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INFINITE LIMITS AND INFINITE LIMIT POINTS OF RANDOM WALKS AND TRIMMED SUMS

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We consider infinite limit points (in probability) for sums and lightly trimmed sums of i.i.d. random variables normalized by a nonstochastic sequence. More specifically, let $X_1, X_2, \ldots$ be independent random variables with common distribution F. Let $M_n^{(r)}$ be the rth largest among $X_1, \ldots, X_n$; also let $X_n^{(r)}$ be the observation with the rth largest absolute value among $X_1, \ldots, X_n$. Set $S_n = \Sigma_1^n X_i$, ${}^{(r)}S_n = S_n - M_n^{(1)} - \cdots - M_n^{(r)}$ and ${}^{(r)}\tilde{S}_n = S_n - X_n^{(1)} - \cdots - X_n^{(r)}$ (${}^{(0)}S_n = {}^{(0)}\tilde{S}_n = S_n$). We find simple criteria in terms of F for ${}^{(r)}S_n/B_n \to_P \pm\infty$ (i.e., ${}^{(r)}S_n/B_n$ tends to ∞ or to $-\infty$ in probability) or ${}^{(r)}\tilde{S}_n/B_n \to_P \pm\infty$ when $r = 0, 1, \ldots$. Here $B_n \uparrow \infty$ may be given in advance, or its existence may be investigated. In particular, we find a necessary and sufficient condition for ${}^{(r)}S_n/n \to_P \infty$. Some equivalences for the divergence of $|{}^{(r)}\tilde{S}_n|/|X_n^{(r)}|$, or of ${}^{(r)}S_n/(X^-)_n^{(s)}$, where $(X^-)_n^{(s)}$ is the sth largest of the negative parts of the X_i, and for the convergence $P\{S_n > 0\} \to 1$, as $n \to \infty$, are also proven. In some cases we treat divergence along a subsequence as well, and one such result provides an equivalence for a generalized iterated logarithm law due to Pruitt.

1. Introduction.

Let the random walk

$$S_n = X_1 + X_2 + \cdots + X_n$$

denote our fortune after playing n games of chance. Under what conditions on the distribution F of the increments X_i will we win, with probability approaching 1, as $n \to \infty$? In other words, when does $P\{S_n > 0\}$ converge to 1 as $n \to \infty$? Somewhat surprisingly, necessary and sufficient conditions for this have not previously been derived. We give such a condition in this paper, and observe that it encapsulates a certain asymmetry aspect of F. We further show that we will win with probability approaching 1 as $n \to \infty$ if and only if, in fact, $S_n \to_P \infty$; in other words, we win a large amount, eventually, in probability.

A natural extension of this result is to study divergences of the form $S_n/B_n \to_P \infty$ with B_n a nonstochastic sequence increasing to ∞. In particular, when does the weak law of large numbers fail, in the sense that $S_n/n \to_P \infty$, $S_n/n \to_P -\infty$ or $|S_n|/n \to_P \infty$? It turns out that these kinds of behavior depend on the dominance of the large over the small values of X_i, or on the dominance of those values of X_i large in modulus. Thus it will be natural also to study the relationships between S_n and the large and small order statistics of $X_1, X_2, \ldots, X_n$;

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in particular, we consider *lightly trimmed sums*, where we delete from S_n a bounded number of the large or small order statistics.

To further motivate and state our results, we require the following notation. We will assume that X_i are independent and identically distributed random variables with distribution F, and let $M_n^{(1)} \geq M_n^{(2)} \geq \cdots \geq M_n^{(n)}$ denote $X_1, X_2, \ldots, X_n$ arranged in decreasing order, with the indices of the $M_n^{(i)}$ taken in increasing order in case of ties. Similarly, let $X_n^{(1)}, \ldots, X_n^{(n)}$ denote the sample arranged in decreasing order of absolute value, with a similar convention for ordering of ties. We will also need a notation for the small values of the sample, and for these it will be convenient to define $(X^-)_n^{(1)} \geq (X^-)_n^{(2)} \geq \cdots \geq (X^-)_n^{(n)}$ as the order statistics of $X_1^-, \ldots, X_n^-$ when $F(0-) > 0$, where

$$X_i^+ = \max(X_i, 0), \qquad X_i^- = X_i^+ - X_i.$$

If $F(0-) = 0$ we take $(X^-)_n^{(j)} = 0$ for $1 \leq j \leq n$.

We also need sums trimmed by removing large values:

$${}^{(r)}S_n = S_n - M_n^{(1)} - \cdots - M_n^{(r)}, \qquad n \geq r \geq 1$$

(with ${}^{(0)}S_n = S_n$), and sums trimmed by removing values large in modulus:

$${}^{(r)}\widetilde{S}_n = S_n - X_n^{(1)} - \cdots - X_n^{(r)}, \qquad n \geq r \geq 1$$

(again with ${}^{(0)}\widetilde{S}_n = S_n$).

Many authors have studied the relationship between S_n and the extreme order statistics. We refer to Kesten and Maller (1992) for a discussion and references to relevant literature. Kesten and Maller (1992) obtained necessary and sufficient conditions for the divergences ${}^{(r)}S_n/M_n^{(r)} \to_P \infty$ and ${}^{(r)}\widetilde{S}_n/|X_n^{(r)}| \to_P \infty$, $r = 1, 2, 3, \ldots$. These, in fact, are equivalent to each other and to the *positive relative stability* of S_n, that is, to the existence of a nonstochastic sequence $B_n \uparrow \infty$ for which $S_n/B_n \to_P 1$. This in turn is equivalent to ${}^{(r)}S_n/B_n \to_P 1$ and to ${}^{(r)}\widetilde{S}_n/B_n \to_P 1$, $r = 1, 2, \ldots$.

We begin by looking for necessary and sufficient conditions for the existence of a nonstochastic sequence $B_n \uparrow \infty$ for which ${}^{(r)}S_n/B_n \to_P \infty$ or, equivalently, as it turns out, for ${}^{(r)}\widetilde{S}_n/B_n \to_P \infty$. In Theorem 2.1 we show that such a sequence exists if and only if ${}^{(r)}S_n$ or ${}^{(r)}\widetilde{S}_n$ dominates (with probability approaching 1) the extreme negative order statistics or, equivalently, if ${}^{(r)}S_n \to_P \infty$ or ${}^{(r)}\widetilde{S}_n \to_P \infty$. These in turn are equivalent to $P\{S_n > 0\} \to 1$ as $n \to \infty$. An analytic condition is also given for these. Theorem 2.2 finds analogous analytic conditions for ${}^{(r)}S_n/B_n \to_P \infty$ or ${}^{(r)}\widetilde{S}_n/B_n \to_P \infty$ when B_n is a fixed sequence given in advance. The special case $B_n = n$ (in Theorem 2.3) gives a necessary and sufficient condition for $S_n/n \to_P \infty$, which solves a problem considered by Baum (1963) and Révész (1968), pages 80 and 81.

In these theorems it turns out that $(X^-)_n^{(1)}$ is small in probability with respect to B_n. When we replace X_i by $-X_i$ in these theorems, we obtain conditions for ${}^{(r)}S_n/B_n \to_P -\infty$ or ${}^{(r)}\widetilde{S}_n/B_n \to_P -\infty$, with $M_n^{(1)}$ small in probability with respect to B_n. Next consider the situation when S_n is in the domain of attraction of the normal distribution with centering and norming sequence A_n and B_n, which we write as

$$\frac{S_n - A_n}{B_n} \to_D N(0, 1).$$

Criteria for convergence of triangular arrays [e.g., Gnedenko and Kolmogorov (1968), Theorem 26.2] show that this implies $nP\{|X| > \varepsilon B_n\} \to 0$ for each $\varepsilon > 0$ and thus $|X_n^{(1)}|/B_n \to_P 0$. Hence $|X_n^{(r)}|$ is small in probability by comparison with B_n. When this occurs we might expect S_n to become large *in modulus* by comparison with the large values in the sample, so we are led to investigate when $|{}^{(r)}S_n|/|X_n^{(r)}| \to_P \infty$ or $|{}^{(r)}\widetilde{S}_n|/|X_n^{(r)}| \to_P \infty$. This is quite different from the one-sided divergence of ${}^{(r)}S_n/|X_n^{(r)}|$ or ${}^{(r)}\widetilde{S}_n/|X_n^{(r)}|$ to ∞ as studied in Kesten and Maller (1992). It turns out to be related to a combination of asymptotic normality and relative stability of S_n, and in quantifying this we give a variant of a principle due to Lévy, that convergence to normality corresponds to dominance of the *centered* sum of the sample over its large values. This "two-sided" divergence is discussed in Section 3, both through the full sequence of natural numbers and through a subsequence $\{n_i\}$. It came as a surprise to us that the existence of a sequence $\{n_i\}$ for which $|{}^{(r)}\widetilde{S}_{n_i}|/|X_{n_i}^{(r)}| \to_P \infty$ is equivalent to a condition of Pruitt's for a generalized law of the iterated logarithm (see Theorem 3.2).

Each of the theorems below gives equivalences for the probabilistic behavior we are interested in, with one or more purely analytic criteria expressed in terms of the tails or some integrals of the tails of F. In fact, our choice of divergence phenomena discussed here has largely been determined by whether we could find such an equivalent analytic condition.

There certainly are many other possible versions in which one may discuss infinite limit points. We merely mention that one may consider divergence to $+\infty$ or to $-\infty$ of ratios such as ${}^{(r)}S_n/B_n$, ${}^{(r)}\widetilde{S}_n/B_n$, $|{}^{(r)}\widetilde{S}_n|/B_n$, ${}^{(r)}S_n/M_n^{(r)}$, ${}^{(r)}S_n/(X^-)_n^{(s)}$, ${}^{(r)}\widetilde{S}_n/|X_n^{(r)}|$ and $|{}^{(r)}\widetilde{S}_n|/|X_n^{(r)}$. Most cases of divergence to $+\infty$ or to $-\infty$ are quite different phenomena in that one cannot merely interchange X^+ and X^-. One may ask for the existence of B_n with the required property, or one may give B_n in advance. One may also investigate subsequential versions. We further limited ourselves here to divergence in probability. Almost each question can also be asked for almost sure divergence. At the moment we know much less about almost sure divergence, but we hope to return to this later. We believe that this agenda of studying divergence of a variety of quantities related to S_n will lead to some surprising and deep properties of random walks. In Table 1 we summarize the cases of divergence in probability which we have treated in Kesten and Maller (1992) and in this paper.

Various functionals of F will appear in our analytical criteria and proofs. For

TABLE 1
Sum dominates large values in probability (i.p.)

	I.p. divergence type	Other i.p. equivalence	Analytic equivalence	Reference	Comments
I	$\frac{{}^{(r)}S_n}{M_n^{(r)}} \to +\infty$	$\exists\, B_n \uparrow \infty$ such that $\frac{{}^{(r-1)}S_n}{B_n} \to +1$	$A(x) > 0$ for $x \ge x_0$ and $\frac{A(x)}{xH(x)} \to \infty$	Kesten and Maller (1992) Theorem 2.1	Holds for $r = 1$ iff if holds for $r > 1$
II	$\frac{{}^{(r)}S_n}{M_n^{(s)}} \to -\infty$	Not equivalent to negative relative stability in probability	When $E(X^+)^2 = \infty$: $A(x) < 0$ for $x \ge x_0$ and $\frac{-A(x)}{x[1-F(x)]} \to \infty$	See Remark (v) following Theorem 2.1 below	Holds for all $r \ge 0$ and $s \ge 1$ if it holds for one pair (r, s)
III	$\frac{\lvert {}^{(r)}S_n \rvert}{M_n^{(r)}} \to \infty$	?			
IV	$\frac{{}^{(r)}\tilde{S}_n}{\lvert X_n^{(r)} \rvert} \to +\infty$	$\exists\, B_n \uparrow \infty$ such that $\frac{{}^{(r-1)}\tilde{S}_n}{B_n} \to +1$	$A(x) > 0$ for $x \ge x_0$ and $\frac{A(x)}{xH(x)} \to \infty$	Kesten and Maller (1992) Theorem 2.1	Holds for $r = 1$ iff it holds for $r > 1$
V	$\frac{{}^{(r)}\tilde{S}_n}{\lvert X_n^{(r)} \rvert} \to -\infty$	Replace X_i by $-X_i$ in IV			
VI	$\frac{\lvert {}^{(r)}\tilde{S}_n \rvert}{\lvert X_n^{(r)} \rvert} \to \infty$	See Theorem 3.1 below	$\frac{x\lvert A(x)\rvert + U(x)}{x^2 H(x)} \to \infty$	See Theorem 3.1 below	
VII	$\frac{{}^{(r)}S_n}{(X^-)_n^{(s)}} \to +\infty$	$P\{{}^{(r)}S_n > 0\} \to 1$	When $E(X^-)^2 = \infty$: $A(x) > 0$ for $x \ge x_0$ and $\frac{A(x)}{xF(-x)} \to \infty$	See Theorem 2.1 below	

future reference we list these here:

$$H(x) = P\{|X| > x\} = 1 - F(x) + F(-x-);$$

$$\nu_+(x) = \int_{[0,x]} y\, dF(y), \qquad \nu_-(x) = -\int_{[-x,0]} y\, dF(y),$$

$$\nu(x) = \nu_+(x) - \nu_-(x) = E\Big(XI\big(|X| \le x\big)\Big);$$

$$A_+(x) = \int_0^x \big(1 - F(y)\big)\, dy, \qquad A_-(x) = \int_{-x}^0 F(y)\, dy,$$

$$A(x) = A_+(x) - A_-(x);$$

$$V_+(x) = \int_{[0,x]} y^2\, dF(y), \qquad V_-(x) = \int_{[-x,0]} y^2\, dF(y),$$

$$V(x) = V_+(x) + V_-(x) = E\Big(X^2 I\big(|X| \le x\big)\Big);$$

$$U_+(x) = 2\int_0^x y\big[1 - F(y)\big]\,dy, \qquad U_-(x) = 2\int_{-x}^0 |y|F(y)\,dy,$$
$$U(x) = U_+(x) + U_-(x) = 2\int_0^x yH(y)\,dy.$$

Here X is any random variable having distribution F. We mention the following relations which are obtained by integrating by parts:

$$(1.1)\qquad V(x) = -x^2H(x) + U(x), \qquad A(x) = x\big[1 - F(x) - F(-x-)\big] + \nu(x).$$

Throughout this paper we assume $P\{|X| > x\} > 0$ *for all* x *so that the* X_i *have unbounded support.*

2. One-sided results. One way of motivating the results in this section is to consider the weak law of large numbers in the form [see, e.g., Feller (1971), page 235]:

$$xH(x) \to 0 \quad \text{if and only if } \frac{S_n}{n} - \nu(n) \to_P 0.$$

(Throughout, we will omit "$x \to \infty$," "$n \to \infty$," etc., when it is obvious.) Thus when $xH(x) \to 0$ and $\nu(x) \to \infty$ we have $S_n/n \to_P \infty$. However, $xH(x) \to 0$, equivalently, $x[1 - F(x)] \to 0$ and $xF(-x) \to 0$, bound both the positive and negative tails of F. Surely, to get $S_n/n \to_P \infty$, we need only have the positive tail dominate the negative tail in some way. Our first result is of this kind.

THEOREM 2.1. *Let* $r = 0, 1, 2, \ldots$ *and* $s = 1, 2, 3, \ldots$. *If* $U_-(\infty) = \infty$, *the following are equivalent*:

$$(2.1)\qquad {}^{(r)}S_n \to_P \infty;$$

$$(2.2)\qquad \text{there exists } B_n \uparrow \infty \text{ such that } \frac{{}^{(r)}S_n}{B_n} \to_P \infty;$$

$$(2.3)\qquad P\big\{{}^{(r)}S_n > 0\big\} \to 1;$$

$$(2.4)\qquad \frac{{}^{(r)}S_n}{(X^-)_n^{(s)}} \to_P \infty;$$

$$(2.5a)\qquad \frac{A(x)}{xF(-x)} \to \infty.$$

If $U_-(\infty) < \infty = U_+(\infty)$ *and* $F(-x) > 0$ *for all* $x > 0$, *then* (2.1) *to* (2.4) *are equivalent to*

$$(2.5b)\qquad A(x) \ge 0 \quad \text{for } x \text{ large enough}.$$

If $U_+(\infty) = \infty$ *and* $F(-x) = 0$ *for some* $x > 0$, *then each of* (2.1) *to* (2.3) *is equivalent to* (2.5b). *If* $EX^2 < \infty$, *then* (2.1) *to* (2.3) *hold if and only if* $EX > 0$. *The theorem remains true if* ${}^{(r)}S_n$ *is replaced by* ${}^{(r)}\tilde{S}_n$ *throughout.*

REMARKS. (i) Conditions (2.1) and (2.2) (for $r = 0$) in Theorem 2.1 are clearly equivalent to the existence of a nonstochastic $B'_n \uparrow \infty$ such that, for $0 < \varepsilon < 1$,

$$P\{S_n \geq (1-\varepsilon)B'_n\} \to 1. \tag{2.6}$$

This is a kind of *one-sided relative stability* of S_n. It is weaker than positive relative stability of S_n, which is equivalent to

$$\frac{A(x)}{xP\{|X| > x\}} \to \infty; \tag{2.7}$$

see Kesten and Maller (1992) for a discussion of relative stability and its equivalence with ${}^{(r)}S_n/M_n^{(r)} \to_P \infty$ and ${}^{(r)}\widetilde{S}_n/|X_n^{(r)}| \to_P \infty$. (2.7) obviously implies (2.5a) or (2.5b). Positive relative stability also implies the following: $A(x)$ is positive for x large enough, is slowly varying as $x \to \infty$ and satisfies $A(x) \sim \nu(x)$, and the sequence B_n for which $S_n/B_n \to_P 1$, equivalently,

$$\frac{S_n - n\nu(B_n)}{B_n} \to_P 0, \tag{2.8}$$

may be chosen to be the restriction to the integers of a function which is regularly varying with index 1; see Bingham, Goldie and Teugels (1987) for definitions and properties of slow and regular variation.

Some of the above properties have useful one-sided analogues. One can show that if (2.2) holds, then B_n can be chosen to satisfy

$$\frac{\sum_{i=1}^n X_i^- - n\nu_-(B_n)}{B_n} \to_P 0, \tag{2.9}$$

while, under (2.5a), $A(x)$ and $\nu(x)$ always satisfy

$$\limsup_{x\to\infty} \frac{A(x\lambda)}{A(x)} \geq 1 \tag{2.10}$$

for each fixed $\lambda \geq 1$, and

$$\limsup_{x\to\infty} \frac{|\nu(x)|}{A(x)} \leq 1. \tag{2.11}$$

Inequality (2.10) is a one-sided version of slow variation while (2.9) is a one-sided version of (2.8).

(ii) Although (2.11) holds, it is not in general true under (2.5a) that $\nu(x) \sim A(x)$ as $x \to \infty$. Take, for example,

$$1 - F(x) = \frac{1}{\log x} \quad \text{and} \quad F(-x) = \frac{1}{(\log x)^2},$$

when x is large. Then

$$A(x) \sim \frac{x}{\log x}, \qquad \nu(x) \sim \frac{x}{(\log x)^2}, \qquad \frac{\nu(x)}{xF(-x)} \to 1.$$

Thus $U_-(\infty) = \infty$ and (2.5a) holds, but $\nu(x)/A(x) \to 0$. Note, however, that if

$$\frac{\nu(x)}{xF(-x)} \to \infty, \tag{2.12}$$

then (2.5a) holds by (1.1). This shows that (2.12) is sufficient but not necessary for (2.5a).

(iii) Theorem 2.1 has an interesting connection with some work of Griffin and McConnell (1994), which was developed quite independently. For $x > 0$ let T_x be the first time S_n exists the interval $[-x, x]$, that is,

$$T_x = \inf\{n\colon |S_n| > x\}.$$

Griffin and McConnell show that $P\{S_{T_x} > 0\} \to 1$, $x \to \infty$ (thus S_n exits with high probability on the positive side of the interval), if and only if

$$\frac{U(x) + x|\nu(x)|}{x^2F(-x)} \to \infty \tag{2.13}$$

and

$$\liminf_{x\to\infty} \frac{xA(x)}{U(x) + x|\nu(x)|} > 0. \tag{2.14}$$

Somewhat surprisingly, these conditions together are equivalent to (2.5a) [provided $U(\infty) = \infty$]. In fact, by multiplying (2.13) and (2.14) one clearly obtains (2.5a). Conversely, (2.5a) implies (2.13), since, by (1.1),

$$\begin{aligned} &U(x) + x|\nu(x)| \\ &\quad = V(x) + x^2[1 - F(x) + F(-x-)] + x\big|A(x) - x[1 - F(x) - F(-x-)]\big| \\ &\quad \geq x|A(x)|. \end{aligned} \tag{2.15}$$

Also (2.5a) implies by (2.11) that $|\nu(x)| \leq (1 + o(1))A(x)$, and by (4.13) below that

$$\liminf_{x\to\infty} \frac{xA(x)}{U(x)} \geq \frac{1}{2}.$$

Thus $U(x) + x|\nu(x)| \leq (2 + o(1))xA(x)$ and certainly (2.14) holds. One may also give a direct probabilistic proof that (2.1) implies $P\{S_{T_x} > 0\} \to 1, x \to \infty$, based on the Markovian property of the stopping time T_x (We remark that Griffin and McConnell's results go well beyond the above-mentioned equivalence, by considering subsequential and higher-dimensional versions.)

(iv) The division of Theorem 2.1 into cases when $U_-(\infty)$ is finite or not is mainly for convenience in exposition. Lemma 4.3 below shows that (2.5a) is equivalent to (2.1) to (2.4) regardless of whether $U_-(\infty)$ is finite or not, provided $F(-x) > 0$ for $x > 0$.

(v) Theorem 2.1 can also be used to obtain conditions for divergence to $-\infty$ in probability. This is obvious for ${}^{(r)}\widetilde{S}_n$ since modulus trimming is independent of the sign of X_i. For ${}^{(r)}S_n \to_P -\infty$ we similarly interchange $-X_i$ and X_i and ask when

$$P\left\{S_n + \sum_{j=1}^{r} (X^-)_n^{(j)} > 0\right\} \to 1, \qquad n \to \infty. \tag{2.16}$$

This is answered in the lines following (4.38); it occurs if and only if $P\{S_n > 0\} \to 1$, equivalently, if the conditions in Theorem 2.1 hold. Thus we obtain, for $r = 0, 1, 2, \dots,$

$$ {}^{(r)}\widetilde{S}_n \to_P -\infty, \quad \text{equivalently, } {}^{(r)}S_n \to_P -\infty,$$

if and only if

$$\frac{A(x)}{x[1-F(x)]} \to -\infty \quad \text{in case } U_+(\infty) = \infty \tag{2.17}$$

or

$$A(x) \le 0 \quad \text{in case } U_+(\infty) < \infty = U_-(\infty) \tag{2.18}$$

or

$$EX < 0 \quad \text{in case } EX^2 < \infty. \tag{2.19}$$

The arguments following (4.38) and Theorem 2.1 also show that (2.16) is equivalent to

$$\frac{S_n + \sum_{j=1}^{r} (X^-)_n^{(j)}}{(X^-)_n^{(s)}} \to_P \infty$$

[if $F(0-) > 0$]. Therefore, necessary and sufficient conditions for

$$\frac{{}^{(r)}S_n}{M_n^{(s)}} \to_P -\infty$$

are given by (2.17) to (2.19).

The next theorem is analogous to the preceding, but now the sequence B_n is given in advance.

THEOREM 2.2. *Let $r = 0, 1, 2, \dots$ and $B_n \uparrow \infty$ be a given sequence. If $U_-(\infty) = \infty$, then*

$$\frac{{}^{(r)}S_n}{B_n} \to_P \infty \tag{2.20}$$

if and only if

$$\frac{A(x)}{xF(-x)} \to \infty \quad \text{and} \quad \frac{nA(B_n)}{B_n} \to \infty. \tag{2.21}$$

If $U_-(\infty) < \infty = U_+(\infty)$, *then* (2.20) *is equivalent to*

$$A(x) \geq 0 \quad \textit{for } x \textit{ large enough and } \frac{nA(B_n)}{B_n} \to \infty. \tag{2.22}$$

If $EX^2 < \infty$, *then* (2.20) *is equivalent to* $EX > 0$ *and* $n/B_n \to \infty$. *The theorem remains true if* ${}^{(r)}S_n$ *is replaced by* ${}^{(r)}\widetilde{S}_n$.

The next theorem allows ${}^{(r)}S_n/n$ or ${}^{(r)}\widetilde{S}_n/n$ to tend in probability to any positive constant, possibly ∞.

THEOREM 2.3. *Let* $r = 0, 1, 2, \dots$ *and* $a \in (0, \infty]$. *If* $U_-(\infty) = \infty$, *then*

$$\frac{{}^{(r)}S_n}{n} \to_P a \tag{2.23}$$

if and only if

$$\frac{A(x)}{xF(-x)} \to \infty \quad \textit{and} \quad A(x) \to a. \tag{2.24}$$

If $U_-(\infty) < \infty$, *then* (2.23) *holds if and only if* $A(x) \to a$. *The theorem remains true if* ${}^{(r)}S_n$ *is replaced by* ${}^{(r)}\widetilde{S}_n$.

REMARKS. (i) For $a = \infty$, Theorem 2.3 is immediate from Theorem 2.2. As we shall see in Section 4.3, the only additional point in Theorem 2.3 is to observe that, for $a < \infty$, (2.24) is equivalent to Feller's conditions for the weak law of large numbers [see Feller (1971), page 565].

(ii) If we replace X_i by $-X_i$ in Theorem 2.3 and use Proposition 4.1 below, then we also obtain a necessary and sufficient condition for ${}^{(r)}S_n/n \to_P -\infty$. For instance, if $U_+(\infty) = \infty$, this is equivalent to

$$\frac{A(x)}{x[1 - F(x-)]} \to -\infty \quad \text{and} \quad A(x) \to -\infty.$$

(iii) In general, $A(x)/xF(-x) \to \infty$ alone does not imply $S_n/n \to_P \infty$. In fact, there exists an F with $U_-(\infty) = \infty$ and mean 0 for which S_n is relatively stable, that is, $S_n/B_n \to_P 1$ for some $B_n \uparrow \infty$ [see, e.g., Breiman (1968), Exercise 3.7.17]. By Theorem 2.1, $A(x)/[xF(-x)] \to \infty$ in this example. Yet $A(x) \to 0$ since F has mean 0. [Note that $S_n/n \to 0$ a.s. here, so that necessarily $B_n = o(n)$.]

(iv) In general, $A(x) \to \infty$ alone does not imply $S_n/n \to_P \infty$. For example, take i.i.d. $Y_i \geq 0$ with tail $P\{Y_i > x\} \sim 1/[x \log x]$ and i.i.d. Z_i symmetric with tail $P\{|Z_i| > x\} \sim x^{-1/2}$. Let $X_i = Y_i + Z_i$. Then $A(x) \sim \log\log x$ since $A(x)$ is mainly determined by the tail of Y_i, yet $A(x)/[xF(-x)] \sim 2\log\log x/x^{1/2} \to 0$. [This example is due to Révész (1968), page 80.]

It would be interesting to find a "subsequential" version of Theorem 2.1, that is, a criterion for the existence of a sequence of integers n_i through which ${}^{(r)}S_{n_i} \to_P \infty$. At present we only have such a criterion when $X \geq 0$ a.s.

THEOREM 2.4. *Suppose $F(0-) = 0$ and $n_i \uparrow \infty$, $B_{n_i} \uparrow \infty$ are given sequences. Then ${}^{(r)}S_{n_i}/B_{n_i} \to_P \infty(0)$ if and only if $n_i A(B_{n_i})/B_{n_i} \to \infty(0)$ as $n_i \to \infty$.*

REMARKS. (i) In each of Theorems 2.1 to 2.4, the analytic condition is independent of r (and, in Theorem 2.1, s). Thus the other properties also hold or fail for all r and s simultaneously. In fact, the same sequence B_n can be used for all r (see Proposition 4.1 below).

(ii) In general, $nA(B_n)/B_n \to 1$ or $n\nu(B_n)/B_n \to 1$ do not imply $S_n/B_n \to_P 1$. Take, for example, a nonnegative X_i whose tail $1 - F(x)$ is slowly varying, say $1 - F(x) \sim L(x) \downarrow 0$. Then $A(x) \sim xL(x)$ and we can choose $B_n \uparrow \infty$ such that $nA(B_n)/B_n = 1$. If $S_n/B_n \to_P 1$ we would have $n[1 - F(B_n)] \to 0$ [cf. Gnedenko and Kolmogorov (1986), page 124], yet $n[1-F(B_n)] \sim nA(B_n)/B_n = 1$. Likewise if B_n is chosen so that $n\nu(B_n)/B_n \to 1$, then $\nu(x)/x[1-F(x)] \to 0$ [which follows from (1.1)] implies $n[1 - F(B_n)] \to \infty$.

3. Two-sided results. Our first result is a two-sided analogue of Theorem 2.1, and is related to results of Lévy [(1937), pages 333–336], who shows that the centered sum dominates the large values in modulus if the centered sum is asymptotically normal. Theorem 3.1 also is, in part, a two-sided analogue of Lemma 3.2 in Kesten and Maller (1992).

THEOREM 3.1. *For $r = 1, 2, \ldots$ the following are equivalent*:

(3.1) $$\frac{|{}^{(r)}\widetilde{S}_n|}{|X_n^{(r)}|} \to_P \infty;$$

(3.2) $$\text{for some } T > 0, \quad P\Big\{\big|{}^{(r)}\widetilde{S}_n\big| \leq T\big|X_n^{(r)}\big|\Big\} \to 0;$$

(3.3) $$\frac{x|A(x)| + U(x)}{x^2 P\{|X| > x\}} \to \infty;$$

(3.4) *there is a nonstochastic sequence $D_n \uparrow \infty$ such that every infinite sequence of integers contains a subsequence $n' \to \infty$ for which ${}^{(r)}\widetilde{S}_{n'}/D_{n'}$ converges in distribution to a normal random variable, possibly degenerate, but not degenerate at 0.*

THEOREM 3.2. *For $r = 1, 2, \ldots$ the following are equivalent*:

(3.5) *there is an infinite sequence of integers n_i such that*
$$\frac{|{}^{(r)}\widetilde{S}_{n_i}|}{|X_{n_i}^{(r)}|} \to_P \infty, \qquad n_i \to \infty;$$

(3.6) *there is an infinite sequence of integers n_i such that, for some* $T > 0$,

$$P\Big\{\big|{}^{(r)}\widetilde{S}_{n_i}\big| \le T\big|X_{n_i}^{(r)}\big|\Big\} \to 0, \qquad n_i \to \infty;$$

(3.7)
$$\limsup_{x\to\infty} \frac{x|A(x)| + U(x)}{x^2P\{|X| > x\}} = \infty;$$

at least one of the following holds:

(3.8a)
$$\limsup_{x\to\infty} \frac{U(x)}{x^2P\{|X| > x\} + x|A(x)|} = \infty$$

or

(3.8b)
$$\limsup_{x\to\infty} \frac{x|A(x)|}{U(x)} = \infty;$$

at least one of the following holds:

(3.9a)
$$\limsup_{x\to\infty} \frac{|U(x)|}{x^2P\{|X| > x\}} = \infty$$

or

(3.9b)
$$\limsup_{x\to\infty} \frac{|A(x)|}{xP\{|X| > x\}} = \infty;$$

(3.10) *there is an infinite sequence of integers n_i and a nonstochastic sequence D_{n_i} such that ${}^{(r)}\widetilde{S}_{n_i}/D_{n_i}$ converges in distribution to a normal random variable, possibly degenerate, but not degenerate at* 0.

REMARKS. (i) Theorem 3.2 has a nice connection with a generalized law of the iterated logarithm due to Pruitt [(1981), Theorem 5.2]. He showed that (3.7) holds if and only if there is a nonstochastic sequence $B_n \uparrow \infty$ such that

$$0 < \limsup_{n\to\infty} \frac{|S_n|}{B_n} < \infty \quad \text{a.s.}$$

The equivalence of (3.7) and (3.8) is due to Pruitt (1981), Lemma 2.6, and (3.9) is due to Martikainen (1980). Equation (3.9a) is equivalent to Lévy's [(1937), page 113] condition for S_n to be in the domain of partial attraction of the normal. See also Lemmas 4.5 and 4.6 below for other interesting sidelights on these conditions. Equation (3.8a) is equivalent to subsequential *uncentered* asymptotic normality [see (4.62)], while (3.8b) is equivalent to subsequential relative stability [see (4.55)].

(ii) In the two-sided case the analogues of (2.1) and (2.2) always hold (except when F is concentrated on the single point 0, but this case was excluded by the requirement that F have unbounded support). This means that we always have

$$\frac{|{}^{(r)}S_n|}{B_n} \to_P \infty \quad \text{and} \quad \frac{|{}^{(r)}\widetilde{S}_n|}{B_n} \to_P \infty \quad \text{for some } B_n \uparrow \infty.$$

However, these do not imply (3.1) to (3.4). We demonstrate this following the proof of Theorem 3.2.

Our next theorem is a convergence rather than a divergence result, giving a criterion for the *relative compactness* of S_n/n. The corollary following it then gives a necessary and sufficient condition for the subsequential divergence of $|{}^{(r)}S_n|/n$.

THEOREM 3.3. *For $r = 0, 1, \dots$, the following are equivalent*:

(3.11)
$$\lim_{x\to\infty} \limsup_{n\to\infty} P\left\{\frac{|{}^{(r)}S_n|}{n} > x\right\} < 1;$$

(3.12)
$$\limsup_{n\to\infty} P\left\{\frac{|{}^{(r)}S_n|}{n} > x\right\} \le \frac{c}{x} \text{ for some } c = c(r) \text{ and all } x \text{ large enough};$$

(3.13)
$$\limsup_{n\to\infty} \frac{x|A(x)| + U(x)}{x} < \infty.$$

The theorem remains true if ${}^{(r)}S_n$ is replaced by ${}^{(r)}\widetilde{S}_n$ throughout.

COROLLARY TO THEOREM 3.3. *There is an infinite sequence of integers n_i such that*

(3.14)
$$\frac{|{}^{(r)}S_{n_i}|}{n_i} \to_P \infty$$

if and only if

(3.15)
$$\limsup_{n\to\infty} \frac{x|A(x)| + U(x)}{x} = \infty.$$

The corollary remains true if ${}^{(r)}S_n$ is replaced by ${}^{(r)}\widetilde{S}_n$.

4. Proofs. We begin with a general proposition which essentially shows that light trimming has no influence in the situations of this paper. The divergence under consideration for ${}^{(r)}S_n$ or ${}^{(r)}\widetilde{S}_n$ for any r is equivalent to divergence for S_n itself. For any subset C of $[-\infty, \infty]$ and $\varepsilon > 0$, we use the notation

(4.1)
$$C^\varepsilon = \{x + y \colon x \in C, \ |y| < \varepsilon\}.$$

PROPOSITION 4.1. *Let C_n be any sequence of Borel sets of* $[-\infty, \infty]$ *and* $B_n \uparrow \infty$ *a sequence of constants. Let* $n_1 < n_2 < \cdots$. *Then, for all* $r, s \geq 0$, *and* $\varepsilon > 0$,

$$\lim_{i\to\infty} P\left\{\frac{{}^{(r)}S_{n_i}}{B_{n_i}} \in C_{n_i}\right\} = 1 \tag{4.2}$$

or

$$\lim_{i\to\infty} P\left\{\frac{{}^{(r)}\widetilde{S}_{n_i}}{B_{n_i}} \in C_{n_i}\right\} = 1 \tag{4.3}$$

implies

$$\lim_{i\to\infty} P\left\{\frac{{}^{(s)}S_{n_i}}{B_{n_i}} \in C^{\varepsilon}_{n_i}\right\} = \lim_{i\to\infty} P\left\{\frac{{}^{(s)}\widetilde{S}_{n_i}}{B_{n_i}} \in C^{\varepsilon}_{n_i}\right\} = 1. \tag{4.4}$$

REMARK. Of particular interest to us will be the special case when ${}^{(r)}S_{n_i}/B_{n_i} \to_P \infty$ or $-\infty$ or ${}^{(r)}\widetilde{S}_{n_i}/B_{n_i} \to_P \pm\infty$ for some $r \geq 0$. For instance, in the first case (4.2) holds for $C_n = [T, \infty)$, for any fixed T. Then (4.4) shows that also, for all $s \geq 0$,

$$\frac{{}^{(s)}S_{n_i}}{B_{n_i}} \to_P \infty \quad \text{and} \quad \frac{{}^{(s)}\widetilde{S}_{n_i}}{B_{n_i}} \to_P \infty.$$

PROOF OF PROPOSITION 4.1. To simplify the notation, we only consider the case where $\{n_i\}$ is the full sequence of natural numbers; the proof for a subsequence is the same. Furthermore, we restrict ourselves to proving that (4.3) implies (4.4)—again there is no essential difference for starting at (4.2).

It is convenient to break ties by introducing an i.i.d. sequence $\{U_i\}_{i\geq 1}$ of random variables, such that each U_i is uniformly distributed on [0, 1] and such that $\{U_i\}_{i\geq 1}$ is independent of $\{X_i\}_{i\geq 1}$. We then regard $|X_i|$ as strictly greater than $|X_j|$ if $|X_i| > |X_j|$ or $|X_i| = |X_j|$ and $U_i > U_j$. In this way the rth largest $|X_i| = |X_n^{(r)}|$ and ${}^{(r)}\widetilde{S}_n$ are uniquely determined. In fact, it is not hard to see that w.p.1, $|X_i| > |X_j|$ if and only if $\widetilde{\gamma}(|X_i|, U_i) < \widetilde{\gamma}(|X_j|, U_j)$, where

$$\widetilde{\gamma}(\ell, u) = P\{|X| > \ell\} + (1-u)P\{|X| = \ell\}. \tag{4.5}$$

For $\ell \geq 0$ and $u \in [0, 1]$ define the events

$$E(i, \ell, u) = \{|X_i| > \ell \text{ or } |X_i| = \ell \text{ and } U_i > u\},$$

and an i.i.d. sequence of random variables $Z_i(\ell, u)$ with the conditional distribution of X_i, given that $E(i, \ell, u)$ fails. Finally,

$$\widetilde{S}_j(\ell, u) = \sum_{i=1}^{j} Z_i(\ell, u).$$

Then for any fixed $\ell \ge 0, u \in [0, 1]$,

$$P\left\{\frac{{}^{(r)}\widetilde{S}_n}{B_n} \notin C_n\right\} \ge P\{E(i, \ell, u) \text{ occurs for exactly } r \text{ values of } i \le n\} \times P\left\{\frac{\widetilde{S}_{n-r}(\ell, u)}{B_n} \notin C_n\right\}.$$

Let $\delta > 0$ and define

$$A_n = \{(\ell, u)\colon \delta \le n\widetilde{\gamma}(\ell, u) \le 1/\delta\}.$$

Then if n is large enough, uniformly for $(\ell, u) \in A_n$,

$$P\{E(i, \ell, u) \text{ occurs for exactly } r \text{ values of } i \le n\} = \binom{n}{r}[\widetilde{\gamma}(\ell, u)]^r[1 - \widetilde{\gamma}(\ell, u)]^{n-r} \ge \frac{1}{2r!}e^{-n\widetilde{\gamma}(\ell, u)}[n\widetilde{\gamma}(\ell, u)]^r \ge \frac{\delta^r e^{-1/\delta}}{2r!},$$

because

$$P\{E(i, \ell, u)\} = \widetilde{\gamma}(\ell, u)$$

and $(\ell, u) \in A_n$ if and only if

$$\delta \le n\widetilde{\gamma}(\ell, u) \le \frac{1}{\delta}. \tag{4.6}$$

It follows from (4.3) that

$$P\left\{\frac{\widetilde{S}_{n-r}(\ell, u)}{B_n} \in C_n\right\} \to 1$$

as $n \to \infty$ and n, ℓ, u vary such that $(\ell, u) \in A_n$ or, equivalently, such that (4.6) holds. Because $\widetilde{S}_{n-s}(\ell, u)$ and $\widetilde{S}_{n-r}(\ell, u)$ differ by $|s - r|$ summands with the distribution of $Z_i(\ell, u)$, and $B_n \to \infty$, this implies further that for $s \ge 0$ and $\varepsilon > 0$

$$P\left\{\frac{\widetilde{S}_{n-s}(\ell, u)}{B_n} \in C_n^{\varepsilon}\right\} \to 1 \quad \text{as } n \to \infty, \tag{4.7}$$

again uniformly under (4.6).

Next, let $j(s) \le n$ be the unique index for which $X_n^{(s)} = X_{j(s)}$. Then, for $s \ge 1$, we have by (4.7) as $n \to \infty$,

$$\begin{aligned} P\left\{\frac{{}^{(s)}\widetilde{S}_n}{B_n} \in C_n^{\varepsilon}\right\} &= \int P\{|X_n^{(s)}| \in d\ell, U_{j(s)} \in du\} P\left\{\frac{\widetilde{S}_{n-s}(\ell, u)}{B_n} \in C_n^{\varepsilon}\right\} \\ &\ge \int_{A_n} P\{|X_n^{(s)}| \in d\ell, U_{j(s)} \in du\}(1 + o(1)) \\ &\ge (1 + o(1))P\left\{\delta \le n\widetilde{\gamma}\left(|X_n^{(s)}|, U_{j(s)}\right) \le 1/\delta\right\}. \end{aligned} \tag{4.8}$$

Now $P\{\widetilde{\gamma}(|X_i|, U_i) < \overline{\gamma}\} = \overline{\gamma}$, by virtue of the definition (4.5); that is, the $\widetilde{\gamma}(|X_i|, U_i)$ are i.i.d. random variables with a uniform distribution on [0, 1]. Also $\widetilde{\gamma}\ (|X_n^{(s)}|, U_{j(s)})$ is the sth smallest value among $\widetilde{\gamma}(|X_1|, U_1), \dots, \widetilde{\gamma}(|X_n|, U_n)$. It follows easily that $n\widetilde{\gamma}(|X_n^{(s)}|, U_{j(s)})$ is tight in $(0, \infty)$ and the probability in the right hand side of (4.8) tends to 1 when $\delta \downarrow 0$, uniformly in n. This gives the second half of (4.4) when $s \geq 1$.

When $s = 0$ take ℓ and u such that $n\widetilde{\gamma}(\ell, u) = \delta$. Then use (4.7) and let first $n \to \infty$ and then $\delta \downarrow 0$ to obtain

$$
\begin{aligned}
P\left\{\frac{S_n}{B_n} \in C_n^\varepsilon\right\} &\geq P\{E(i, \ell, u) \text{ fails for all } i \leq n\} P\left\{\frac{\widetilde{S}_n(\ell, u)}{B_n} \in C_n^\varepsilon\right\} \\
&= \left[1 - \widetilde{\gamma}(\ell, u)\right]^n P\left\{\frac{\widetilde{S}_n(\ell, u)}{B_n} \in C_n^\varepsilon\right\} \to 1.
\end{aligned} \tag{4.9}
$$

Now we can repeat the above argument with (4.9) taking the place of (4.3), and with $\widetilde{\gamma}(\ell, u)$ replaced by $\gamma(\ell, u) = P\{X > \ell\} + (1-u)P\{X = \ell\}$. This means that the X_i are ranked according to increasing values of $\gamma(X_i, U_i)$. Moreover, $|X_n^{(s)}|$ is replaced by $M_n^{(s)}$. This yields

$$
\liminf_{n\to\infty} P\left\{\frac{{}^{(s)}S_n}{B_n} \in C_n^{2\varepsilon}\right\} = 1,
$$

thus completing the proof of (4.4). □

4.1. *Proof of Theorem* 2.1. The equivalence between (2.1) and (2.2) is trivial; (2.2) obviously implies (2.1), while if (2.1) holds we can find a sequence $C_n \to \infty$ such that $P\{{}^{(r)}S_n \geq C_n^2\} \to 1$. Then $B_n := \inf\{C_k : k \geq n\} \uparrow \infty$ and since, for any $T > 0$, $TB_n \leq C_n^2$ for large enough n,

$$
P\left\{\frac{{}^{(r)}S_n}{B_n} \geq T\right\} \geq P\{{}^{(r)}S_n \geq C_n^2\} \to 1.
$$

Thus (2.2) holds.

The remaining equivalences are proved via a series of lemmas. The key ingredients in the proof are a Chebyshev-like upper bound for the probability that S_n remains *small*, and a corresponding lower bound derived from results of Kesten and Lawler (1992) and Kesten and Maller (1992). Note that Theorem 2.1 is essentially trivial when $P\{X \geq 0\} = 1$ or $P\{X \leq 0\} = 1$. We therefore *restrict ourselves to distributions with* $P\{X > 0\} > 0$ *and* $P\{X < 0\} > 0$.

LEMMA 4.2. *Fix $T \in \mathbb{R}$ and $r = 0, 1, 2, \dots$, and suppose that $x_+ > 0, x_- > 0$ are such that*

$$
n\left\{\nu_+(x_+) - \nu_-(x_-) + x_+\left[1 - F(x_+)\right]\right\} > T + rx_+. \tag{4.10}
$$

Then

$$P\left\{ {}^{(r)}S_n \le T, \sum_{i=1}^{n} X_i I(X_i < -x_-) = 0 \right\} \le \frac{n\left\{V_+(x_+) + V_-(x_-) + x_+^2[1 - F(x_+)]\right\}}{\left\{n\left[\nu_+(x_+) - \nu_-(x_-) + x_+[1 - F(x_+)]\right] - (T + rx_+)\right\}^2} \tag{4.11}$$

and

$$P\{{}^{(r)}S_n \le T\} \le \frac{n\left\{V_+(x_+) + V_-(x_-) + x_+^2[1 - F(x_+)]\right\}}{\left\{n\left[\nu_+(x_+) - \nu_-(x_-) + x_+[1 - F(x_+)]\right] - (T + rx_+)\right\}^2} + nF(-x_-). \tag{4.12}$$

PROOF. Let

$$T_n = \sum_{i=1}^{n} \{X_i I(-x_- \le x_i \le x_+) + x_+ I(X_i > x_+)\}$$

and let ${}^{(r)}T_n$ be the trimmed sum obtained by removing the r largest summands from T_n. Then, for sample points for which $\sum X_i I(X_i < -x_-) = 0$, we have

$${}^{(r)}S_n \ge {}^{(r)} T_n \ge T_n - rx_+.$$

Next, we can easily calculate

$$E\{X_i I(-x_- \le X_i \le X_+) + x_+ I(X_i > x_+)\} = \nu_+(x_+) - \nu_-(x_-) + x_+[1 - F(x_+)]$$

and

$$E\{X_i I(-x_- \le X_i \le x_+) + x_+ I(X_i > x_+)\}^2 = V_+(x_+) + V_-(x_-) + x_+^2[1 - F(x_+)].$$

So by Chebyshev's inequality

$$\begin{aligned} P&\left\{ {}^{(r)}S_n \le T, \sum_{i=1}^{n} X_i I(X_i < -x_-) = 0 \right\} \\ &\le P\{T_n \le T + rx_+\} \\ &= P\Big\{T_n - n\left\{\nu_+(x_+) - \nu_-(x_-) + x_+[1 - F(x_+)]\right\} \\ &\qquad \le (T + rx_+) - n\left\{\nu_+(x_+) - \nu_-(x_-) + x_+[1 - F(x_+)]\right\}\Big\} \\ &\le \frac{n\left\{V_+(x_+) + V_-(x_-) + x_+^2[1 - F(x_+)]\right\}}{\left\{n\left[\nu_+(x_+) - \nu_-(x_-) + x_+[1 - F(x_+)]\right] - (T + rx_+)\right\}^2}, \end{aligned}$$

provided (4.10) holds. This proves (4.11), and then (4.12) follows from the bound

$$P\left\{\sum_{i=1}^{n} X_i I(X_i < -x_-) \neq 0\right\} \leq nF(-x_-). \qquad \square$$

LEMMA 4.3. *The conditions $U_-(\infty) = \infty$ and $A(x)/xF(-x) \to \infty$, or $U_-(\infty) < \infty = U_+(\infty)$ and $A(x) \geq 0$, for x large enough, imply*

$$\liminf_{x\to\infty} \frac{xA(x)}{U(x)} \geq \frac{1}{2}, \tag{4.13}$$

$$xA(x) \to \infty \tag{4.14}$$

and

$$\frac{xA(x)}{U_-(x)} \to \infty. \tag{4.15}$$

Also $U_-(\infty) < \infty = U_+(\infty)$, $A(x) \geq 0$ for x large enough and $F(-x) > 0$ for all $x > 0$ imply $A(x)/[xF(-x)] \to \infty$.

PROOF. Suppose first that $U_-(\infty) = \infty$, so $F(-x) > 0$ for all $x > 0$, and suppose also that $A(x)/xF(-x) \to \infty$. Then, for given $\varepsilon > 0, xF(-x) \leq \varepsilon A(x)$ if $x \geq x_0 = x_0(\varepsilon)$. Thus $A(x) > 0$ for $x \geq x_0$. Note that then

$$\begin{aligned} U_-(x) &= 2\int_0^x yF(-y)\,dy \leq 2\varepsilon \int_{x_0}^x A(y)\,dy + O(1) \\ &= 2\varepsilon \int_0^x A(y)\,dy + O(1). \end{aligned}$$

Also

$$\begin{aligned} U_-(x) &= 2\int_0^x yF(-y)\,dy = 2\int_0^x y\,dA_-(y) \\ &= 2xA_-(x) - 2\int_0^x A_-(y)\,dy \end{aligned}$$

and similarly

$$U_+(x) = 2xA_+(x) - 2\int_0^x A_+(y)\,dy.$$

Thus, since $U_-(\infty) = \infty$, we have, for large x,

$$\begin{aligned} U_-(x) &\leq 2\varepsilon \int_0^x [A_+(y) - A_-(y)]\,dy + O(1) \\ &= \varepsilon\{2xA_+(x) - U_+(x) - 2xA_-(x) + U_-(x)\} + O(1) \\ &\leq \varepsilon\{2xA(x) + U_-(x)\} + O(1) \\ &\leq 2\varepsilon xA(x) + 2\varepsilon U_-(x). \end{aligned}$$

It follows that

$$\frac{xA(x)}{U_-(x)} \geq \frac{1-2\varepsilon}{2\varepsilon}$$

and so $xA(x)/U-(x) \to \infty$, which is (4.15). In turn, this implies $xA(x) \to \infty$, that is, (4.14), since $U_-(\infty) = \infty$. Also

$$\begin{aligned} U_+(x) - U_-(x) &= 2\left\{xA_+(x) - \int_0^x A_+(y)\,dy - xA_-(x) + \int_0^x A_-(y)\,dy\right\} \\ &= 2\left\{xA(x) - \int_0^x A(y)\,dy\right\} \\ &= 2\left\{xA(x) - \int_{x_0}^x A(y)\,dy\right\} + O(1) \\ &\leq 2xA(x) + O(1). \end{aligned} \tag{4.16}$$

Thus $xA(x) \to \infty$ gives

$$U(x) \leq \big[2 + o(1)\big]xA(x) + 2U_-(x) = \big[2 + o(1)\big]xA(x),$$

since $xA(x)/U_-(x) \to \infty$. Thus (4.13) follows.

Next consider the case $U_-(\infty) < \infty = U_+(\infty)$ and $A(x) \geq 0$ for $x \geq x_0$. We still have (4.16), and, together with $U_-(\infty) < \infty = U_+(\infty)$, this gives

$$U_+(x) \leq 2xA(x) + O(1) = 2xA(x) + o\big(U_+(x)\big) \tag{4.17}$$

so

$$\liminf_{x\to\infty} \frac{xA(x)}{U_+(x)} \geq \frac{1}{2}.$$

This means $xA(x) \to \infty$, which is (4.14), and then that $U_-(x)/[xA(x)] \to 0$, that is, (4.15), since $U_-(\infty) < \infty$. Thus we again obtain (4.13).

Finally, if $U_-(\infty) < \infty = U_+(\infty), A(x) \geq 0$ for $x \geq x_0$ and $F(-x) > 0$ for $x > 0$, then, as we saw, $xA(x) \to \infty$, and, in addition, $x^2F(-x) \to 0$ as a consequence of $U_-(\infty) < \infty$. Thus $A(x)/[xF(-x)] \to \infty$, completing the proof of the lemma. □

LEMMA 4.4. *Let F be continuous and $0 < F(0-) \leq F(0) < 1$. For $0 < \varepsilon < 1$, let L_+ and L_- satisfy*

$$F\big(-L_-(\varepsilon)\big) = \varepsilon = 1 - F\big(L_+(\varepsilon)\big).$$

[Thus $-L_-(\varepsilon)$ and $L_+(\varepsilon)$ are ε- and $(1-\varepsilon)$-quantiles of F, respectively.] Then there exists a constant $K < \infty$ and, for $l \geq 0$, $\sigma \geq 1$, $\rho > 0$, constants $C(l, \sigma, \rho, F) > 0$ and $n_0(l, \sigma, \rho, F) < \infty$, such that, for all $\lambda \in [\sigma^{-1}, \sigma]$,

$$\begin{aligned} P\left\{S_n \leq n\left[\nu_+\left(L_+\left(\frac{\lambda}{n}\right)\right) - \nu_-\left(L_-\left(\frac{\rho\lambda}{n}\right)\right)\right] + KL_+\left(\frac{\lambda}{n}\right) - lL_-\left(\frac{\rho\lambda}{n}\right)\right\} \\ \geq C(l, \sigma, \rho, F) > 0 \end{aligned} \tag{4.18}$$

for $n \geq n_0(l, \sigma, \rho, F)$.

PROOF. This is essentially a consequence of Lemma 3.1 in Kesten and Maller (1992). See also Lemmas 1 and 2 in Kesten and Lawler (1992). Write

$$\alpha = P\{X_i \geq 0\}.$$

By assumption $0 < \alpha < 1$. Let N, n be such that

$$\alpha n - n^{-1/2} \leq N \leq \alpha n.$$

We shall use the above-mentioned lemma to show that there exists a constant $K > 0$, and for each $\lambda_0 > 0$, $\rho > 0$, $l > 0$, there exist constants $C_i = C_i(l, \lambda_0, \rho, F) > 0, i = 1, 2$, such that for $\lambda_0/2 \leq \lambda \leq 2\lambda_0$, we have

$$P\left\{\sum_1^N X_i \leq \frac{N}{\alpha}\nu_+\left(L_+\left(\frac{\lambda}{n}\right)\right) + KL_+\left(\frac{\lambda}{n}\right) \middle| X_i \geq 0 \text{ for } 1 \leq i \leq N\right\} \geq C_1 > 0 \tag{4.19}$$

and

$$P\left\{\sum_1^{n-N}(-X_i) \geq \frac{(n-N)}{(1-\alpha)}\nu_-\left(L_-\left(\frac{\rho\lambda}{n}\right)\right) + lL_-\left(\frac{\rho\lambda}{n}\right) \middle| X_i < 0 \text{ for } 1 \leq i \leq n-N\right\} \geq C_2 > 0 \tag{4.20}$$

for all large n. The proof will then be completed by the same estimates as used in (3.5) and (3.6) and following lines of Kesten and Maller (1992).

First we note that

$$P\left\{X_i \geq L_+\left(\frac{\lambda}{n}\right) \middle| X_i \geq 0\right\} = \frac{1}{\alpha}P\left\{X_i \geq L_+\left(\frac{\lambda}{n}\right)\right\} = \frac{\lambda}{\alpha n} \tag{4.21}$$

and

$$E\left\{X_i \middle| 0 \leq X_i \leq L_+\left(\frac{\lambda}{n}\right)\right\} = \left(\alpha - \frac{\lambda}{n}\right)^{-1}\nu_+\left(L_+\left(\frac{\lambda}{n}\right)\right). \tag{4.22}$$

From these relations and the central limit theorem, it is easy to obtain (4.19) when

$$E\{X_i^2 \,|\, X_i \geq 0\} < \infty.$$

We may therefore assume this second moment to be infinite, so that, for large enough n,

$$E\left\{X_i^2 I\left(X_i \leq L_+\left(\frac{\lambda_0}{2n}\right)\right) \middle| X_i \geq 0\right\} \geq 16B_2^2, \tag{4.23}$$

when B_2 is fixed such that

$$P\{X_i \geq B_2 \,|X_i \geq 0\} \leq \frac{1}{16}. \tag{4.24}$$

We now apply (3.4) of Kesten and Maller (1992) when $W_i^{(N)}$ has the conditional distribution of X_i, given $X_i \geq 0$, and with $\delta = N\lambda/(\alpha n)$. As in the proof of Theorem 2.1 of Kesten and Maller (1992), we have

$$P\left\{W_1^{(N)} \geq L_+\left(\frac{\lambda}{n}\right)\right\} = P\left\{X_i \geq L_+\left(\frac{\lambda}{n}\right) \middle| X_i \geq 0\right\} = \frac{\lambda}{\alpha n} = \frac{\delta}{N}.$$

Thus the $L(N, \delta)$ of Kesten and Maller (1992) [which is just a $(1-\delta/N)$ quantile of $W_1^{(N)}$] can be chosen as $L_+(\lambda/n)$, that is,

$$L(N, \delta) = L_+\left(\frac{\lambda}{n}\right).$$

Still in the notation of Kesten and Maller (1992), we have $G^{(N)}$ for the distribution of $W_i^{(N)}$, and (in the case of a continuous F and $G^{(N)}$)

$$m(N, \delta) = \int_0^{L(N,\delta)} x\, dG^{(N)}(x) = \frac{1}{\alpha}\nu_+(L(N, \delta)) = \frac{1}{\alpha}\nu_+\left(L_+\left(\frac{\lambda}{n}\right)\right)$$

and

$$s^2(N, \delta) = \frac{1}{\alpha}V_+\left(L_+\left(\frac{\lambda}{n}\right)\right).$$

Therefore, by (3.4) of Kesten and Maller (1992) [but $K(T)$ there should be $K(T, r)$] with $T = 0$,

$$\begin{aligned}
&P\left\{\sum_1^N X_i \leq \frac{N}{\alpha}\nu_+\left(L_+\left(\frac{\lambda}{n}\right)\right) + K(0,0)L_+\left(\frac{\lambda}{n}\right) \middle| X_i \geq 0 \text{ for } 1 \leq i \leq N\right\} \\
&\quad = P\left\{\sum_1^N W_i^{(N)} \leq \frac{N}{\alpha}\nu_+\left(L_+\left(\frac{\lambda}{n}\right)\right) + K(0,0)L_+\left(\frac{\lambda}{n}\right)\right\} \\
&\quad \geq P\left\{\sum_1^N W_i^{(N)} I(W_i^{(N)} \leq L(N,\delta)) \leq Nm(N,\delta) + K(0,0)L(N,\delta) \text{ and } \right. \\
&\qquad\qquad \left. W_i^{(N)} > L(N,\delta) \text{ for no value of } i \leq N\right\} \\
&\quad \geq C_1 > 0.
\end{aligned} \tag{4.25}$$

Here C_1 is the $C(0, \lambda_0, 0, B_2, 0)$ of Kesten and Maller (1992), with B_2 as in (4.24) and (4.23), which is just (3.2) of that reference when $B_1 = 0$. Inequality (4.25) proves (4.19) with $K = K(0, 0)$.

In the same way, we can apply (3.3) of Kesten and Maller (1992) with N replaced by $n-N$ and with each of the $W_i^{(N)}$ having the conditional distribution of $-X_i$ given $X_i < 0$. This gives (4.20) with $C_2 = C(l, \lambda_0, 0, B_2, 0)$, where now B_2 and n are such that

$$P\{X_i \le -B_2 | X_i < 0\} \le \frac{1}{16}$$

and

$$E\left\{X_i^2 I\left(X_i \ge -L_-\left(\frac{\rho\lambda_0}{2n}\right)\right)\Big| X_i < 0\right\} \ge 16B_2^2. \tag{4.26}$$

Again (4.26) will hold for large n if $E\{X_i^2 I(X_i < 0)\} = \infty$, while (4.20) follows directly from the central limit theorem if the second moment is finite. Thus (4.20) is also proven.

Finally, we must prove (4.18) from (4.19) and (4.20). Write Γ for the event

$$S_n \le n\left[\nu_+\left(L_+\left(\frac{\lambda}{n}\right)\right) - \nu_-\left(L_-\left(\frac{\rho\lambda}{n}\right)\right)\right] + KL_+\left(\frac{\lambda}{n}\right) - lL_-\left(\frac{\rho\lambda}{n}\right),$$

and Λ for the random set of indices $1 \le i \le n$ with $X_i \ge 0$. Then $\Gamma \supset (\Gamma_1 \cap \Gamma_2 \cap \Gamma_3)$, where

$$\Gamma_1 = \left\{\sum_1^n X_i^+ \le \frac{|\Lambda|}{\alpha}\nu_+\left(L_+\left(\frac{\lambda}{n}\right)\right) + KL_+\left(\frac{\lambda}{n}\right)\right\},$$

$$\Gamma_2 = \left\{\sum_1^n X_i^- \ge \frac{(n-|\Lambda|)}{(1-\alpha)}\nu_-\left(L_-\left(\frac{\rho\lambda}{n}\right)\right) + lL_-\left(\frac{\rho\lambda}{n}\right)\right\},$$

($|\Lambda|$ denotes the cardinality of Λ) and

$$\Gamma_3 = \{\alpha n - n^{-1/2} \le |\Lambda| \le \alpha n\}.$$

Therefore, if we condition on the set Λ, we have

$$\begin{aligned} P\{\Gamma\} &\ge P\{\Gamma_1 \cap \Gamma_2 \cap \Gamma_3\} \\ &\ge \sum_{\alpha n - n^{-1/2} \le |\Lambda| \le \alpha n} P\{X_i \ge 0 \text{ for } i \in \Lambda, X_i < 0 \text{ for } i \notin \Lambda\} \\ &\qquad \times P\{\Gamma_1 | X_i \ge 0 \text{ for } i \in \Lambda\} P\{\Gamma_2 | X_i < 0 \text{ for } i \notin \Lambda\}, \end{aligned}$$

where the sum is over all subsets Λ of $\{1,\dots,n\}$ with $\alpha n - n^{-1/2} \le |\Lambda| \le \alpha n$. But since the X_i are independent and identically distributed, (4.19) shows that

$$\begin{aligned} &P\{\Gamma_1 | X_i \ge 0 \text{ for } i \in \Lambda\} \\ &\quad = P\left\{\sum_1^N X_i \le \frac{N}{\alpha}\nu_+\left(L_+\left(\frac{\lambda}{n}\right)\right) + KL_+\left(\frac{\lambda}{n}\right)\Big| X_i \ge 0 \text{ for } 1 \le i \le N\right\} \ge C_1 \end{aligned}$$

on $\{|\Lambda| = N\}$ when $\alpha n - n^{-1/2} \le |\Lambda| \le \alpha n$.

Similarly, (4.20) shows that

$$P\{\Gamma_2 \mid X_i < 0 \text{ for } i \notin \Lambda\} \geq C_2.$$

Finally, we note that $|\Lambda|$ has a binomial distribution with parameters n and α so that $P\{\Gamma_3\} \geq C_3$ for large n and some constant $C_3 = C_3(\alpha) > 0$. Thus

$$P\{\Gamma\} \geq P\{\Gamma_1 \cap \Gamma_2 \cap \Gamma_3\} \geq C_1 C_2 P\{\Gamma_3\} \geq C_1 C_2 C_3 > 0$$

for large n. This proves (4.18) when λ is restricted to an interval $[\lambda_0/2, \lambda_0]$. Since the interval $[\sigma^{-1}, \sigma]$ can be covered by finitely many such intervals, (4.18) also holds uniformly for $\lambda \in [\sigma^{-1}, \sigma]$. This completes the proof of Lemma 4.4. □

We now return to the proof of Theorem 2.1. So far we have shown that (2.1) and (2.2) are equivalent, and each of these trivially implies (2.3). We now proceed by proving, when $U(\infty) = \infty$, that (2.3) implies (2.5) and (2.5) implies (2.4). That (2.4) implies (2.1) is trivial. The finite variance case is then easily dealt with.

We first show that (2.3) implies (2.5) when $F(-x) > 0$ for $x > 0$. We restrict ourselves to continuous F. A short remark for general, not necessarily continuous, F will be given at the end of this part of the proof. Until further notice we also assume $1 - F(x) > 0$ for all $x > 0$. Note that then $L_+(\varepsilon)$ and $L_-(\varepsilon)$ as defined in Lemma 4.4 are positive for ε small enough.

Assume (2.3) holds for some $r = 0, 1, \ldots$. Then $P\{S_n > 0\} \to 1$. Our first step is to prove that this implies

$$\frac{[\nu(x)]^-}{x(1 - F(x))} \to 0. \tag{4.27}$$

[Note that $[\nu(x)]^- = \max(0, -\nu(x))$; this is not the same as $\nu_-(x)$.] To see this, we apply Lemma 4.4 with $l = 1$. This shows that, uniformly for $\lambda \in [\sigma^{-1}, \sigma]$ and $\rho > 0, n \geq n_0(1, \sigma, \rho, F)$,

$$P\left\{S_n \leq n\left[\nu_+\left(L_+\left(\frac{\lambda}{n}\right)\right) - \nu_-\left(L_-\left(\frac{\rho\lambda}{n}\right)\right)\right] + KL_+\left(\frac{\lambda}{n}\right) - L_-\left(\frac{\rho\lambda}{n}\right)\right\}$$
$$\geq C(1, \sigma, \rho, F) > 0.$$

Since $P\{S_n > 0\} \to 1$ and $L_-(\rho\lambda/n) \geq 0$, this forces

$$n\left[\nu_+\left(L_+\left(\frac{\lambda}{n}\right)\right) - \nu_-\left(L_-\left(\frac{\rho\lambda}{n}\right)\right)\right] + KL_+\left(\frac{\lambda}{n}\right) \geq 0 \tag{4.28}$$

for large n. Now note that

$$\nu_-\left(L_-\left(\frac{\rho\lambda}{n}\right)\right) \geq \nu_-\left(L_+\left(\frac{\lambda}{n}\right)\right) - \frac{\rho\lambda}{n} L_+\left(\frac{\lambda}{n}\right). \tag{4.29}$$

This is obvious if $L_-(\rho\lambda/n) \geq L_+(\lambda/n)$ since $\nu_-(x)$ is increasing. If, on the other hand, $L_-(\rho\lambda/n) < L_+(\lambda/n)$, then

$$\begin{aligned}\nu_-\left(L_+\left(\frac{\lambda}{n}\right)\right) &= \int_{[-L_+(\lambda/n),\,0]} |x|\,dF(x)\\ &= \nu_-\left(L_-\left(\frac{\rho\lambda}{n}\right)\right) + \int_{[-L_+(\lambda/n),\,-L_-(\rho\lambda/n))} |x|\,dF(x)\\ &\leq \nu_-\left(L_-\left(\frac{\rho\lambda}{n}\right)\right) + L_+\left(\frac{\lambda}{n}\right)P\left\{X < -L_-\left(\frac{\rho\lambda}{n}\right)\right\}\\ &\leq \nu_-\left(L_-\left(\frac{\rho\lambda}{n}\right)\right) + L_+\left(\frac{\lambda}{n}\right)\frac{\rho\lambda}{n},\end{aligned}$$

using the definition of L_- in the last step. Thus (4.29) holds, and substitution into (4.28) shows that

$$n\left[\nu_+\left(L_+\left(\frac{\lambda}{n}\right)\right) - \nu_-\left(L_+\left(\frac{\lambda}{n}\right)\right)\right] + (K+\rho\lambda)L_+\left(\frac{\lambda}{n}\right) \geq 0$$

or

$$\frac{\nu\big(L_+(\lambda/n)\big)}{(\lambda/n)L_+(\lambda/n)} \geq -\rho - \frac{K}{\lambda}. \tag{4.30}$$

This quickly implies (4.27). We take $\sigma = (2K/\rho) + 2$. Then for large x we can choose $\lambda \in [1 \vee K/\rho, \sigma]$ and n such that

$$1 - F(x) = \frac{\lambda}{n}.$$

With this choice we may take x for $L_+(\lambda/n)$, and hence (4.30) gives

$$\frac{\nu(x)}{x\big(1-F(x)\big)} \geq -\rho - \frac{K}{\lambda} \geq -2\rho.$$

Thus, for any $\rho > 0$,

$$\liminf_{x\to\infty} \frac{\nu(x)}{x\big(1-F(x)\big)} \geq -2\rho,$$

whence

$$\liminf_{x\to\infty} \frac{\nu(x)}{x\big(1-F(x)\big)} \geq 0.$$

Statement (4.27) follows.

We turn now to the main step in the proof of (2.5). Assume that (2.5a) fails. In this case there exists a sequence $x_k \uparrow \infty$ and a constant $D < \infty$ such that

$$\frac{A(x_k)}{x_k F(-x_k)} = \frac{\nu(x_k) + x_k\big(1 - F(x_k)\big)}{x_k F(-x_k)} - 1 \le D - 1.$$

Consequently,

$$\nu(x_k) + x_k\big(1 - F(x_k)\big) \le D x_k F(-x_k). \tag{4.31}$$

We shall show by another application of Lemma 4.4 that this contradicts $P\{S_n > 0\} \to 1$ and thus (2.3). This time we choose

$$n_k = \left\lfloor \frac{K}{2DF(-x_k)} \right\rfloor.$$

Now, since $1 - F(x) > 0$ for all x, we have, for large k,

$$\begin{aligned} P\{X > x_k\} &= 1 - F(x_k) \\ &\le \frac{1}{x_k}\left\{x_k\big(1 - F(x_k)\big) + \big[\nu(x_k)\big]^+\right\} \\ &= \frac{1}{x_k}\left\{x_k\big(1 - F(x_k)\big) + \nu(x_k) + \big[\nu(x_k)\big]^-\right\} \\ &\le \frac{1}{x_k}\left\{D x_k F(-x_k) + \frac{1}{2} x_k\big(1 - F(x_k)\big)\right\} \end{aligned}$$

by (4.31) and (4.27). Of course, this estimate is also valid if $1 - F(x) = 0$ for large x, so we can drop the assumption that $1 - F(x) > 0$ at this stage. Thus, by our choice of n_k,

$$P\{X > x_k\} \le 2DF(-x_k) \le \frac{K}{n_k}.$$

We may therefore take

$$L_+\left(\frac{K}{n_k}\right) \le x_k. \tag{4.32}$$

We now take $\lambda = K, \sigma = K \vee K^{-1}$ and $\rho = 1/(4D)$. Then

$$F(-x_k) \sim \frac{K}{2Dn_k} = \frac{2\rho\lambda}{n_k}$$

and we may therefore take

$$L_-\left(\frac{\rho\lambda}{n_k}\right) \ge x_k. \tag{4.33}$$

Then

$$\nu_-\left(L_-\left(\frac{\rho\lambda}{n_k}\right)\right) \geq \nu_-(x_k)$$

and hence

$$\nu_+\left(L_+\left(\frac{\lambda}{n_k}\right)\right) - \nu_-\left(L_-\left(\frac{\rho\lambda}{n_k}\right)\right) \leq \nu(x_k). \tag{4.34}$$

Finally, Lemma 4.4, together with (4.32) to (4.34), gives

$$P\{S_{n_k} \leq n_k\nu(x_k) + (K-l)x_k\} \geq C(l,\sigma,\rho,F) > 0 \tag{4.35}$$

for large k. But for $l > 3K/2$, we have

$$n_k\nu(x_k) + (K-l)x_k \leq n_kDx_kF(-x_k) + (K-l)x_k \leq \left(\frac{3K}{2} - l\right)x_k < 0$$

by (4.31) and the choice of n_k. Thus (4.35) contradicts (2.3) when $l > 3K/2$. This proves that (2.3) implies (2.5a) when $F(-x) > 0$ for all x and F is continuous. Of course, (2.5a) implies $A(x) \geq 0$ for large x in this case, and thus (2.5b) also holds.

We will show that (2.5) implies (2.4). Suppose that $U_-(\infty) = \infty$ and $A(x)/[xF(-x)] \to \infty$, that is, (2.5a) holds, or that $U_-(\infty) < \infty = U_+(\infty)$ and $A(x) \geq 0$ for x large enough, and $F(-x) > 0$ for all x, which is (2.5b). Then, by Lemma 4.3, $A(x)/[xF(-x)] \to \infty$, so $xF(-x) = o(A(x))$ in both cases. Also, by Lemma 4.3, $xA(x) \to \infty$, so, in fact, $A(x) > 0$ for x large enough, say $x \geq x_1$. We will show that $P\{^{(r)}S_n \leq T(X^-)_n^{(1)}\} \to 0$ for fixed $T > 0$; this will prove (2.4) for $s = 1$ and thus for $s = 2, 3, \ldots$ as well. To do this, take $T_1 > T+r$ and define $B_n = B_n(T_1)$ by

$$B_n = \sup\left\{x \geq x_1 \colon \frac{A(x)}{x} \geq \frac{T_1}{n}\right\}.$$

Since $A(x) > 0$ for $x \geq x_1$, this gives a sequence $B_n \uparrow \infty$ satisfying, by the continuity of $A(x)$, $nB_n^{-1}A(B_n) = T_1$ for large n. Then $xF(-x) = o(A(x))$ implies

$$n\frac{\{\nu(B_n) + B_n[1-F(B_n)]\}}{B_n} = n\frac{\{A(B_n) + B_nF(-B_n)\}}{B_n}$$
$$\sim \frac{nA(B_n)}{B_n} = T_1 > T + r,$$

so condition (4.10) of Lemma 4.2 is satisfied for n large enough when $x_+ = x_- = B_n$ and T is replaced by TB_n. We also have

$$nF(-B_n) = \left\{\frac{nA(B_n)}{B_n}\right\}\left\{\frac{B_nF(-B_n)}{A(B_n)}\right\} \to 0$$

and

$$\limsup_{n\to\infty} n\frac{\left\{V(B_n)+B_n^2[1-F(B_n)]\right\}}{B_n^2} \le \limsup_{n\to\infty}\frac{nU(B_n)}{B_n^2}$$
$$= \limsup_{n\to\infty}\left\{\frac{U(B_n)}{B_nA(B_n)}\right\}\left\{\frac{nA(B_n)}{B_n}\right\}$$
$$\le 2T_1$$

by (4.13) of Lemma 4.3. Thus, by (4.12) of Lemma 4.2,

$$P\{{}^{(r)}S_n \le TB_n\} \le \frac{n\left\{V(B_n)+B_n^2[1-F(B_n)]\right\}/B_n^2}{\left\{n\left[\nu(B_n)+B_n[1-F(B_n)]/B_n\right]-(T+r)\right\}^2}+nF(-B_n),$$

which gives

$$\limsup_{n\to\infty} P\{{}^{(r)}S_n \le TB_n\} \le \frac{2T_1}{\{T_1-(T+r)\}^2}.$$

Note that, since $nF(-B_n)\to 0$,

$$P\{(X^-)_n^{(1)} < B_n\} = P^n(X_1^- < B_n) = [1-F(-B_n)]^n \to 1,$$

so

$$P\left\{\frac{{}^{(r)}S_n}{(X^-)_n^{(1)}} \le T\right\} \le P\left\{\frac{{}^{(r)}S_n}{B_n} \le \frac{T(X^-)_n^{(1)}}{B_n}, \frac{(X^-)_n^{(1)}}{B_n} < 1\right\}$$
$$+P\{(X^-)_n^{(1)} \ge B_n\}$$
$$\le P\left\{\frac{{}^{(r)}S_n}{B_n} \le T\right\}+o(1).$$

Hence

$$\limsup_{n\to\infty} P\left\{\frac{{}^{(r)}S_n}{(X^-)_n^{(1)}} \le T\right\} \le \frac{2T_1}{\{T_1-(T+r)\}^2},$$

and letting $T_1\to\infty$ completes the proof that (2.5) implies (2.4). This completes the proof when $F(-x)>0$ for $x>0$ and F is continuous.

Now consider the case when $F(-x_0)=0$ for some $x_0>0$ and $U_+(\infty)=\infty$. We prove that (2.3) implies (2.5b). But now, by (1.1), even without assuming F to be continuous,

$$\nu(x) = A(x) - x(1-F(x)) \le A(x)$$

for $x \ge x_0$. Thus, if $A(x)<0$ for arbitrarily large x, then since X is bounded below

$$EX = \lim_{x\to\infty}\nu(x) \le 0.$$

In this case we would have

$$\liminf_{n\to\infty} P\{S_n \le 0\} > 0$$

by virtue of Lemma 1 in Kesten and Lawler (1992), giving a contradiction.

Conversely, let (2.5b) hold, so $A(x_1) \ge 0$ for some $x_1 \ge x_0$. Then, for $x > x_1$,

$$A(x) = A(x_1) + \int_{x_1}^{x} \bigl(1 - F(y)\bigr)\,dy > 0,$$

since $F(y) < 1$ for all y (recall that we assumed the support of F unbounded, while it is bounded below in the present case). Also

$$EX_1 = \lim_{x\to\infty} A(x) \ge \int_{x_1}^{\infty} \bigl(1 - F(y)\bigr)\,dy > 0$$

(EX_1 may be $+\infty$). Thus (2.1) to (2.3) hold by the strong law of large numbers. This argument works for $U_+(\infty)$ finite or infinite.

Finally, if $EX^2 < \infty$, then $EX < 0$ is incompatible with (2.3) by the weak law of large numbers, since ${}^{(r)}S_n/n \to_P EX$. Also $EX = 0$ contradicts (2.3) by the central limit theorem, since then ${}^{(r)}S_n/n^{1/2}$ is asymptotically normal.

This proves the full theorem for ${}^{(r)}S_n$ when F is continuous. Continuity of F was assumed only for the implication from (2.3) to (2.5a) when $EX^2 = \infty$. This implication for general F can be proven by replacing X_i by $Y_i = X_i + U_i$, where each U_i has a uniform distribution on $[-1, +1]$ and all the X_i and U_j, $i \ge 1, j \ge 1$, are independent. By means of Proposition 4.1 and by Theorem 3.1 of Esseen (1968), one can then show that (2.3) implies

$$\frac{1}{\sqrt{n}} \sum_1^n Y_i \to_P \infty. \tag{4.36}$$

By what we have proved already, (4.36) implies the analogue of (2.5a) for the distribution of the Y_i. It is tedious and unilluminating to derive (2.5a) itself from this, and we skip the details.

Finally, we must show that ${}^{(r)}S_n$ may be replaced by ${}^{(r)}\widetilde{S}_n$. We shall write $\widetilde{(2.i)}$ for (2.i) when ${}^{(r)}S_n$ is replaced by ${}^{(r)}\widetilde{S}_n$, $1 \le i \le 4$. Since none of $\widetilde{(2.1)}$ to $\widetilde{(2.4)}$ or (2.5) can occur when $P\{X \ge 0\} = 0$, we may, as before, assume for the remainder of this proof that

$$P\{X \ge 0\} > 0$$

Now recall that ${}^{(r)}\widetilde{S}_n$ is obtained from S_n by removing the j smallest and $(r-j)$ largest X_i's for some $0 \le j \le r$, while ${}^{(r)}S_n$ is obtained by removing the r largest observations. From this it is not hard to see that, for $n \ge r$,

$${}^{(r)}S_n \le {}^{(r)}\widetilde{S}_n \le S_n + \sum_{j=1}^{r} (X^-)_n^{(j)}. \tag{4.37}$$

Now (4.37) shows immediately that (2.1) to (2.4) imply $(\widetilde{2.1})$ to $(\widetilde{2.4})$. For the converse, note that again each of $(\widetilde{2.1})$, $(\widetilde{2.2})$ and $(\widetilde{2.4})$ imply $(\widetilde{2.3})$. By virtue of (4.37) it therefore suffices to show that (2.1) with $r = 0$ follows from

$$P\left\{S_n + \sum_{j=1}^{r}(X^-)_n^{(j)} > 0\right\} \to 1. \tag{4.38}$$

However, it is easy to deduce (2.1) from (4.38) and Proposition 4.1. To see this, note that (2.1) is trivial if $F(0-) = 0$, while for $F(0-) > 0$, (4.38) just says that

$$P\left\{\sum_{1}^{n}(-X_i) - (r \text{ largest terms among } -X_1, \dots, -X_n) < 0\right\} \to 1.$$

We then also have

$$P\left\{n^{-1/4}\left[\sum_{1}^{n}(-X_i) - (r \text{ largest terms among } -X_1, \dots, -X_n)\right] < 0\right\} \to 1$$

and hence, by Proposition 4.1,

$$P\left\{n^{-1/4}\sum_{1}^{n}(-X_i) \le 1\right\} \to 1,$$

or, equivalently,

$$P\{S_n \ge -n^{1/4}\} \to 1.$$

This implies (2.1) with $r = 0$, because by a general concentration function inequality [Esseen (1968), Theorem 3.1],

$$P\{-n^{1/4} \le S_n \le n^{1/4}\} \to 0$$

unless X is a constant a.s. However, in this last case (4.38) forces $X > 0$ a.s. and then (2.1) is trivial. This completes the proof of Theorem 2.1. □

4.2. *Proof of Theorem* 2.2. Suppose $U_-(\infty) = \infty$ and $A(x)/[xF(-x)] \to \infty$ or $U_-(\infty) < \infty = U_+(\infty)$ and $A(x) \ge 0$ for x large enough. Suppose also that $F(-x) > 0$ for $x > 0$, so $A(x)/[xF(-x)] \to \infty$ also in the latter case by Lemma 4.3. Then by Theorem 2.1, ${}^{(r)}S_n/(X^-)_n^{(1)} \to_P \infty$, so, for $T > 0$,

$$\begin{aligned} P\{{}^{(r)}S_n \le TB_n\} &\le P\{{}^{(r)}S_n \le TB_n, (X^-)_n^{(1)} \le B_n\} + P\{{}^{(r)}S_n \le T(X^-)_n^{(1)}\} \\ &\le P\{{}^{(r)}S_n \le TB_n, \Sigma X_i I(X_i < -B_n) = 0\} + o(1). \end{aligned} \tag{4.39}$$

If, on the other hand, $F(-x) = 0$ for some $x > 0$, then (4.39) holds trivially since $\Sigma X_i I(X_i < -B_n) = 0$ a.s. for n large enough. Under the further assumption that $nA(B_n)/B_n \to \infty$, we have, by (1.1),

$$\frac{n\left\{\nu(B_n) + B_n[1 - F(B_n)]\right\}}{B_n} = \frac{n\{A(B_n) + B_nF(-B_n-)\}}{B_n} \ge \frac{nA(B_n)}{B_n} \to \infty.$$

Thus we can apply the bound (4.11) with $x_+ = x_- = B_n$ and T replaced by TB_n to obtain

$$\begin{aligned} P\{{}^{(r)}S_n \le TB_n, \Sigma X_i I(X_i < -B_n) = 0\} \\ \le \frac{n\{V(B_n) + B_n^2[1 - F(B_n)]\}}{\{n[\nu(B_n) + B_n[1 - F(B_n)] - (T + r)B_n\}^2} \\ \le [1 + o(1)]\frac{\{V(B_n) + B_n^2[1 - F(B_n)]\}}{nA^2(B_n)}. \end{aligned}$$

By Lemma 4.3 we also know that $\limsup_{x\to\infty} U(x)/[xA(x)] \le 2$ so

$$\frac{V(B_n) + B_n^2[1 - F(B_n)]}{nA^2(B_n)} \le \frac{U(B_n)}{nA^2(B_n)} = \left\{\frac{U(B_n)}{B_n A(B_n)}\right\}\left\{\frac{B_n}{nA(B_n)}\right\} \to 0.$$

Thus, via (4.39), we have $P\{{}^{(r)}S_n \le TB_n\} \to 0$ or ${}^{(r)}S_n/B_n \to_P \infty$.

Conversely, suppose ${}^{(r)}S_n/B_n \to_P \infty$. Then $S_n \to_P \infty$ and we know from Theorem 2.1 that $A(x)/[xF(-x)] \to \infty, x \to \infty$, when $U_-(\infty) = \infty$, or $A(x) \ge 0$, for large x, when $U_-(\infty) < \infty = U_+(\infty)$. It remains to show that $nA(B_n)/B_n \to +\infty$. Suppose this fails, so there is a sequence $n_i \uparrow \infty$ such that

$$\frac{n_i A(B_{n_i})}{B_{n_i}} \to a < \infty. \tag{4.40}$$

If we now define

$$T_n = \sum_{j=1}^{n}\big((X_j \wedge B_n) \vee (-B_n)\big),$$

then

$$\begin{aligned} E\left(\frac{T_{n_i}}{B_{n_i}}\right) &= \frac{n_i A(B_{n_i})}{B_{n_i}} = a + o(1), \\ \operatorname{Var}\left(\frac{T_{n_i}}{B_{n_i}}\right) &\le \frac{n_i U(B_{n_i})}{B_{n_i}^2} \le 2a + o(1) \quad \text{(by Lemma 4.3)} \end{aligned}$$

and

$$n_i H(B_{n_i}) \le \frac{n_i U(B_{n_i})}{B_{n_i}^2} \le 2a + o(1).$$

It follows that, for $T > a$,

$$\begin{aligned} P\{S_{n_i} \ge TB_{n_i}\} &\le P\{T_{n_i} \ge TB_{n_i}\} + P\{T_{n_i} \ne S_{n_i}\} \\ &\le \frac{2a}{(T-a)^2} + 1 - e^{-2a} + o(1). \end{aligned}$$

On the other hand, it follows from ${}^{(r)}S_n/B_n \to_P \infty$ that $S_{n_i}/B_{n_i} \to_P \infty$ (e.g., by Proposition 4.1). This contradiction shows that (4.40) is impossible and this completes the proof of Theorem 2.2 when $EX^2 = \infty$.

When $EX^2 < \infty$, the weak law of large numbers gives ${}^{(r)}S_n/n \to_P EX$, so if $EX > 0$ and $n/B_n \to \infty$, then ${}^{(r)}S_n/B_n \to_P \infty$. Conversely, if ${}^{(r)}S_n/B_n \to_P \infty$, then $EX > 0$ by Theorem 2.1 and so $n/B_n \to \infty$.

This completes the proof for ${}^{(r)}S_n$. For ${}^{(r)}\widetilde{S}_n$ we merely have to observe that (2.20) for ${}^{(r)}S_n$ and for ${}^{(r)}\widetilde{S}_n$ are equivalent by Proposition 4.1. □

4.3. *Proof of Theorem* 2.3. When $a = \infty$ the result is immediate from Theorem 2.2 when $EX^2 = \infty$, while if $EX^2 < \infty$, then neither ${}^{(r)}S_n/n \to_P \infty$ nor ${}^{(r)}\widetilde{S}_n/n \to_P \infty$ can occur, by the weak law of large numbers. Moreover $\lim_{x\to\infty} A(x)$ is finite in this case. So we need only consider the case $0 < a < \infty$.

Now assume (2.23) holds with $0 < a < \infty$ and $EX^2 = \infty$. (The result is trivial if $EX^2 < \infty$.) Then $S_n/n \to_P a$ [Kesten and Maller (1992), Theorem 2.1, or Proposition 4.1 above], and, equivalently [Feller (1971), page 565], $\nu(x) \to a$ and $x[1-F(x)+F(-x-)] \to 0$, so $A(x) \to a$. Since $a > 0$ and $xF(-x) \to 0$, (2.24) follows. Conversely, let $A(x) \to a \in (0, \infty)$. If $U_-(\infty) = \infty$, suppose also that $A(x)/[xF(-x)] \to \infty$. Then $xF(-x) \to 0$. If $U_-(\infty) < \infty$, then $x^2F(-x) \to 0$, so again $xF(-x) \to 0$. But then

$$\begin{aligned} A(2x) - A(x) &= \int_x^{2x} \big[1 - F(y) - F(-y)\big]\, dy \\ &\geq x\big[1 - F(2x)\big] - xF(-x) = x\big[1 - F(2x)\big] + o(1). \end{aligned}$$

Since $A(2x) - A(x) \to 0$, it also follows that $x[1 - F(x)] \to 0$ and, by (1.1),

$$\lim_{x\to\infty} \nu(x) = \lim_{x\to\infty} A(x) = a.$$

By Feller (1971), page 565, this implies (2.23) for $r = 0$, and by Theorem 2.1 of Kesten and Maller (1992) or Proposition 4.1, (2.23) for any r follows.

Again (2.23) for ${}^{(r)}S_n$ and for ${}^{(r)}\widetilde{S}_n$ are equivalent by Proposition 4.1. □

4.4. *Proof of Theorem* 2.4. We shall just prove this for the whole sequence n since the general case is no different. We have $S_n/B_n \to_P \infty$ (or to 0) if and only if $E(e^{-\lambda S_n/B_n}) \to 0$ (or to 1) for all $\lambda > 0$, equivalently, if

$$n \int_{[0,\infty)} (1 - e^{-\lambda x/B_n})\, dF(x) \to \infty \quad \text{(or to 0)}.$$

Using

$$ye^{-1} \leq 1 - e^{-y} \leq y \quad \text{for } 0 \leq y \leq 1$$

and

$$1 - e^{-1} \leq 1 - e^{-y} \leq 1 \quad \text{for } y \geq 1$$

and (1.1), it is easy to show that this is equivalent to $nA(B_n)/B_n \to \infty$ (respectively, 0). This proves the theorem for S_n, and for ${}^{(r)}S_n$ it then follows from Proposition 4.1. We remark that ${}^{(r)}\widetilde{S}_n = {}^{(r)}S_n$ since $X_i \geq 0$ in this theorem. □

4.5. *Proof of Theorem* 3.1. Clearly (3.1) implies (3.2). Suppose then that (3.2) holds for some $T > 0$, and without loss of generality take $T \le 1$. We shall show that (3.3) holds. Choose $\delta \in (0, T^2/6)$ and then choose $\eta \in (0,1)$ so that $\delta < (1-\eta)T^2/6$. Define a sequence D_n by

$$D_n = \sup\left\{x > 0\colon \frac{x|A(x)| + U(x)}{x^2} \ge \frac{\delta}{n}\right\}. \tag{4.41}$$

Then $D_n < \infty$, since $U(x)/x^2 \to 0$ and $A(x)/x \to 0$ as $x \to \infty$. Also $D_n \uparrow \infty$ because $U(x) > 0$ for all $x > 0$. Introduce the following notation:

$$h(y) = 1 - H(y-) = P\{|X| < y\},$$
$$S_n(y) = \sum_{i=1}^{n} Z_i(y),$$

where the $Z_i(y), i \ge 1$, are i.i.d., each with the conditional distribution of X, given $|X| < y$. Then we can write

$$\begin{aligned} P\left\{\left|{}^{(r)}\widetilde{S}_n\right| \le T\left|X_n^{(r)}\right|\right\} &\ge P\left\{\left|{}^{(r)}\widetilde{S}_n\right| \le T\left|X_n^{(r)}\right|,\ \left|X_n^{(r+1)}\right| < \left|X_n^{(r)}\right|\right\} \\ &\ge \binom{n}{r}\int_{[D_n,\infty)} P\left\{\min_{1\le j\le r}|X_j| \in dy\right\} \\ &\qquad\times\left[P\{|X| < y\}\right]^{n-r} P\{|S_{n-r}(y)| \le Ty\}. \end{aligned} \tag{4.42}$$

Note that the last integral is restricted to $y \ge D_n$. For such y, we have, by (1.1),

$$\frac{nV(y)}{y^2} \le \frac{nU(y)}{y^2} \le \delta \tag{4.43}$$

and

$$\frac{n|\nu(y-)|}{y} \le \frac{n|A(y)|}{y} + nH(y-) \le \delta + \frac{nU(y)}{y^2} \le 2\delta.$$

Assume that n is large enough for $h(D_n) \ge 1 - \eta$. Then

$$T - (n-r)\frac{|\nu(y-)|}{yh(y)} \ge T - \frac{2\delta}{1-\eta} > 0.$$

Note now that $EZ_i(y) = \nu(y-)/h(y)$ and $\mathrm{Var}(Z_i(y)) \le V(y)/h(y)$. Therefore, by Chebyshev's inequality,

$$\begin{aligned} P\{|S_{n-r}(y)| > Ty\} &\le P\left\{\left|S_{n-r}(y) - (n-r)\frac{\nu(y-)}{h(y)}\right| + (n-r)\frac{|\nu(y-)|}{h(y)} > Ty\right\} \\ &\le \frac{(n-r)V(y)/h(y)}{y^2\left[T - (n-r)|\nu(y-)|/(yh(y))\right]^2} \\ &\le \frac{\delta/(1-\eta)}{\left[T - 2\delta/(1-\eta)\right]^2} =: c. \end{aligned}$$

It is easily checked that $c < 1$ because of the choice of δ and η. From (4.42) we now deduce that

$$
\begin{aligned}
P\Big\{\big|{}^{(r)}\tilde{S}_n\big| \le T\big|X_n^{(r)}\big|\Big\} &\ge (1-c)\binom{n}{r}\int_{[D_n,\infty)} P\Big\{\min_{1\le j\le r}|X_j| \in dy\Big\} \\
&\quad\times\Big[P\big\{|X| < y\big\}\Big]^{n-r} \\
&\ge c_1 n^r\Big[P\big\{|X| < D_n\big\}\Big]^{n-r} P\Big\{\min_{1\le j\le r}|X_j| \ge D_n\Big\} \qquad (4.44)\\
&= c_1 n^r\Big[P\big\{|X| < D_n\big\}\Big]^{n-r}\Big[P\big\{|X| \ge D_n\big\}\Big]^r \\
&\ge c_1\Big[nP\big\{|X| \ge D_n\big\}\Big]^r e^{-nP\{|X|\ge D_n\}/P\{|X|<D_n\}}
\end{aligned}
$$

for some $c_1 > 0$. Since, by (4.43),

$$nP\big\{|X| \ge D_n\big\} = nP\big\{|X| > D_n\big\} + nP\big\{|X| = D_n\big\} \le \frac{nU(D_n)}{D_n^2} + \frac{nV(D_n)}{D_n^2} \le 2\delta,$$

(4.44) and (3.2) imply that

$$nH(D_n) = nP\big\{|X| > D_n\big\} \to 0, \qquad n \to \infty. \tag{4.45}$$

Now, by continuity of $U(x)$ and $A(x)$, we have

$$\delta D_n^2 = n\big(D_n|A(D_n)| + U(D_n)\big).$$

Thus (4.45) gives

$$\frac{D_n|A(D_n)| + U(D_n)}{D_n^2 H(D_n)} \to \infty, \qquad n \to \infty. \tag{4.46}$$

This proves (3.3) along the sequence D_n. To show that this implies the full (3.3), consider the left continuous function

$$g(x) := \frac{x|A(x)| + U(x)}{x^2 H(x-)}.$$

If (3.3) fails, then in view of (4.46) there must exist sequences $n_1 < n_2 < \cdots$ and $x_k \in (D_{n_k-1}, D_{n_k})$ and constant $T \ge 5$ such that

$$g(x) \ge T \ge 5 \quad \text{for } x \in \big(x_k, D_{n_k}\big] \text{ and } g(x_k) \le T. \tag{4.47}$$

Thus it suffices to show that (4.45) and (4.47) are incompatible. However, it is not hard to deduce from $|A(x+dx)| - |A(x)| \le H(x)\,dx$ and a similar relation for U and the first relation in (4.47) that

$$\frac{x|A(x)| + U(x)}{x^2}$$

is decreasing on $[x_k, D_{n_k}]$. But then

$$g(x_k) \geq \frac{1}{H(x_k-)}\frac{D_{n_k}|A(D_{n_k})| + U(D_{n_k})}{D_{n_k}^2}$$
$$\geq \frac{1}{H(D_{n_k-1})}\frac{\delta}{n_k} \to \infty \quad [\text{by } (4.45)].$$

This contradicts the second relation in (4.47), so that we have proven (3.3).

We now prove that (3.3) implies (3.4). Suppose (3.3) holds, and define D_n by (4.41) with $\delta = 1$, so that

$$\frac{n[D_n|A(D_n)| + U(D_n)]}{D_n^2} = 1. \tag{4.48}$$

If $0 < x < 1$ we have by (3.3) that, for $0 < \varepsilon < x^2$ and large n,

$$nH(xD_n) \leq \frac{\varepsilon}{x^2}\left[\frac{n|A(xD_n)|}{D_n} + \frac{nU(xD_n)}{D_n^2}\right]$$
$$\leq \frac{\varepsilon}{x^2}\left[\frac{n|A(D_n)|}{D_n} + \frac{nU(D_n)}{D_n^2} + \frac{|n\int_{xD_n < y \leq D_n}[1 - F(y) - F(-y)]\,dy|}{D_n}\right]$$
$$\leq \frac{\varepsilon + \varepsilon nH(xD_n)}{x^2}.$$

This shows that $nH(xD_n) \leq \varepsilon/(x^2 - \varepsilon)$ and so $nH(xD_n) \to 0, n \to \infty$ for $0 < x < 1$ and hence for $x > 0$. Given any sequence $n' \uparrow \infty$ of integers, take a further subsequence if necessary so that, as $n' \to \infty$,

$$\frac{n'V(D_{n'})}{D_{n'}^2} \to a' \quad \text{and} \quad \frac{n'\nu(D_{n'})}{D_{n'}} \to b'.$$

By (4.48), (1.1) and $nH(D_n) \to 0$, we have $a' + |b'| = 1$. Again, since $n'H(xD_{n'}) \to 0$ for $x > 0$, we see that, as $n' \to \infty$,

$$\frac{n'V(xD_{n'})}{D_{n'}^2} = \frac{n'V(D_{n'})}{D_{n'}^2} + \frac{O\{n'\int_{\min(x,\,1)D_{n'} \leq |y| \leq \max(x,\,1)D_{n'}} y^2\,dF(y)\}}{D_{n'}^2}$$
$$= a' + o(1) + O\Big\{\max(1, x^2)n'H\big(\min(1, x)D_{n'}\big)\Big\}$$
$$\to a'.$$

Similarly,

$$\frac{n'\nu(xD_{n'})}{D_{n'}} \to b'$$

for $x > 0$. By the criteria for convergence to the normal or degenerate distribution [Gnedenko and Kolmogorov (1968), Theorems 25.1, 26.2 and 27.2], we thus have

$$\frac{S_{n'} - n'\nu(D_{n'})}{D_{n'}} \to_D N(0, a')$$

and, in fact, that

$$\frac{S_{n'}}{D_{n'}} \to_D N(b', a'). \tag{4.49}$$

Here $N(b', a')$ stands for a normal random variable with mean b' and variance a'; if $a' = 0$ we interpret (4.49) as $S_{n'}/D_{n'} \to_P b'$. Note that if $a' = 0$, then $|b'| = 1$. Also, since $n'H(xD_{n'}) \to 0$ for $x > 0$, we have $X_{n'}^{(1)}/D_{n'} \to_P 0$ and hence also

$$\frac{{}^{(r)}\widetilde{S}_{n'}}{D_{n'}} \to_D N(b', a'). \tag{4.50}$$

Thus we have proved (3.4).

Finally, if (3.4) holds, then any sequence of integers has a subsequence n' for which (4.50) holds with $a' + |b'| > 0$. By Mori [(1984), Proof of Theorem 3], one then also has (4.49) and, as above, $X_{n'}^{(1)}/D_{n'} \to_P 0$. If $a' > 0$ and $T > 0, \delta > 0$, then

$$\begin{aligned} \limsup_{n'} P\left\{ \frac{|{}^{(r)}\widetilde{S}_{n'}|}{|X_{n'}^{(r)}|} \le T \right\} &\le \limsup_{n'} P\left\{ \frac{|{}^{(r)}\widetilde{S}_{n'}|}{D_{n'}} \le T\delta, \frac{|X_{n'}^{(r)}|}{D_{n'}} \le \delta \right\} \\ &\quad + \limsup_{n'} P\left\{ \frac{|X_{n'}^{(r)}|}{D_{n'}} > \delta \right\} \\ &= P\left\{ |N(b', a')| \le T\delta \right\} \to 0, \qquad \delta \to 0, \end{aligned} \tag{4.51}$$

and so $|{}^{(r)}\widetilde{S}_{n'}|/|X_{n'}^{(r)}| \to_P \infty$. If, on the other hand, $a' = 0$, then $|b'| > 0$ and

$$\frac{|{}^{(r)}\widetilde{S}_{n'}|}{|X_{n'}^{(r)}|} = \frac{|{}^{(r)}\widetilde{S}_{n'}|}{D_{n'}} \frac{D_{n'}}{|X_{n'}^{(r)}|} \sim |b'| \frac{D_{n'}}{|X_{n'}^{(r)}|} \to_P \infty, \tag{4.52}$$

so again $|{}^{(r)}\widetilde{S}_{n'}|/|X_{n'}^{(r)}| \to_P \infty$. Since this convergence holds for all subsequences, we do indeed have $|{}^{(r)}\widetilde{S}_n|/|X_n^{(r)}| \to_P \infty$. This proves (3.1). □

4.6. *Proof of Theorem* 3.2. Clearly, (3.5) implies (3.6), so let (3.6) hold. Then, for some $T > 0$ and some $n_1 < n_2 < \cdots$,

$$P\left\{ |{}^{(r)}\widetilde{S}_{n_i}| \le T|X_{n_i}^{(r)}| \right\} \to 0.$$

Then we obtain exactly as in the preceding proof of (3.3) from (3.2) that

$$n_i H(D_{n_i}) \to 0$$

and

$$\frac{D_{n_i}|A(D_{n_i})| + U(D_{u_i})}{D_{n_i}^2 H(D_{n_i})} \to \infty.$$

This implies (3.7).

The proof that (3.7) implies (3.8) is virtually identical to that of Lemma 2.6 of Pruitt (1981), so we do not produce it here.

Now it is obvious that (3.8) implies (3.9), if we take into account that $U(x) \geq x^2 H(x)$. Clearly, either of (3.9a) or (3.9b) implies (3.7). Thus we see that (3.7) to (3.9) are equivalent.

For the remainder of the proof, the following two lemmas are useful. They are also of interest in themselves since they give necessary and sufficient conditions for *uncentered* subsequential convergence to normality or for subsequential relative stability.

LEMMA 4.5. *If* $r = 0, 1, 2, \dots$ *the following are equivalent*: *there are sequences* $n_i \uparrow \infty$ *and* $B_{n_i} \uparrow \infty$ *such that*

$$\frac{|{}^{(r)}\widetilde{S}_{n_i}|}{B_{n_i}} \to_P 1 \quad \text{or} \quad \frac{|{}^{(r)}S_{n_i}|}{B_{n_i}} \to_P 1; \tag{4.53}$$

$$\limsup_{x\to\infty} \frac{\nu^2(x)}{H(x)V(x)} = \infty; \tag{4.54}$$

$$\limsup_{x\to\infty} \frac{x|A(x)|}{x^2H(x)+V(x)} = \infty. \tag{4.55}$$

PROOF. First, we show that (4.53) is equivalent to the following property: each subsequence of n_i has a further subsequence $\{m_j\}$ such that

$$\frac{S_{m_j}}{B_{m_j}} \to_P 1 \quad \text{or} \quad \frac{S_{m_j}}{B_{m_j}} \to_P -1. \tag{4.56}$$

If (4.56) holds, then by the degenerate convergence criterion [Gnedenko and Kolmogorov (1968), Theorem 27.2]

$$\frac{|X_{m_j}^{(1)}|}{B_{m_j}} \to_P 0$$

and hence also, for each r,

$$\frac{{}^{(r)}\widetilde{S}_{m_j}}{B_{m_j}} \to_P \pm 1 \quad \text{and} \quad \frac{{}^{(r)}S_{m_j}}{B_{m_j}} \to_P \pm 1. \tag{4.57}$$

This easily implies (4.53).

Conversely, if (4.53) holds for some r, then by Proposition 4.1 also

$$\frac{|S_{n_i}|}{B_{n_i}} \to_P 1. \tag{4.58}$$

Therefore any subsequence of the n_i contains a further subsequence $\{m_j\}$ such that S_{m_j}/B_{m_j} converges in distribution to some random variable Z, which must be infinitely divisible with $P\{|Z| \le 1\} = 1$. From Feller (1971), page 177, we know that then $P\{Z = c\} = 1$ for some constant c. Of course, we must have $c = \pm 1$, by virtue of (4.58). Thus (4.56) holds.

Now assume first that $EX^2 = \infty$. If (4.53) holds, then (4.56) holds along some subsequence $\{m_j\}$. By the degenerate convergence criterion again [Gnedenko and Kolmogorov (1968), Theorem 27.2], we then also have

$$(4.59) \qquad m_j H(B_{m_j}) \to 0, \qquad \frac{m_j \nu(B_{m_j})}{B_{m_j}} \to \pm 1, \qquad \frac{m_j V(B_{m_j})}{B_{m_j}^2} \to 0.$$

Conditions (4.59) easily imply (4.54) and (4.55) [use (1.1) again to obtain (4.55)].

Conversely, let (4.54) hold and take $x_i \uparrow \infty$ so that $[H(x_i)V(x_i)]/\nu^2(x_i) \to 0$. Define n_i as the integer part of

$$\left\{\frac{V(x_i)}{H(x_i)\nu^2(x_i)}\right\}^{1/2}.$$

A standard proof using the Cauchy–Schwarz inequality shows that $\nu^2(x) = o(V(x))$, $x \to \infty$, when $EX^2 = \infty$. Thus $n_i \to \infty$. Now

$$n_i^2 H^2(x_i) \sim \frac{V(x_i)H(x_i)}{\nu^2(x_i)} \to 0$$

and

$$\frac{V(x_i)}{n_i \nu^2(x_i)} \sim n_i H(x_i) \to 0.$$

Thus

$$P\left\{\left|\sum_{j=1}^{n_i} X_j I(|X_j| \le x_i) - n_i \nu(x_i)\right| > \varepsilon n_i \nu(x_i)\right\} \le \frac{V(x_i)}{\varepsilon^2 n_i \nu^2(x_i)} \to 0,$$

while

$$P\left\{\sum_{j=1}^{n_i} X_j I(|X_j| \le x_i) \ne \sum_{j=1}^{n_i} X_j\right\} \le n_i P\{|X_i| > x_i\} = n_i H(x_i) \to 0.$$

These give $S_{n_i}/[n_i \nu(x_i)] \to_P 1$, which, by the first part of the proof, implies (4.53) if we take $B_{n_i} = [V(x_i)/H(x_i)]^{1/2} \sim n_i |\nu(x_i)|$. Note that B_{n_i} indeed increases to ∞, since x_i increases to ∞.

Next let (4.55) hold and choose $x_i \uparrow \infty$ so that

$$\frac{|A(x_i)|}{x_i H(x_i)} \to \infty \quad \text{and} \quad \frac{x_i |A(x_i)|}{V(x_i)} \to \infty.$$

The first relation here together with (1.1) shows that $\nu(x_i) \sim A(x_i)$. Therefore

$$\frac{|\nu(x_i)|}{x_i H(x_i)} \to \infty \quad \text{and} \quad \frac{x_i |\nu(x_i)|}{V(x_i)} \to \infty.$$

Multiplying these gives (4.54).

Now let $EX^2 < \infty$. If $EX \neq 0$, then (4.53) to (4.55) are trivial since $\nu(x) \to EX$, as $x \to \infty$ and we may take $n_i = n$ and $B_n = n|EX|$ in (4.53). So suppose $EX = 0$. Then

$$(4.60) \qquad x|\nu(x)| = x\left|\int_{|u|>x} u\,dF(u)\right| \le \int_{|u|>x} u^2\,dF(u) \to 0, \qquad x \to \infty.$$

Since $E(X^2) < \infty, x^2H(x) \to 0$, so by (1.1), $x|A(x)| \to 0$. It follows that

$$\frac{V(x)}{x|A(x)|} \to \infty$$

and (4.55) cannot hold. Also, by Schwarz's inequality

$$\begin{aligned}\nu^2(x) &= \left[\int_{|u|>x} u\,dF(u)\right]^2 \le \left[\int_{|u|>x} u^2\,dF(u)\right]H(x),\\ &= o\big[H(x)\big],\end{aligned}$$

so $H(x)/\nu^2(x) \to \infty$ and (4.54) cannot hold. Also (4.53) cannot hold since then, by Rogozin (1976), (4.56) would imply

$$B_{m_j}^2 \sim m_j\big|\nu\big(B_{m_j}\big)\big|B_{m_j} = o(m_j),$$

because $x\nu(x) \to 0$. But $S_{m_j}/m_j^{1/2} \to_D N(0,1)$, so

$$\frac{|S_{m_j}|}{B_{m_j}} = \left(\frac{|S_{m_j}|}{m_j^{1/2}}\right)\left(\frac{m_j^{1/2}}{B_{m_j}}\right) \to_P \infty.$$

Thus none of (4.53), (4.54) or (4.55) holds when $EX^2 < \infty, EX = 0$. This completes the proof. □

LEMMA 4.6. *For* $r = 0, 1, 2, \dots$ *there are sequences* $n_i \uparrow \infty, C_{n_i} \uparrow \infty$, *such that*

$$(4.61) \qquad \frac{{}^{(r)}\widetilde{S}_{n_i}}{C_{n_i}} \to_D N(0,1) \quad \textit{or} \quad \frac{{}^{(r)}S_{n_i}}{C_{n_i}} \to_D N(0,1)$$

if and only if

$$(4.62) \qquad \limsup_{x\to\infty} \frac{U(x)}{x^2H(x) + x\big|A(x)\big|} = \infty.$$

PROOF. Assume that (4.61) holds for some $r \ge 1$. It then follows from the proof of Theorem 3 in Mori (1984) that also

$$(4.63) \qquad \frac{S_{n_i}}{C_{n_i}} \to_D N(0,1).$$

[Strictly speaking, Mori only proves this from ${}^{(r)}\widetilde{S}_{n_i}/C_{n_i} \to_D N(0,1)$, but a similar proof works when ${}^{(r)}S_{n_i}/C_{n_i} \to_D N(0,1)$; see also Kesten (1993).] For the time being assume also that $EX^2 = \infty$, so that $|\nu(x)|^2 = o(V(x))$ as $x \to \infty$. (4.63) is equivalent, by Gnedenko and Kolmogorov (1968), Theorem 25.1, to

$$n_i H(xC_{n_i}) \to 0, \qquad \frac{n_i \nu(C_{n_i})}{C_{n_i}} \to 0, \qquad \frac{n_i V(xC_{n_i})}{C_{n_i}^2} \to 1 \tag{4.64}$$

for all $x > 0$. Thus, using (1.1), the necessity of (4.62) is obvious.

Conversely, let (4.62) hold and take $x_i \uparrow \infty$ so that

$$\frac{x_i^2 H(x_i)}{U(x_i)} \to 0 \quad \text{and} \quad \frac{x_i |A(x_i)|}{U(x_i)} \to 0.$$

Define n_i as the integer part of

$$\min\left\{\sqrt{\frac{x_i^2}{H(x_i)U(x_i)}}, \sqrt{\frac{x_i^3}{|A(x_i)|U(x_i)}}\right\}.$$

Since $H(x) \to 0$, $U(x)/x^2 \to 0$ and $|A(x)|/x \to 0$ as $x \to \infty$, we have $n_i \to \infty$. Also

$$n_i^2 H^2(x_i) \le \frac{x_i^2 H(x_i)}{U(x_i)} \to 0$$

and

$$\frac{n_i^2 A^2(x_i)}{x_i^2} \le \frac{x_i |A(x_i)|}{U(x_i)} \to 0,$$

while, by (1.1),

$$\frac{n_i V(x_i)}{x_i^2} = \frac{n_i U(x_i)}{x_i^2} + o(1) \sim \frac{1}{n_i H(x_i)} \to \infty \left\{\text{if } n_i^2 \sim \frac{x_i^2}{H(x_i)U(x_i)}\right\}$$

or

$$\frac{n_i V(x_i)}{x_i^2} = \frac{n_i U(x_i)}{x_i^2} + o(1) \sim \frac{x_i}{n_i |A(x_i)|} \to \infty \left\{\text{if } n_i^2 \sim \frac{x_i^3}{|A(x_i)|U(x_i)}\right\}.$$

Now let

$$C_{n_i}^2 = n_i V(x_i).$$

Then $C_{n_i}/x_i \to \infty$ and so $n_i H(xC_{n_i}) = O(n_i H(x_i)) \to 0$ for $x > 0$. Also

$$\frac{n_i V(xC_{n_i})}{C_{n_i}^2} = \frac{n_i V(x_i)}{C_{n_i}^2} + \frac{n_i \int_{x_i < |u| \le xC_{n_i}} u^2\, dF(u)}{C_{n_i}^2} \to 1,$$

while, by (1.1), $n_i \nu(x_i)/x_i \to 0$ since $n_i A(x_i)/x_i \to 0$. Thus

$$\frac{n_i \nu(C_{n_i})}{C_{n_i}} = \frac{n_i \nu(x_i)}{C_{n_i}} + \frac{n_i \int_{x_i < |u| \le xC_{n_i}} u\, dF(u)}{C_{n_i}} \to 0.$$

By (4.64) these imply $S_{n_i}/C_{n_i} \to_D N(0,1)$ and $X_{n_i}^{(1)}/C_{n_i} \to_P 0$. Hence

$$\frac{{}^{(r)}S_{n_i}}{C_{n_i}} \to_D N(0,1) \quad \text{and} \quad \frac{{}^{(r)}\widetilde{S}_{n_i}}{C_{n_i}} \to_D N(0,1).$$

Finally, if $EX^2 < \infty$ and $EX = 0$, both conditions (4.61) and (4.62) hold, since $x|\nu(x)| \to 0$ as we showed in (4.60). If $EX^2 < \infty$ and $EX \neq 0$, it is easy to see that neither condition can hold. □

We now complete the proof of Theorem 3.2. Suppose (3.8) holds. If (3.8a), that is, (4.62), holds, by Lemma 4.6, we can choose integers n_i and a sequence D_{n_i} such that ${}^{(r)}\widetilde{S}_{n_i}/D_{n_i} \to_D N(0,1)$. If (3.8b), that is, (4.55), holds, by Lemma 4.5 and its proof, we can choose n_i and D_{n_i} so that ${}^{(r)}\widetilde{S}_{n_i}/D_{n_i} \to_P \pm 1$ [see (4.57)]. Thus (3.10) holds, since we interpret degenerate convergence as convergence to a degenerate normal random variable.

If (3.10) holds, then the proof from (3.4) to (3.1) [see (4.51) and (4.52)] can again be used to deduce (3.5). This completes the proof of Theorem 3.2. □

4.7. *Proof of Remark* (ii) *to Theorem* 3.2. A general concentration function inequality [see Esseen (1968), Theorem 3.1] shows that $|S_n| \to_P \infty$ when F is not concentrated on one point. If F is concentrated on one point, which is different from 0, then $|S_n| \to_P \infty$ is even more obvious. We can therefore find some $B_n \uparrow \infty$ such that $|S_n|/B_n \to_P \infty$ in the same way as in the proof of (2.2) from (2.1). Moreover, $|S_n|/B_n \to_P \infty$, $|{}^{(r)}\widetilde{S}_n|/B_n \to_P \infty$ and $|{}^{(r)}S_n|/B_n \to_P \infty$ are all equivalent by Proposition 4.1. Hence all these relations hold if F is not concentrated on $\{0\}$. On the other hand, it is clear that (3.3) or (3.4) does fail for some distributions not concentrated on $\{0\}$. □

4.8. *Proof of Theorem* 3.3. Suppose (3.11) holds for some r. By Proposition 4.1, (3.11) then also holds for $r = 0$. Assume that (3.13) fails so that there is a sequence $x_i \to \infty$ such that $|A(x_i)| + U(x_i)/x_i \to \infty$. If $\limsup_{i\to\infty} U(x_i)/x_i = \infty$ we can take a subsequence so that $U(x_i)/x_i \to \infty$ as $i \to \infty$. By the argument of Lemma 1 of Erickson and Kesten (1974), we then have $P\{|S_{x_i}|/x_i \le T\} \to 0$ for $T \ge 1$, or $|S_{x_i}|/x_i \to_P \infty$, $i \to \infty$, which contradicts (3.11) for $r = 0$. Alternatively, $U(x_i)/x_i$ is bounded, so we can assume $|A(x_i)| \to \infty$. Then $V(x_i)/x_i$ and $x_i H(x_i)$ are bounded, so $|\nu(x_i)| \to \infty$ and $\nu^2(x_i)/H(x_i)V(x_i) \to \infty$. Defining n_i as the integer part of

$$\left\{\frac{V(x_i)}{H(x_i)\nu^2(x_i)}\right\}^{1/2},$$

we obtain, as in the proof of (4.53) from (4.54), that $S_{n_i}/n_i\nu(x_i) \to_P 1$. This means

$$\frac{|S_{n_i}|}{n_i} \sim_P |\nu(x_i)| \to \infty$$

as $i \to \infty$, and contradicts (3.11). Thus (3.13) holds.

Next suppose (3.13) holds, so $|A(x)| + U(x)/x \leq c$ for x large enough, and thus, since $U(x) \geq x^2 H(x), xH(x) \leq c$ for such x. Hence by (1.1), $|\nu(x)| \leq 2c$. By truncation at $n\lambda$ and Chebyshev's inequality,

$$P\left\{|S_n - n\nu(n\lambda)| > \frac{n\lambda}{2}\right\} \leq \frac{4nV(n\lambda)}{n^2\lambda^2} + nH(n\lambda) \leq \frac{4c}{\lambda} + \frac{c}{\lambda} = \frac{5c}{\lambda}.$$

Furthermore,

$$\begin{aligned} P\left\{\left|{}^{(r)}S_n - n\nu(n\lambda)\right| > \frac{3n\lambda}{4}\right\} &= P\left\{\left|S_n - n\nu(n\lambda) - \sum_1^r M_n^{(i)}\right| > \frac{3n\lambda}{4}\right\} \\ &\leq P\left\{|S_n - n\nu(n\lambda)| > \frac{n}{2}\lambda\right\} + P\left\{|X_n^{(i)}| > \frac{n\lambda}{4r}\right\} \\ &\leq \frac{5c}{\lambda} + nH\left(\frac{n\lambda}{4r}\right) \leq \frac{5+4r}{\lambda}c. \end{aligned}$$

So if we choose $x > 2(5+4r)c \geq (5+4r)|\nu(nx)|$, then we obtain

$$\begin{aligned} P\left\{\left|\frac{{}^{(r)}S_n}{n}\right| > x\right\} &= P\left\{\left|{}^{(r)}S_n - n\nu(nx) + n\nu(nx)\right| > nx\right\} \\ &\leq P\left\{\left|{}^{(r)}S_n - n\nu(nx)\right| > n\big[x - |\nu(nx)|\big]\right\} \\ &\leq P\left\{\left|{}^{(r)}S_n - n\nu(nx)\right| > \frac{3nx}{4}\right\} \leq \frac{5+4r}{x}c < 1, \end{aligned}$$

which proves (3.12). Clearly, (3.12) implies (3.11). This proves Theorem 3.3 for ${}^{(r)}S_n$ and the proof for ${}^{(r)}\widetilde{S}_n$ is the same. □

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