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STABLE PROCESSES WITH SAMPLE PATHS IN ORLICZ SPACES

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Let $X = \{X(t);\ t \in T\}$ be a measurable symmetric α-stable process, $0 < \alpha < 2$. In this paper necessary and sufficient conditions for X to have almost all sample paths in an Orlicz space $\mathbb{L}_\psi(T, \mu)$ with a function ψ satisfying the Δ_2-condition are given.

1. Introduction and results. Let $X = \{X(t);\ t \in T\}$ be a measurable symmetric α-stable (SαS) stochastic process, $0 < \alpha < 2$. If T is a separable metric space, then [see Samorodnitsky and Taqqu (1994)] every such process admits an integral representation

$$\{X(t);\ t \in T\} =_d \left\{ \int_E h(t,x) M(dx);\ t \in T \right\}, \tag{1.1}$$

where M is an SαS random measure on $(E, \mathcal{E})$ with a σ-finite control measure m, $h\colon T \times E \to \mathbb{R}$ is a jointly measurable function such that

$$\sigma_\alpha(t) := \left(\int_E |h(t,x)|^\alpha m(dx) \right)^{1/\alpha} < +\infty \qquad \forall\ t \in T, \tag{1.2}$$

and $=_d$ denotes equality in distribution. We say that a function $\psi\colon \mathbb{R}^+ \to \mathbb{R}^+$ is an *M-function* if it is nondecreasing, continuous, $\psi(u) = 0$ if and only if $u = 0$ and $\psi(u) \to \infty$ as $u \to \infty$. A function $\psi\colon \mathbb{R}^+ \to \mathbb{R}^+$ is said to satisfy the Δ_2-*condition* if there is a finite constant A such that

$$\psi(2u) \le A\psi(u) \qquad \forall\ u > 0. \tag{1.3}$$

Throughout the paper μ is a σ-finite measure on the Borel σ-algebra of subsets of T.

The problem we study is as follows: Given a measurable SαS process X with the integral representation (1.1), a measure μ on T and an M-function ψ satisfying the Δ_2-condition, is it true that

$$\int_T \psi\big(|X(t)|\big) \mu(dt) < +\infty \quad \text{a.s.?} \tag{1.4}$$

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If $\psi(u) = u^p$, $0 < p < +\infty$, then the following theorem is true [see Samorodnitsky (1992)]:

THEOREM A. *Let X be a measurable SαS process with the integral representation* (1.1), $0 < \alpha < 2$. *Let* (T, μ) *be a σ-finite measure space and* $p > 0$. *Then*

$$\int_T |X|^p d\mu < +\infty \quad a.s.$$

if and only if

$$\int_E \|h(\cdot, x)\|_p^\alpha m(dx) = \int_E \left(\int_T |h(t,x)|^p \mu(dt) \right)^{\alpha/p} m(dx) < +\infty \quad when\ p > \alpha,$$

$$\int_T \sigma_\alpha^p(t)\mu(dt) = \int_T \left(\int_E |h(t,x)|^\alpha m(dx) \right)^{p/\alpha} \mu(dt) < +\infty \quad when\ p < \alpha,$$

$$\int_T \int_E |h(t,x)|^\alpha \left(1 + \log_+ \frac{|h(t,x)|^\alpha \int_E \int_T |h(u,v)|^\alpha m(dv)\mu(du)}{\int_E |h(t,v)|^\alpha m(dv) \int_T |h(u,x)|^\alpha \mu(du)} \right) \mu(dt)m(dx) < +\infty$$
$$when\ p = \alpha.$$

Let $\mathbb{L}_0 = \mathbb{L}_0(T, \mu)$ be the linear space of equivalence classes of μ-measurable functions defined and finite μ-a.e. on T. Define

$$\mathbb{L}_\psi = \mathbb{L}_\psi(T, \mu) := \left\{ f \in \mathbb{L}_0(T, \mu) \colon \rho_\psi(f) := \int_T \psi(|f|)\, d\mu < +\infty \right\}.$$

$\mathbb{L}_\psi$ is a complete linear metric space called a (generalized) Orlicz space when endowed with the metric

$$d(f, g) = \inf\left\{ u > 0 \colon \rho_\psi\big((f - g)/u\big) \le u \right\}$$

[see Rao and Ren (1991), Chapter X]. The d-convergence $f_n \to 0$ in $\mathbb{L}_\psi$ is equivalent to convergence $\rho_\psi(f_n) \to 0$. The space $\mathbb{L}_\psi$ is separable if the measure space (T, μ) is. Define the quantity

$$\|f\|_\psi := \inf\left\{ u > 0 \colon \rho_\psi(f/u) \le 1 \right\}. \tag{1.5}$$

In general, if $\rho_\psi(f_n) \to 0$, then also $\|f_n\|_\psi \to 0$, but the reverse implication is not always true. The question (1.4) reduces then to the question of whether or not almost all sample functions of X belong to the space $\mathbb{L}_\psi$. This question (including the $\mathbb{L}_p$-case described in Theorem A) can also be viewed as a part of the general problem of describing α-stable probability measures on function spaces.

The following theorem is the main result of this paper.

THEOREM 1.1. *Let X be a measurable SαS process with the integral representation* (1.1), $0 < \alpha < 2$. *Let* (T, μ) *be a σ-finite measure space and let* ψ *be an*

M-function satisfying the Δ_2-condition. Then (1.4) *holds if and only if*

$$\int_E \|h(\cdot,x)\|_\psi^\alpha m(dx) < +\infty, \tag{1.6}$$

$$\int_T \psi\big(\sigma_\alpha(t)\big)\mu(dt) < +\infty \tag{1.7}$$

and for some (equivalently, every) constant $c_0 > 0$,

$$\int_T \int_E |h(t,x)|^\alpha m(dx)\mu(dt) \int_{c_0\sigma_\alpha(t)}^{|h(t,x)|/\|h(\cdot,x)\|_\psi} \psi(r)\frac{dr}{r^{1+\alpha}} < +\infty, \tag{1.8}$$

where the innermost integral is set equal to zero when its lower limit exceeds its upper limit. Here $\{\sigma_\alpha(t);\ t \in T\}$ *is defined in* (1.2) *and*

$$\|h(\cdot,x)\|_\psi = \inf\left\{u > 0\colon \int_T \psi\big(|h(t,x)|/u\big)\mu(dt) \le 1\right\}, \qquad x \in E. \tag{1.9}$$

An important ingredient in our proofs is the approach for describing convergent series in sequence Orlicz spaces developed by Kwapien and Woyczynski (1987). A direct extension of this result to SαS random series in a (not necessarily sequence) Orlicz space is stated in subsequent text. Let $\{\xi_i;\ i \ge 1\}$ be a sequence of independent real rv's with the common stable distribution $S_\alpha(1,0,0)$ and let $f = \{f_i;\ i \ge 1\}$ be a sequence of real-valued measurable functions defined on T and such that

$$\sigma_\alpha^f(t) := \left(\sum_{i=1}^\infty |f_i(t)|^\alpha\right)^{1/\alpha} < +\infty \qquad \forall\, t \in T.$$

Then a stochastic process $S = \{S(t);\ t \in T\}$ defined by

$$S(t) = \sum_{i=1}^\infty \xi_i f_i(t), \qquad t \in T,$$

is a measurable SαS process with an integral representation (1.1), where the SαS random measure M is defined on $E = \mathbb{N}$ with the counting measure as control measure m. By Theorem 1.1, $S \in \mathbb{L}_\psi(T,\mu)$ a.s. if and only if

$$\sum_{i=1}^\infty \|f_i\|_\psi^\alpha < +\infty, \tag{1.10}$$

$$\int_T \psi\big(\sigma_\alpha^f(t)\big)\mu(dt) < +\infty \tag{1.11}$$

and for some (equivalently, every) constant $c_0 > 0$,

$$\sum_{i=1}^\infty \int_T |f_i(t)|^\alpha \mu(dt) \int_{c_0\sigma_\alpha^f(t)}^{|f_i(t)|/\|f_i\|_\psi} \psi(r)\frac{dr}{r^{1+\alpha}} < +\infty. \tag{1.12}$$

This statement in conjunction with Propositions 2.2 and 2.4 yields the following corollary.

COROLLARY 1.2. *Let* $\{\xi_i;\, i \geq 1\}$ *and* $\{f_i;\, i \geq 1\}$ *be as before. Let* (T, μ) *be a* σ*-finite measure space and let* ψ *be an* M*-function satisfying the* Δ_2*-condition. Then the series*

$$\sum_{i=1}^{\infty} \xi_i f_i \text{ converges a.s. in } \mathbb{L}_\psi(T, \mu) \tag{1.13}$$

if and only if (1.10), (1.11) *and* (1.12) *hold.*

Under additional growth conditions on ψ we show that only one (or two) of the conditions (1.6), (1.7) and (1.8) are necessary and sufficient for (1.4).

COROLLARY 1.3. *Under the conditions of Theorem* 1.1 *assume in addition that there is a real number* $p \in (\alpha, +\infty)$ *such that for a finite constant* C,

$$\psi(xu) \leq Cx^p \psi(u) \qquad \forall\, x \leq 1,\ u > 0. \tag{1.14}$$

Then (1.4) *is equivalent to* (1.6).

It would not be out of place to compare this statement with known results in probability in Banach spaces. Note that $\mathbb{L}_\psi(T, \mu)$ is a separable Banach function space with a function norm $\|\cdot\|_\psi$ defined by (1.4) if the M-function ψ satisfying the Δ_2-condition is convex, that is, ψ is a *Young function*, and the measure space (T, μ) is separable. Recall that a Banach space $\mathbb{L}_\psi$ has stable type α provided that (1.10) implies (1.13). Hence, by Corollaries 1.2 and 1.3, $\mathbb{L}_\psi$ has stable type α if the Young function ψ satisfies the condition (1.14). In this case (ψ is a Young function) the converse statement to Corollary 1.3 is true. Namely, if (1.6) implies (1.4), then $\mathbb{L}_\psi$ has stable type α and therefore has stable type $\alpha + \varepsilon$ for some $\varepsilon > 0$ [see Maurey and Pisier (1976)]. Now under additional conditions on the measure space (T, μ), the growth condition (1.14) follows from Corollary 10 in Kaminska and Turett (1990).

Moreover, if ψ is a Young function, then the statement of Corollary 1.3 also can be derived from general results in probability in Banach spaces as follows. By Theorem 7.5.1 in Linde (1986) under the assumption (1.6) for a kernel operator K from a dual space $\mathbb{L}_\psi^\star(T, \mu)$ into $\mathbb{L}_\alpha(E, m)$ defined by

$$Kf(x) := \int_T h(t, x) f(t) \mu(dt), \qquad x \in E,$$

there is a probability measure on $\mathbb{L}_\psi(T, \mu)$ with the characteristic function

$$\exp\left\{-\int_E |Kf(x)|^\alpha m(dx)\right\}, \qquad f \in \mathbb{L}_\psi^\star, \tag{1.15}$$

if and only if $\mathbb{L}_\psi$ has stable type α. Now (1.4) will follow from (1.6) under the condition (1.14) for some $p > \alpha$ if the measurable SαS process X given by (1.1) induces a cylindrical probability on $\mathbb{L}_\psi$ with the chracteristic function (1.15). Under the condition (1.7) this may be verified using Theorem 3.3 of Norvaiša (1992). We are done because by the proof of Corollary 1.3, (1.6) implies (1.7) and because by Proposition 6.4.5 in Linde (1986), (1.4) always implies (1.7).

The following statement describes functions ψ for which the condition (1.7) is dominating.

COROLLARY 1.4. *Under the conditions of Theorem* 1.1 *assume in addition that there is a real number* $p \in (0, \alpha)$ *such that for a finite constant* C,

$$\psi(xu) \le Cx^p\psi(u) \qquad \forall\, x \ge 1,\ u > 0. \tag{1.16}$$

Then (1.4) *is equivalent to* (1.7).

Of course, the condition (1.7) has no meaning in a general Banach space and hence Corollary 1.4 (for a Young function ψ) cannot be derived in a similar manner as the previous one from what we know about probability in Banach spaces. Nevertheless, it seems to be an interesting question to compare different notions of stable cotype α, $0 < \alpha < 2$, of Banach spaces [see Chapter 8 in Linde (1986) for a review of some of them] with the class of Orlicz spaces in which the statement of Corollary 1.4 holds.

Observe that the condition (1.7) does have a meaning in an arbitrary Banach function space (or more generally in a modular function space). According to the main result of Norvaiša [(1992), Theorem 3.7], in the Banach function space $\mathbb{L}_\psi$ with an order continuous dual there is a probability measure with the characteristic function (1.15) provided (1.7) holds and $\mathbb{L}_\psi$ is p-concave, for some $1 \le p < \alpha$. Using the characterization of M-functions satisfying (1.16) due to Matuszewska (1962) (see also our Lemma 4.2), one can prove that the growth condition (1.16) of ψ for some $1 \le p < +\infty$ implies that $\mathbb{L}_\psi$ is a p-concave Banach function space [cf. Proposition 3.6(2) of Norvaiša (1993)]. This represents another point of view on Corollary 1.4.

It is easy to see that Corollaries 1.3 and 1.4 yield Theorem A when $p > \alpha$ and $p < \alpha$, respectively. When $p = \alpha$, that is, when $\psi(u) = u^\alpha, u > 0$, then the condition (1.8) with the constant

$$c_0 = \left(\int_T \int_E |h(t,x)|^\alpha m(dx)\mu(dt)\right)^{-1}$$

in conjunction with (equivalent) conditions (1.6) and (1.7) also yields Theorem A. Therefore, Theorem A is a simple consequence of Theorem 1.1.

We conclude with the following corollary.

COROLLARY 1.5. *Under the conditions of Theorem* 1.1, *assume in addition that there is a finite constant* C *such that the inequality*

$$\psi(xu) \le C\psi(x)\psi(u) \tag{1.17}$$

holds for all $x, u > 0$ *and either*

$$\int_0^1 \frac{\psi(r)\,dr}{r^{1+\alpha}} < +\infty \quad or \quad \int_1^\infty \frac{\psi(r)\,dr}{r^{1+\alpha}} < +\infty \tag{1.18}$$

are satisfied. Then (1.4) *is equivalent to* (1.6) *and* (1.7).

It is easy to check that the functions

$$\psi(u) = u^\alpha \log^p(1+u), \qquad p > 0,\, u > 0,$$

and

$$\psi(u) = u^\alpha \log^p\left(1+u^{-1}\right), \qquad p > 0,\, u > 0,$$

satisfy the condition (1.17) and one of the two conditions in (1.18), respectively.

The proof of Theorem 1.1 is based on a series representation for α-stable random measures [see Section 3.9 in Samorodnitsky and Taqqu (1994)] and on a reduction of this representation to a convergence of Rademacher series in the Orlicz space $\mathbb{L}_\psi$. The basic tools we use are Hoffmann–Jørgensen type inequalities [see, e.g., Kwapien and Woyczynski (1987), Section 2] and Lévy-type inequalities (Lemma 2.3). The latter result in conjunction with Proposition 2.2 allows us to characterize convergent series in a modular function space without appealing to the convergence of characteristic functions as in the Itô-Nisio theorem [see Itô and Nisio (1968)].

All preliminary results are contained in the next section. Section 3 is devoted to the proof of Theorem 1.1 and the proofs of Corollaries 1.3, 1.4 and 1.5 are given in Section 4.

2. Preliminaries. In this section we consider random variables with values in a linear metric space of measurable functions $\mathbb{L} = \mathbb{L}_\rho$ defined by a function ρ as follows. Recall that $\mathbb{L}_0 = \mathbb{L}_0(T, \mu)$ denotes a linear space (of equivalence classes) of μ-measurable functions defined and μ-a.e. finite on T. Consider a function ρ: $\mathbb{L}_0 \to [0, +\infty]$ such that:

[M1] $\rho(f) = 0$ iff $f = 0$ (μ-a.e.).
[M2] $\rho(-f) = \rho(f)$.
[M3] $\rho(f+g) \le \rho(f) + \rho(g)$ whenever $|f| \wedge |g| = 0$ (μ-a.e.).
[M4] $|f| \le |g|$ (μ-a.e.) implies $\rho(f) \le \rho(g)$.
[M5] $\rho(\mathbb{I}_{T_0}) < \infty$ for every finite μ-measure subset T_0 of T.

From [M3] and [M4] it follows that

$$\rho(uf + vg) \le \rho(f) + \rho(g) \quad \text{for } u, v \ge 0,\ u + v = 1.$$

We call the function ρ modular and the set $\mathbb{L}_\rho$ defined by

$$\mathbb{L}_\rho = \mathbb{L}_\rho(T, \mu) := \left\{ f \in \mathbb{L}_0(T, \mu) \colon \lim_{u \to 0} \rho(uf) = 0 \right\} \tag{2.1}$$

a modular space [see Rolewicz (1972)]. We consider a modular ρ, which in addition to the foregoing conditions [M1]–[M5] also satisfies the following conditions:

[$\Delta 2$] There exists a finite constant A such that $\rho(2f) \le A\rho(f)$, $\forall f \in \mathbb{L}_\rho$.

[μ] The convergence $\rho(f_n) \to 0$ implies the convergence $f_n \to 0$ in μ-measure.

[m] $\rho(X)$ is a measurable function if $X = \{X(t);\ t \in T\}$ is a measurable stochastic process and $\rho(X) =_d \rho(Y)$ for any two measurable stochastic processes $X = \{X(t);\ t \in T\}$ and $Y = \{Y(t);\ t \in T\}$ such that $\rho(X) < \infty$ a.s., $\rho(Y) < \infty$ a.s. and $X =_d Y$.

The set $\mathbb{L}_\rho$ is a linear metric space and under the conditon [$\Delta 2$] the convergence $f_n \to 0$ in $\mathbb{L}_\rho$ is equivalent to the convergence $\rho(f_n) \to 0$. In passing, for $\mathbb{L}_\rho$ to be a linear metric space and for the characterization of the convergence in $\mathbb{L}_\rho$ in terms of the modular convergence, a weaker variant of the condition [$\Delta 2$] is sufficient [see Musielak (1983)]. We use the present version of it because we apply the results of this section to the modular ρ_ψ defined by

$$\rho_\psi(f) := \int_T \psi(|f|)\,d\mu, \tag{2.2}$$

where ψ is an M-function satisfying the Δ_2-condition (1.3), which obviously implies condition [$\Delta 2$] for ρ_ψ with the same constant A. It is easy to see that ρ_ψ also satisfies conditions [M1]–[M5], [μ] and the first part of the condition [m]. The fact that the second part of condition [m] is satisfied by ρ_ψ seems to belong to mathematical folklore. Because it is important to us, we give a proof.

PROPOSITION 2.1. *The function modular ρ_ψ defined by* (2.2) *satisfies the condition* [m].

PROOF. Let $X = \{X(t);\ t \in T\}$ and $Y = \{Y(t);\ t \in T\}$ be two measurable stochastic processes on probability spaces $(\Omega_X, \mathcal{F}_X, P_X)$ and $(\Omega_Y, \mathcal{F}_Y, P_Y)$, respectively, such that $\rho_\psi(X) < \infty$ a.s., $\rho_\psi(Y) < \infty$ a.s. and $X =_d Y$. Let $\hat{\mu}$ be a probability measure on T such that $\mu(dt) = \phi(t)\hat{\mu}(dt)$ for some nonnegative measurable function ϕ. Define on a separate probability space a sequence $\{t_i;\ i \ge 1\}$ of independent T-valued rv's with the common distribution $\hat{\mu}$ on T. By the strong law of large numbers for P_X-a.a. $\omega \in \Omega_X$, we have,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \psi(|X(t_i,\omega)|)\phi(t_i) = \int_T \psi(|X(\cdot,\omega)|)\phi\,d\hat{\mu}, \tag{2.3}$$

$\hat{\mu}$-a.e. By Fubini's theorem for $\hat{\mu}$-a.a. realizations of $\{t_i;\ t \in T\}$ (2.3) holds for P_X-a.a. $\omega \in \Omega_X$. In the same way one can conclude that for $\hat{\mu}$-a.a. realizations of $\{t_i;\ i \ge 1\}$ we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \psi(|Y(t_i,\omega)|)\phi(t_i) = \int_T \psi(|Y(\cdot,\omega)|)\phi\,d\hat{\mu} \tag{2.4}$$

for P_Y-a.a. $\omega \in \Omega_Y$. Therefore, there exists a common realization $\{t_i;\ i \geq 1\}$ such that (2.3) and (2.4) hold. Because X and Y have the same finite dimensional distributions, it follows that the right-hand sides of (2.3) and (2.4) have the same distribution too. □

Another example of a modular ρ satisfying condition [M1]–[M5], [Δ2] and [μ] is given by a function norm as defined, for example, by Luxemburg (1955). The first part of the condition [m] for a function norm is known among analysts as a Luxemburg–Gribanov theorem [see Theorem 99.2 in Zaanen (1983)]. The second part of the condtion [m] is introduced here due to a lack, in general, of a nontrivial dual space of $\mathbb{L}_\rho$, which is usually used to identify distributions of induced Banach space rv's.

A modular ρ satisfying, in addition, the conditions [Δ2], [μ] and [m] will be called a *function modular* and the corresponding space $\mathbb{L}_\rho$ defined by (2.1) will be called a *modular function space* (MFS). This terminology differs somewhat from that used recently by Kozlowski (1988). It remains largely a matter of future investigation to decide how much different assumptions on ρ affect the structure of the function space $\mathbb{L}_\rho$.

If, in addition, a modular ρ satisfies the condition

[oc] $$\rho\left(f\mathbb{I}_{T_n}\right) \to 0 \text{ whenever } f \in \mathbb{L}_\rho \text{ and } T_n \downarrow \emptyset,$$

then we call ρ and $\mathbb{L}_\rho$ an order continuous function modular and an order continuous MFS, respectively. One can prove as in Luxemburg [(1955), Lemma 1.2.2] that the following variant of the dominated convergence theorem holds in an order continuous MFS $\mathbb{L}_\rho$:

$$\text{if } |f_n| \leq g,\ g \in \mathbb{L}_\rho \text{ and } f_n \to 0\ \mu\text{-a.e., then } \rho(f_n) \to 0.$$

It is easy to see that an Orlicz space $\mathbb{L}_\psi$ is an order continuous MFS if an M-function ψ satisfies the Δ_2-condition [see Theorem 2.3.3 in Luxemburg (1955) for the converse statement].

We are now ready to continue our study of sums of random variables with values in a MFS $\mathbb{L}_\rho$.

PROPOSITION 2.2. *Let $\mathbb{L}_\rho(T,\mu)$ be a MFS. Consider measurable stochastic processes $X = \{X(t);\ t \in T\}$ and $S_n = \{S_n(t);\ t \in T\}$, $n = 1, 2, \dots$, and an $\mathbb{L}_\rho$-valued rv S defined on a probability space $(\Omega, \mathcal{F}, P)$ such that*:

(i) *$S_n \in \mathbb{L}_\rho$ a.s. for all $n = 1, 2, \dots$ and*

$$\lim_{n\to\infty} \rho(S_n - S) = 0 \quad a.s.$$

(ii) *For μ-a.a. $t \in T$,*

$$\lim_{n\to\infty} S_n(t) = X(t) \quad a.s.$$

Then X has a.a. sample paths in $\mathbb{L}_\rho$ and $X = S$ a.s.

PROOF. Let $S_n(t,\omega) \not\to X(t,\omega)$ mean that $S_n(t,\omega)$ does not converge to $X(t,\omega)$ as $n \to \infty$. By Fubini's theorem, we have

$$\begin{aligned}\int_\Omega \mu\Big(\{t \in T\colon\ S_n(t,\omega) \not\to X(t,\omega)\}\Big)P(d\omega) &= \int_T P\Big(\{\omega \in \Omega\colon S_n(t,\omega) \not\to X(t,\omega)\}\Big)\mu(dt)\\ &= 0 \quad \text{by assumption (ii).}\end{aligned}$$

Hence, there is an event $\Omega_1 \in \mathcal{F}$ with $P(\Omega_1) = 0$ such that for every $\omega \notin \Omega_1$,

$$\lim_{n\to\infty} S_n(\cdot,\omega) = X(\cdot,\omega), \qquad \mu\text{-a.e.} \tag{2.5}$$

Consider the $\mathbb{L}_\rho$-valued rv S. For an $\omega \in \Omega$, pick an arbitrary element $S(\omega)(\cdot)$ from the equivalence class $S(\omega)$ and defines $S(t,\omega) := S(\omega)(t)$, $t \in T$. By assumption (i), there is another event $\Omega_2 \in \mathcal{F}$ with $P(\Omega_2) = 0$ such that for every $\omega \notin \Omega_2$,

$$\lim_{n\to\infty} \rho\big(S_n(\cdot,\omega) - S(\cdot,\omega)\big) = 0$$

so that

$$\lim_{n\to\infty} S_n(\cdot,\omega) = S(\cdot,\omega) \quad \text{in } \mu\text{-measure.}$$

Therefore, for every $\omega \notin \Omega_2$ there is a subsequence $\{n_k(\omega)\}$ such that

$$\lim_{k\to\infty} S_{n_k(\omega)}(\cdot,\omega) = S(\cdot,\omega), \qquad \mu\text{-a.e.} \tag{2.6}$$

Let $\Omega_0 := \Omega_1 \cup \Omega_2$. Then $P(\Omega_0) = 0$ and according to (2.5) and (2.6), for every $\omega \notin \Omega_0$,

$$\mu\Big(\{t \in T\colon X(t,\omega) \neq S(t,\omega)\}\Big) = 0.$$

Hence, $X(\cdot,\omega) \in \mathbb{L}_\rho$ for every $\omega \notin \Omega_0$, which completes the proof. □

The following lemma is a version of Lévy's inequality.

LEMMA 2.3. *Let ρ be a function modular on $\mathbb{L}_0(T,\mu)$ and let $X_i = \{X_i(t);\ t \in T\}$, $i = 1,2,\dots$, be a sequence of independent symmetric measurable stochastic processes with all sample paths in a MFS $\mathbb{L}_\rho$. Let $S = \{S(t);\ t \in T\}$ be a measurable stochastic process with all sample paths in $\mathbb{L}_\rho$ and such that for μ-a.a. $t \in T$, the partial sums $S_n(t) := \sum_{i=1}^n X_i(t)$ converge a.s. to the limit $S(t)$. Then for all $y > 0$ and $0 \le K < +\infty$, we have*

$$P\bigg(\Big\{\sup_n \rho\Big(S_n \mathbf{1}(|S_n| > K)\Big) > y\Big\}\bigg) \le 2P\bigg(\Big\{\rho\Big(S\mathbf{1}(|S| > K)\Big) > y/2\Big\}\bigg) \tag{2.7}$$

and

$$(2.8)\quad P\left(\left\{\sup_n \rho\left(R_n\mathbb{I}(|R_n| > K)\right) > y\right\}\right) \le 2P\left(\left\{\rho\left(S\mathbb{I}(|S| > K)\right) > y/2\right\}\right),$$

where $R_n = S - S_n$, $n = 1, 2, \ldots$.

PROOF. We use the notation $X^K := X\mathbb{I}(|X| > K)$. For (2.7) it is enough to prove that the inequality

$$(2.9)\qquad P\left(\left\{\max_{1\le n\le N}\rho\,(S_n^K) > y\right\}\right) \le 2P\left(\left\{\rho\,(S^K) > y/2\right\}\right)$$

holds for any fixed integer $N \ge 1$. For every $n \ge 1$, by the property [M3] of ρ we have

$$(2.10)\qquad \begin{aligned} \rho\,(S_n^K) &\le \rho\Big((S_n - R_n)^K\mathbb{I}\big(|S_n - R_n| > |S_n + R_n|\big) \\ &\qquad + (S_n + R_n)^K\mathbb{I}\big(|S_n + R_n| > |S_n - R_n|\big)\Big) \\ &\le \rho\big((S_n + R_n)^K\big) + \rho\big((S_n - R_n)^K\big). \end{aligned}$$

let $\tau := \inf\{n \ge 1: \rho\,(S_n^K) > y\}$. Then by (2.10) we have

$$(2.11)\qquad \begin{aligned} P\left(\left\{\max_{1\le n\le N}\rho\,(S_n^K) > y\right\}\right) &= \sum_{n=1}^N P(\{\tau = n\}) \\ &\le \sum_{n=1}^N P\left(\left\{\tau = n,\ \rho\,(S^K) > y/2\right\}\right) \\ &\quad + \sum_{n=1}^N P\left(\left\{\tau = n,\ \rho\big((S_n - R_n)^K\big) > y/2\right\}\right). \end{aligned}$$

Because R_n has the same distribution as $-R_n$ and $\{\tau = n\}$ only depends on $X_1, \ldots, X_n$, we conclude that corresponding probabilities in two sums in (2.11) coincide. From this, (2.9) follows. To prove (2.8), one may argue in a similar way using $\tau' = \sup\{1 \le n \le N: \rho(R_n^K) > y\}$ instead of τ. The proof of Lemma 2.3 is complete. □

PROPOSITION 2.4. *Let* $\mathbb{L}_\rho(T, \mu)$ *be an order continuous MFS and let* $X_i = \{X_i(t);\ t \in T\}$ *be a sequence of independent symmetric measurable stochastic processes with all sample paths in* $\mathbb{L}_\rho$. *Assume that for* μ*-a.a.* $t \in T$, *the partial sums* $S_n(t) = \sum_{i=1}^n X_i(t)$ *converge a.s. to a measurable stochastic process* $S = \{S(t);\ t \in T\}$, *with all sample paths in* $\mathbb{L}_\rho$. *Then*

$$\lim_{n\to\infty}\rho\,(S_n - S) = 0 \quad a.s.$$

PROOF. Fix an arbitrary $\varepsilon > 0$. Due to the order continuity of ρ, there exists $T_0 \subset T$ with $\mu(T_0) < +\infty$ and such that

$$P\Big(\Big\{\rho\,(S\mathbb{I}_{T/T_0}) > \varepsilon\Big\}\Big) \leq \varepsilon. \tag{2.12}$$

By Lemma 2.3 we get

$$P\Big(\Big\{\sup_{n\geq 1}\rho\,(S_n\mathbb{I}_{T/T_0}) > 2\varepsilon\Big\}\Big) \leq 2\varepsilon. \tag{2.13}$$

Let $\rho_0 := \rho(\,\cdot\,\mathbb{I}_{T_0})$. Using once again the order continuity of ρ, one can find a finite constant K such that

$$P\Big(\Big\{\rho_0(S^K) > \varepsilon\Big\}\Big) \leq \varepsilon. \tag{2.14}$$

Lemma 2.3 implies that

$$P\Big(\Big\{\sup_{n}\rho_0(S_n^K) > 2\varepsilon\Big\}\Big) \leq 2\varepsilon \tag{2.15}$$

and

$$P\Big(\Big\{\sup_{n}\rho_0(R_n^K) > 2\varepsilon\Big\}\Big) \leq 2\varepsilon, \tag{2.16}$$

where $R_n = S - S_n$. Using the property [M3] we have

$$\begin{aligned}\rho\,(S_n - S) &\leq \rho\big((S_n - S)\mathbb{I}_{T/T_0}\big) + \rho_0\Big((S_n - S)\mathbb{I}\big(|S_n| \leq 2K\big)\Big)\\ &\quad + \rho_0\Big((S_n - S)\mathbb{I}\big(|S| > K\big)\mathbb{I}\big(|S_n| > 2K\big)\Big) + \rho_0\big(R_n^K\big)\\ &=: \sum_{i=1}^{4} I_i(n).\end{aligned}$$

Denote by Ω_i, $i = 1, \dots, 5$, the events whose probabilities appear in the left-hand sides of (2.12)–(2.16), respectively. Then on the event $\bigcap_{i=1}^{5}\Omega_i^c$ we have

$$\begin{aligned}&\sup_{n\geq 1} I_1(n) \leq 3A\varepsilon,\\ &\sup_{n\geq 1} I_3(n) \leq 3A\varepsilon,\\ &\sup_{n\geq 1} I_4(n) \leq 2\,\varepsilon.\end{aligned}$$

By the order continuity of ρ_0, it follows that

$$\lim_{n\to\infty} I_2(n) = 0.$$

Hence

$$P\left(\left\{\limsup_{n\to\infty} \rho(S_n - S) > 6A\varepsilon + 2\varepsilon\right\}\right) \le 8\varepsilon.$$

Because ε is arbitrary, the proof is complete. □

For an $f \in \mathbb{L}_\rho$ denote

$$[f](t) := f(t)/\big(1 \vee \|f\|_\rho\big), \qquad t \in T,$$

where [cf. (1.5)]

$$\|f\|_\rho = \inf\Big\{u > 0\colon \rho\big(f/u\big) \le 1\Big\}.$$

The following lemma can be proved along the same lines as Lemma 3.1 of Kwapien and Woyczynski (1987).

LEMMA 2.5. *Let $\{f_i;\, i \ge 1\}$ be a sequence of functions from a MFS $\mathbb{L}_\rho$, and let $\{r_i;\, i \ge 1\}$ be a Rademacher sequence, that is, a sequence of symmetric iid real rv's with only two values +1 and −1. The following two conditions are equivalent*:

(i) *The series $\sum_i r_i f_i$ converges a.s. in $\mathbb{L}_\rho$.*

(ii) *For some (equivalently, every) $\theta > 0$, the series $\sum_i r_i[\theta f_i]$ converges a.s. in $\mathbb{L}_\rho$ and $\sharp\{i \ge 1\colon \|f_i\|_\rho > \theta^{-1}\} < +\infty$.*

3. Proof of Theorem 1.1. Without loss of generality we may assume that m is a probability measure on $(E, \mathcal{E})$. Indeed, left $\widehat{m}$ be a probability measure on $(E, \mathcal{E})$ such that $m(dx) = k^\alpha(x)\widehat{m}(dx)$ for some nonnegative measurable function k and let $\widehat{h}(t,x) := h(t,x)k(x), t \in T,\ x \in E$. Then the conditions (1.6), (1.7) and (1.8) for the pairs h, m and $\widehat{h}, \widehat{m}$ are equivalent. Moreover, we have

$$\left\{\int_E \widehat{h}(t,x)\widehat{M}(dx);\ t \in T\right\} =_d \left\{\int_E h(t,x)M(dx);\ t \in T\right\},$$

where $\widehat{M}$ is an SαS random measure on $(E, \mathcal{E})$ with the control measure $\widehat{m}$. This in conjunction with Proposition 2.1 allows us to assume that m is a probability measure.

The advantage of this reduction to a probability measure is that now we can use a series representation for the SαS random measure M. Let $\Gamma = \{\Gamma_i;\, i \ge 1\}$ be a sequence of arrival times of a Poisson process with unit arrival rate defined on a probability space $(\Omega_1, \mathcal{F}_1, P_1)$, let $V = \{V_i;\, i \ge 1\}$ be a sequence of independent rv's, defined on a probability space $(\Omega_2, \mathcal{F}_2, P_2)$, with values in E and with the common distribution m and let $r = \{r_i;\, i \ge 1\}$ be a Rademacher sequence defined on a probability space $(\Omega_3, \mathcal{F}_3, P_3)$. In that case [Samorodnitsky and Taqqu (1994)],

$$\left\{\int_E h(t,x)M(dx);\ t \in T\right\} =_d \big\{S(t);\ t \in T\big\}, \tag{3.1}$$

where for every $t \in T$, $S(t)$ is an a.s. convergent series defined on $(\Omega, \mathcal{F}, P) = (\Omega_1 \times \Omega_2 \times \Omega_3, \mathcal{F}_1 \otimes \mathcal{F}_2 \otimes \mathcal{F}_3, P_1 \times P_2 \times P_3)$, given by

$$S(t) = c_\alpha^{1/\alpha} \sum_{i=1}^{\infty} r_i \Gamma_i^{-1/\alpha} h(t, V_i), \qquad t \in T, \tag{3.2}$$

and

$$c_\alpha = \left(\int_0^\infty x^{-\alpha} \sin x \, dx \right)^{-1}.$$

Let us outline the proof of the theorem before giving the full story. In the sufficiency part we show that under conditions (1.6), (1.7) and (1.8), the partial sums of the series S induce a convergent series in $\mathbb{L}_\psi(T, u)$. Then (1.4) will follow by Proposition 2.2. In the necessity part we will use Proposition 2.4 to conclude that if (1.4) holds, then S induces a convergent series in $\mathbb{L}_\psi(T, \mu)$ and then (1.6), (1.7) and (1.8) will be derived. However, to use Proposition 2.4, we have to show first that

$$\int_T \psi\big(|h(t,x)|\big) \mu(dt) < +\infty \quad \text{for } m\text{-a.a. } x \in E. \tag{3.3}$$

By (1.4) and (3.1) we have

$$\int_T \psi\big(|S(t)|\big) \mu(dt) < +\infty \quad \text{a.s.},$$

implying by Proposition 2.1,

$$\int_T \psi\big(|\widetilde{S}(t)|\big) \mu(dt) < +\infty \quad \text{a.s.},$$

where

$$\widetilde{S}(t) = c_\alpha^{1/\alpha} \left(r_1 \Gamma_1^{-1/\alpha} h(t, V_1) - \sum_{i=2}^{\infty} r_i \Gamma_i^{-1/\alpha} h(t, V_i) \right), \qquad t \in T.$$

Inequality (3.3) now follows from the Δ_2-condition for ψ.

Now we are ready for the proof of necessity.

Necessity. (1.4) $\Rightarrow$ (1.6). By (3.1), (3.3) and Propositions 2.1 and 2.4, the series

$$\sum_{i=1}^{\infty} r_i \Gamma_i^{-1/\alpha} h(\cdot, V_i) \text{ converges a.s. in } \mathbb{L}_\psi(T, \mu). \tag{3.4}$$

By Fubini's theorem, (3.4) holds for a.a. fixed $\omega_1 \in \Omega_1$ and $\omega_2 \in \Omega_2$. Then by Lemma 2.5, it follows for every $\theta > 0$,

$$P\Big(\big\{ \theta \Gamma_i^{-1/\alpha} \|h(\cdot, V_i)\|_\psi > 1 \text{ i.o.} \big\} \Big) = 0. \tag{3.5}$$

Note that $\Gamma_i = \sum_{j=1}^{i} e_j$, where e_j's are iid exponential rv's with $Ee_j = 1$. By the strong law of large numbers, we have

$$\lim_{i \to \infty} \Gamma_i / i = 1 \quad \text{a.s.} \tag{3.6}$$

Hence, for all $\theta > 0$,

$$P\Big(\big\{\|h(\cdot, V_i)\|_\psi > i^{1/\alpha}/\theta \text{ i.o.}\big\}\Big) = 0.$$

By the Borell–Cantelli lemma, it then follows that

$$\sum_{i=1}^{\infty} P\Big(\big\{\|h(\cdot, V_i)\|_\psi > i^{1/\alpha}/\theta\big\}\Big) < +\infty,$$

which is obviously equivalent to (1.6).

(1.4) $\Rightarrow$ (1.7). By (3.4), Fubini's theorem and Lemma 2.5, we conclude that for *a.a.* fixed $\omega_1 \in \Omega_1$ and $\omega_2 \in \Omega_2$ and all $\theta > 0$, the series

$$\widetilde{S} = \sum_{i=1}^{\infty} r_i \big[\theta \Gamma_i^{-1/\alpha} h(\cdot, V_i)\big] \text{ converges } P_3\text{-a.s. in } \mathbb{L}_\psi. \tag{3.7}$$

By Fubini's theorem once again, one can conclude that for a.a. fixed $\omega_1 \in \Omega_1$, the series in (3.8) converges $P_{2,3}(:= P_2 \times P_3)$-a.s. in $\mathbb{L}_\psi$. Then by Corollary 2.1 of Kwapien and Woyczynski (1987) for a.a. fixed $\omega_1 \in \Omega_1$ and all $\theta > 0$,

$$E_{2,3} \int_T \psi\left(\left|\sum_{i=1}^{\infty} r_i \big[\theta \Gamma_i^{-1/\alpha} h(\cdot, V_i)\big](t)\right|\right) \mu(dt) < +\infty. \tag{3.8}$$

By the strong law of large numbers (3.6), there are numbers $\gamma_1, \gamma_2 \in (0, \infty)$ such that

$$P_1\left(\left\{\sup_{i \geq 1} \Gamma_i / i \leq \gamma_1\right\}\right) \leq 15/16 \tag{3.9}$$

and

$$P_1\left(\left\{\sup_{i \geq 1} i / \Gamma_i \leq \gamma_2\right\}\right) \geq 15/16. \tag{3.10}$$

Let us denote by $\Omega_1^{(1)}$ the event defined by the left-hand side of (3.9). It follows from (3.5) that

$$P\left(\left\{\sup_{i \geq 1} \Gamma_i^{-1/\alpha} \|h(\cdot, V_i)\|_\psi < +\infty\right\}\right) = 1.$$

Therefore, by Fubini's theorem there is an event $\Omega_1^{(2)}$ of full probability such that for all $\omega_1 \in \Omega_1^{(2)}$,

$$P_2\left(\left\{\sup_{i \geq 1} \Gamma_i^{-1/\alpha} \|h(\cdot, V_i)\|_\psi < +\infty\right\}\right) = 1.$$

Fix any $\widetilde{\omega}_1 \in \Omega_1^{(1)} \cap \Omega_1^{(2)}$ and choose $\theta_0 = \theta_0(\widetilde{\omega}_1) > 0$ so small that

$$P_2\left(\left\{\sup_{i\geq 1}\Gamma_i(\widetilde{\omega}_1)^{-1/\alpha}\|h(\cdot, V_i)\|_\psi > \theta_0^{-1}\right\}\right) \leq 1/16. \tag{3.11}$$

Finally choose $\varepsilon > 0$ so small that

$$P\Big(\big\{|\xi| > \varepsilon(c_\alpha\gamma_1\gamma_2)^{1/\alpha}/\theta_0\big\}\Big) \geq 3/4, \tag{3.12}$$

where ξ is a real rv with the stable distribution $S_\alpha(1,0,0)$. Taking into account the monotonicity of ψ, we have for every $t \in T$,

$$\begin{aligned}
&E_{2,3}\psi\left(\left|\sum_{i=1}^{\infty} r_i\left[\theta_0\Gamma_i(\widetilde{\omega}_1)^{-1/\alpha}h(\cdot, V_i)\right](t)\right|\right)\\
&\quad\geq \psi(\varepsilon\sigma_\alpha(t))P_{2,3}\left(\left\{\left|\sum_{i=1}^{\infty} r_i\left[\theta_0\Gamma_i(\widetilde{\omega}_1)^{-1/\alpha}h(\cdot, V_i)\right](t)\right| > \varepsilon\sigma_\alpha(t)\right\}\right)\\
&\quad\geq \psi(\varepsilon\sigma_\alpha(t))\left(P_{2,3}\left(\left\{\left|\sum_{i=1}^{\infty} r_i\theta_0\Gamma_i(\widetilde{\omega}_1)^{-1/\alpha}h(t, V_i)\right| > \varepsilon\sigma_\alpha(t)\right\}\right)\right.\\
&\qquad\qquad\left. - P_{2,3}\left(\left\{\sup_{i\geq 1}\Gamma_i(\widetilde{\omega}_1)^{-1/\alpha}\|h(\cdot, V_i)\|_\psi > \theta_0^{-1}\right\}\right)\right).
\end{aligned} \tag{3.13}$$

Furthermore, using the contraction principle for probabilities twice, we get by (3.10) and (3.12),

$$\begin{aligned}
&P_{2,3}\left(\left\{\left|\sum_{i=1}^{\infty} r_i\theta_0\Gamma_i(\widetilde{\omega}_1)^{-1/\alpha}h(t, V_i)\right| > \varepsilon\sigma_\alpha(t)\right\}\right)\\
&\quad\geq 2^{-1}P_{2,3}\left(\left\{\left|\sum_{i=1}^{\infty} r_i i^{-1/\alpha}h(t, V_i)\right| > \varepsilon\gamma_1^{1/\alpha}\sigma_\alpha(t)/\theta_0\right\}\right)\\
&\quad\geq 2^{-1}P\left(\left\{\left|\sum_{i=1}^{\infty} r_i i^{-1/\alpha}h(t, V_i)\right| > \varepsilon\gamma_1^{1/\alpha}\sigma_\alpha(t)/\theta_0,\ i \leq \gamma_2\Gamma_i\ \forall\, i\right\}\right)\\
&\quad\geq 4^{-1}\left(P\left(\left\{\left|\sum_{i=1}^{\infty} r_i\Gamma_i^{-1/\alpha}h(t, V_i)\right| > \varepsilon(\gamma_1\gamma_2)^{1/\alpha}\sigma_\alpha(t)/\theta_0\right\}\right)\right.\\
&\qquad\qquad\left. - P\left(\left\{\sup_{i\geq 1} i/\Gamma_i > \gamma_2\right\}\right)\right)\\
&\quad\geq 4^{-1}\Big(P\Big(\big\{|\xi| > \varepsilon(c_0\gamma_1\gamma_2)^{1/\alpha}/\theta_0\big\}\Big) - 1/16\Big) \geq 11/64.
\end{aligned} \tag{3.14}$$

Integrating (3.13) and using (3.11) and (3.14), we conlcude by (3.8) that

$$\infty+ > \int_T E_{2,3}\psi\left(\left|\sum_{i=1}^{\infty} r_i\left[\theta_0\Gamma_i(\widetilde{\omega}_1)^{-1/\alpha}h(t, V_i)\right]\right|\right)\mu(dt) \geq \frac{7}{64}\int_T \psi(\varepsilon\sigma_\alpha(t))\mu(dt).$$

Hence (1.7) holds.

(1.4) $\Rightarrow$ (1.8). Note that if (1.6) and (1.7) hold, then the condition (1.8) is equivalent to

$$\int_T \int_E \int_0^\infty \psi\Big([r|h(\cdot,x)|](t)\Big)\mathbb{I}\Big([r|h(\cdot,x)|](t) > c_0\sigma_\alpha(t)\Big)\frac{dr\,m(dx)\mu(dt)}{r^{1+\alpha}} < +\infty.$$

Because the rv Γ_i has a gamma distribution with i degrees of freedom, we have for each $t \in T$ and $\theta > 0$,

$$\int_E \int_0^\infty \psi\Big([r|h(\cdot,x)|](t)\Big)\mathbb{I}\Big([r|h(\cdot,x)|](t) > c_0\sigma_\alpha(t)\Big)\frac{dr\,m(dx)}{r^{1+\alpha}}$$

$$= \frac{1}{(1+\alpha)\theta^\alpha}\sum_{i=1}^\infty E\Big\{\psi\Big([\theta\Gamma_i^{-1/\alpha}|h(\cdot,V_i)|](t)\Big)\mathbb{I}\Big([\theta\Gamma_i^{-1/\alpha}|h(\cdot,V_i)|] > c_0\sigma_\alpha(t)\Big)\Big\}.$$

Therefore, it is sufficient to prove that for some $\theta > 0$,

$$\int_T \sum_{i=1}^\infty E\Big\{\psi\Big([\theta\Gamma_i^{-1/\alpha}|h(\cdot,V_i)|]\Big)\mathbb{I}\Big([\theta\Gamma_i^{-1/\alpha}|h(\cdot,V_i)|] > c_0\sigma_\alpha\Big)\Big\}\,d\mu < +\infty. \tag{3.15}$$

We know that for a.a. fixed $\omega_1 \in \Omega_1$, the series $\widetilde{S}$ in (3.7) converges in $\mathbb{L}_\psi$ $P_{2,3}$-a.s. for any $\theta > 0$. By Corollaries 2.1 and 2.2 of Kwapien and Woyczynski (1987), for a.a. fixed $\omega_1 \in \Omega_1$, we have

$$\int_T E_2 \sup_{i\ge 1}\psi\Big([\theta\Gamma_i^{-1/\alpha}|h(\cdot,V_i)|](t)\Big)\mu(dt) < +\infty \tag{3.16}$$

for all $\theta > 0$. First we prove that we may replace E_2 in (3.16) by $E_{1,2}$. By the strong law of large numbers (3.6), (3.16) yields

$$\int_T E_2 \sup_{i\ge 1}\psi\Big([\theta i^{-1/\alpha}|h(\cdot,V_i)|](t)\Big)\mu(dt) < +\infty \tag{3.17}$$

for all $\theta > 0$. Now for every positive integer i_0 put

$$M_{i_0} := \inf_{i\ge i_0}\big(\Gamma_i/i\big)^{1/\alpha}. \tag{3.18}$$

It is a simple computation to check that for any $H > 0$ there is an integer $i_0(H) > 0$ such that

$$EM_{i_0(H)}^{-H} < +\infty. \tag{3.19}$$

Observe that for any $i > i_0$ and all $t \in T$,

$$[\theta i^{-1/\alpha}|h(\cdot,V_i)|](t) \ge \big(M_{i_0}\wedge 1\big)\,[\theta\Gamma_i^{-1/\alpha}|h(\cdot,V_i)|](t)$$

and so for every $i > i_0$ and all $t \in T$, we have

$$\psi\Big(\big[\theta i^{-1/\alpha}|h(\cdot, V_i)|\big](t)\Big) \geq A^{-1}\big(M_{i_0} \wedge 1\big)^H \psi\Big(\big[\theta \Gamma_i^{-1/\alpha}|h(\cdot, V_i)|\big](t)\Big), \tag{3.20}$$

where $H = \log_2 A$ [see (1.3)]. We conclude by (3.17) and (3.19) that

$$\begin{aligned} &\int_T E_{1,2} \sup_{i > i_0(H)} \psi\Big(\big[\theta\Gamma_i^{-1/\alpha}|h(\cdot, V_i)|\big](t)\Big)\mu(dt) \\ &\quad \leq A E_1\big(M_{i_0(H)} \wedge 1\big)^{-H} \int_T E_2 \sup_{i > i_0(H)} \psi\Big(\big[\theta i^{-1/\alpha}|h(\cdot, V_i)|(t)\big]\Big)\mu(dt) \\ &\quad < +\infty \end{aligned} \tag{3.21}$$

for all $\theta > 0$. Furthermore, for every positive integer i_0 we have

$$\begin{aligned} &\int_T E_{1,2} \max_{1 \leq i \leq i_0} \psi\Big(\big[\theta\Gamma_i^{-1/\alpha}|h(\cdot, V_i)|\big](t)\Big)\mu(dt) \\ &\quad \leq \sum_{i=1}^{i_0} E_2 \int_T \psi\big(|h(t, V_i)/\|h(\cdot, V_i)\|_\psi\big)\mu(dt) \leq i_0 < +\infty. \end{aligned}$$

This in conjunction with (3.21) yields

$$\int_T E_{1,2} \sup_{i \geq 1} \psi\Big(\big[\theta\Gamma_i^{-1/\alpha}|h(\cdot, V_i)|\big](t)\Big)\,\mu(dt) < +\infty \tag{3.22}$$

for all $\theta > 0$. Therefore, for every integer i_0 and all $\theta > 0$, we have

$$\begin{aligned} &\int_T \sum_{i=1}^{i_0} E\bigg[\psi\Big(\big[\theta\Gamma_i^{-1/\alpha}|h(\cdot, V_i)|(t)\big]\Big) \\ &\qquad \times \mathbb{I}\Big(\big[\theta\Gamma_i^{-1/\alpha}|h(\cdot, V_i)|(t)\big] > c_0\sigma_\alpha(t)\Big)\bigg]\mu(dt) < +\infty. \end{aligned} \tag{3.23}$$

To prove finiteness of the remainder of the series in (3.15), we proceed as follows. By Proposition 2.2 of Kwapien and Woyczynski (1987), for every $t \in T$, $\theta > 0$ and $y > 0$, we have

$$\begin{aligned} &P_2\bigg(\Big\{\sup_{i \geq 1}\big[\theta i^{-1/\alpha}|h(\cdot, V_i)|\big](t) > y\Big\}\bigg) \\ &\quad \geq \Bigg(1 - 2P_{2,3}\bigg(\bigg\{\bigg|\sum_{i=1}^{\infty} r_i\big[\theta i^{-1/\alpha}h(\cdot, V_i)\big](t)\bigg| > y\bigg\}\bigg)\Bigg) \\ &\qquad \times \sum_{i=1}^{\infty} P_2\Big(\Big\{\big[\theta i^{-1/\alpha}|h(\cdot, V_i)\big](t) > y\Big\}\Big). \end{aligned}$$

For γ_1 as in (3.9) choose $\theta > 0$ so small that

$$P\Big(\{|\xi| > c_\alpha^{1/\alpha} c_0/2\theta\gamma_1^{1/\alpha}\}\Big) \le 1/32,$$

where ξ is a real rv with the stable distribution $S_\alpha(1,0,0)$. Using the contraction principle for probabilities, we get for all $y \ge c_0\sigma_\alpha(t)/2$,

$$\begin{aligned}
&P_{2,3}\left(\left\{\left|\sum_{i=1}^{\infty} r_i\left[\theta i^{-1/\alpha} h(\cdot, V_i)\right](t)\right| > y\right\}\right)\\
&\quad\le 2P_{2,3}\left(\left\{\left|\sum_{i=1}^{\infty} r_i i^{-1/\alpha} h(t, V_i)\right| > y/\theta\right\}\right)\\
&\quad\le 4P\left(\left\{\left|\sum_{i=1}^{\infty} r_i \Gamma_i^{-1/\alpha} h(t, V_i)\right| > y/\theta\gamma_1^{1/\alpha}\right\}\right) + 2P_1\left(\left\{\sup_{i\ge 1}\Gamma_i/i > \gamma_1\right\}\right)\\
&\quad\le 4P\Big(\{|\xi| > c_0 c_\alpha^{1/\alpha}/2\theta\gamma_1^{1/\alpha}\}\Big) + 1/8 \le 1/4.
\end{aligned}$$

Therefore, for all $t \in T$ and $y \ge c_0\sigma_\alpha(t)/2$, we have

$$P_2\left(\left\{\sup_{i\ge 1}\left[\theta i^{-1/\alpha}|h(\cdot, V_i)|\right](t) > y\right\}\right) \ge \tfrac{1}{2}\sum_{i=1}^{\infty} P_2\left(\left\{\left[\theta i^{-1/\alpha}|h(\cdot, V_i)|\right](t) > y\right\}\right).$$

Observe that for all $t \in T$ and $0 < y < c_0\sigma_\alpha(t)/2$,

$$\begin{aligned}
&P_2\left(\left\{\left[\theta i^{-1/\alpha}|h(\cdot, V_i)|\right](t)\mathbb{I}\left(\left[\theta i^{-1/\alpha}|h(\cdot, V_i)|\right](t) > c_0\sigma_\alpha/2\right) > y\right\}\right)\\
&\qquad = P_2\left(\left\{\left[\theta i^{-1/\alpha}|h(\cdot, V_i)|\right](t) > c_0\sigma_\alpha/2\right\}\right).
\end{aligned}$$

Therefore, for all $t \in T$ and $y > 0$, we have

$$\begin{aligned}
&P_2\left(\left\{\sup_{i\ge 1}\left[\theta i^{-1/\alpha}|h(\cdot, V_i)|\right](t) > y\right\}\right)\\
&\quad\ge \tfrac{1}{2}\sum_{i=1}^{\infty} P_2\left(\left\{\left[\theta i^{-1/\alpha}|h(\cdot, V_i)|\right](t)\right.\right.\\
&\qquad\left.\left.\times\ \mathbb{I}\left(\left[\theta i^{-1/\alpha}|h(\cdot, V_i)|\right](t) > c_0\sigma_\alpha(t)/2\right) > y\right\}\right).
\end{aligned}\tag{3.24}$$

Let ν_ψ be a Lebesgue–Stieltjes measure such that

$$\psi(u) = \int_0^u \nu_\psi(dy) \qquad \forall\, u \geq 0.$$

Then (3.24) yields

$$\begin{aligned}
&E_2\psi\Big(\sup_{i\geq 1}\big[\theta i^{-1/\alpha}|h(\cdot,V_i)|\big](t)\Big)\\
&\quad= \int_0^\infty P_2\bigg(\Big\{\sup_{i\geq 1}\big[\theta i^{-1/\alpha}|h(\cdot,V_i)|\big](t) > y\Big\}\bigg)\nu_\psi(dy)\\
&\quad\geq \tfrac{1}{2}\sum_{i=1}^\infty \int_0^\infty P_2\bigg(\Big\{\big[\theta i^{-1/\alpha}|h(\cdot,V_i)|\big](t)\\
&\qquad\qquad \times \mathbb{I}\Big(\big[\theta i^{-1/\alpha}|h(\cdot,V_i)|\big](t) > c_0\sigma_\alpha(t)\Big) > y\Big\}\bigg)\nu_\psi(dy)\\
&\quad= \tfrac{1}{2}\sum_{i=1}^\infty E_2\psi\Big(\big[\theta i^{-1/\alpha}|h(\cdot,V_i)|\big](t)\mathbb{I}\Big(\big[\theta i^{-1/\alpha}|h(\cdot,V_i)|\big](t) > c_0\sigma_\alpha(t)\Big)\Big)
\end{aligned}$$

for every $t \in T$. In the notation of (3.18) with $H = \log_2 A$, we have now for every $i > i_0(2H)$,

$$\begin{aligned}
&E_{1,2}\Big[\psi\Big(\big[\theta\Gamma_i^{-1/\alpha}|h(\cdot,V_i)|\big](t)\Big)\mathbb{I}\Big(\big[\theta\Gamma_i^{-1/\alpha}|h(\cdot,V_i)|\big](t) > c_0\sigma_\alpha(t)\Big)\Big]\\
&\quad\leq AE_{1,2}\bigg\{\big(M_{i_0(2H)}\wedge 1\big)^{-H}\psi\Big(\big[\theta i^{-1/\alpha}|h(\cdot,V_i)|\big](t)\Big)\\
&\qquad\qquad \times \mathbb{I}\Big(\big[\theta\Gamma_i^{-1/\alpha}|h(\cdot,V_i)|\big](t) > c_0\sigma_\alpha(t)\Big)\\
&\qquad\qquad \times \Big[\mathbb{I}\Big(\big[\theta i^{-1/\alpha}|h(\cdot,V_i)|\big](t) > c_0\sigma_\alpha(t)/2\Big)\\
&\qquad\qquad\qquad + \mathbb{I}\Big(\big[\theta i^{-1/\alpha}|h(\cdot,V_i)|\big](t) \leq c_0\sigma_\alpha(t)/2\Big)\Big]\bigg\} \qquad (3.25)\\
&\quad\leq AE_1\big(M_{i_0(2H)}\wedge 1\big)^{-H}E_2\Big[\psi\Big(\big[\theta i^{-1/\alpha}|h(\cdot,V_i)|\big](t)\Big)\\
&\qquad\qquad \times \mathbb{I}\Big(\big[\theta i^{-1/\alpha}|h(\cdot,V_i)|\big](t) > c_0\sigma_\alpha(t)/2\Big)\Big]\\
&\qquad + A\Big(E_1\big(M_{i_0(2H)}\wedge 1\big)^{-2H}\Big)^{1/2}\psi\big(c_0\sigma_\alpha(t)/2\big)\Big(P_1\Big(\big\{i/\Gamma_i > 2^\alpha\big\}\Big)\Big)^{1/2}
\end{aligned}$$

Integrating (3.25), we obtain

$$\int_T \sum_{i > i_0(2H)} E_{1,2}\Big\{\psi\Big(\big[\theta\Gamma_i^{-1/\alpha}|h(\cdot, V_i)|\big](t)\Big) \times \mathbb{I}\Big(\big[\theta\Gamma_i^{-1/\alpha}|h(\cdot, V_i)|\big](t) > c_0\sigma_\alpha(t)\Big)\Big\}\mu(dt)$$

$$\leq C_1 \int_T E_2\psi\Big(\sup_{i\geq 1}\big[\theta i^{-1/\alpha}|h(\cdot, V_i)|\big](t)\Big)\mu(dt) + C_2 \sum_{i > i_0(2H)} \Big(P_1\big(\{\Gamma_i < 2^{-\alpha}i\}\big)\Big)^{1/2} \int_T \psi(c_0\sigma_\alpha(t)/2)\mu(dt) < +\infty$$

by (3.22) and (1.7) for some finite constants C_1 and C_2. This in conjunction with (3.23) yields (3.15). Hence (1.8) holds and the proof of the necessity is complete.

Sufficiency. By Propositon 2.2 it is enough to prove that the series (3.2) converges a.s. in $\mathbb{L}_\psi$. It is enough to show P_3-a.s. convergence of this series for a.a. fixed $\omega_1 \in \Omega_1$ and $\omega_2 \in \Omega_2$. This allows us to use Lemma 2.5 in the opposite direction. Because the condition (1.6) implies (3.5), it remains to show that for a.a. fixed $\omega_1 \in \Omega_1$ and $\omega_2 \in \Omega_2$,

$$\int_T E_3\psi\left(\left|\sum_{i=1}^{\infty} r_i\big[\theta\Gamma_i^{-1/\alpha}h(\cdot, V_i)\big](t)\right|\right)\mu(dt) < +\infty$$

for some $\theta > 0$. This will obviously follow once we prove that for a.a. fixed $\omega_1 \in \Omega_1$,

$$\int_T E_{2,3}\psi\left(\left|\sum_{i=1}^{\infty} r_i\big[\theta\Gamma_i^{-1/\alpha}h(\cdot, V_i)\big](t)\right|\right)\mu(dt) < +\infty$$

for some $\theta > 0$. By Proposition 2.1 of Kwapien and Woyczynski (1987), for any fixed $\omega_1 \in \Omega_1$ and for all $t \in T$ and $\theta > 0$, we have

$$E_{2,3}\psi\left(\left|\sum_{i=1}^{\infty} r_i\big[\theta\Gamma_i^{-1/\alpha}h(\cdot, V_i)\big](t)\right|\right) \leq \frac{A E_{2,3}\sup_{i\geq 1}\psi\Big(\big[\theta\Gamma_i^{-1/\alpha}|h(\cdot, V_i)|\big](t)\Big) + 8A^2\psi\big(\sigma_\alpha(t)\big)}{1/3 - 4A^2 P_{2,3}\left(\left\{\psi\Big(\big|\sum_{i=1}^{\infty} r_i\big[\theta\Gamma_i^{-1/\alpha}h(\cdot, V_i)\big](t)\big|\Big) > \psi\big(\sigma_\alpha(t)\big)\right\}\right)}. \tag{3.26}$$

By the strong law of large numbers (3.6), the event

$$\Omega_1^{(3)} := \Big\{\lim_{n\to\infty}\Gamma_n/n = 1\Big\} \cap \{\Gamma_1 > 0\}$$

has P_1-probability 1. Fix any $\widetilde{\omega} \in \Omega_1^{(3)}$. Choose a finite number $\gamma_3 = \gamma_3(\widetilde{\omega})$ such that

$$P_1\left(\left\{\sup_{i\geq 1} \Gamma_i/\Gamma_i(\widetilde{\omega}) > \gamma_3\right\}\right) \leq 1/96A^2.$$

For this γ_3 choose a positive number $\theta_0 = \theta_0(\widetilde{\omega})$ such that

$$P_1\left(\left\{|\xi| > c_\alpha^{1/\alpha}/\theta_0\gamma_3^{1/\alpha}\right\}\right) \leq 1/96A^2,$$

where ξ is a real rv with a stable distribution $S_\alpha(1,0,0)$. Now by the contraction principle for probabilities, we have for each $t \in T$,

$$\begin{aligned}
P_{2,3}&\left(\left\{\psi\left(\left|\sum_{i=1}^{\infty} r_i\left[\theta_0\Gamma_i(\widetilde{\omega})^{-1/\alpha}h(\cdot,V_i)\right](t)\right|\right) > \psi\big(\sigma_\alpha(t)\big)\right\}\right)\\
&= P_{2,3}\left(\left\{\left|\sum_{i=1}^{\infty} r_i\left[\theta_0\Gamma_i(\widetilde{\omega})^{-1/\alpha}h(\cdot,V_i)\right](t)\right| > \sigma_\alpha(t)\right\}\right)\\
&\leq 2P_{2,3}\left(\left\{\left|\sum_{i=1}^{\infty} r_i\theta_0\Gamma_i(\widetilde{\omega})^{-1/\alpha}h(t,V_i)\right| > \sigma_\alpha(t)\right\}\right)\\
&\leq 2P\left(\left\{\left|\sum_{i=1}^{\infty} r_i\Gamma_i^{-1/\alpha}h(t,V_i)\right| > \sigma_\alpha(t)/\theta_0\gamma_3^{1/\alpha}\right\}\right)\\
&\quad + 2P_1\left(\left\{\sup_{i\geq 1}\Gamma_i/\Gamma_i(\widetilde{\omega}) > \gamma_3\right\}\right) \leq 1/24A^2.
\end{aligned}$$

By (3.26) we conclude that for every $\widetilde{\omega} \in \Omega_1^{(3)}$ and $t \in T$,

$$\begin{aligned}
E_{2,3}&\psi\left(\left|\sum_{i=1}^{\infty} r_i\left[\theta_0\Gamma_i(\widetilde{\omega})^{-1/\alpha}h(\cdot,V_i)\right](t)\right|\right)\\
&\leq 6A\left\{E_2\sup_{i\geq 1}\psi\Big(\left[\theta_0\Gamma_i(\widetilde{\omega})^{-1/\alpha}|h(\cdot,V_i)|\right](t)\Big) + 8A\psi\big(\sigma_\alpha(t)\big)\right\}.
\end{aligned}$$

By (1.7) and the Δ_2-condition, it is enough to show that

$$\int_T E_{1,2}\sup_{i\geq 1}\psi\Big(\left[\theta\Gamma_i^{-1/\alpha}|h(\cdot,V_i)|\right](t)\Big)\mu(dt) < +\infty \tag{3.27}$$

for any $\theta > 0$. Observe that for all $t \in T$ and $i \geq 1$,

$$\begin{aligned}
\psi&\Big(\left[\theta\Gamma_i^{-1/\alpha}|h(\cdot,V_i)|\right](t)\Big)\\
&\leq \psi\big(c_0\sigma_\alpha(t)\big) + \psi\Big(\left[\theta\Gamma_i^{-1/\alpha}|h(\cdot,V_i)|\right](t)\Big)\\
&\qquad\times \mathbb{I}\Big(\left[\theta\Gamma_i^{-1/\alpha}|h(\cdot,V_i)|\right](t) > c_0\sigma_\alpha(t)\Big).
\end{aligned}$$

Therefore,

$$\begin{aligned}\int_T E_{1,2} \sup_{i \geq 1} \psi\Big(\big[\theta\Gamma_i^{-1/\alpha}|h(\cdot, V_i)|\big](t)\Big)\mu(dt) \\ \leq \int_T \psi\big(c_0\sigma_\alpha(t)\big)\mu(dt) \\ + \int_T \sum_{i=1}^{\infty} E\Big\{\psi\Big(\big[\theta\Gamma_i^{-1/\alpha}|h(\cdot, V_i)|\big](t)\Big) \\ \times \mathbb{I}\Big(\big[\theta\Gamma_i^{-1/\alpha}|h(\cdot, V_i)|\big](t) > c_0\sigma_\alpha(t)\Big)\Big\}\mu(dt).\end{aligned}$$

Because condition (1.8) is equivalent to (3.15) [under (1.6) and (1.7)], the right-hand side of the last inequality is finite and this implies (3.27).

The proof of Theorem 1.1 is now complete. □

4. Some consequences of Theorem 1.1. To prove Corollaries 1.3 and 1.4 we need some characterization results of M-functions due to Matuszewska (1962). Recall that a function φ is *equivalent* to a function ψ if there are positive constants a, b, k_1, k_2 such that

$$a\varphi(k_1 u) \leq \psi(u) \leq b\varphi(k_2 u) \qquad \forall\ u > 0.$$

LEMMA 4.1. *Let ψ be an M-function and let $0 < p < +\infty$. The following are equivalent*:

(i) *There are finite constants C_1, C_2 such that*

$$\psi(xu) \leq C_1 x^p \psi(C_2 u); \tag{4.1}$$

for all $x \leq 1$ and $u > 0$.

(ii) *There exists an M-function φ equivalent to ψ and such that the function $u \to \varphi(u^{1/p})$ is convex.*

PROOF. The property (i) of ψ is equivalent to $\psi(u) = u^p\chi(u)$ for all $u > 0$, where χ is a pseudo-increasing function, that is, there are positive constants m, n such that

$$\chi(v) \geq m\chi(nu) \quad \text{for all } v \geq u > 0.$$

The proof that this is equivalent to property (ii) of ψ is the same as the proof of the statement 2.7.3 of Matuszewska (1962) for large u. □

Now we are ready to prove Corollary 1.3.

PROOF OF COROLLARY 1.3. It is enough to show that (1.6) implies (1.8) and (1.7). Let I_3 denote the integral (1.8). By the assumption (1.14), we have

$$\begin{aligned}
I_3 &= \int_E \int_T \|h(\cdot,x)\|_\psi^\alpha \int_0^1 \psi\left(\frac{r|h(t,x)|}{\|h(\cdot,x)\|_\psi}\right) \mathbb{I}\left(\frac{r|h(t,x)|}{\|h(\cdot,x)\|_\psi} > c_0\sigma_\alpha(t)\right) \frac{dr\, \mu(dt) m(dx)}{r^{1+\alpha}} \\
&\le C \int_0^1 r^{p-\alpha-1}\, dr \int_E \|h(\cdot,x)\|_\psi^\alpha \int_T \psi\left(\frac{|h(t,x)|}{\|h(\cdot,x)\|_\psi}\right) \mu(dt)\, m(dx) \\
&\le \frac{C}{p-\alpha} \int_E \|h(\cdot,x)\|_\psi^\alpha m(dx).0
\end{aligned}$$

Thus (1.8) follows from (1.6). Turning to this integral I_2 in (1.7), let φ be an M-function equivalent to ψ from Lemma 4.1 and let I_1 denote the integral in (1.6). Using Jensen's inequality twice and the Δ_2-condition (1.3), we get

$$\begin{aligned}
I_2 &\le b \int_T \varphi\big(k_2 \sigma_\alpha(t)\big) \mu(dt) \\
&= b \int_T \varphi\left(k_2 \left(\int_E \frac{I_1 |h(t,x)|^\alpha}{\|h(\cdot,x)\|_\psi^\alpha} \frac{\|h(\cdot,x)\|_\psi^\alpha}{I_1} m(dx)\right)^{1/\alpha}\right) \mu(dt) \\
&\le b \int_T \varphi\left(\left(\int_E \left(\frac{k_2 I_1^{1/\alpha} |h(t,x)|}{\|h(\cdot,x)\|_\psi}\right)^p \frac{\|h(\cdot,x)\|_\psi^\alpha}{I_1} m(dx)\right)^{1/p}\right) \mu(dt) \\
&\le b \int_E \int_T \varphi\left(\frac{k_2 I_1^{1/\alpha} |h(t,x)|}{\|h(\cdot,x)\|_\psi}\right) \mu(dt) \frac{\|h(\cdot,x)\|_\psi^\alpha}{I_1} m(dx) \\
&\le \frac{b}{a} A \left(\frac{k_2}{k_1} I_1^{1/\alpha}\right)^{\log_2 A}
\end{aligned}$$

Therefore, (1.6) also implies (1.7) and the proof of Corollary 1.3 is complete. □

LEMMA 4.2. *Let ψ be an M-function and let $0 < p < +\infty$. The following statements are equivalent*:

(i) *There are finite constants C_1, C_2 such that the inequality* (4.1) *holds for all $x \ge 1$ and $u > 0$.*

(ii) *There exists an M-function φ equivalent to ψ and such that the function $u \to \varphi(u^{1/p})$ is concave.*

The proof of Lemma 4.2 is the same as the proof of Lemma 4.1 with the function χ being pseudo-decreasing instead of pseudo-increasing.

PROOF OF COROLLARY 1.4. It is enough to show that (1.7) implies (1.8) and (1.6). Let I_3 be again the integral (1.8). By the assumption (1.16), we have

$$\begin{aligned} I_3 &\le \int_T \sigma_\alpha^\alpha(t) \int_{c_0\sigma_\alpha(t)}^\infty \psi(r) \frac{dr\,\mu(dt)}{r^{1+\alpha}} \\ &= \frac{1}{c_0^\alpha} \int_T \int_1^\infty \psi\big(rc_0\sigma_\alpha(t)\big) \frac{dr\,\mu(dt)}{r^{1+\alpha}} \\ &\le \frac{1}{c_0^\alpha(\alpha-p)} \int_T \psi\big(c_0\sigma_\alpha(t)\big)\mu(dt). \end{aligned}$$

Thus (1.8) follows from (1.7). Turning to the integral I_1 in (1.6), observe first that under (1.7), $\|h(\cdot,x)\|_\psi < \infty$ for m-a.a. $x \in E$. Indeed, by (1.9) this is equivalent to saying that

$$\int_T \psi\big(|h(t,x)|\big)\mu(dt) < +\infty, \quad m\text{-a.e.,} \tag{4.2}$$

but (1.7) implies that

$$\begin{aligned} \infty + &> \int_T \varphi\left(k_1\left(\int_E |h(t,x)|^\alpha m(dx)\right)^{1/\alpha}\right)\mu(dt) \\ &= \int_T \varphi\left(k_1\left(\int_E |h(t,x)|^\alpha g^{-1}(x) g(x)\, m(dx)\right)^{1/\alpha}\right)\mu(dt) \\ &\ge \int_T \varphi\left(k_1\left(\int_E |h(t,x)|^p g^{-p/\alpha}(x) g(x)\, m(dx)\right)^{1/p}\right)\mu(dt) \\ &\ge \int_T \left(\int_E \varphi\big(k_1|h(t,x)| g^{-1/\alpha}(x)\big)\, g(x)\, m(dx)\right)\mu(dt) \\ &\ge \frac{1}{b}\int_E \left(\int_T \psi\left(\frac{k_1}{k_2}|h(t,x)| g^{-1/\alpha}(x)\right)\mu(dt)\right) g(x)\, m(dx), \end{aligned}$$

where $g > 0$ is such that $\int_E g\,dm = 1$ and φ is the function from Lemma 4.2 equivalent to ψ. By the Δ_2-property of ψ, (4.2) follows. Now, for every integer $n \ge 1$ denote

$$E_n := \{x \in A_n : \|h(\cdot,x)\|_\psi^\alpha \le n\},$$

where $\{A_n;\ n \ge 1\}$ is a sequence of measurable sets of finite m-measure increasing to E and

$$I_1(n) := \int_{E_n} \|h(\cdot,x)\|_\psi^\alpha m(dx).$$

Therefore, it is enough to show that (1.7) and the assumption that $I_1(n)$ tends to infinity as $n \to \infty$ lead to a contradiction. Using again Jensen's inequality twice, we get

$$\begin{aligned}
&\int_T \psi\left(\frac{k_2\sigma_\alpha(t)}{k_1 I_1^{1/\alpha}(n)}\right)\mu(dt) \\
&\geq a\int_T \varphi\left(k_2\left(\int_{E_n} \frac{|h(t,x)|^\alpha}{\|h(\cdot,x)\|_\psi^\alpha}\frac{\|h(\cdot,x)\|_\psi^\alpha}{I_1(n)}m(dx)\right)^{1/\alpha}\right)\mu(dt) \qquad (4.3)\\
&\geq a\int_{E_n}\int_T \varphi\left(\frac{k_2|h(t,x)|}{\|h(\cdot,x)\|_\psi}\right)\mu(dt)\frac{\|h(\cdot,x)\|_\psi^\alpha}{I_1(n)}m(dx) \geq \frac{a}{b}.
\end{aligned}$$

By the Δ_2-condition of ψ and (1.7), the left-hand side of (4.3) tends to zero if $I_1(n)$ tends to infinity as $n \to \infty$. This contradiction completes the proof of Corollary 1.4. □

Proof of Corollary 1.5. We need to show that (1.6) and (1.7) imply (1.8). With I_3 again denoting the integral (1.8), we have, by the assumption (1.17),

$$\begin{aligned}
I_3 &= \int_E \|h(\cdot,x)\|_\psi^\alpha \int_T\int_0^1 \psi\left(\frac{r|h(t,x)|}{\|h(\cdot,x)\|_\psi}\right)\mathbb{I}\left(\frac{r|h(t,x)|}{\|h(\cdot,x)\|_\psi} > c_0\sigma_\alpha(t)\right)\frac{dr\,\mu(dt)m(dx)}{r^{1+\alpha}}\\
&\leq C\int_E \|h(\cdot,x)\|_\psi^\alpha m(dx)\int_0^1 \frac{\psi(r)\,dr}{r^{1+\alpha}}.
\end{aligned}$$

Under the first assumption in (1.18), (1.8) follows from (1.6). Under the second assumption in (1.18), we get by (1.17),

$$\begin{aligned}
I_3 &\leq \int_T \sigma_\alpha^\alpha(t)\int_{c_0\sigma_\alpha(t)}^\infty \psi(r)\frac{dr\,\mu(dt)}{r^{1+\alpha}}\\
&= c_0^{-\alpha}\int_T\int_1^\infty \psi\big(rc_0\sigma_\alpha(t)\big)\frac{dr\,\mu(dt)}{r^{1+\alpha}}\\
&\leq c_0^{-\alpha}\int_T \psi\big(c_0\sigma_\alpha(t)\big)\mu(dt)\int_1^\infty \frac{\psi(r)\,dr}{r^{1+\alpha}}.
\end{aligned}$$

Thus (1.8) follows from (1.7). □

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