The Annals of Probability
1994, Vol. 22, No. 2, 820–832

CURRENT FLUCTUATIONS FOR THE ASYMMETRIC SIMPLE EXCLUSION PROCESS[1]

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We compute the diffusion coefficient of the current of particles through a fixed point in the one-dimensional nearest neighbor asymmetric simple exclusion process in equilibrium. We find $D = |p - q|\rho(1-\rho)|1-2\rho|$, where p is the rate at which the particles jump to the right, q is the jump rate to the left and ρ is the density of particles. Notice that D vanishes if $p = q$ or $\rho = 1/2$. Laws of large numbers and central limit theorems are also proven. Analogous results are obtained for the current of particles through a position travelling at a deterministic velocity r. As a corollary we get that the equilibrium density fluctuations at time t are a translation of the fluctuations at time 0. We also show that the current fluctuations at time t are given, in the scale $t^{1/2}$, by the initial density of particles in an interval of length $|(p-q)(1-2\rho)|t$. The process is isomorphic to a growth interface process. Our result means that the equilibrium growth fluctuations depend on the general inclination of the surface. In particular, they vanish for interfaces roughly perpendicular to the observed growth direction.

1. Introduction. The nearest neighbor one-dimensional simple exclusion process is the Markov process $\eta_t \in \{0,1\}^{\mathbb{Z}}$ with generator given by

$$Lf(\eta) = \sum_{x\in\mathbb{Z}} \sum_{y=x\pm 1} p(x,y)\eta(x)\big(1-\eta(y)\big)\left[f(\eta^{x,y}) - f(\eta)\right],$$

where f is a continuous function,

$$p(x,y) = \begin{cases} p, & \text{if } y = x+1, \\ q, & \text{if } y = x-1, \\ 0, & \text{otherwise,} \end{cases}$$

$p + q = 1$ and

$$\eta^{x,y}(z) = \begin{cases} \eta(z), & \text{if } z \neq x, y, \\ \eta(x), & \text{if } z = y, \\ \eta(y), & \text{if } z = x. \end{cases}$$

A convenient way to describe the process is the so-called graphical construction. At most one particle is admitted at each site $x \in \mathbb{Z}$. Each pair of sites $(x, x+1)$ has associated two Poisson process with rates p and q, respectively. An arrow

Received September 1992; revised January 1993.

[1]Work partially supported by FAPESP, Projeto Temático, 90/3918-5 and CNPq.

AMS 1991 *subject classifications.* 60K35, 82C22, 82C24, 82C41.
Key words and phrases. Asymmetric simple exclusion, current fluctuations, driven interface.

pointing from x to $x+1$ is attached to each event of the process with parameter p, and arrows pointing from $x+1$ to x are attached to events of the process with parameter q. All these Poisson processes are independent and the null event "two arrows occur at the same time" is neglected. When an arrow appears pointing from x to y, if there is a particle at x and no particle at y, then at that time the particle jumps to the empty site. For any other configuration nothing happens. This process was introduced by Spitzer (1970) and has received a great deal of attention. The existence of the process and the ergodic properties were studied by Liggett (1976, 1985). The set of invariant measures is the set of convex combinations of the product measures ν_ρ and blocking measures. In the case $p > q$ the blocking measures concentrate on a denumerable set of configurations and have asymptotic density 0 and 1 to the left and right of the origin, respectively. When $p = q$ there are no blocking invariant measures. The hydrodynamical limit was studied by Andjel and Vares (1987) and extended by Benassi, Fouque, Saada and Vares (1991) for monotone initial density profiles. Rezakhanlou (1990) proposed a general approach to prove a law of large numbers for the density fields of attractive particle systems that works for general initial density profiles. Landim (1993) uses this law of large numbers to prove local equilibrium.

The current through rt at time t is defined by $J_{rt,t}$ = number of particles to the left of the origin at time zero and to the right of rt at time t minus number of particles to the right of the origin at time zero and to the left of rt at time t. Let X_t^x be the position of a tagged particle initially located at x. Then we define formally the current as the random process depending on the initial configuration η given by

$$J_{rt,t}(\eta) = \sum_{x\le 0} \eta(x) 1\{X_t^x > rt\} - \sum_{x>0} \eta(x) 1\{X_t^x \le rt\}.$$

We assume that the distribution of the initial configuration is the stationary measure ν_ρ, the product measure with density ρ. Under this initial distribution,

$$EJ_{rt,t} = \big((p-q)\rho(1-\rho) - r\rho\big)t. \tag{1.1}$$

The identity holds if rt is integer, which we assume without loss of generality [if not, the difference is $O(1)$]. Our main result is the following. It holds for any $p, q, p+q = 1$.

THEOREM 1. *Law of large numbers:*

$$\lim_{t\to\infty} \frac{J_{rt,t}}{t} = \big((p-q)\rho(1-\rho) - r\rho\big), \qquad \textit{almost surely}. \tag{1.2}$$

Central limit theorem: Let $G(0, D)$ be a centered normal random variable with variance D. Then

$$\lim_{t\to\infty} \frac{J_{rt,t} - EJ_{rt,t}}{\sqrt{t}} = G(0, D_J), \tag{1.3}$$

in distribution, where $D_J = \lim_{t\to\infty}(VJ_{rt,t}/t)$ *and* V *is the variance. Furthermore,*

$$\lim_{t\to\infty}\frac{VJ_{rt,t}}{t} = \rho(1-\rho)\big|(p-q)(1-2\rho)-r\big|. \tag{1.4}$$

Dependence on the initial configuration:

$$\lim_{t\to\infty}\frac{E\big(J_{rt,t}-N_{th(r,\rho)}-(p-q)\rho^2 t\big)^2}{t} = 0, \tag{1.5}$$

where $h(r,\rho) = r-(1-2\rho)(p-q)$, $N_r(\eta) = -\sum_{x=0}^{r}\eta(x)$ *for* $r > 0$ *and* $N_r(\eta) = \sum_{x=r}^{0}\eta(x)$ *for* $r \le 0$. *The quantity* $N_{th}(\eta)$ *depends only on the initial configuration* η.

REMARK. Notice that for $p = q$ and $r = 0$ or for $r = (p-q)(1-2\rho)$, $D_J = 0$. The first fact can be proven using Arratia (1983) or a formula given in De Masi and Ferrari (1985a). Indeed, De Masi and Ferrari (1985b) showed that for $p = 1/2$ and all ρ,

$$\lim_{t\to\infty}\frac{VJ_{0,t}}{t^{1/2}} = \sqrt{\frac{2}{\pi}}\rho(1-\rho) \tag{1.6}$$

and that

$$\lim_{t\to\infty} t^{-1/4}J_{0,t} = N\left(0, \sqrt{\frac{2}{\pi}}\rho(1-\rho)\right).$$

The fact that $D_J = 0$ for $r = (p-q)(1-2\rho)$ is more surprising. For $p = 1$ and $r = (1-2\rho)$, we show that

$$VJ_{(1-2\rho)t,t} = \rho(1-\rho)E\big|R_t^0 - (1-2\rho)t\big|, \tag{1.7}$$

where R_t^0 is the position of a second class particle initially located at the origin. For $p = 1$, a second class particle interacts with the other particles in the following way: It jumps to empty sites to the right at rate 1 and interchanges positions with ("first class") particles to its left at rate 1. Spohn (1991) gives heuristic arguments suggesting that VR_t^0 behaves as $t^{4/3}$. This would imply that the variance of the current through $(1-2\rho)t$ behaves as $t^{2/3}$.

An important corollary of (1.4) is that it allows one to show that the equilibrium fluctuations translate rigidly in time. More precisely, let ξ_t^ε be the fluctuations fields defined by

$$\xi_t^\varepsilon(\Phi) = \varepsilon^{1/2}\sum_x \Phi(\varepsilon x)\big[\eta_{\varepsilon^{-1}t}(x) - E\eta_{\varepsilon^{-1}t}(x)\big], \tag{1.8}$$

for smooth integrable functions Φ. We prove in Section 6 that, letting $\bar r = (p-q)(1-2\rho)$,

$$\lim_{\varepsilon\to 0}E\big(\xi_t^\varepsilon - \tau_{\varepsilon^{-1}\bar r t}\xi_0^\varepsilon\big)^2 = 0, \tag{1.9}$$

where the translation τ is defined by $\tau_y\xi_t^\varepsilon(\phi) = \xi_t^\varepsilon(\tau_y\Phi)$ and $\tau_y\Phi(x) = \Phi(x+y)$.

In Section 2 we give some results on the behavior of tagged and second class particles. In Section 3 we compute the current fluctuations (1.4). The law of large numbers (1.2) is shown in Section 4. The dependence on the initial configuration (1.5) and the central limit theorem (1.3) are shown in Section 5. In Section 7 we discuss consequences of our results on the motion of an interface model related to the simple exclusion process.

2. The motion of tagged and second class particles. We recall briefly some results concerning the motion of a tagged particle and show a lemma relating the tagged particle to a second class particle. We assume that the initial distribution of η_t is the equilibrium measure ν_ρ. At time 0, a particle is put at a fixed site x, regardless of the value of the configuration $\eta_0(x)$. This particle is tagged and followed. It interacts by exclusion with the other particles. Its position is denoted X_t^x. The joint process (η_t, X_t^x) is Markov and the measure $\nu'_\rho = \nu_\rho\big(\cdot \mid \eta(0) = 1\big)$ is extremal invariant for the (Markov) process $\tau_{X_t^x}\eta_t$. Under this distribution,

$$EX_t^0 = (1-\rho)(p-q)t. \tag{2.1}$$

Kipnis (1986) proved the following law of large numbers:

$$\lim_{t\to\infty} \frac{X_t^0}{t} = (1-\rho)(p-q); \tag{2.2}$$

and the following central limit theorem:

$$\lim_{t\to\infty} \frac{X_t^0 - (1-\rho)(p-q)t}{\sqrt{t}} = G(0, D_X), \tag{2.3}$$

in distribution. The variance D_X is given by

$$D_X = \lim_{t\to\infty} \frac{VX_t^0}{t} = (1-\rho)(p-q). \tag{2.4}$$

The limit was computed by De Masi and Ferrari (1985a). These results also follow from a recent extension of Burke's theorem due to Ferrari and Fontes (1992) that states the following. Assume that the initial distribution of η_t is given by ν'_ρ. Then there exist random variables $K \geq 0$ with a finite exponential moment [i.e., for some positive θ, $E\exp(\theta K) < \infty$] and K_t satisfying $P(|K_t| \geq k) \leq P(K \geq k)$ for all $k \geq 0$ (i.e., $|K_t| \leq K$ stochastically), such that

$$X_t^0 = N_t + K_t, \tag{2.5}$$

for all $t \geq 0$, where N_t is a Poisson process with parameter $(1-\rho)(p-q)$. In particular, this implies that if $r < (1-\rho)(p-q)$, then

$$\lim_{t\to\infty} \frac{E\Big(\big(X_t^0 - rt\big)^2 1\{X_t^0 \leq rt\}\Big)}{t} = 0. \tag{2.6}$$

Now we recall the definition of the so-called second class particle and some results concerning its asymptotic behavior. Let η^x be the configuration η modified at x, that is, $\eta^x(x) = 1 - \eta(x)$, $\eta^x(y) = \eta(y)$ for $y \neq x$. Let η_t^x be the process with initial configuration η^x. Then, using the graphical construction, the proceses η_t and η_t^x can be realized simultaneously with the same arrows. In this way the number of sites where the two configurations disagree is exactly 1 for all t. This is the basic coupling of Liggett (1976, 1985). Calling R_t^x the site where the configurations disagree by time t, one can show that the process (η_t, R_t^x) is Markovian and that R_t^x can be described as a second class particle: It jumps over nearest neighbor empty sites at rates q and p to the right and left, respectively, and exchanges positions with (first class) nearest neighbor particles at rates q and p to the right and left, respectively. Details can be found in Ferrari (1992), as well as the following law of large numbers:

$$\lim_{t\to\infty} \frac{R_t^0}{t} = (p-q)(1-2\rho), \quad \text{a.s.} \tag{2.7}$$

Since the absolute value of the position of a second class particle is dominated above by a Poisson process of rate 1, R_t^0/t is uniformly integrable. Then

$$\lim_{t\to\infty} \frac{E(R_t^0 - rt)^+}{t} = \begin{cases} 0, & \text{if } r > (p-q)(1-2\rho), \\ (p-q)(1-2\rho) - r, & \text{otherwise.} \end{cases} \tag{2.8}$$

We also have, for all ρ and $p \geq q$,

$$\lim_{t\to\infty} \frac{E\left(R_t^0 - X_t^0\right)^+}{t} = 0. \tag{2.9}$$

Next we show a technical identity needed in the computation of the current fluctuations. Fix a configuration η with infinitely many particles to the right and left of the origin and with a particle at the origin. Let U_t^y be the position at time t of a tagged particle initially at y for the configuration η. Let Z_t^y be the position at time t of a tagged particle initially at y for η^0, the configuration η without the particle at the origin.

LEMMA 2.10. *For all $r \in \mathbb{R}$ it holds that*

$$\sum_{y<0} \eta(y) 1\{Z_t^y > r, U_t^y \leq r\} = 1\{R_t^0 \leq r, X_t^0 > r\} \quad a.s. \tag{2.11}$$

PROOF. Let $\{y_i : i \in \mathbb{Z}\}$ be the ordered occupied sites of η such that $y_0 = 0$. Let $\{z_i : i \in \mathbb{Z} \setminus \{0\}\}$ be the ordered occupied sites of η^0, in such a way that $y_i = z_i$ for all $i \neq 0$. Let π_t^i denote the label of the η_t^0 particle that at time t is in the position $y_i(t) = U_t^{y_i}$, if there is such a particle. Assign to π_t^i the symbol $\emptyset$ otherwise. In this way, $\{(i, \pi_t^i) : i \in \mathbb{Z}\}$ tells us how the particles of the processes η_t and η_t^0 are coupled. Assuming $v_0 = 0$ and $T_0 = 0$ define, for $n \geq 1$,

$$T_n = \inf\{t > T_{n-1} : \pi_t^{v_{n-1}} \neq \emptyset\}, \quad v_n = i, \text{ for } i \text{ satisfying } \pi_{T_n}^{v_{n-1}} = i.$$

There is always a discrepancy of particles between η_t and η_t^0, and η_t has one more particle. Initially the (only) discrepancy is located at 0 and $\pi_0^0 = \emptyset$ but this location changes in time. The time of the nth change is T_n, while v_n is the index of the new location. At time t the discrepancy is located at $y_i(t)$ if $\pi_t^i = \emptyset$. It holds by induction on n that if $t \in [T_n, T_{n+1})$, then

$$\pi_t^{v_n} = \emptyset, \tag{2.12a}$$

$$\text{if } v_n \geq 0, \quad \text{then } \pi_t^i = \begin{cases} i, & \text{if } i \in [0, v_n]^c, \\ i+1, & \text{if } i \in [0, v_n); \end{cases} \tag{2.12b}$$

$$\text{if } v_n \leq 0, \quad \text{then } \pi_t^i = \begin{cases} i, & \text{if } i \in [v_n, 0]^c, \\ i-1, & \text{if } i \in (v_n, 0]. \end{cases} \tag{2.12c}$$

Now, we have

$$X_t^0 = U_t^{y_0}, \tag{2.13}$$

and all (2.12) is saying is that, for $t \in [T_n, T_{n+1})$,

$$R_t^0 = U_t^{y_{v_n}}; \tag{2.14a}$$

$$\text{if } v_n \geq 0, \quad \text{then } U_t^{y_i} = \begin{cases} Z_t^{y_i}, & \text{if } i \in [0, v_n]^c, \\ Z_t^{y_{i+1}}, & \text{if } i \in [0, v_n); \end{cases} \tag{2.14b}$$

$$\text{if } v_n \leq 0, \quad \text{then } U_t^{y_i} = \begin{cases} Z_t^{y_i}, & \text{if } i \in [v_n, 0]^c, \\ Z_t^{y_{i-1}}, & \text{if } i \in (v_n, 0]. \end{cases} \tag{2.14c}$$

The exclusion interaction implies that, for $j < i$,

$$Z_t^{y_j} < Z_t^{y_i} \quad \text{and} \quad U_t^{y_j} < U_t^{y_i}. \tag{2.15}$$

So, for $i < 0, Z_t^{y_i} > r$, and $U_t^{y_i} \leq r$ imply by (2.14b, c) that $t \in [T_n, T_{n+1})$ for which $v_n < 0$. This, (2.14c) and (2.15) imply that, for all $j \neq i$, either $Z_t^{y_j} \leq r$ and $U_t^{y_j} \leq r$ or $Z_t^{y_i} > r$ and $U_t^{y_j} > r$. Hence,

$$\begin{aligned} \sum_{i<0} 1\{Z_t^{y_i} > r, U_t^{y_t} \leq r\} &= 1\left\{ \bigcup_{i<0} \{Z_t^{y_i} > r, U_t^{y_i} \leq r\} \right\} \\ &\leq 1\{R_t^0 \leq r, X_t^0 > r\}, \end{aligned}$$

where the inequality holds by (2.13)–(2.15). For the reverse inequality observe that if $t \in [T_n, T_{n+1})$, then

$$U_t^{y_0} \geq r, U_t^{y_{v_n}} < r \quad \text{implies} \quad Z_t^{y_i} > r, U_t^{y_i} \leq r$$

for some $i < 0$, by taking $i = \min\{k \leq 0 \colon U_t^{y_k} \geq r\} - 1$. This proves the lemma. □

3. Current fluctuations. In this section we prove (1.4). Recall that X_t^x denotes the position of a tagged particle that at time 0 is put at x. For a fixed initial configuration η, we write $J_{rt,t}(\eta) = (J_{rt,t}(\eta))^+ - (J_{rt,t}(\eta))^-$ where

$$(J_{rt,t}(\eta))^+ = \sum_{x\le 0}\eta(x)1\{X_t^x > rt\}, \quad (J_{rt,t}(\eta))^- = \sum_{x>0}\eta(x)1\{X_t^x \le rt\}. \tag{3.1}$$

By translation invariance,

$$\begin{aligned} E(J_{rt,t})^+ &= E\left(\sum_{x\le 0}\eta(x)1\{X_t^x > rt\}\right)\\ &= \rho\sum_{x\le 0}P\{X_t^x > rt\} = \rho E\left(X_t^0 - rt\right)^+,\\ E(J_{rt,t})^- &= E\left(\sum_{x>0}\eta(x)1\{X_t^x \le rt\}\right)\\ &= \rho\sum_{x>0}P\{X_t^x \le rt\} = \rho E\left(X_t^0 - rt\right)^-. \end{aligned} \tag{3.2}$$

Since $J^+J^- \equiv 0$,

$$VJ_{rt,t} = V(J_{rt,t})^+ + V(J_{rt,t})^- + 2E(J_{rt,t})^+E(J_{rt,t})^-. \tag{3.3}$$

We compute now $V(J_{rt,t})^+ = E((J_{rt,t})^+)^2 - (E(J_{rt,t})^+)^2$. We have

$$\begin{aligned} E\left((J_{rt,t})^+\right)^2 &= \rho E(X_t^0 - rt)^+ + 2\sum_{y<x\le 0}E\left(\eta(x)\eta(y)1\{X_t^x > rt\}1\{X_t^y > rt\}\right)\\ &= \rho E(X_t^0 - rt)^+ + 2\rho^2\sum_{y<x\le 0}P(X_t^y > rt)\\ &\quad + 2\sum_{y<x\le 0}\left(E\left(\eta(x)\eta(y)1\{X_t^y > rt\}\right) - \rho^2P\left(X_t^y > rt\right)\right)\\ &= A_1(t) + A_2(t) + A_3(t). \end{aligned} \tag{3.4}$$

Reordering the sum in the second term of (3.4),

$$A_2(t) = \rho^2E\left((X_t^0 - rt)^+\right)^2 - \rho^2E(X_t^0 - rt)^+.$$

The third term in (3.4) is

$$A_3(t) = 2\rho\sum_{y<x\le 0}\left[P\left(X_t^y > rt, \eta(y) = 1|\eta(x) = 1\right) - P\left(X_t^y > rt, \eta(y) = 1\right)\right].$$

Let A, B and B^c, the complement of B, be events with positive probability. Then $P(A|B) - P(A) = P(B^c)(P(A|B) - P(A|B^c))$. Hence we write

$$\begin{aligned} A_3(t) &= -2\rho(1-\rho)\sum_{y<x\le 0}\left[P\left(X_t^y > rt, \eta(y) = 1|\eta(x) = 0\right)\right.\\ &\qquad\qquad \left.- P\left(X_t^y > rt, \eta(y) = 1|\eta(x) = 1\right)\right]\\ &= -2\rho(1-\rho)\sum_{y<x\le 0}E\left(\eta(y)1\{Z_t^{y,x} > rt, U_t^{y,x} \le rt\}\right), \end{aligned} \tag{3.5}$$

where $U_t^{y,x}$ (respectively, $Z_t^{y,x}$) is the position of the tagged particle starting at y for the system where a particle is present at x (respectively, is not present at x). In order to compute the last line of (3.5), we couple two processes that start with a configuration chosen according to ν_ρ, but one of them has a particle at site x while the other has a hole at x. We choose the basic coupling for which the number of discrepancies is always one [see the discussion before (2.7)]. Denote R_t^x the position at time t of the disprepancy initially at x. By (2.11),

$$
\begin{aligned}
A_3(t) &= -2\rho(1-\rho)\sum_{x\le 0} P\big(X_t^x > rt, R_t^x \le rt\big)\\
&= -2\rho(1-\rho)\sum_{x\le 0} P\big(X_t^x > rt\big) + 2\rho(1-\rho)\sum_{x\le 0} P\big(R_t^x > rt, X_t^x > rt\big)\\
&= -2\rho(1-\rho)E\big(X_t^0 - rt\big)^+ + 2\rho(1-\rho)\sum_{x\le 0} P\big(R_t^0 - rt > x, X_t^0 - rt > x\big)\\
&= -2\rho(1-\rho)E\big(X_t^0 - rt\big)^+ + 2\rho(1-\rho)\Big(E\big(R_t^0 - rt\big)^+ - L_{rt}^+\Big),
\end{aligned}
\tag{3.6}
$$

where

$$
L_{rt}^+ = \sum_{x\ge 0} P\left(R_t^0 - rt > x, X_t^0 - rt \le x\right).
$$

For the identity of the second terms of the second and third lines of (3.6), we have used translation invariance. From (3.4),

$$
\begin{aligned}
E\big((J_{rt,t})^+\big)^2 &= \rho E\big(X_t^0 - rt\big)^+ + \rho^2 E\Big(\big(X_t^0 - rt\big)^+\Big)^2 - \rho^2 E\big(X_t^0 - rt\big)^+\\
&\quad -2\rho(1-\rho)E\big(X_t^0 - rt\big)^+ + 2\rho(1-\rho)\Big(E\big(R_t^0 - rt\big)^+ - L_{rt}^+\Big)
\end{aligned}
$$

and, using (3.2),

$$
\begin{aligned}
V(J_{rt,t})^+ &= \rho^2 V\big(X_t^0 - rt\big)^+ - \rho(1-\rho)E\big(X_t^0 - rt\big)^+\\
&\quad + 2\rho(1-\rho)\Big(E\big(R_t^0 - rt\big)^+ - L_{rt}^+\Big).
\end{aligned}
\tag{3.7}
$$

Now we compute the variance of $(J_{rt,t})^-$:

$$
\begin{aligned}
E\big((J_{rt,t})^-\big)^2 &= \rho E\big(X_t^0 - rt\big)^- + 2\sum_{0<x<y} E\big(\eta(x)\eta(y)1\{X_t^y \le rt\}\big)\\
&= \rho E\big(X_t^0 - rt\big)^- + 2\rho^2\sum_{0<x<y} P\big(X_t^y \le rt\big)\\
&\quad +2\sum_{0<x<y}\Big(E\big(\eta(x)\eta(y)1\{X_t^y \le rt\}\big) - \rho^2 P(X_t^y \le rt)\Big)\\
&= B_1(t) + B_2(t) + B_3(t).
\end{aligned}
\tag{3.8}
$$

The second term in the last line of (3.8) is, analogously to $A_2(t)$,

$$
B_2(t) = \rho^2 E\Big(\big(X_t^0 - rt\big)^-\Big)^2 - \rho^2 E\big(X_t^0 - rt\big)^-,
$$

and in a similar way to the computation of $A_3(t)$ in (3.5),

$$B_3(t) = -2\rho(1-\rho)L_{rt}^-,$$

where

$$L_{rt}^- = \sum_{x<0} P\big(R_t^0 - rt > x,\ X_t^0 - rt \le x\big).$$

Then,

$$V(J_{rt,t})^- = \rho^2 V\big(X_t^0 - rt\big)^- + \rho(1-\rho)E(X_t^0 - rt)^- - 2\rho(1-\rho)L_{rt}^-. \tag{3.9}$$

Now

$$L_{rt}^+ + L_{rt}^- = \sum_x P(R_t^0 - rt > x, X_t^0 - rt \le x) = E(R_t^0 - X_t^0)^+.$$

We can now put everything together and compute the variance of the current. Substitute (3.2), (3.7) and (3.9) in (3.3) to obtain

$$\begin{aligned} VJ_{rt,t} &= \rho^2\Big(V\big(X_t^0 - rt\big)^+ + V\big(X_t^0 - rt\big)^- + 2E\big(X_t^0 - rt\big)^+E\big(X_t^0 - rt\big)^-\Big)\\ &\quad -\rho(1-\rho)\Big(E\big(X_t^0 - rt\big)^+ - E\big(X_t^0 - rt\big)^-\Big)\\ &\quad +2\rho(1-\rho)\Big(E\big(R_t^0 - rt\big)^+ - E\big(R_t^0 - X_t^0\big)^+\Big)\\ &= \rho^2 VX_t^0 - \rho(1-\rho)E\big(X_t^0 - rt\big)\\ &\quad +2\rho(1-\rho)\Big(E\big(R_t^0 - rt\big)^+ - E\big(R_t^0 - X_t^0\big)^+\Big). \end{aligned} \tag{3.10}$$

Taking the limit as $t \to \infty$ and using (2.1), (2.4), (2.8) and (2.9),

$$\lim_{t\to\infty} \frac{VJ_{rt,t}}{t} = \rho(1-\rho)\big|(p-q)(1-2\rho) - r\big|.$$

This shows (1.4). In order to show (1.7), we assume $p = 1$. In this case it is known that X_t^0 is a Poisson process of rate $(1-\rho)$ [Spitzer (1970), Liggett (1985)] for which

$$E\big(X_t^0\big) = V\big(X_t^0\big) = (1-\rho)t \quad \text{and} \quad \big(R_t^0 - X_t^0\big)^+ \equiv 0.$$

Using the fact that the current through $-rt$ when the density is $1-\rho$ has the same law as $J_{rt,t}$, (3.10) reads

$$VJ_{(1-2\rho)t,t} = \rho(1-\rho)E\big|R_t^0 - (1-2\rho)t\big|.$$

Observe that (3.10) works also for $p = 1/2$: From (3.10) and $VX_t^0 = \sqrt{2t/\pi}(1-\rho)/\rho + o(\sqrt{t})$ [Arratia (1983)] one can deduce (1.6). The key point is that a second class particle in symmetric exclusion behaves just as a simple symmetric random walk.

4. Law of large numbers. We prove now the law of large numbers. It holds that

$$\lim_{t\to\infty} \frac{J_{rt,t}}{t} = (p-q)\rho(1-\rho) - r\rho \tag{4.1}$$

a.s. with respect to the process with initial distribution ν_ρ. The proof of (4.1) would be a consequence of the ergodic theorem if one knew that the product measures ν_ρ are extremal invariant for the process $\tau_{[rt]}\eta_t$, where $[\cdot]$ is the integer part [see Kipnis (1986)]. It is not clear to us how to show this extremality. To overcome the difficulty consider a Poisson process $U(t)$ with rate λ, independent of η_t. It is not hard to show that the invariant measures for the process $\tau_{U(t)}\eta_t$ are translation invariant. Then use Liggett's (1976, 1985) techniques to show that the set of extremal invariant measures for $\tau_{U(t)}\eta_t$ is $\{\nu_\rho: 0 \le \rho \le 1\}$. Hence $J_{U(t),t}$, the current through $U(t)$ satisfies a law of large numbers.

$$\lim_{t\to\infty} \frac{J_{U(t),\,t}}{t} = (p-q)\rho(1-\rho) - \lambda\rho \tag{4.2}$$

a.s. with respect to the process with initial distribution ν_ρ. Now use that $U(t)/t$ converges to λ almost surely and the fact that the current is a decreasing function of r to conclude the proof of (4.1). This argument was used by Ferrari (1992) to show a law of large numbers for a second class particle.

5. Dependence on the initial configuration. Since N_{th} is a sum of independent random variables, (1.5) implies the central limit theorem (1.3). To show (1.5) for $r < (p-q)(1-\rho)$, write

$$\begin{aligned}
&J_{rt,t} - N_{th(r,\rho)} - (p-q)\rho^2 t \\
&\quad = \sum_{x<0} \eta(x)1\{X_t^x > rt\} - \sum_{x\ge 0} \eta(x)1\{X_t^x \le rt\} - \sum_{x=th}^{0} \eta(x) - (p-q)\rho^2 t \\
&\quad = \sum_{x=th}^{0} \eta(x)\big(1\{X_t^x > rt\} - 1\big) \\
&\qquad + \left(\sum_{x<th} \eta(x)1\{X_t^x > rt\} - (p-q)\rho^2 t - \sum_{x\ge th} \eta(x)1\{X_t^x < rt\}\right) \\
&\qquad - \sum_{x\ge 0} \eta(x)1\{X_t^x \le rt\} + \sum_{x\ge th} \eta(x)1\{X_t^x < rt\} \\
&\quad = C_1(t) + C_2(t) + C_3(t) + C_4(t).
\end{aligned} \tag{5.1}$$

It suffices to show that $\lim_{t\to\infty}(EC_i(t)^2/t) = 0$. Now for $\bar r = (p-q)(1-2\rho)$

$$|C_1(t)| \le 1\{X_t^{th} \le rt\}\big|X_t^{th} - rt\big| + 1 = 1\{X_t^0 \le \bar r t\}\big|X_t^0 - \bar r t\big| + 1,$$

in distribution. Since $\bar r < (p-q)(1-\rho)$, the above inequality and (2.6) imply that $\lim_{t\to\infty}(EC_1(t)^2/t) = 0$. The same argument applies to $C_3(t)$ and $C_4(t)$, which has the same law as $C_3(t)$ for $r = \bar{r}$. On the other hand, $C_2(t)$ has the same distribution as $J_{\bar{r}t,t} - EJ_{\bar{r}t,t}$, whose limiting variance vanishes when divided by t in the limit $t \to \infty$ by (1.4). This shows (1.5) for $r < (p-q)(1-\rho)$. Changing the role of particles and holes, $J_{rt,t}$ has the same law as the current through $-rt$ when the density is $1-\rho$. This shows (1.5) for all r.

6. Density fluctuations. We define the fluctuations density fields by

$$\xi_t^\varepsilon = \varepsilon^{1/2} \sum_x \Phi(\varepsilon x)\left[\eta_{\varepsilon^{-1}t}(x) - E\eta_{\varepsilon^{-1}t}(x)\right]. \tag{6.1}$$

Since we consider only the equilibrium case, the expected value is taken with respect to the initial measure ν_ρ. Hence $E\eta_{\varepsilon^{-1}t}(x) = \rho$. We prove that, as $\varepsilon \to \infty$, the fluctuations fields converge to a Gaussian field that translates rigidly in time, as predicted by Spohn [(1991), Section 6.3]. For $t = 0$,

$$\lim_{\varepsilon \to 0} \xi_t^\varepsilon(\Phi) = \xi(\Phi), \tag{6.2}$$

where $\xi(\Phi)$ is Gaussian white noise with mean zero and covariance

$$E\big(\xi(\Psi)\xi(\Phi)\big) = \rho(1-\rho)\int dr \Psi(r)\Phi(r). \tag{6.3}$$

Let $\xi_t(r) = \tau_r \xi_t$ and $\bar{r} = (p-q)(1-2\rho)$.

THEOREM 6.4. *As $\varepsilon \to 0$ the equilibrium fluctuation fields ξ_t^ε defined in* (6.1) *converge to the solution ξ_t of the linear equation*

$$\frac{\partial}{\partial t}\xi_t(r) = -\bar{r}\frac{\partial}{\partial r}\xi_t(r), \tag{6.5}$$

with initial condition ξ_0, the Gaussian field with zero mean and convariance given by (6.3).

PROOF. The theorem says that the fluctuations in equilibrium just translate at velocity $\bar{r}$, the average velocity of a second class particle. To prove the result, we consider indicator functions of intervals. The extension to general functions is standard. Let

$$\Phi(w) = \Phi_{[0,u]}(w) = 1\{0 \le w \le u\},$$

and let $\tau_r\Phi(w) = \Phi(r+w)$ be the translation by r. Since the variation of the number of particles can occur only at the boundaries of the interval, we have

$$\varepsilon E\big(\xi_0^\varepsilon(\tau_{-\bar{r}t\varepsilon^{-1}}\Phi) - \xi_t^\varepsilon(\Phi)\big)^2 = \varepsilon E(\tau_{-\bar{r}t\varepsilon^{-1}} J^\varepsilon_{\bar{r}t,t} - \tau_{(-\bar{r}t+u)\varepsilon^{-1}} J^\varepsilon_{\bar{r}t,t})^2, \tag{6.6}$$

where $J^\varepsilon_{rt,t} = J_{\varepsilon^{-1}rt,\varepsilon^{-1}t}$. Since the distribution of $\tau_{rt} J^\varepsilon_{r',t}$ is independent of r, by summing and subtracting $EJ^\varepsilon_{\bar{r}t,t}$ we have that the right-hand side of (6.6) is bounded above by $2\varepsilon^2 VJ^\varepsilon_{\bar{r},t}$, which converges to zero as $\varepsilon \to 0$, by (1.4). □

7. An interface model. The one-dimensional nearest neighbors simple exclusion process is isomorphic to a two-dimensional interface model. See, for instance, Rost (1982) and De Masi, Ferrari and Vares (1989). We first define the model. Let $\xi_t \in \{\xi \in \mathbb{Z}^{\mathbb{Z}}: |\xi(x) - \xi(x+1)| = 1\}$ be the process with generator

$$\bar{L}f(\xi) = \sum_{x\in\mathbb{Z}} \big(q 1\{\xi(x-1) + \xi(x+1) - 2\xi(x) > 0\}\,[f(\xi^{x,+}) - f(\xi)] \\ + p 1\{\xi(x-1) + \xi(x+1) - 2\xi(x) < 0\}\,[f(\xi^{x,-}) - f(\xi)]\big),$$

where $\xi^{x,\pm}(x) = \xi(x) \pm 2$ and $\xi^{x,\pm}(y) = \xi(y)$ otherwise. In words, interpreting $\xi(y)$ as the height of a surface at x, the process can be described by saying that at rate q the surface at x increases two units if both heights at $x-1$ and $x+1$ are bigger than the height at x. Analogously, at rate p the surface decreases two units if both neighbor heights are smaller than the height at x. For a given configuration $\eta \in \{0,1\}^{\mathbb{Z}}$, define $\xi = T_\eta \in \mathbb{Z}^{\mathbb{Z}}$ by

$$\xi(x) = \sum_{y=0}^{x} \big(2\eta(y) - 1\big).$$

Letting ξ_t^ξ and η_t^η denote the interface and the simple exclusion processes with initial configuration ξ and η, respectively, it holds that

$$\xi_t^{T_\eta}(x) = (T\eta_t^\eta)(x) + J_{0,t}.$$

Hence

$$\xi_t(0) \equiv J_{0,t}.$$

The density in the simple exclusion process gives the general inclination of the surface. Density 1/2 gives a surface parallel to the x axis (flat). Our results on the current mean that in equilibrium the diffusion coefficient for a flat surface scales in a different way than the diffusion coefficient for an inclined surface, no matter for which inclination. For the flat surface the correct normalization would be $t^{2/3}$. Our interpretation is that a flat surface has "more memory" than an inclined surface. In this last, one sees a flux of particles falling down the hill and picks the space fluctuations of the initial configuration. This does not happen in the flat case.

Alexander, Cheng, Janowsky and Lebowitz (1993) studied a two-dimensional asymmetric simple exclusion process. For this process the transition function is given by $p((x,y),(x,y+1)) = 1/2, p((x,y),(x\pm 1,y)) = \frac{1}{4}$ and $p((x,y),(z,w)) = 0$ otherwise. The process starts with a product measure with density 0 and $\rho > 0$ in the half planes $\{y > 0\}$ and $\{y \le 0\}$, respectively (flat initial surface). Defining $Y(t)$ as the first coordinate of the leftmost particle on the y axis, they found via simulations that the variance of $Y(t)$ behaves as $t^{1/4}$. Is this normalization correct for inclined surfaces? Why does the normalization factor in the flat case depend on the model?

Acknowledgments. We thank K. Ravishankar for bringing to our attention the paper by Alexander, Cheng, Janowsky and Lebowitz, and Tom Mountford for reading the manuscript and making useful remarks.

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