level of a *one-particle motion* (underlying Brownian motion process).

For $t \in I = [0, T]$ and a path $w \in \mathbb{C} = \mathbf{C}[I, \mathbb{R}]$, write $\tilde{w}_t := \{w_{s \wedge t}; \ s \in I\}$ for the corresponding path in $\mathbb{C}$ stopped at time t, and $\mathbb{C}^t$ for the set of all those *stopped paths*. Note that $\mathbb{C}^t$ is a *closed* subspace of the Banach space $\mathbb{C}$, that $\mathbb{C}^s \subseteq \mathbb{C}^t$ for $s \leq t$, that $\mathbb{C}^T = \mathbb{C}$ and that $\mathbb{C}^0$ can be identified with $\mathbb{R}$. Given a path $w \in \mathbb{C}$, we interpret $\tilde{w} := \{\tilde{w}; \ t \in I\}$ as a *path trajectory*. Because, for $s \leq t$,

$$(2.2.1) \qquad \|\tilde{w}_t - \tilde{w}_s\|_\infty = \sup_{s \leq r \leq t} |w_r - w_s| \xrightarrow[t-s \to 0]{} 0,$$

$\tilde{w}$ belongs to the Banach space $\mathbf{C}[I, \mathbb{C}]$, hence, to its *closed* subspace

$$(2.2.2) \qquad \hat{\mathbf{C}}[I, \mathbb{C}] := \{\omega \in \mathbf{C}[I, \mathbb{C}]; \ \omega_t \in \mathbb{C}^t \ \forall t \in I\}.$$

Moreover,

$$(2.2.3) \qquad \|\tilde{w}' - \tilde{w}''\|_\infty = \|w' - w''\|_\infty, \qquad w', w'' \in \mathbb{C},$$

implies that $w \mapsto \tilde{w}$ maps $\mathbb{C}$ continuously into $\hat{\mathbf{C}}[I, \mathbb{C}]$. Note, the graph $\{[s, \omega_s]; \ s \in I\}$ of $\omega \in \hat{\mathbf{C}}[I, \mathbb{C}]$ is a subset of $I \mathbin{\hat{\times}} \mathbb{C}^{\bullet}$ [recall (2.1.1)] and that $I' \mathbin{\hat{\times}} \mathbb{C}^{\bullet}$ is a *closed* subset of $I \times \mathbb{C}$ provided that I' is a closed subinterval of I.

For each $z \in \mathbb{R}$, denote by Π_z the law on $\mathbb{C}$ of a *Brownian path* W with diffusion constant κ and starting at $W_0 = z$ (we normalize in such a way that $\kappa \Delta$ is the generator of this strong Markov process). Given the Brownian path W we denote its continuous *Brownian local time measure* at the catalyst's position c by $L^c(W, dt)$ and note that formally

$$(2.2.4) \qquad L^c(W, dt) = \delta_c(W_t)\, dt.$$

Using the continuous map $w \mapsto \tilde{w}$ of $\mathbb{C}$ into $\hat{\mathbf{C}}[I, \mathbb{C}]$ introduced previously, we arrive at the so-called *Brownian path process* $\widetilde{W} = [\widetilde{W}, L^c, \tilde{\Pi}_{s,w}, s \in I, w \in \mathbb{C}^s]$ on I, which is a time-inhomogeneous strong Markov process realized in the Banach space $\hat{\mathbf{C}}[I, \mathbb{C}]$ [defined in (2.2.2)]. Note that we adjoined the additive functional $L^c = L^c(W)$. This makes sense because given the Brownian path process $\widetilde{W}$, we can recover the Brownian path W itself by *projection*: $W_t = (\widetilde{W}_t)_t$, $t \in I$.

The *semigroup* of $\widetilde{W}$ will be denoted by

$$(2.2.5) \quad \tilde{S}_{s,t}\varphi(w) := \tilde{\Pi}_{s,w}\varphi(\widetilde{W}_t), \qquad 0 \le s \le t \le T, \ w \in \mathbb{C}^s, \ \varphi \in \mathbf{B}[\mathbb{C}, \mathbb{R}],$$

and the related *generator* by $\{\tilde{A}_s; \ s \in I\}$. (Note that as a rule occurrences of the capital letter W refer to random objects, whereas related lowercase letters denote fixed elements in path spaces such as $\mathbb{C}$ or $\mathbf{C}[I, \mathbb{C}]$.)

In subsequent text we will use the continuous (nonnegative) additive functional $L^c(W)$ as a clock to govern the branching of a particle whose motion is described by W. Actually, L^c will serve as a probabilistic refinement of the rough characteristics of the super-Brownian motion $\mathfrak{X}$, which is provided by the branching rate δ_c. More precisely L^c will be exploited for the description of the historical point-catalytic super-Brownian motion $\tilde{\mathfrak{X}}$ we will deal with in the next subsection (Dynkin's additive functional approach).

2.3. Historical point-catalytic super-Brownian motion $\tilde{\mathfrak{X}}$.

Let $\mathcal{M}_f^I$ denote the set of all finite (nonnegative) measures μ defined on $\mathbb{C} = \mathbf{C}[I, \mathbb{R}]$, equipped with the topology of *weak convergence*. Write $\|\mu\|$ for the *total mass* $\mu(\mathbb{C})$ of a measure $\mu \in \mathcal{M}_f^I$. Introduce the closed subsets $\mathcal{M}_f^t := \{\mu \in \mathcal{M}_f^I; \ \mu(\mathbb{C}\backslash\mathbb{C}^t) = 0\}$ of measures on paths stopped at time $t \in I = [0, T]$. Note that $\mathcal{M}_f^T = \mathcal{M}_f^I$.

Now we are in a position to formulate as a proposition the existence and characterization of the *historical point-catalytic super-Brownian motion* $\tilde{\mathfrak{X}}$ related to the super-Brownian motion $\tilde{\mathfrak{X}}$, which was mentioned in the introduction.

PROPOSITION 2.3.1 (Historical point-catalytic super-Brownian motion $\tilde{\mathfrak{X}}$).
There exists a time-inhomogeneous right Markov process $\tilde{\mathfrak{X}} = [\tilde{\mathfrak{X}}, \tilde{\mathbb{P}}_{s,\mu}, s \in I, \mu \in \mathcal{M}_f^s]$ *with states* $\tilde{\mathfrak{X}}_t \in \mathcal{M}_f^t, t \in I$, *having the Laplace transition functional*

$$(2.3.2) \qquad \tilde{\mathbb{P}}_{s,\mu}\exp(\tilde{\mathfrak{X}}_t, -\varphi) = \exp(\mu, -u_\varphi(s, \cdot, t)),$$

$0 \le s \le t \le T, \mu \in \mathscr{M}_f^s, \varphi \in \mathbf{B}_+[\mathbb{C}^t, \mathbb{R}]$, *where* $u_\varphi(\cdot, \cdot, t)$ *is the unique* $\mathscr{B}([0, t] \hat{\times}$
$\mathbb{C}^\cdot)$-*measurable [recall (2.1.1)] bounded nonnegative solution of the nonlinear integral equation*

$$(2.3.3) \qquad u_\varphi(s, w, t) = \tilde{\Pi}_{s,w}\left[\varphi(\widetilde{W}_t) - \int_s^t L^c(W, dr)\, u_\varphi^2(r, \widetilde{W}_r, t)\right],$$

$0 \le s \le t, w \in \mathbb{C}^s$, *or symbolically*,

$$(2.3.4) \quad -\frac{\partial}{\partial s}u_\varphi(s, w, t) = \tilde{A}_s u_\varphi(s, w, t) - \delta_c(w_s)\, u_\varphi^2(s, w, t), \quad 0 \le s \le t,\ w \in \mathbb{C}^s,$$

with terminal condition $u_\varphi(t, \cdot, t) = \varphi$. *The following expectation and variance formulas hold:*

$$(2.3.5) \qquad\qquad \tilde{\mathbb{P}}_{s,\mu}(\tilde{\mathscr{X}}_t, \varphi) = \int \mu(dw)\ \tilde{\Pi}_{s,w}\varphi(\widetilde{W}_t),$$

$$(2.3.6) \quad \tilde{\mathbb{V}}\mathrm{ar}_{s,\mu}(\tilde{\mathscr{X}}_t, \varphi) = 2\int \mu(dw)\ \tilde{\Pi}_{s,w}\int_s^t L^c(W, dr)[\tilde{\Pi}_{r,\widetilde{W}_r}\varphi(\widetilde{W}_t)]^2,$$

$0 \le s \le t \le T, \mu \in \mathbf{B}[\mathbb{C}^t, \mathbb{R}]$.

Consequently, the historical population $\tilde{\mathscr{X}}_t$ at time t is now a measure on paths stopped at time t, which, in contrast to $\mathscr{X}_t$, includes information on which routes in space the masses present at time t had followed up to t. Of course, by the *projection* $\tilde{\mathscr{X}}_t(\{w \in \mathbb{C}^t;\ w_t \in \cdot\}) = \mathscr{X}_t$, from $\tilde{\mathscr{X}}$ we can deduce the super-Brownian motion $\mathscr{X}$ (but not its path continuity established in DF).

Note that (2.3.4) *formally* follows from (2.3.3) by using the representation (2.2.4) of the Brownian local time measure $L^c(W, dr)$, by writing (2.3.3) in terms of the semigroup of the Markov process $\widetilde{W}$ [recall (2.2.5)], and then by a formal differentiation to the time variable s; compare with (1.1.1), where we could use a forward formulation because of time-homogeneity.

REMARKS ON THE PROOF OF PROPOSITION 2.3.1. The construction of the process is ·essentially a special case of the general construction of Dynkin [see, e.g., Dynkin (1991a, b)]. In our case the Brownian path process $\widetilde{W}$ serves as the underlying right Markov process ξ (motion component), the Brownian local time measure $L^c(W, dt)$ at the catalyst's position as the continuous additive functional K of ξ and the branching mechanism ψ is specialized to

$$\psi(u)(s, w) := u^2(s, w), \qquad 0 \le s \le T,\ w \in \mathbb{C}^s,\ u \in \mathbf{B}[I \times \mathbb{C}, \mathbb{R}_+].$$

The right property (and consequently right continuity and strong Markov property) of $\tilde{\mathscr{X}}$ follows from the arguments in Dynkin (1993), Theorem 2.1, together with Section 1.6.

2.4. *Historical point-catalytic occupation time measures $\tilde{\mathfrak{Y}}$.* In this subsection we introduce an historical analog of the point-catalytic occupation time process $\mathfrak{Y}$ of (1.1.3). For this purpose, for fixed $0 \leq s \leq t \leq T$, let

$$(2.4.1) \qquad (\tilde{\mathfrak{Y}}_{s,t}, \psi) := \int_s^t dr (\tilde{\mathfrak{X}}_r, \psi(r, \cdot)), \qquad \psi \in \mathbf{B}_+[[0,t] \hat{\times} \mathbb{C}^{\bullet}, \mathbb{R}],$$

getting a finite random measure $\tilde{\mathfrak{Y}}_{s,t}$ defined on $[s,t] \hat{\times} \mathbb{C}^{\bullet}$. In fact, by the Proposition 2.3.1, $\tilde{\mathfrak{X}}$ is right continuous and hence the integral in (2.4.1) makes sense; see also Theorem 3.1 in Dynkin (1993).

We call $\tilde{\mathfrak{Y}} := \{\tilde{\mathfrak{Y}}_{s,t}; \ 0 \leq s \leq t \leq T\}$ the *historical point-catalytic occupation time process related to* $\tilde{\mathfrak{X}}$. By standard arguments [see also Theorem 1.2 in Dynkin (1991b)] we get the following proposition.

PROPOSITION 2.4.2 (historical point-catalytic occupation time measures $\tilde{\mathfrak{Y}}$).
The Laplace transition functional of $[\tilde{\mathfrak{X}}, \tilde{\mathfrak{Y}}]$ *is given by*

$$(2.4.3) \qquad \tilde{\mathbb{P}}_{s,\mu} \exp[-(\tilde{\mathfrak{X}}_t, \varphi) - (\tilde{\mathfrak{Y}}_{s,t}, \psi)] = \exp(\mu, -u_{\varphi,\psi}(s, \cdot, t)),$$

$0 \leq s \leq t \leq T, \mu \in \mathscr{M}_f^s, \varphi \in \mathbf{B}_+[\mathbb{C}^t, \mathbb{R}], \psi \in \mathbf{B}_+[[0,t] \hat{\times} \mathbb{C}^{\bullet}, \mathbb{R}]$, *where* $u_{\varphi,\psi}(\cdot, \cdot, t)$ *is the unique* $\mathscr{B}([0,t] \hat{\times} \mathbb{C}^{\bullet})$-*measurable bounded nonnegative solution of the nonlinear integral equation*

$$(2.4.4) \quad \begin{aligned} u_{\varphi,\psi}&(s, w, t) \\ &= \tilde{\Pi}_{s,w}\Big[\varphi(\widetilde{W}_t) + \int_s^t dr\, \psi(r, \widetilde{W}_r) - \int_s^t L^c(W, dr) u_{\varphi,\psi}^2(r, \widetilde{W}_r, t)\Big], \end{aligned}$$

$0 \leq s \leq t, \ w \in \mathbb{C}^s$, *or, more formally,*

$$(2.4.5) \qquad -\frac{\partial}{\partial s} u_{\varphi,\psi}(s, w, t) = \tilde{A}_s u_{\varphi,\psi}(s, w, t) + \psi(s, w) - \delta_c(w_s) u_{\varphi,\psi}^2(s, w, t),$$

with terminal condition $u_{\varphi,\psi}(t, \cdot, t) = \varphi$. *The following expectation formula holds:*

$$(2.4.6) \qquad \tilde{\mathbb{P}}_{s,\mu}(\tilde{\mathfrak{Y}}_{s,t}, \psi) = \int \mu(dw) \int_s^t dr\, \tilde{\Pi}_{s,w} \psi(r, \widetilde{W}_r),$$

$0 \leq s \leq t \leq T, \ \mu \in \mathscr{M}_f^s, \ \psi \in \mathbf{B}_+[[0,t] \hat{\times} \mathbb{C}^{\bullet}, \mathbb{R}]$.

$\tilde{\mathfrak{Y}}_{s,t}(d[r, w])$ measures the increment of the historical process $\tilde{\mathfrak{X}}$ at time r at the path w stopped at time r. Note that again by projection, $\tilde{\mathfrak{Y}}_{0,t}([0,t] \times \{w \in \mathbb{C}^t; \ w_t \in \cdot\}) = \mathfrak{Y}_t$, from $\tilde{\mathfrak{Y}}$ we can deduce the point-catalytic occupation time process $\mathfrak{Y}$ related to $\mathfrak{X}$.

2.5. *Historical super-Brownian local time measures.* The next step in our program is to turn to the *historical super-Brownian local time measures* (*historical occupation density measures*) $\tilde{\lambda}^z_{s,t}$, which can loosely be defined by

$$(2.5.1) \qquad (\tilde{\lambda}^z_{s,t}, \psi) := \int \tilde{\mathcal{Y}}_{s,t}(d[r,w])\, \psi(r,w)\, \delta_z(W_r),$$

$z \in \mathbb{R}$, $0 \le s \le t \le T$, $\psi \in \mathbf{B}_+[[s,t]\,\hat{\times}\,\mathbb{C}^\bullet, \mathbb{R}]$. Roughly speaking, $\tilde{\lambda}^z_{s,t}(d[r,w])$ measures the increment of the *density* of those paths w concerning the historical process $\tilde{\mathcal{X}}$ that at time r are stopped at z.

To be more precise, we introduce the set

$$(2.5.2) \qquad \mathbb{C}^{t,z} := \{w \in \mathbb{C}^t, w_t = z\}, \qquad t \in I,\ z \in \mathbb{R},$$

of continuous paths on I stopped at time t at z. Note that $I' \,\hat{\times}\, \mathbb{C}^{\bullet,z}$ (for any $z \in \mathbb{R}$) is a *closed* subset of $I \times \mathbb{C}$ if I' is a closed subinterval of I. Now we are ready to rigorously define the historical occupation density measures.

PROPOSITION 2.5.3 (Historical super–Brownian local time measures $\tilde{\lambda}^z_{s,t}$). *Fix* $z \in \mathbb{R}$, $0 \le s \le t \le T$ *and* $\mu \in \mathcal{M}^s_f$. *Then there is a finite random measure* $\tilde{\lambda}^z_{s,t}$ *defined on* $[s,t]\,\hat{\times}\,\mathbb{C}^{\bullet,z}$ *[recall (2.5.2) and (2.1.1)] having Laplace functional*

$$(2.5.4) \quad \tilde{\mathbb{P}}_{s,\mu} \exp(\tilde{\lambda}^z_{s,t}, -\psi) = \exp(\mu, -u_{\psi,z}(s,\cdot,t)), \qquad \psi \in \mathbf{B}[[0,t]\,\hat{\times}\,\mathbb{C}^{\bullet,z}, \mathbb{R}],$$

where $u_{\psi,z}(\cdot,\cdot,t)$ *is the unique* $\mathscr{B}([0,t]\,\hat{\times}\,\mathbb{C}^\bullet)$-*measurable bounded nonnegative solution of the nonlinear integral equation*

$$(2.5.5) \quad \begin{aligned} &u_{\psi,z}(s,w,t) \\ &= \tilde{\Pi}_{s,w}\Big[\int_s^t L^z(W,dr)\, \psi(r,\widetilde{W}_r) - \int_s^t L^c(W,dr)\, u^2_{\psi,z}(r,\widetilde{W}_r,t) \Big], \end{aligned}$$

$0 \le s \le t \le T$, $w \in \mathbb{C}^s$, *or, more formally,*

$$(2.5.6) \quad -\frac{\partial}{\partial s} u_{\psi,z}(s,w,t) = \tilde{A}_s u_{\psi,z}(s,w,t) + \delta_z(w_s)\psi(s,w) - \delta_c(w_s)u^2_{\psi,z}(s,w,t),$$

with terminal condition $u_{\psi,z}(t,\cdot,t) = 0$. *The random measure* $\tilde{\lambda}^z_{s,t}$ *is* $\tilde{\mathbb{P}}_{s,\mu}$-*a.s. diffuse, that is, it does not carry mass at any single point set. Finally, the following expectation formula holds:*

$$(2.5.7) \qquad \tilde{\mathbb{P}}_{s,\mu}(\tilde{\lambda}^z_{s,t}, \psi) = \int \mu(dw)\, \tilde{\Pi}_{s,w}\Big[\int_s^t L^z(W,dr)\, \psi(r,\widetilde{W}_r) \Big],$$

$\psi \in \mathbf{B}_+[[0,t]\,\hat{\times}\,\mathbb{C}^{\bullet,z}, \mathbb{R}]$.

Note that according to this proposition $\tilde{\lambda}^z_{s,t}$ is defined only for fixed z, s, t (and not as a family of random measures on a common probability space). This is

sufficient for the purpose of proving Theorem 1.1.4. In fact, the original super-Brownian local time measures λ^z restricted to the interval $[s, t]$ and then denoted by $\lambda^z_{s,t}$ coincide in law with the marginal of $\tilde{\lambda}^z_{s,t}$:

$$(2.5.8) \qquad \lambda^z_{s,t}(B) \stackrel{\mathscr{L}}{=} \tilde{\lambda}^z_{s,t}(B \times \mathbb{C}), \qquad z \in \mathbb{R}, \quad B \in \mathscr{B}(I), \quad 0 \le s \le t \le T.$$

REMARKS ON THE PROOF OF PROPOSITION 2.5.3. As in the nonhistorical case, one can derive the Laplace functional of $\tilde{\lambda}^z_{s,t}$ by approximating the δ-function δ_z in the heuristic expression (2.5.1) by the smooth functions $p(\varepsilon, \cdot - z)$, $\varepsilon \to 0$. Here p denotes the continuous Brownian transition density function (with generator $\kappa\Delta$):

$$(2.5.9) \qquad p(r, y) := (4\pi\kappa r)^{-1/2} \exp[-y^2/4\kappa r], \qquad r > 0, \quad y \in \mathbb{R}.$$

Because the original super-Brownian local time measures λ^z are a.s. diffuse (Theorem 1.2.4 in DF), the a.s. diffuseness of $\tilde{\lambda}^z_{s,t}$ follows from (2.5.8) by contradiction. The derivation of the expectation formula is standard.

Note that the laws of the random measures $\tilde{\lambda}^z_{s,t}$ are *infinitely divisible;* in fact, in the representation (2.5.4) pass from μ to μ/n, $n \ge 1$.

3. A Palm representation formula.

The purpose in this section is to derive a *probabilistic representation* of the Laplace functional concerning the Palm canonical clusters in the Lévy–Khintchine representation of the infinitely divisible random measure $\tilde{\lambda}^c_{s,T}$ (see Theorem 3.3.9).

3.1. *Specialization.* *From now on we assume* in this section that, for $s \in I = [0, T]$, the starting measure $\mu \in \mathscr{M}^s_f$ of $\tilde{\mathcal{X}}$ at time s is a unit measure $\delta_{\mathbf{w}}$ concentrated at $\mathbf{w} \in \mathbb{C}^{s,c}$, a path, stopped at time s at the catalyst; recall (2.5.2). (Note that we reserve the boldface letter $\mathbf{w}$, for a starting path.) For convenience, we write $\tilde{\mathbb{P}}_{s,\mathbf{w}}$ instead of $\tilde{\mathbb{P}}_{s,\mu}$. Also, let

$$(3.1.1) \qquad \mathbb{C}^{r,c}_{s,\mathbf{w}} := \{w \in \mathbb{C}^{r,c}; \; w \text{ is a continuous extension of } \mathbf{w}\},$$

$0 \le s \le t \le T$, $\mathbf{w} \in \mathbb{C}^{s,c}$, denote the set of all c-*bridge paths* w on $[s, r]$, that is, bridge paths on $[s, r]$.that start and end at the catalyst's position c. (The path w before time s is determined by $\mathbf{w}$, whereas after time r it is constantly c, but the behavior of w outside $[s, r]$ is not relevant in most cases.) Note that the subsets $[s, T] \hat{\times} \mathbb{C}^{\cdot,c}_{s,\mathbf{w}}$ of $I \times \mathbb{C}$ are *closed.*

By Proposition 2.5.3, with respect to $\tilde{\mathbb{P}}_{s,\mathbf{w}}$, the finite random measures $\tilde{\lambda}^c_{s,T}$ are concentrated on $[s, T] \hat{\times} \mathbb{C}^{\cdot,c}_{s,\mathbf{w}}$ and have Laplace functionals

$$(3.1.2) \qquad \tilde{\mathbb{P}}_{s,\mathbf{w}} \exp(\tilde{\lambda}^c_{s,T}, -\psi) = \exp[-u_{\psi,c}(s, \mathbf{w}, T)],$$

$s \in I$, $\mathbf{w} \in \mathbb{C}^{s,c}$, $\psi \in \mathbf{B}_+[[0, T] \hat{\times} \mathbb{C}^{\cdot,c}, \mathbb{R}]$. Here $u_{\psi,c}(\cdot, \cdot, T)$ is the unique bounded nonnegative solution of the nonlinear integral equation

$$(3.1.3) \qquad u_{\psi,c}(s, w, T) = \tilde{\Pi}_{s,w} \int_s^T L^c(W, dr)[\psi(r, \widetilde{W}_r) - u^2_{\psi,c}(r, \widetilde{W}_r, T)],$$

$s \in I$, $w \in \mathbb{C}^{s,c}$, or, as a formal shorthand,

$$(3.1.4) \quad -\frac{\partial}{\partial s}u_{\psi,c}(s,w,T) = \tilde{A}_s u_{\psi,c}(s,w,T)$$
$$+ \delta_c(w_s)\psi(s,w) - \delta_c(w_s)u_{\psi,c}^2(s,w,T),$$

$s \in I$, $w \in \mathbb{C}^{s,c}$, with terminal condition $u_{\psi,c}(T,\cdot,T) = 0$.

3.2. *Vanishing deterministic part of* $\tilde{\lambda}_{s,T}^c$. In this subsection we show that the historical super-Brownian local time measures $\tilde{\lambda}_{s,T}^c$ do *not* contain a deterministic part.

LEMMA 3.2.1 (Vanishing deterministic component of $\tilde{\lambda}_{s,T}^c$). *Fix* $s \in I$ *and* $w \in \mathbb{C}^{s,c}$. *If* $\tilde{\mathbb{P}}_{s,w}(\tilde{\lambda}_{s,T}^c \geq \nu) = 1$ *for some deterministic measure* ν *on* $I \times \mathbb{C}$, *then* $\nu = 0$.

PROOF. Take ν as in the assumption of the lemma, and suppose that $m := \nu(I \times \mathbb{C}) > 0$. Hence, $\tilde{\lambda}_{s,T}^c(I \times \mathbb{C}) \geq m$ with $\tilde{\mathbb{P}}_{s,w}$-probability 1. Therefore, by the representation (3.1.2) with $\psi \equiv \theta$, $\theta > 0$,

$$(3.2.2) \quad \exp[-\theta m] \geq \tilde{\mathbb{P}}_{s,w}\exp[-\theta\tilde{\lambda}_{s,T}^c(I \times \mathbb{C})] = \exp[-u_\theta(T-s)],$$

where, by (3.1.3), $u_\theta \geq 0$ solves the simplified forward equation

$$(3.2.3) \quad u_\theta(t) = \theta\int_0^t dr\, p(r,0) - \int_0^t dr\, p(t-r,0)\, u_\theta^2(r), \qquad t \geq 0,$$

related to the formal equation

$$(3.2.4) \quad \frac{\partial}{\partial t}u_\theta = \kappa\Delta u_\theta + \theta\delta_c - \delta_c\, u_\theta^2, \qquad t \geq 0, \quad u_\theta\big|_{t=0+} = 0$$

(for convenience, by time-homogeneity, we switched to a *forward* setting). In the next lemma, we will show that $u_\theta(t) \sim \sqrt{\theta}$ as $\theta \to \infty$ holds, for fixed $t > 0$. Then from (3.2.2) we conclude $\theta m \leq u_\theta(T-s) \sim \sqrt{\theta}$ as $\theta \to \infty$, for fixed $s < T$, which is an obvious contradiction, because $m > 0$ by assumption. On the other hand, if $s = T$, then use $u_\theta(0) = 0$ to again derive a contradiction using (3.2.2). The proof will be finished after verifying the following lemma. $\square$

LEMMA 3.2.5. *The (nonnegative) solutions to (3.2.3) satisfy* $u_\theta(t) \sim \sqrt{\theta}$ *as* $\theta \to \infty$, *for each fixed* $t > 0$.

PROOF. Setting $\nu_\theta(s) := \theta^{-1/2}u_\theta(\theta^{-1}s)$, $s > 0$, from (3.2.3) we get

$$\nu_\theta(s) = \int_0^s dr\, p(r,0) - \int_0^s dr\, p(s-r,0)\, \nu_\theta^2(r).$$

Hence, by uniqueness, $v_\theta = u_1$. Therefore, $u_\theta(t) = \theta^{1/2} u_1(\theta t)$ holds. However, by DF, Lemma 4.3.1(ii) (where $\mathfrak{c} = 0$ without loss of generality), we have $u_1(s) \uparrow 1$ as $s \uparrow \infty$. Then the claim follows because $t > 0$ by assumption. $\square$

3.3. *The Palm representation formula.*

Recall that each historical occupation density measure $\tilde{\lambda}^{\mathfrak{c}}_{s,t}$ has an *infinitely divisible distribution*. Therefore, we can apply the so-called *Lévy–Khintchine representation*, which, for convenience we state in the following lemma (where, by Lemma 3.2.1, the deterministic component is dropped).

LEMMA 3.3.1 (Lévy–Khintchine representation). *Fix $s \in I$ and $\mathbf{w} \in \mathbb{C}^{s,\mathfrak{c}}$. There is a uniquely determined σ-finite measure $Q_{s,\mathbf{w}}$ defined on the set of all nonvanishing finite measures χ on $[s,T] \,\hat{\times}\, \mathbb{C}^{\bullet;\mathfrak{c}}_{s,\mathbf{w}}$ [recall (3.1.1)] such that*

$$(3.3.2) \qquad \tilde{\mathbb{P}}_{s,\mathbf{w}} \exp(\tilde{\lambda}^{\mathfrak{c}}_{s,T}, -\psi) = \exp\left[-\int Q_{s,\mathbf{w}}(d\chi)(1 - \exp(\chi, -\psi)) \right],$$

$\psi \in \mathbf{B}_+[[s,T] \,\hat{\times}\, \mathbb{C}^{\bullet;\mathfrak{c}}_{s,\mathbf{w}}, \mathbb{R}]$, *where the latter integrals are finite.*

Roughly speaking, the *Lévy–Khintchine measure* $Q_{s,\mathbf{w}}$ describes canonical clusters χ, which by a "Poissonian superposition" can be added up to form $\tilde{\lambda}^{\mathfrak{c}}_{s,T}$. In this case a canonical cluster $\chi =: \int \chi(d[r,w]) \, \delta_{[r,w]}$ is interpreted as a collection of weighted pairs $[r,w]$ referring to $\mathfrak{c}$-bridge paths w on $[s,r]$ [recall (3.1.1)]. In particular, the "randomness" of the paths w given r concerns the behavior of w *before time r*.

Next we want to determine the *intensity* measure $\overline{Q}_{s,\mathbf{w}}$ (first moment measure) of the Lévy–Khintchine measure $Q_{s,\mathbf{w}}$ defined by

$$(3.3.3) \quad (\overline{Q}_{s,\mathbf{w}}, \psi) = \int Q_{s,\mathbf{w}}(d\chi) \, (\chi, \psi), \qquad \psi \in \mathbf{B}_+[[s,T] \,\hat{\times}\, \mathbb{C}^{\bullet;\mathfrak{c}}_{s,\mathbf{w}}, \mathbb{R}].$$

Because $\tilde{\lambda}^{\mathfrak{c}}$ does not have a deterministic component, by (2.5.7) we have

$$(3.3.4) \qquad (\overline{Q}_{s,\mathbf{w}}, \psi) = \tilde{\mathbb{P}}_{s,\mathbf{w}}(\tilde{\lambda}^{\mathfrak{c}}_{s,T}, \psi) = \tilde{\Pi}_{s,\mathbf{w}} \int_s^T L(W, dr) \, \psi(r, \widetilde{W}_r).$$

For convenience, we introduce the $\mathfrak{c}$-*Brownian bridge laws*

$$(3.3.5) \qquad \tilde{\Pi}^{r;\mathfrak{c}}_{s,\mathbf{w}}(A) := \Pi_{s,\mathbf{w}}(\widetilde{W}_r \in A \mid W_r = \mathfrak{c}),$$
$$0 \leq s \leq t \leq T, \mathbf{w} \in \mathbb{C}^s, A \in \mathscr{B}(\mathbb{C}).$$

Note that $\tilde{\Pi}^{r;\mathfrak{c}}_{s,\mathbf{w}}$ is a law on the path space $\mathbb{C}$ (and not on $\mathbf{C}[I, \mathbb{C}]$), so we could write $\Pi^{r;\mathfrak{c}}_{s,\mathbf{w}}$ as well. Using this notation, from (3.3.4) we make the following conclusion.

LEMMA 3.3.6 (Intensity measure $\overline{Q}_{s,\mathbf{w}}$). *Fix $s \in I$ and $w \in \mathbb{C}^{s,c}$. The intensity measure $\overline{Q}_{s,\mathbf{w}}$ of $Q_{s,\mathbf{w}}$ is a finite measure on $[s,T] \hat{\times} \mathbb{C}^{\bullet,c}_{s,\mathbf{w}}$ given by*

$$(3.3.7) \qquad (\overline{Q}_{s,\mathbf{w}}, \psi) = \int_s^T dr \; p(r-s, 0) \int \tilde{\Pi}^{r,c}_{s,\mathbf{w}}(dw) \psi(r, w),$$

$\psi \in \mathbf{B}_+[[s,T] \hat{\times} \mathbb{C}^{\bullet,c}_{s,\mathbf{w}}, \mathbb{R}].$

Roughly speaking, $\overline{Q}_{s,\mathbf{w}}(d[r,w])$ selects pairs $[r,w]$ in such a way that r is absolutely continuous distributed, and conditioned on r, the path w is a c-bridge on $[s,r]$ with law $\tilde{\Pi}^{r,c}_{s,\mathbf{w}}$.

For almost all w with respect to $\tilde{\Pi}^{r,c}_{s,\mathbf{w}}$, the *Brownian bridge local time measure* $L^c(w, dt)$ makes sense (note that $\tilde{\Pi}^{r,c}_{s,\mathbf{w}}$ describes a semi-Martingale). It will be involved in the Palm representation formula that follows.

For $\overline{Q}_{s,\mathbf{w}}$-almost all $[r,w] \in [s,T] \hat{\times} \mathbb{C}^{\bullet,c}_{s,\mathbf{w}}$ we can build the *Palm distributions* $Q^{r,w}_{s,\mathbf{w}}$ formally defined by *disintegration:*

$$(3.3.8) \qquad Q_{s,\mathbf{w}}(d\chi) \; \chi(dr, dw) = \overline{Q}_{s,\mathbf{w}}(dr, dw) \; Q^{r,w}_{s,\mathbf{w}}(d\chi).$$

Roughly speaking, $Q^{r,w}_{s,\mathbf{w}}$ describes a canonical cluster χ according to $Q_{s,\mathbf{w}}$, but given that it contains the pair $[r, w]$. Recall that by (3.3.7) the bridge local time measure $L^c(w)$ makes sense.

THEOREM 3.3.9 (Palm representation formula). *Fix $s \in I$ and $w \in \mathbb{C}^{s,c}$. For $\overline{Q}_{s,\mathbf{w}}$-almost all $[r,w] \in [s,T] \hat{\times} \mathbb{C}^{\bullet,c}_{s,\mathbf{w}}$, the Palm distribution $Q^{r,w}_{s,\mathbf{w}}$ has Laplace functional*

$$(3.3.10) \quad \int Q^{r,w}_{s,\mathbf{w}}(d\chi) \exp(\chi, -\psi) = \exp\Big[-2 \int_s^r L^c(w, dt) \, u_{\psi,c}(t, w_{\cdot \wedge t}, T)\Big],$$

$\psi \in \mathbf{B}_+[[0,T] \hat{\times} \mathbb{C}^{\bullet,c}, \mathbb{R}]$, *where $u_{\psi,c}(\cdot, \cdot, T) \geq 0$ is the unique bounded solution of* (3.1.3).

Consequently, given the Palm time point r and a corresponding c-bridge path w, the Laplace functional of $Q^{r,w}_{s,\mathbf{w}}$ is expressed with the help of the (deterministic) bridge local time measure $L^c(w)$ and solutions of the equation related to $\tilde{\lambda}^c_{s,T}$. The proof of this theorem will immediately follow.

3.4. *Proof of Theorem 3.3.9.* To get (3.3.10), we must show that, for functions $\varphi, \psi \in \mathbf{B}_+[[0,t] \hat{\times} \mathbb{C}^{\bullet,c}, \mathbb{R}]$,

$$
\begin{aligned}
(3.4.1) \quad & \int \overline{Q}_{s,\mathbf{w}}(dr, dw) \; \varphi(r, w) \int Q^{r,w}_{s,\mathbf{w}}(d\chi) \; \exp(\chi, -\psi) \\
& = \int \overline{Q}_{s,\mathbf{w}}(dr, dw) \; \varphi(r, w) \exp\Big[-2 \int_s^r L^c(w, dt) \, u_{\psi,c}(t, w_{\cdot \wedge t}, T)\Big].
\end{aligned}
$$

Using (3.3.8) and (3.3.4), this can be written as

$$\int Q_{s,\mathbf{w}}(d\chi)\, (\chi, \varphi)\exp(\chi, -\psi)$$

(3.4.2)
$$= \tilde{\Pi}_{s,\mathbf{w}} \int_s^T L^c(W, dr)\, \varphi(r, \widetilde{W}_r)$$

$$\times \exp\Big[-2\int_s^r L^c(W, dt)\, u_{\psi,c}(t, \widetilde{W}_t, T)\Big].$$

Here $u_{\psi,c}(\cdot, \cdot, T) \geq 0$ solves (3.1.3). To prove (3.4.2) we shall first reformulate both sides of (3.4.2) separately in order to show that they satisfy the same equation [(3.4.5)]. Starting with the l.h.s., it equals

(3.4.3) $\dfrac{\partial}{\partial\varepsilon}\int Q_{s,\mathbf{w}}(d\chi)(1 - \exp(\chi, -(\psi + \varepsilon\varphi)))\big|_{\varepsilon=0+} =: v_{\psi,\varphi}(s, \mathbf{w}, T),$

which by (3.3.2), (3.1.2) and (3.1.3) coincides with $(\partial/\partial\varepsilon)u_\varepsilon(s, \mathbf{w}, T)|_{\varepsilon=0+}$. Here $u_\varepsilon(\cdot, \cdot, T) \geq 0$ (for given $\psi + \varepsilon\varphi$) is a solution to

(3.4.4) $u_\varepsilon(s, w, T) = \tilde{\Pi}_{s,w}\displaystyle\int_s^T L^c(W, dr)[(\psi + \varepsilon\varphi)(r, \widetilde{W}_r) - u_\varepsilon^2(r, \widetilde{W}_r, T)],$

$s \in I$, $w \in \mathbb{C}^{s,c}$, that is, (3.1.3) with ψ replaced by $\psi + \varepsilon\varphi$. Consequently, $v_{\psi,\varphi}(\cdot, \cdot, T)$ solves

(3.4.5)
$$v(s, w, T) = \tilde{\Pi}_{s,w}\int_s^T L^c(W, dr)\varphi(r, \widetilde{W}_r)$$

$$- 2\tilde{\Pi}_{s,w}\int_s^T L^c(W, dr)\, u_{\psi,c}(r, W_r, T)v(r, \widetilde{W}_r, T),$$

$s \in I$, $W \in \mathbb{C}^{s,c}$, with $u_{\psi,\varphi}$ taken from (3.1.3). In other words, this equation describes the "Laplace transform" of the *Campbell measure* of $Q_{s,\mathbf{w}}$ as written at the left-hand side of (3.4.2). By the way, replacing formally φ by $\delta_{r,w}$ we get the equation related to the Laplace transform of the Palm distribution $Q_{s,\mathbf{w}}^{r,w}$. We also mention that by a formal differentiation of (3.4.5),

(3.4.6)
$$-\frac{\partial}{\partial s}v(s, w, T) = \tilde{A}_s v(s, w, T) + \delta_c(w_s)\varphi(s, w)$$

$$- 2\delta_c(w_s)\, u_{\psi,c}(s, w, T)\, v(s, w, T), \quad s \in I, \ w \in \mathbb{C}^{s,c},$$

with terminal condition $v(T, \cdot, T) = 0$. Therefore, the r.h.s. of (3.4.2) can formally be thought of as a *Feynman–Kac solution* of (3.4.5).

To complete the proof, by uniqueness it remains to show that the r.h.s. of (3.4.2) also satisfies (3.4.5). We change the notation from $[s, \mathbf{w}]$ to $[r, \widetilde{W}_r]$ and put the r.h.s. of (3.4.2) into the second term of the r.h.s. of (3.4.5) in place of $v(r, \widetilde{W}_r, T)$. Then, for $s \in I$, $w \in \mathbb{C}^{s,c}$, that second term equals

$$-2\tilde{\Pi}_{s,w}\int_s^T L^c(W, dr)\, u_{\psi,c}(r, \widetilde{W}_r, T)$$

$$\times \tilde{\Pi}_{r,\widetilde{W}_r}\int_r^T L^c(w, dr')\, \varphi(r', \widetilde{W}_r)\exp\Big[-2\int_r^{r'} L^c(W, dt)\, u_{\psi,c}(t, \widetilde{W}_t, T)\Big].$$

Applying the Markov property at time r to the Brownian path process $\widetilde{W}$, this simplifies to

$$-2\tilde{\Pi}_{s,w} \int_s^T L^c(W, dr) \, u_{\psi,c}(r, \widetilde{W}_r, T)$$

$$\times \int_r^T L^c(w, dr') \, \varphi(r', \widetilde{W}_r) \exp\left[-2 \int_r^{r'} L^c(W, dt) \, u_{\psi,c}(t, \widetilde{W}_t, T)\right].$$

Change the order of integration to get

$$-2\tilde{\Pi}_{s,w} \int_s^T L^c(W, dr') \, \varphi(r', \widetilde{W}_{r'})$$

$$\times \int_s^{r'} L^c(W, dr) \, u_{\psi,c}(r, \widetilde{W}_r, T) \exp\left[-2 \int_r^{r'} L^c(W, dt) \, u_{\psi,c}(t, \widetilde{W}_t, T)\right].$$

The latter expression compensates the two remaining terms

$$\tilde{\Pi}_{s,w} \int_s^T L^c(W, dr) \, \varphi(r, \widetilde{W}_r),$$

$$-\tilde{\Pi}_{s,\mathbf{w}} \int_s^T L^c(W, dr') \, \varphi(r', \widetilde{W}_{r'}) \exp\left[-2 \int_s^{r'} L^c(W, dt) \, u_{\psi,c}(t, \widetilde{W}_t, T)\right]$$

of (3.4.5), where we used the r.h.s. of (3.4.2) instead of $v(s, w, T)$. In fact,

$$\int_s^{r'} L^c(W, dr) \, 2u_{\psi,c}(r, \widetilde{W}_r, T) \exp\left[-2 \int_r^{r'} L^c(W, dt) \, u_{\psi,c}(t, \widetilde{W}_t, T)\right]$$

equals

$$\int_s^{r'} L^c(W, dr) \, \frac{d}{L^c(W, dr)} \, \exp\left[-2 \int_r^{r'} L^c(W, dt) \, u_{\psi,c}(t, \widetilde{W}_t, T)\right]$$

$$= 1 - \exp\left[-2 \int_r^s L^c(W, dt) \, u_{\psi,c}(t, \widetilde{W}_t, T)\right].$$

This completes the proof of Theorem 3.3.9. $\quad\square$

4. Singularity of the Catalyst's Occupation Density Measure.

In this final section we shall carry out the following program. The main point will be to show that the Palm canonical clusters χ according to $Q_{s,\mathbf{w}}^{r,w}$, more precisely their marginals $\chi^* := \chi(\cdot \times \mathbb{C})$ on $[0, T]$, have an infinite left upper density $\bar{b}(\chi^*)$ at r [see definition (4.2.1) and Theorem 4.2.2]. This implies the analogous property for the "Palm realizations" of the historical super-Brownian local time measures $\tilde{\lambda}_{s,T}^c$. Therefore, $\lambda_{s,T}^c \stackrel{\ell}{=} \tilde{\lambda}_{s,T}^c(\cdot \times \mathbb{C})$ also has infinite left upper densities at a random exceptional set of times that carries $\lambda_{s,T}^c$ (Corollary 4.3.1). Because $\lambda_{s,T}^c$ is diffuse and for an absolutely continuous measure ν the left upper density $\bar{b}(\nu)$ is finite at ν a.e. r, this finally implies that $\lambda_{s,T}^c$ is singular, for any $0 \leq s \leq T$, thus proving Theorem 1.1.4.

4.1. *A simple equation estimate.*   Fix a constant $b > 0$. To realize the program outlined above we need a *lower estimate* of the solution to the following (ordinary) integral equation:

$$(4.1.1) \quad u_\varepsilon(t) = (b\varepsilon)^{-1} \int_0^{t \wedge b\varepsilon} ds \, p(t-s,0) - \int_0^t ds \, p(t-s,0) \, u_\varepsilon^2(s),$$

$t \geq 0$, where $\varepsilon > 0$. [This is related to the formal equation

$$\frac{\partial}{\partial t} u_\varepsilon(t,z) = \kappa \Delta u_\varepsilon(t,z) + (b\varepsilon)^{-1} \mathbf{1}_{(0,b\varepsilon)}(t)$$

$$- \delta_\mathfrak{c}(z) u_\varepsilon^2(t,z), \quad t \geq 0, \quad z \in \mathbb{R}, u_\varepsilon \big|_{t=0+} = 0.]$$

LEMMA 4.1.2.   *Recall that $b > 0$ is fixed. For each constant $a \in (0,b]$, there is a constant $\delta > 0$ such that for the solution u_ε of (4.1.1),*

$$(4.1.3) \qquad\qquad u_\varepsilon(t) \geq \delta \, \varepsilon^{-1/2}, \qquad t \in [\varepsilon a, \varepsilon b], \ \varepsilon > 0.$$

PROOF.   As in the proof of Lemma 3.2.5 we rescale by setting $v(t) := \sqrt{\varepsilon} u_\varepsilon(\varepsilon t)$, $\varepsilon, t > 0$. Then we immediately see that v solves the same equation as u_1. By uniqueness we conclude that $v = u_1$. Thus we only need to show that u_1 is bounded away from 0 on the interval $[a,b]$. Restricting our attention to $t \leq b$, we can exploit the fact that $u_1(t)$ is related to the super-Brownian local time $\mathfrak{y}_t(\mathfrak{c}) = \lambda^\mathfrak{c}([0,t])$ [recall (2.5.8)]:

$$\mathbb{E}_{0,\mathfrak{c}} \exp[-b^{-1} \mathfrak{y}_t(\mathfrak{c})] = \exp[-u_t(t)], \qquad 0 \leq t \leq b$$

(cf. Proposition 2.5.3 or DF, Theorem 1.2.4). Hence u_1 is *monotone* nondecreasing. Therefore, it suffices to show that the solution to

$$(4.1.4) \quad v(t) = b^{-1} \int_0^t ds \, p_1(s,0) - \int_0^t ds \, p_1(t-s,0) \, v^2(s), \qquad 0 < t < b,$$

is strictly positive for sufficiently small $t \in (0,a]$.

The first term on the r.h.s. of (4.1.4) equals $k\sqrt{t}$ with some constant $k > 0$. We use it to bound $v^2(s) \leq k^2 s \leq k^2 t$ in the second term to arrive at

$$v(t) \geq k\sqrt{t} - k^3 \sqrt{t^3}, \qquad t < b,$$

which is certainly strictly positive for all $t > 0$ sufficiently small. This completes the proof.   □

4.2. *Infinite left upper densities of Palm canonical clusters.*   Let ν be a fixed measure on $I = [0,T]$. The *left upper density* $\overline{\mathfrak{d}}_r(\nu)$ at $\mathrm{r} \in I$ of ν is defined in the following way:

$$(4.2.1) \qquad\qquad \overline{\mathfrak{d}}_r(\nu) := \limsup_{\varepsilon \downarrow 0} \varepsilon^{-1} \nu((t-\varepsilon, t)).$$

In the following text, this notation will be applied to the *marginal* $\chi^* :=$ $\chi(\cdot \times \mathbb{C})$ of a measure χ on $I \times \mathbb{C}$. Recall the definition (3.3.3) of the intensity measures $\overline{Q}_{s,\mathbf{w}}$ as well as (3.3.8) of the Palm distributions $Q_{s,\mathbf{w}}^{r,w}$. Now we show that the left upper densities blow up at Palm time points.

THEOREM 4.2.2 (Infinite left upper densities at Palm times). *Fix* $s \in I = [0, T]$ *and* $\mathbf{w} \in \mathbb{C}^{s,c}$. *For* $\overline{Q}_{s,\mathbf{w}}$-*almost all pairs* $[r, w] \in (s, T] \hat{\times} \mathbb{C}_{s,\mathbf{w}}^{\bullet,c}$,

(4.2.3) $$Q_{s,\mathbf{w}}^{r,w}(\overline{\mathsf{b}}_r(\chi^*) = +\infty) = 1.$$

PROOF. Given $[r, w]$, it suffices to verify that

$$Q_{s,\mathbf{w}}^{r,w} \exp(\chi, (b\varepsilon)^{-1}\mathbf{1}\{(r - b\varepsilon, r) \times \mathbb{C}^{\bullet,c}\}) \longrightarrow 0$$

along some sequence $\varepsilon_n \downarrow 0$ (depending on $[r, w]$), where we set $b = r/2\kappa$ (recall that $k > 0$ is the diffusion constant). However, according to the Palm representation formula (3.3.10), the latter Laplace functional expression equals

$$\exp\left[-2 \int_s^r L^c(w, dt)\, u_{\psi,c}(t, w_{\cdot \wedge t}, T)\right],$$

where $u_{\psi,c}(\cdot, \cdot, T) \geq 0$ is the unique bounded solution of our equation (3.1.3) with $\psi = (b\varepsilon)^{-1}\mathbf{1}\{(r - b\varepsilon, r) \hat{\times} \mathbb{C}^{\bullet,c}\}$. Because this ψ is constant in the second variable (belonging to $\mathbb{C}$), by uniqueness of solutions we conclude that $u_{\psi,c}(\cdot, \cdot, T) =: v_\varepsilon$ only depends on the time variable and satisfies the (ordinary) equation

$$v_\varepsilon(t) = \int_t^T d\tau\, p(\tau - t, 0)\, (b\varepsilon)^{-1}\mathbf{1}\{r - b\varepsilon < \tau < r\} - \int_t^T d\tau\, p(\tau - t, 0)\, v_\varepsilon^2(\tau),$$

$s \leq t \leq T$. Hence $v_\varepsilon(t) = 0$ for $t \geq r$, and the upper limit in the integrals on the right-hand side can be replaced by r. Setting now $v_\varepsilon(t) =: u_\varepsilon(r - t)$, $s \leq t \leq r$, we can easily verify that u_ε solves the simplified forward equation (4.1.1) on $[0, r - s]$. Consequently, it suffices to prove that

(4.2.4) $$\lim_{n \to \infty} \int_s^r L^c(w, dt)\, u_{\varepsilon_n}(r - t) = \infty$$

for some sequence $\varepsilon_n \downarrow 0$ (depending on $[r, w]$) for $\overline{Q}_{s,\mathbf{w}}$-almost all $[r, w] \in [s, T] \hat{\times} \mathbb{C}_{s,\mathbf{w}}^{\bullet,c}$.

Recall that according to (3.3.7), given r, such a c-bridge path w on $[s, r]$ has the law $\tilde{\Pi}_{s,\mathbf{w}}^{r,c}$. In view of homogeneity there is no loss of generality in setting $s = 0$ and $c = 0$ in order to prove (4.2.4). By reversibility in law of the 0-Brownian bridge, the statement (4.2.4) will follow if, for $0 < r \leq T$,

$$\limsup_{\varepsilon \downarrow 0} \int_0^r L^0(w, dt)\, u_\varepsilon(t) = \infty$$

for $\tilde{\Pi}_{0,0}^{r,0}$-almost all 0-Brownian bridge paths w on $[0, r]$, where u_ε solves (4.1.1). Restrict the domain of integration additionally to $t \in [\varepsilon a, \varepsilon b]$ with $a = r/5\kappa$

(recall that $b = r/2\kappa$). Then by Lemma 4.1.2 it suffices to show that

(4.2.5) $$\limsup_{\varepsilon \downarrow 0} \varepsilon^{-1/2} L^0(W^r, [\varepsilon a, \varepsilon b]) = \infty \quad \text{a.s.,}$$

where W^r denotes a 0-Brownian bridge on $[0, r]$ distributed according to $\tilde{\Pi}^{r,0}_{0,0}$.

Let $\{B_t;\ t \geq 0\}$ be a *standard* Brownian motion (i.e., the diffusion constant κ equals $1/2$) starting at 0. Then the 0-Brownian bridge $\{W^r_t;\ 0 \leq t < r\}$ has the same law as $\{(1 - t/r)B_{2\kappa t/(r-t)};\ 0 \leq t < r\}$ [cf., for instance, Revuz and Yor (1991), Exercise 1.3.10]. Hence, for $0 < \varepsilon < \kappa$, in (4.2.5) we may replace the 0-Brownian bridge local time

$$L^0(W^r, [\varepsilon a, \varepsilon b]) = \int_{\varepsilon a}^{\varepsilon b} dt\, \delta_0(W^r_t)$$

by

$$\int_{\varepsilon a}^{\varepsilon b} dt\, \delta_0(B_{2\kappa t/(r-t)}).$$

(For convenience, we proceed formally with the δ-functions setting.) Substitute $s = 2\kappa t/(r - t)$ and by recalling that $\varepsilon < \kappa$, the latter integral can be bounded below by

$$\geq 2\kappa r \int_{\varepsilon/2}^{\varepsilon} ds\, (2\kappa + s)^{-2} \delta_0(B_s) \geq \text{const } L^0(B, [\varepsilon/2, \varepsilon])$$

(recall that r and κ are fixed). Setting $L(t) := L^0(B, [0, t])$, for (4.2.5) now it suffices to show that there exists a sequence $\varepsilon_n \downarrow 0$ of random times such that a.s.

(4.2.6) $$\varepsilon_n^{-1/2}[L(\varepsilon_n) - L(\varepsilon_n/2)] \xrightarrow[n \to \infty]{} \infty.$$

Recall that the standard Brownian local time (at 0) process $\{L(\varepsilon);\ \varepsilon \geq 0\}$ has the same law as $\{\max_{0 \leq x \leq 1} B_{x\varepsilon};\ \varepsilon \geq 0\}$; see, for instance, Revuz and Yor (1991), Theorem 6.2.3.

Strassen's law of the iterated logarithm says that with probability 1 the set of limit points of $\{[2\varepsilon \log\log(1/\varepsilon)]^{-1/2}B(x\varepsilon);\ 0 \leq x \leq 1\}$ in $\mathbb{C}$ (now with $T = 1$) as $\varepsilon \downarrow 0$ is given by the set of all absolutely continuous functions g on $[0, 1]$ satisfying $g(0) = 0$ and $\int_0^1 dx[g'(x)]^2 \leq 1$.

In fact, this version (for ε tending to 0) can be concluded, for instance, from Theorem 1 in Mueller (1981) by setting $t := 1/\varepsilon$, $h(t) := \log\log t$ and using the single point set $\mathscr{P}(t) := \{(0, 1/t)\}$. (Note that then $dA_1(t) = t^{-1}dt$.)

Applying Strassen's law for the identity function $g(x) = x$, we can find a sequence $\varepsilon_n \downarrow 0$ of (random) times such that for $0 \leq s \leq 1$,

(4.2.7) $$[2\varepsilon_n \log\log(1/\varepsilon_n)]^{-1/2} \max_{0 \leq x \leq s} B_{x\varepsilon_n} \longrightarrow s \quad \text{as } n \to \infty.$$

Exploiting this for $s = 1$ and $s = 1/2$, we see that the expressions in (4.2.6) are of order $[\log\log(1/\varepsilon_n)]^{1/2} \xrightarrow[n \to \infty]{} \infty$. This completes the proof. $\square$

4.3. *Completion of the Proof of Theorem* 1.1.4. Fix $s \in I = [0, T]$ and $w \in \mathbb{C}^{s,c}$. According to Theorem 4.2.2,

$$\int \overline{Q}_{s,\mathbf{w}}(d[r, w]) \int Q_{s,\mathbf{w}}^{r,w}(d\chi) \, \mathbf{1}\{\tilde{\mathfrak{d}}_r(\chi^*) < \infty\} = 0.$$

Using the fact that the Palm canonical clusters χ distributed by $Q_{s,\mathbf{w}}^{r,w}$ related to the Lévy–Khintchine measure $Q_{s,\mathbf{w}}$ concerning $\tilde{\lambda}_{s,T}^c$ are *stochastically smaller* than the Palm clusters χ distributed by $P_{s,\mathbf{w}}^{r,w}$, say, related to the law of $\tilde{\lambda}_{s,T}^c$ itself [see Lemma 10.6 in Kallenberg (1983)], the previous statement implies [recalling (3.3.4)]

$$\tilde{\mathbb{P}}_{s,\mathbf{w}} \int \tilde{\lambda}_{s,T}^c(d[r, w]) \int P_{s,\mathbf{w}}^{r,w}(d\chi) \, \mathbf{1}\{\tilde{\mathfrak{d}}_r(\chi^*) < \infty\} = 0.$$

From a *disintegration* formula analogous to (3.3.8),

$$\tilde{\mathbb{P}}_{s,\mathbf{w}} \int \tilde{\lambda}_{s,T}^c(dr, \mathbb{C}) \, \mathbf{1}\{\tilde{\mathfrak{d}}_r((\tilde{\lambda}_{s,T}^c)^*) < \infty\} = 0, \qquad s \in I, \mathbf{w} \in \mathbb{C}^{s,c}.$$

Consequently, we derived the following result, writing $[0, c]$ instead of $[s, \mathbf{w}]$ in the special case $s = 0$.

COROLLARY 4.3.1 (Infinite left upper densities at exceptional times). *With* $\tilde{\mathbb{P}}_{0,c}$-*probability* 1, $\tilde{\lambda}_{0,T}^c(\cdot \times \mathbb{C})$ *has infinite left upper density at* $\tilde{\lambda}_{0,T}^c(\cdot \times \mathbb{C})$-*almost every time.*

Now it is very easy to *verify Theorem* 1.1.4: Recalling (2.5.8), combine the previous corollary with the fact that the original super-Brownian local time measure $\lambda_{0,T}^c$ at the catalyst's location is *diffuse* a.s. (Theorem 1.2.4 in DF), and take into account that $T > 0$ is arbitrary. □

REFERENCES

DAWSON, D. A. and FLEISCHMANN, K. (1994). A super-Brownian motion with a single point catalyst. *Stochastic Process Appl.* **49** 3–40.

DAWSON, D. A. and PERKINS, E. A. (1991). Historical processes. *Mem. Amer. Math. Soc.* **93** 1–179.

DYNKIN, E. B. (1991a). Branching particle systems and superprocesses. *Ann. Probab.* **19** 1157–1194.

DYNKIN, E. B. (1991b). Path processes and historical superprocesses. *Probab. Theory Related Fields* **90** 1–36.

DYNKIN, E. B. (1993). On regularity of superprocesses. *Probab. Theory Related Fields* **95** 263–281.

FLEISCHMANN, K. (1994). Superprocesses in catalytic media. In *Measure-Valued Processes, Stochastic Partial Differential Equations, and Interacting Systems* (D. A. Dawson, ed.) 99–110. Amer. Math. Soc., Providence, RI.

FLEISCHMANN, T. and LEGALL, J. F. (1994). New approach to the single point catalytic super-Brownian motion. Unpublished manuscript.

KALLENBERG, O. (1983). *Random Measures*, 3rd. ed. Akademie, Berlin.

KERSTAN, J., MATTHES, K. and MECKE, J. (1982). *Infinitely Divisible Point Processes*. Nauka, Moscow (in Russian).

KONNO, N. and SHIGA, T. (1988). Stochastic differential equations for some measure-valued dif-
      fusions. *Probab. Theory Related Fields* **79** 201–225.
MUELLER, C. (1981). A unification of Strassen's law and Lévy's modulus of continuity. *Z. Wahrsch.
      Verw. Gebiete* **56** 163–179.
REIMERS, M. (1989). One dimensional stochastic partial differential equations and the branching
      measure diffusion. *Probab. Theory Related Fields* **81** 319–340.
REVUZ, D. and YOR, M. (1991). *Continuous Martingales and Brownian Motion.* Springer, Berlin.

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