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STATIONARY PROCESSES INDEXED BY A HOMOGENEOUS TREE

BY JEAN-PIERRE ARNAUD

Université Paul Sabatier

Let T be the set of vertices of a homogeneous tree and let $(X_t)_{t\in T}$ be a second-order real or complex-valued process such that the expected value $\mathbb{E}(X_s\overline{X}_t)$ depends only on the distance between the vertices s and t. In this paper we construct a measure space $(K, \mathscr{H}, m)$ and an isometry of the closed subspace of $L^2_{\mathbb{C}}(\Omega, \mathscr{A}, P)$ spanned by $(X_t)_{t\in T}$ onto $L^2(m)$.

1. Introduction. A homogeneous tree T of order $q \geq 1$ is an infinite connected graph (the edges are not oriented) without nontrivial closed loops, such that every vector belongs to exactly $(q+1)$ edges. Let $d(s,t)$ be the distance between vertices s and t, that is, the number of edges in the shortest path from s to t. For simplicity, T will denote the set of vertices (with the above structure). An automorphism of T is a permutation g of T such that if $\{t, s\}$ is any edge, then $\{g(t), g(s)\}$ is also an edge. The set G of all automorphisms of T forms a group (with composition as group operation) acting on T. Clearly, G is the group of isometries of T: For all s and t in T and all g in G, $d(g(s), g(t)) = d(s,t)$.

DEFINITION 1.1. Let $(\Omega, \mathscr{A}, P)$ be a probability space, let T be a homogeneous tree and let $t \to X_t$: $T \to L^2_{\mathbb{C}}(\Omega, \mathscr{A}, P)$ be a second-order complex stochastic process $X = (X_t)_{t\in T}$. The process X will be said to be stationary if there exists a function φ: $\mathbb{N} \to \mathbb{R}$ such that

$$\mathbb{E}\big(X_s\overline{X}_t\big) = \varphi(d(s,t)) \tag{1.0}$$

for all s and t in T.

From this definition it follows that for any integer m and any sequence (t_j, λ_j) in $T \times \mathbb{C}$ one has

$$\sum_{i,j=1}^{m} \varphi\big(d(t_i,t_j)\big)\lambda_i\overline{\lambda}_j = \left(E\left|\sum_{j=1}^{m}\lambda_j X_j\right|^2\right) \geq 0.$$

When $q = 1$ these sequences $\varphi(d(t_i,t_j))_{i,j\in\mathbb{Z}}$ are described by the celebrated Herglotz theorem. Using a generalization of the Herglotz theorem, in [1] we deduced the following spectral representation theorem for the stationary process $(X_t)_{t\in T}$.

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THEOREM 1.2. *Let $X = (X_t)_{t \in T}$ be a stationary process on the tree T of order q. There exists a unique positive measure (called the $\mathbb{R}$ spectral measure) μ_X on $[-1, +1]$ such that*

$$E(X_s \overline{X}_t) = \int_{-1}^{+1} P_{d(s,t)}(x) \mu_X(dx),$$

where $(P_n(z))_{n=0}^{\infty}$ is a sequence of polynomials defined by $P_0(z) = 1$, $P_1(z) = z$, and for $n \geq 1$,

$$zP_n(z) = \frac{q}{q+1} P_{n+1}(z) + \frac{1}{q+1} P_{n-1}(z).$$

REMARK. Equivalent objects to the polynomials $\{P_n\}$ (also called Dunau polynomials) appear in the literature without names, with other normalizations. See [6], [16], [15]. An equivalent of Theorem 1.2 appears in [6] and [11]. More precisely, the class of Dunau polynomials forms a hypergroup [10] so that, following [11], we have

$$\varphi(n) = \int_{D_s} P_n(x) \mu(dx),$$

with

$$D_s = \{x \in \mathbb{R};\, |P_n(x)| \leq 1 \text{ for } n \in \mathbb{N}\}.$$

We now raise the problem solved in this present paper: Given a stationary process X on the tree T, find a measure space $(K, \mathscr{H}, m\}$ and an isometry ψ from the closed subspace $H(X)$ of $L^2_{\mathbb{C}}(P)$ generated by the $\{X_t,\ t \in T\}$ onto $L^2_{\mathbb{C}}(m)$.

Typically, our problem is a particular case of stationary processes on a Gelfand pair (see [12] for a general description of these processes). The solution of our problem is in the ends of the tree, which is the subject of the following section.

2. The ends of the tree. Let t be a vertex of the tree T. Then an infinite chain with origin t is an infinite sequence $w = [s_0, s_1, \ldots, s_n, \ldots]$ of vertices of T such that $s_0 = t$, $s_i \neq s_j$ if $j \neq i$ and for $n \geq 0$, $\{s_n, s_{n+1}\}$ is an edge of T. Let B_t be the set of all the infinite chains of T with origin t and let S_n^t be the set of the vertices s of T such that $d(s,t) = n$. Therefore, B_t is a subset of $\Pi_{i \in \mathbb{N}} S_i^t$. The sets S_i^t are finite with cardinality Π_i, where

(2.1) $$\Pi_0 = 1 \quad \text{and} \quad \Pi_i = q^{i-1}(q+1) \quad \text{if } i > 0.$$

If we equip each S_i^t with the discrete topology, then $\Pi_{i \in \mathbb{N}} S_i^t$ with product topology is compact and the Cantor procedure allows us to deduce that B_t is compact. For s and t in T, let

$$\Omega_s^t = \{w = [s_n]_{n \geq 0} \in B_t | s_n = s \text{ if } n = d(s,t)\}$$

The sets $\{\Omega_s^t,\ s \in T\}$ form a basis for the topology of B_t. For two vertices s

and s' in T, we define the relation

$$s \leq_t s' \text{ if and only if } \Omega^t_{s'} \subset \Omega^t_s \tag{2.2}$$

Thus, for a fixed $t \in T$, $(T, \leq_t)$ is a partially ordered set and we have the following properties:

$$s \leq_t s' \Rightarrow \Omega^t_s \cap \Omega^t_{s'} = \Omega^t_{s'}, \tag{2.3}$$

$$s \nleq_t s' \Rightarrow \Omega^t_s \cap \Omega^t_{s'} = \varnothing. \tag{2.4}$$

On $\bigcup_{t \in T} B_t$, we now introduce an equivalence relation R between infinite chains w and w' as follows: wRw' if and only if the symmetric difference between the two sets w and w' is finite. The equivalence classes for R are called ends, and B will denote the set of all ends of T.

For any vertex s of T and any end b of B, there exists a unique infinite chain w, in the equivalence class b, which has s for origin, and we shall denote it by $[s, b)$. If we consider two infinite chains with origin s and s', in the equivalence class $b \in B$, then for each vertex $t \in [s, b) \cap [s', b)$ the difference $d(s,t) - d(s',t)$ does not depend on t, and so we let

$$\delta_b(s, s') = d(s,t) - d(s',t) \tag{2.5}$$

following Cartier's terminology [5].

This definition implies naturally that if s, s' and s'' are three vertices of T, then

$$\delta_b(s, s'') = \delta_b(s, s') + \delta_b(s', s''). \tag{2.6}$$

From the identity (2.6) we conclude easily that the relation $\delta_b(s, s') = 0$ is an equivalence relation between the vertices s and s' of T; the equivalence classes are called the *horocycles* associated with b.

We can construct a natural topology on B. Denote by i_s the canonical inclusion $B_s \subset \bigcup_{t \in T} B_t$ and define on $\bigcup_{t \in T} B_t$ the final topology for the (i_s) family. We can then define on B the quotient topology. With this topology, B_s is homeomorphic to B and we denote by $\varphi_s \colon B_s \to B$ this homeomorphism.

We can now construct on B a natural collection of measures $(\nu_t)_{t \in T}$.

Let K_t be the subgroup of G consisting of all the automorphisms g of G such that $g(t) = t$. We define on G the pointwise convergence topology. More precisely, we have $g_n \to g$ if and only if for each vertex s of T there exists a number N_s such that for $n \geq N_s$ we have $g_n(s) = g(s)$. The group G is then a locally compact group, and K_t is a compact subgroup of G.

Proposition 2.1.

(i) *The subgroup K_t acts transitively on B_t, that is, for each infinite chain b_1 and b_2 of B_t, there exists an automorphism g of K_t such that $g(b_1) = b_2$.*

(ii) *There exists on B_t a unique probability measure ν_t invariant under K_t. It is defined by*

$$\nu_t(\Omega^t_s) = \Pi^{-1}_{d(s,t)}. \tag{2.7}$$

PROOF.

(i) A detailed proof can be found in [7].

(ii) (a) *Existence of the measure* ν_t. Denote by dg the measure on K_t such that $dg(K_t) = 1$. Let f be a continuous function on B_t. Then $\int_{K_t} f(g^{-1}b)\, dg$ does not depend on the chain b [according to (i)] and is equal to a constant $\sigma(f)$.

We have then defined a linear form σ on the space of continuous functions on B_t such that $\sigma(1) = 1$.

From the Riesz theorem there exists a probability measure ν_t invariant under K_t such that

$$\sigma(f) = \int_{B_t} f(b)\nu_t(db) \quad \text{and} \quad \nu_t(\Omega_s^t) = \int_{K_t} 1_{\Omega_s^t}(g^{-1}b)\, dg.$$

Let s' be a vertex such that $d(s',t) = d(s,t)$. Then $\nu_t(\Omega_{s'}^t) = \nu_t(\Omega_s^t)$ since dg is invariant under left and right multiplication on K_t. Hence, $\nu_t(\Omega_s^t) = \Pi_{d(s,t)}^{-1}$.

(b) *Uniqueness of* ν_t. Denote by ν_t' another probability measure on B_t invariant under the action of K_t. Hence,

$$\int_{B_t} f(g^{-1}b)\nu_t'(db) = \int_{B_t} f(b)\nu_t'(db).$$

It then suffices to apply Fubini's theorem to obtain

$$\int_{K_t}\int_{B_t} f(g^{-1}b)\nu_t'(db)\, dg = \int_{B_t} \nu_t'(db)\int_{K_t} f(g^{-1}b)\, dg,$$

and then

$$\nu_t'(f) = \int_{B_t} f(b)\nu_t'(db) = \sigma(f)\nu_t'(B_t) = \sigma(f). \qquad \square$$

It is now necessary to fix a reference vertex 0 until the end of this paper. Henceforth, we write $|t|$ for $d(0,t)$, Ω_t for Ω_t^0 and S_n for S_n^0, and we shall denote respectively by $\delta_b(0,\cdot)$: $T \to \mathbb{Z}$ and by $\delta_.(0,t)$: $B \to \mathbb{Z}$ the function whose value in $t \in T$ and respectively in $b \in B$ is equal to $\delta_b(0,t)$.

We are now ready to define the measures ν_t on B from the basis of open sets $(\varphi_0(\Omega_s))_{s\in T}$ by

$$\nu_t(\varphi_0(\Omega_s)) = \nu_t(\varphi_t^{-1}(\varphi_0(\Omega_s))).$$

We keep the same symbol ν_t as a measure on B_t and as a measure on B. The measures ν_t on B are mutually absolutely continuous with densities given by the following expression [6]:

$$\frac{d\nu_t}{d\nu_s}(b) = q^{\delta_b(s,t)}. \tag{2.8}$$

For this part of the paper we thank Pierre Cartier who gave us some ideas, in particular, the consideration of the following spaces denoted below by V_n.

DEFINITION 2.2. For each $t \in T$, we denote by 1_{Ω_t} the characteristic function of Ω_t and by V_n the subspace of $\mathbb{C}^B$ defined by

$$V_n = \left\{ f = \sum_{t \in S_n} f_t 1_{\Omega_t};\ f_t \in \mathbb{C} \right\}.$$

From this definition it follows that V_n is a finite-dimensional linear space whose dimension is the cardinality of S_n, that is, Π_n. It is easy to verify that $V_n \subset V_{n+1}$ and that for $z \neq 0$, $z^{\delta.(0,t)}$ is in $V_{|t|}$.

We shall now work on the subspace of the linear space $\mathbb{C}^B$ defined by $\mathscr{E}(B) = \overline{\bigcup_{n=0}^{\infty} V_n}$ equipped with the inner product space $\langle f, g \rangle = \int_B f(b)\overline{g(b)}\nu_0(db)$. Our approach to solving the problem is to represent $P_{d(s,t)}(x) = \langle D_t(x, \cdot), D_s(x, \cdot) \rangle$ with the appropriate function $b \to D_t(x, b)$ so that by Theorem 1.2 we have

$$E(X_s \overline{X}_t) = \int_{-1}^{+1} \int_B D_t(x, b)\, D_s(x, b) \mu_X \otimes \nu_0(dx, db),$$

which leads us to the isometry.

When $x \in [-\rho_q, \rho_q]$, where $\rho_q = 2\sqrt{q}/(q+1)$, the representation of $P_{d(s,t)}(x)$ has been derived in [3], and then we have $D_t(x, b) = \sqrt{q}(\exp i\theta)^{\delta_b(0,t)}$ with $\theta = \arccos(x/\rho_q)$.

This result can be reformulated in a more convenient manner by using the Joukovsky transform. Following a terminology employed in [13], we shall call the Joukovsky transform the function J: $\mathbb{C}^* \to \mathbb{C}$ defined by $J(v) = (\rho_q/2)(v + v^{-1})$.

According to Theorem 1.2, we shall only have to consider v in the set $J^{-1}[-1, +1] = E_{\mathbb{C}} \cup E_{\mathbb{R}}$ with

$$E_{\mathbb{C}} = \{v \in \mathbb{C};\ |v| = 1\} \xrightarrow{J} [-\rho_q, \rho_q],$$

$$E_{\mathbb{R}} = \{v \in \mathbb{R};\ q^{-1/2} \le |v| \le q^{1/2}\} \xrightarrow{J} [-1, -\rho_q] \cup [1, \rho_q].$$

The representation of $P_{d(s,t)}(J(v))$ when $v \in E_{\mathbb{C}}$ is then given by

$$P_{d(s,t)}(J(v)) = \int_B e_t(b)\overline{e_s(b)}\nu_0(db) = \langle e_t, e_s \rangle, \tag{2.9}$$

with $e_t(b) = (\sqrt{q}\,v)^{\delta_b(0,t)}$.

To obtain the representation of $P_{d(s,t)}(J(v))$ when $v \in E_{\mathbb{R}}$, we have to define a family of inner products $\langle\ ,\ \rangle_v$ on $\mathscr{E}(B)$ which are equivalent to $\langle\ ,\ \rangle$ on each V_n.

PROPOSITION 2.3. *Let v be a real number such that $|v| > q^{-1/2}$. Then*

$$\operatorname{Rank}\{e_t;\ |t| \le n\} = \operatorname{Rank}\{e_t;\ |t| = n\} = \Pi_n.$$

PROOF. To prove Proposition 2.3, we need the following lemma.

LEMMA 2.4. *Let $(\alpha_n)_{n=1}^{\infty}$ be a real sequence and let $(B_n)_{n=0}^{\infty}$ be a matrix sequence such that $B_0 = (1)$ and, for $n > 0$, B_n is the square matrix of order q^n given by the recurrence formula*

$$B_n = \begin{bmatrix} B_{n-1} & & \alpha_n \\ & B_{n-1} & \\ \alpha_n & & B_{n-1} \end{bmatrix} \qquad (q \times q \textit{ square blocks of order } q^{n-1}).$$

Let M_n be the square matrix of order $q^{n-1}(q+1)$:

$$M_n = \begin{bmatrix} B_{n-1} & & \alpha_n \\ & B_{n-1} & \\ \alpha_n & & B_{n-1} \end{bmatrix}$$

$$\left[(q+1) \times (q+1) \textit{ square blocks of order } q^{n-1}\right],$$

and let

$$\mu_k = 1 + \sum_{i=1}^{k} (q-1) q^{i-1} \alpha_i \quad \text{and} \quad \gamma_k = \mu_k - q^k \alpha_k.$$

Then

$$\det M_n = \left(\mu_{n-1} + q^n \alpha_n\right)\left(\mu_{n-1} - q^{n-1}\alpha_n\right) \prod_{i=1}^{n-1} \gamma_{n-i}^{(q^{i+1} - q^{i-1})}.$$

PROOF. We can prove the lemma either by using linear combinations on rows or columns or by referring to the following classical result.

If $(M_{ij})_{i,j=1}^{n}$ are $p \times p$ matrices on $\mathbb{C}$ which commute pairwise, then

$$\det \begin{bmatrix} M_{11} & \cdots & M_{1n} \\ \vdots & & \vdots \\ M_{n1} & \cdots & M_{nn} \end{bmatrix} = \det\left(\sum_{\sigma \in S_n} \varepsilon(\sigma) \prod_{j=1}^{n} M_{\sigma(j)j} \right),$$

where S_n is the permutation group of $\{1, \dots, n\}$ and $\varepsilon(\sigma)$ is the signature of the permutation σ. The proof of this property is simple. We can show it directly if the M_{ij} are diagonal, or diagonalizable, by using the fact they commute, and we can prove the general case by density. □

PROOF OF PROPOSITION 2.3. The set $\{1_{\Omega_t}, t \in S_n\}$ is a basis of V_n, and

$$e_t = \sum_{s \in S_n} \left(\sqrt{q}\, v\right)^{n - d(s,t)} 1_{\Omega_s}. \tag{2.10}$$

It then suffices to compute the rank of the matrix

$$N = \left(\left(\sqrt{q}\, v\right)^{-d(s,t)}\right)_{s,t \in S_n}.$$

If we number the vertices of S_n such that $d(s_i, s_j) = 2$ with $(k-1)q < i <$

$j \le kq$, we can apply Lemma 2.4 with $\alpha_i = (\sqrt{q}\,v)^{-2i}$ and then show that $\det N > 0$.

This is a consequence of the inequalities

$$\gamma_1 = 1 - \left(\sqrt{q}\,v\right)^{-2} > 0,$$

$$\gamma_{k+1} - \gamma_k = q^k\left(\sqrt{q}\,v\right)^{-2k}\left(1 - \left(\sqrt{q}\,v\right)^{-2}\right) > 0,$$

$$\mu_{n-1} + q^n\left(\sqrt{q}\,v\right)^{-2n} > 0,$$

$$\mu_{n-1} - q^{n-1}\left(\sqrt{q}\,v\right)^{-2n} = \gamma_{n-1} + q^{n-1}\left(\sqrt{q}\,v\right)^{-2(n-1)}\left(1 - \left(\sqrt{q}\,v\right)^{-2}\right) > 0. \quad \square$$

We have thus proved that $\{e_t;\ |t| = n\}$ spans V_n. Let us now consider the operators $D_n\colon \mathbb{C}^T \to \mathbb{C}^T$ defined by

$$(D_n f)(t) = \Pi_n^{-1} \sum_{d(s,t)=n} f(s),$$

$$(D_0 f)(t) = t.$$

With the operators D_n we can simplify the integration of functions of $\mathscr{E}(B)$ thanks to formula (2.11).

THEOREM 2.5. *Let t be a fixed vertex of T and let h be any function in $\mathbb{C}^z$. Then $h(\delta_\cdot(0,t))$ is in $V_{|t|}$ and we have*

$$\int_B h(\delta_\cdot(0,t))\nu_0(db) = D_{|t|}\big(h(\delta_b(0,\cdot))\big)(0). \tag{2.11}$$

PROOF. Let $s \in S_{|t|}$ and $b, b' \in \Omega_s$. Since $\delta_b(0,t) = \delta_{b'}(0,t)$, we have $h(\delta_\cdot(0,t)) \in V_{|t|}$. Thus

$$\int_B h(\delta_b(0,t))\nu_0(db) = \sum_{s\in S_{|t|}} \int_{\Omega_s} h(\delta_b(0,t))\nu_0(db).$$

Let us select a vertex $s \in S_{|t|}$ and an end b_s in $\Omega_{s'}$. Then for each end b in Ω_s we have $\delta_b(0,t) = \delta_{b_s}(0,t)$, and thus

$$\int_{\Omega_s} h(\delta_b(0,t))\nu_0(db) = h\big(\delta_{b_s}(0,t)\big)\nu_0(\Omega_s).$$

Hence,

$$\int_B h(\delta_b(0,t))\nu_0(db) = \sum_{s\in S_{|t|}} h\big(\delta_{b_s}(0,t)\big)\Pi_{|t|}^{-1}.$$

Notice that $\delta_{b_s}(0,t) = d(s,0) - d(s,t)$ and $\delta_{b_t}(0,s) = d(t,0) - d(s,t)$. It follows that for each $s \in S_{|t|}$ we have $\delta_{b_t}(0,s) = \delta_{b_s}(0,t)$. Thus,

$$\begin{aligned}\int_B h(\delta_b(0,t))\nu_0(db) &= \Pi_{|t|}^{-1} \sum_{s \in S_{|t|}} h\big(\delta_{b_s}(0,t)\big) \\ &= \Pi_{|t|}^{-1} \sum_{s \in S_{|t|}} h\big(\delta_{b_t}(0,s)\big) \\ &= D_{|t|} h\big(\delta_{b_t}(0,\cdot)\big)(0). \qquad \square\end{aligned}$$

With the following proposition (see also [6]), we make explicit the role of the Dunau polynomials P_n.

PROPOSITION 2.6. *For each nonnegative integer n, we have*

$$D_n = P_n(D_1). \tag{2.12}$$

PROOF. The identity (2.12) is equivalent to the following:

$$(q+1)D_1(D_n(f)) = qD_{n+1}(f) + D_{n-1}(f), \qquad \forall f \in \mathbb{C}^T, \forall\, n \geq 1.$$

It suffices then to verify the following equality for $f = 1_u$, where 1_u denotes the characteristic function of $\{u\}$ for $u \in T$:

$$D_n(1_u)(t) = \Pi_n^{-1} \sum_{d(s,t)=n} 1_u(s) = \Pi_n^{-1} 1_{S_n^u}(t),$$

where $1_{S_n^u}$ denotes the characteristic function of S_n^u. Hence,

$$D_1(D_n(1_u))(t) = \Pi_1^{-1} \sum_{d(s,t)=1} \Pi_n^{-1} 1_{S_n^u}(s) = \Pi_1 \Pi_n^{-1}\big(1_{S_{n+1}^u}(t) + q 1_{S_{n-1}^u}(t)\big).$$

Then

$$\begin{aligned}(q+1)D_1(D_n(1_u))(t) &= \Pi_n^{-1}\big(1_{S_{n+1}^u}(t) + q1_{S_{n-1}^u}(t)\big) \\ &= qD_{n+1}(f)(t) + D_{n-1}(f)(t). \qquad \square\end{aligned}$$

As a corollary of Proposition 2.6, if f is an eigenvector of D_1 corresponding to the eigenvalue $\lambda \in \mathbb{C}$, then f is an eigenvector of D_n corresponding to the eigenvalue $P_n(\lambda)$.

PROPOSITION 2.7. *The function $e_\cdot(b)$ is an eigenvector of D_n corresponding to the eigenvalue $P_n(J(v))$.*

PROOF. We have only to verify

$$\begin{aligned}(q+1)D_1(e_\cdot(b))(t) &= \sum_{d(s,t)=1} e_s(b)\\ &= \sum_{d(s,t)=1} \left(\sqrt{q}\,v\right)^{\delta_b(0,s)}\\ &= \left(\sqrt{q}\,v\right)^{\delta_b(0,t)+1} + \left(\sqrt{q}\,v\right)^{\delta_b(0,t)-1}\\ &= \sqrt{q}\,(v+v^{-1})\left(\sqrt{q}\,v\right)^{\delta_b(0,t)}.\end{aligned}$$

□

COROLLARY 2.8.

$$\int_B e_t(b)\nu_0(db) = P_{|t|}(J(v)) = \Pi_n^{-1} \sum_{d(0,s)=|t|} \left(\sqrt{q}\,v\right)^{\delta_b(0,s)}. \tag{2.13}$$

PROOF. If we apply Theorem 2.5 with $h(n) = (\sqrt{q}\,v)^n$, then $h(\delta_\cdot(0,t)) = e_t(\cdot)$ and thus

$$\begin{aligned}\int_B e_t(b)\nu_0(db) = D_{|t|}(e_\cdot(b))(0) &= P_{|t|}(J(v))e_0(b)\\ &= P_{|t|}(J(v)).\end{aligned}$$

□

The family $\{e_t,\ t \in T\}$ spans $\mathscr{E}(B)$, and for a fixed $v \in \mathring{E}_{\mathbb{R}}$ (that is, $v \in R$ and $q^{-1/2} < |v| < q^{1/2}$) we shall define on $\mathscr{E}(B)$ an inner product $\langle\ ,\ \rangle_v$ which will satisfy

$$\langle e_t, e_{t'}\rangle_v = P_{d(t,t')}(J(v)). \tag{2.14}$$

We check first the positive definiteness of (2.14) on V_n.

PROPOSITION 2.9. *Let $v \in \mathring{E}_{\mathbb{R}} = \{v \in \mathbb{R},\ q^{-1/2} < |v| < q^{1/2}\}$. If we construct on V_n the bilinear form $\langle\ ,\ \rangle_v$ by its matrix in the basis $\{e_t,\ t \in S_n\}$ defined by $(P_{d(t,t')}(J(v)))_{t,t' \in S_n}$, then $\langle\ ,\ \rangle_v$ is an inner product on V_n.*

PROOF. We just have to prove that the matrix $M = (P_{d(t,t')}(J(v))_{t,t' \in S_n})$ is positive definite when $q^{-1/2} < |v| < q^{1/2}$. The matrix M is positive semi-definite ([1], Theorem 2). It then suffices to prove that $\det M \neq 0$ if $v \in \mathring{E}_{\mathbb{R}}$. If we apply Lemma 2.4, we get

$$\begin{aligned}\det M = &\left(\gamma_1^{q^{n-2}}\gamma_2^{q^{n-3}}\cdots\gamma_{n-2}^{q}\gamma_{n-1}\right)^{q^2-1}\\ &\times\left(\mu_{n-1} + q^n P_{2n}(J(v))\right)\left(\mu_{n-1} - q^{n-1}P_{2n}(J(v))\right).\end{aligned} \tag{2.15}$$

Let us then show that all the factors in (2.15) are different from 0:

$$\gamma_1 = ((q+1)/q)\left(1 - J(v)^2\right) > 0,$$

$$\gamma_{k+1} - \gamma_k = q^k\left(P_{2k}(J(v)) - P_{2(k+1)}(J(v))\right).$$

Using Proposition 2.7, we can write

$$\begin{aligned}
&P_{2k}(J(v)) - P_{2(k+1)}(J(v)) \\
&\quad = \Pi_{2(k+1)}^{-1}\left(q^2 \sum_{s\in S_{2k}^0} \left(\sqrt{q}\,v\right)^{\delta_b(0,s)} - \sum_{t\in S_{2(k+1)}^0} \left(\sqrt{q}\,v\right)^{\delta_b(0,t)}\right) \\
&\quad = \Pi_{2(k+1)}^{-1}\left(\sum_{s\in S_{2k}^0} \sum_{d(t,s)=2} \left(\left(\sqrt{q}\,v\right)^{\delta_b(0,s)} - \left(\sqrt{q}\,v\right)^{\delta_b(0,t)}\right)\right) \\
&\quad = \Pi_{2(k+1)}^{-1}\left(1 - \left(\sqrt{q}\,v\right)^{-2}\right)\left((\Pi_{2k} - 1) + (q-1)q^2v^2 - q^2v^4\right) > 0
\end{aligned}$$

and then $\gamma_{k+1} > \gamma_k > \cdots > \gamma_1 > 0$.

For the last two factors in (2.15), we have just to prove that $P_{2k}(J(v)) > 0$ for each integer k, by using Proposition 2.7 and writing

$$(D_n e_.(b))(0) = P_n(J(v))e_0(b) = P_n(J(v)),$$

and thus

$$P_{2k}(J(v)) = \Pi_{2k}^{-1} \sum_{d(s,0)=2k} \left(\sqrt{q}\,v\right)^{\delta_b(0,s)},$$

with

$$\delta_b(0,s) = |t| - d(s,t) = 2k - d(s,t), \qquad b \in \Omega_t,\, t \in S_{2k},$$

since $s, t \in S_{2k} \Rightarrow d(s,t)$ is even and $(\sqrt{q}\,v)^{\delta_b(0,s)} > 0$. Thus,

$$\mu_{n-1} - q^{n-1}P_{2n}(J(v)) = \gamma_{n-1} + q^{n-1}\left(P_{2(n-1)}(J(v)) - P_{2n}(J(v))\right) > 0,$$

$$\mu_{n-1} + q^{n-1}P_{2n}(J(v)) = 1 + \sum_{k=1}^{n} (q-1)q^k P_{2k}(J(v)) + q^n P_{2n}(J(v)) > 0.$$

□

With Proposition 2.9, we have shown that for each integer n the linear spaces $(V_n, \langle\ ,\ \rangle_v)$ are Euclidean. Since $(V_n)_{n=0}^{\infty}$ is an increasing sequence, in order to define an inner product on $\mathscr{E}(B) = \bigcup_{n\geq 0} V_n$, we have to verify the consistency of the identity (2.14) on the different linear spaces V_n.

PROPOSITION 2.10. *$(\mathscr{E}(B), \langle\ ,\ \rangle_v)$ is an inner product space.*

PROOF. Let us choose $t, t' \in T$ such that $|t'| > |t|$. Let $n = |t'|$. Then we have $e_t \in V_{|t|} \subset V_n$ and we can write

$$e_t = \sum_{s\in S_n} \alpha_s e_s \quad \text{with } \alpha_s \in \mathbb{R}. \tag{2.16}$$

In the linear space $(V_n, \langle\ ,\ \rangle_v)$ we can write

$$\langle e_t, e_{t'}\rangle_v = P_{d(t,t')}(J(v)).$$

We then have to prove that

$$P_{d(t',t)}(J(v)) = \sum_{s \in S_n} \alpha_s P_{d(t',s)}(J(v)).$$

Multiplying the identity (2.16) by $(\sqrt{q}\,v)^{\delta_b(t',0)}$ and using relation (2.6), we have

$$\left(\sqrt{q}\,v\right)^{\delta_b(t',t)} = \sum_{s \in S_n} \alpha_s \left(\sqrt{q}\,v\right)^{\delta_b(t',s)} \quad \text{for each } b \in B.$$

Thus,

$$\int_B \left(\sqrt{q}\,v\right)^{\delta_b(t',t)} \nu_{t'}(db) = \sum_{s \in S_n} \alpha_s \int_B \left(\sqrt{q}\,v\right)^{\delta_b(t',s)} \nu_{t'}(db).$$

Employing Proposition 2.7 and Theorem 2.5, in which we replace the origin 0 by the origin t', we then have the result. □

PROPOSITION 2.11. *The inner product $\langle\ ,\ \rangle_v$ in $\mathscr{E}(B)$ is invariant under the action of the subgroup K_0 of isometries $g \in G$, such that $g(0) = 0$.*

PROOF. Let $g \in K_0$ and let us compute $\langle e_t \cdot g, e_{t'} \cdot g\rangle_v$. Choose $b \in B$. Then, by definition of the action of K_0 on $\mathscr{E}(B)$,

$$e_t \cdot g(b) = \left(\sqrt{q}\,v\right)^{\delta_{g(b)}(0,t)} = \left(\sqrt{q}\,v\right)^{\delta_b(0,gt)} = e_{gt}(b)$$

and then

$$\begin{aligned}\langle e_t \cdot g, e_{t'} \cdot g\rangle_v &= \langle e_{gt}, e_{gt'}\rangle_v \\ &= P_{d(gt,gt')}(J(v)) = P_{d(t,t')}(J(v)) = \langle e_t, e_{t'}\rangle_v.\end{aligned}$$ □

We shall now introduce a sequence $(W_n)_{n \geq 0}$ of subspaces of $\mathscr{E}(B)$ orthogonal with respect to both $\langle\ ,\ \rangle$ and $\langle\ ,\ \rangle_v$.

DEFINITION 2.12. Let W_n be the sequence of subspaces of $\mathscr{E}(B)$, such that for each integer n and m we have $W_n \perp W_m$ in $(\mathscr{E}(B), \langle\ ,\ \rangle)$, and such that the following direct sum relations hold:

(2.17) $\quad W_0 = V_0, \qquad W_0 \oplus W_1 = V_1, \qquad W_0 \oplus W_1 \oplus \cdots \oplus W_n = V_n.$

In order to study the sequence $(W_n)_{n=0}^{\infty}$ and prove that it is also orthogonal for $\langle\ ,\ \rangle_v$, we define *brooms*.

DEFINITION 2.13. A broom with origin s in T is the set $b(s)$ defined by $b(s) = \{t \in T;\ s \leq t,\ d(s,t) = 1\}$ and shown in Figure 1.

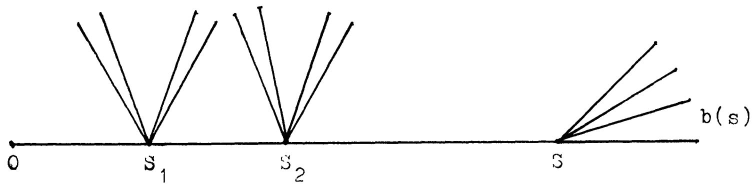

FIG. 1.

PROPOSITION 2.14. *If $q \geq 2$ we have*

$$W_0 = V_0 \quad \textit{and} \quad \dim W_0 = 1,$$

$$W_1 = \left\{ f \in V_1;\ f = \sum_{t \in S_1} f_t 1_{\Omega_t} \middle| \sum_{t \in S_1} f_t = 0 \right\} \quad \textit{and} \quad \dim W_1 = q.$$

If $n \geq 2$, $\dim W_n = (q^2 - 1)q^{n-2}$ and

$$W_n = \left\{ f \in V_n;\ f = \sum_{t \in S_n} f_t 1_{\Omega_t} \middle| \forall\, s \in S_{n-1},\ \sum_{t \in b(s)} f_t = 0 \right\}.$$

PROOF. Let $f = \Sigma_{t \in S_n} f_t 1_{\Omega_t}$ be an element of $W_n \subset V_n$ and let $g = \Sigma_{t \in S_{n-1}} g_t 1_{\Omega_t}$ be an element of V_{n-1}. Then

$$\langle f, g \rangle = \int_B f(b) \bar{g}(b) \nu_0(db) = \sum_{s \in S_{n-1}} \int_{\Omega_s} f(b) \bar{g}(b) \nu_0(db)$$

$$= \sum_{s \in S_{n-1}} \bar{g}_s \int_{\Omega_s} f(b) \nu_0(db) = \sum_{s \in S_{n-1}} \bar{g}_s \sum_{t \in b(s)} f_t \nu_0(\Omega_t) = 0.$$

This equality is true for $g \in V_{n-1}$, in particular, when $g = 1_{\Omega_{s_0}}$ with $s_0 \in S_{n-1}$, and this implies $\Sigma_{t \in b(s_0)} f_t = 0$. The reverse implication is evident. □

COROLLARY 2.15. *For $n \geq 1$ we have in $(\mathscr{C}(B), \langle\ ,\ \rangle)$ the orthogonal decomposition $W_n = \oplus_{s \in S_{n-1}} U_s$ with*

$$U_s = \left\{ f \in W_n,\ f = \sum_{t \in b(s)} f_t 1_{\Omega_t} \middle| \sum_{t \in b(s)} f_t = 0 \right\}, \qquad \dim U_s = q - 1.$$

PROOF. It is evident from Proposition 2.14. It is also easy to verify that for $q \geq 2$, $s \in T$ and $t_0 \in b(s)$ fixed, the family $\{1_{\Omega_{t_0}} - 1_{\Omega_t} | t \in b(s),\ t \neq t_0\}$ is a basis of U_s and then $\dim U_s = q - 1$. □

THEOREM 2.16. *For each* $v \in \mathring{E}_{\mathbb{R}} = \{v \in \mathbb{R};\ q^{-1/2} < |v| < q^{1/2}\}$, *the sequence of subspaces* $(W_n)_{n=0}^{\infty}$ *is also an orthogonal sequence for* $\langle\ ,\ \rangle_v$. *Furthermore, there exists a sequence* $(w_n(v))_{n=0}^{\infty}$ *of real positive numbers such that*

$$\langle f, g\rangle_v = w_n(v)\langle f, g\rangle. \tag{2.18}$$

PROOF. Let us denote by (W_n^v) the sequence of orthogonal subspaces of $(\mathscr{E}(B), \langle\ ,\ \rangle_v)$ defined by

$$W_0^v = V_0, \qquad W_1^v \oplus_v V_0 = V_1, \ldots, W_n^v \oplus_v V_{n-1} = V_n.$$

We shall show that for each integer n, $W_n^v = W_n$; or equivalently that $W_n \subset W_n^v$ and $\dim W_n^v = \dim W_n$.

In fact, we shall verify that, for $s \in S_{n-1}$, $U_s \subset W_n^v$, that is, $U_s \perp_v V_{n-1}$. Let us begin by showing that $U_0 \perp_v V_0$. Let $1_{\Omega_{t_0}} - 1_{\Omega_t}$ be a basis vector of U_0 and let 1_B be a basis vector of V_0. Then

$$\langle 1_{\Omega_{t_0}} - 1_{\Omega_t}, 1_B\rangle_v = \langle 1_{\Omega_{t_0}}, 1_B\rangle_v - \langle 1_{\Omega_t}, 1_B\rangle_v = 0$$

by the invariance of $\langle\ ,\ \rangle_v$ under K_0. Let us continue by showing that $U_s \perp_v V_{n-1}$, $s \in S_{n-1}$, $n \geq 2$. Let $1_{\Omega_{t_0}} - 1_{\Omega_t}$ be a basis vector of U_s and let $1_{\Omega_{s_0}}$ be a basis vector of V_{n-1}. Then

$$\langle 1_{\Omega_{t_0}} - 1_{\Omega_t}, 1_{\Omega_{s_0}}\rangle_v = \langle 1_{\Omega_{t_0}}, 1_{\Omega_t}\rangle_v - \langle 1_{\Omega_t}, 1_{\Omega_{s_0}}\rangle_v. \tag{2.19}$$

If $s_0 = s$, then $t_0, t \in b(s_0)$ and there exists $g \in K_0$ such that $g(t_0) = t$, $g(t) = t_0$, $g(s_0) = s_0$, and, by the invariance of $\langle\ ,\ \rangle_v$ under K_0, we conclude that the left-hand side of (2.19) is equal to 0. If $s_0 \neq s$ there exists $g \in K_0$ such that $g(s_0) = s_0$, $g(s) = s$, $g(t_0) = t$, $g(t) = t_0$ and then the left-hand side of (2.19) is also equal to 0.

PROOF OF THE EXISTENCE OF $w_n(v)$. The inner product $\langle\ ,\ \rangle_v$ on V_n is a linear form in the first variable, and there exists a unique endomorphism E_n of V_n such that

$$\langle f, g\rangle_v = \langle E_n(f), g\rangle.$$

Let $\{(A_n(t_i, t_j)),\ i, j \leq \Pi_n\}$ be the matrix of E_n for the basis $B = (1_{\Omega_{t_i}}, i \leq \Pi_n)$. If $f = \sum_{i \leq \Pi_n} f_{t_i} 1_{\Omega_{t_i}}$, then

$$\langle E_n(f), g\rangle = \Pi_n^{-1} \sum_{i, j \leq \Pi_n} A_n(t_i, t_j) f_{t_j} \bar{g}_{t_i}$$

and then

$$A_n(t_i, t_j) = \langle 1_{\Omega_{t_i}}, 1_{\Omega_{t_j}}\rangle_v \Pi_n.$$

This implies that $(A_n(t_i, t_j))$ is invariant under K_0, since $\langle\ ,\ \rangle_v$ is invariant under K_0 by Proposition 2.11. Thus, $A_n(t,s)$ depends only on the distance $d(s,t)$, and we can write $A_n(t,s) = \psi_n(d(s,t))$ for a function $\psi_n\colon \mathbb{N} \to \mathbb{C}$ which vanishes outside of $\{0,2,4,6,\dots,2n\}$. Therefore, we have the identity

$$\langle f, g\rangle_v = \Pi_n^{-1} \sum_{i,j\le \Pi_n} \psi_n\big(d(t_i,t_j)\big) f_{t_j}\bar{g}_{t_i}. \tag{2.20}$$

Let us verify that for f and $g \in W_n$ we can write

$$\langle f, g\rangle_v = w_n(v)\langle f, g\rangle.$$

It suffices to show this equality for $f = 1_{\Omega_{t_0}} - 1_{\Omega_t}$, $t_0, t \in b(s)$ and $s \in S_{n-1}$. We have

$$\begin{aligned} E_n\big(1_{\Omega_{t_0}} - 1_{\Omega_t}\big) &= \sum_{t_i\in S_n} \big(\psi_n(d(t_i,t_0)) - \psi_n(d(t_i,t))\big)1_{\Omega_{t_i}} \\ &= \big(\psi_n(0) - \psi_n(2)\big)\big(1_{\Omega_{t_0}} - 1_{\Omega_t}\big), \end{aligned}$$

since $d(t_i,t_0) = d(t_i,t)$ when $t_i \notin \{t_0,t\}$ with $t_0 \in b(s)$, $t \in b(s)$. Thus, $w_n(v) = \psi_n(0) - \psi_n(2)$. Furthermore,

$$\begin{aligned} \Big\langle E_n\big(1_{\Omega_{t_0}} - 1_{\Omega_t}\big), 1_{\Omega_{t_0}} - 1_{\Omega_t}\Big\rangle &= \|1_{\Omega_{t_0}} - 1_{\Omega_t}\|_v^2 \\ &= w_n(v)\|1_{\Omega_{t_0}} - 1_{\Omega_t}\|^2 = 2\Pi_n^{-1} w_n(v). \end{aligned}$$

This implies that $w_n(v) \ge 0$. □

Recall that the family $\{e_t,\ t\in S_n\}$ defined by $e_t(b) = (\sqrt{q}\,v)^{\delta_b(0,t)}$ forms a basis of V_n. Let us denote by $e_t^{(n)}$ the projection of e_t on W_n. We saw already that $e_t \in V_{|t|}$. We can then write

$$e_t = \sum_{k=0}^{|t|} e_t^{(k)}. \tag{2.21}$$

DEFINITION 2.17. For $t \in T$ let us denote by $D_t\colon (\mathring{E}_{\mathbb{R}} \cup E_{\mathbb{C}}) \times B \to \mathbb{C}$ the following function:

$$D_t(v,b) = \begin{cases} e_t(b), & \text{if } v \in E_{\mathbb{C}}, \\ \sum_{k=0}^{|t|} \sqrt{w_k(v)}\, e_t^{(k)}(b), & \text{if } v \in \mathring{E}_{\mathbb{R}}. \end{cases} \tag{2.22}$$

THEOREM 2.18. *For $v \in \mathring{E}_{\mathbb{R}} \cup E_{\mathbb{C}}$ and for $s, t \in T$, we have*

$$P_{d(s,t)}(J(v)) = \int_B D_t(v,b)\overline{D_s(v,b)}\nu_0(db). \tag{2.23}$$

PROOF. If $v \in E_{\mathbb{C}}$, (2.23) is equation (2.9). If $v \in \mathring{E}_{\mathbb{R}}$,

$$\begin{aligned}
P_{d(s,t)}(J(v)) &= \langle e_t, e_s \rangle_v = \left\langle \sum_{k=1}^{|t|} e_t^{(k)}, \sum_{k=1}^{|s|} e_s^{(k)} \right\rangle_v \\
&= \sum_{k=1}^{\min(|t|,|s|)} \left\langle e_t^{(k)}, e_s^{(k)} \right\rangle_v \\
&= \sum_{k=1}^{\min(|t|,|s|)} w_k(v) \left\langle e_t^{(k)}, e_s^{(k)} \right\rangle \\
&= \left\langle D_t(v,\cdot), D_s(v,\cdot) \right\rangle \\
&= \int_B D_t(v,b) \overline{D_s(v,b)} \nu_0(db).
\end{aligned}$$

□

With the following proposition we are able to give an explicit formulation for $e^{(k)}$.

PROPOSITION 2.19. *Let $t \in T$ and let $(t_0 = 0, t_1, \dots, t_m)$ be the unique path from 0 to t. Then, for $n \geq |t|$,*

$$\begin{aligned}
e_t^{(0)} &= P_{|t|}(J(v)) 1_B, \\
e_t^{(1)} &= \alpha_1 \sum_{\substack{t \in S_1 \\ t \neq t_1}} \left(1_{\Omega_t} - 1_{t_1}\right), \\
&\vdots \\
e_t^{(n)} &= \alpha_n \sum_{\substack{t \in b(t_{n-1}) \\ t \neq t_n}} \left(1_{\Omega_t} - 1_{t_n}\right),
\end{aligned}$$

with

$$\alpha_k = \left(\sqrt{q}\, v\right)^{2n-|t|} \left(\frac{1 - qv^2}{2qv^2} \right).$$

PROOF. Since $e_t^{(0)} \in W_0 = V_0$ we can write $e_t^{(0)} = C 1_B$, and thus

$$\left\langle e_t - e_t^{(0)}, 1_B \right\rangle = 0 \Rightarrow C = \int e_t \nu_0(db) = P_{|t|}(J(v)).$$

Let us calculate $e_t^{(n)}$ with $n \geq 1$. In (2.10) we wrote

$$e_t = \sum_{s \in S_n} \left(\sqrt{q}\, v\right)^{n - d(s,t)} 1_{\Omega_s}.$$

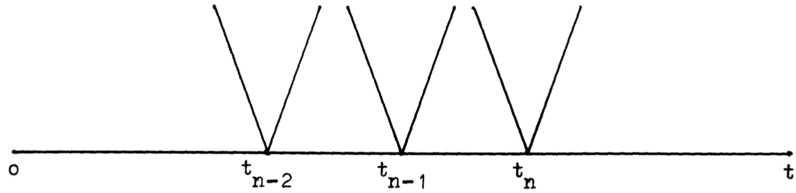

FIG. 2.

Thus for any $f = \sum_{t \in S_n} f_t 1_{\Omega_t}$ in W_n, we have

$$\langle e_t, f \rangle = \sum_{s \in S_n} \left(\sqrt{q}\, v\right)^{n-d(s,t)} \Pi_n^{-1} \bar{f}_s.$$

If $s \in b(t_{n-1})$, $s \neq t_n$, then $d(s,t)$ is a constant and $\sum_{s \in b(t_{n-1})} \bar{f}_s = -\bar{f}_{t_n}$. (See figure 2.) If $r \in S_{n-1}$, $r \neq t_{n-1}$, then for any $s \in b(r)$, $d(s,t)$ is a constant and thus $\sum_{s \in b(r)} \bar{f}_s = 0$. With these remarks we can conclude that

$$\begin{aligned} \langle e_t, f \rangle &= \Pi_n^{-1} \left\{ \left(\sqrt{q}\, v\right)^{n-(d(t_n,t)+2)} \left(-\bar{f}_{t_n}\right) + \left(\sqrt{q}\, v\right)^{n-d(t_n,t)} \bar{f}_{t_n} \right\} \\ &= \Pi_n^{-1} \bar{f}_{t_n} \left(\sqrt{q}\, v\right)^{2n-|t|} \left(1 - \left(\sqrt{q}\, v\right)^{-2}\right). \end{aligned}$$

We now have to carry out a similar calculation with the desired formula for $e_t^{(n)}$:

$$\begin{aligned} &\left\langle \alpha_n \sum_{\substack{t \in b(t_{n-1}) \\ t \neq t_n}} \left(1_{\Omega_t} - 1_{\Omega_{t_n}}\right), f \right\rangle \\ &\quad = \left\langle \alpha_n \sum_{\substack{t \in b(t_{n-1}) \\ t \neq t_n}} \left(1_{\Omega_t} - 1_{\Omega_{t_n}}\right), \sum_{t \in b(t_{n-1})} f_t 1_{\Omega_t} \right\rangle \\ &\quad = \alpha_n \sum_{\substack{t \in b(t_{n-1}) \\ t \neq t_n}} \bar{f}_t \left\langle 1_{\Omega_t}, 1_{\Omega_t} \right\rangle - \alpha_n \bar{f}_{t_n} \left\langle 1_{\Omega_{t_n}}, 1_{\Omega_{t_n}} \right\rangle \\ &\quad = \alpha_n \left(-\bar{f}_{t_n} - \bar{f}_{t_n}\right) \Pi_n^{-1} = -2\alpha_n \Pi_n^{-1} \bar{f}_{t_n}. \end{aligned}$$

If we take

$$\alpha_n = \tfrac{1}{2} \left(\sqrt{q}\, v\right)^{2n-|t|} \left(\left(\sqrt{q}\, v\right)^{-2} - 1\right),$$

then

$$\left\langle e_t - \alpha_n \sum_{\substack{s \in b(t_{n-1}) \\ s \neq t_n}} \left(1_{\Omega_s} - 1_{\Omega_{t_n}}\right), f \right\rangle = 0 \qquad \forall f \in W_n.$$

Since the projection $e_t^{(n)}$ is unique, the proof is complete. □

3. The construction of the fundamental isometry. Let us introduce the basis set K which we call the *Cantor sphere of order* q.

DEFINITION 3.1. Let K' be the set $K' = B \times]-1, +1[$ equipped with the product topology, and let $K = K' \cup \{-1, +1\}$ be the compactification of K' for which a basis of neighborhoods of $+1$ is defined by

$$\{]\alpha, 1[\times \Omega_s | -1 < \alpha < 1, s \in T\}$$

and a basis of neighborhoods of -1 is defined by

$$\{]-1, \alpha[\times \Omega_s | -1 < \alpha < 1, s \in T\}.$$

REMARK. Generally, if B were replaced by a metrizable compact set C, this construction would give a compactification of K_C by two infinite points of $]-1, +1[\times C$. Thus, if C is the unit sphere S_{d-1} of the Euclidean space of dimension d, then K_c is homeomorphic to S_d. If $q > 1$, B has two points, and $K = K_B$ is then homeomorphic to S_1. If $q > 1$, B is homeomorphic to a Cantor set. The point of these remarks is to justify the term "Cantor sphere."

We shall now introduce a collection of remarkable functions $(\Delta_t)_{t \in T}$ which will be the images of X_t by the isomorphism ψ, that is, $\psi(X_t) = \Delta_t$, such that $\int_K \overline{\Delta}_t \Delta_s \, dm = E(\overline{X}_t X_s)$. These functions Δ_t will be the analog of the exponentials e^{ixt} in time series.

DEFINITION 3.2. The functions Δ_t: $K \to \mathbb{C}$ are defined by

(3.1) $$\Delta_t(1) = 1, \qquad \Delta_t(-1) = (-1)^{|t|},$$

(3.2) if $(z, b) \in \,]-\rho_q, \rho_q[\times B$, then there exists $v \in E_{\mathbb{C}}$ such that $J(v) = z$, $\operatorname{Re} v > 0$ and we define $\Delta_t(z, b) = D_t(v, b)$ [see (2.22)],

(3.3) if $(z, b) \in (]-1, -\rho_q] \cup [\rho_q, 1[) \times B$, then there exists $v \in \mathring{E}_{\mathbb{R}}$ such that $J(v) = J(v^{-1}) = z$ and we define $\Delta_t(z, b) = D_t(v, b)$ [see (2.22)].

REMARK. Recall that $D_t(v^{-1}, b) = \overline{D_t(v, b)}$ and that $D_t(v, b)$ is real if $v \in \mathring{E}_{\mathbb{R}}$. If in (3.3) we choose v such that $\operatorname{Re} v < 0$, this will lead us to work with the conjugate of the above Δ_t.

In order to prove the continuity of D_t on K, we need the following lemma.

LEMMA 3.3.

$$\lim_{v\to q^{-1/2}} D_t(v,b) = \lim_{v\to q^{1/2}} D_t(v,b) = 1, \tag{3.4}$$

$$\lim_{v\to -q^{-1/2}} D_t(v,b) = \lim_{v\to -q^{1/2}} D_t(v,b) = (-1)^{|t|}. \tag{3.5}$$

PROOF. Let us choose $v \in \mathring{E}_{\mathbb{R}} \cup E_{\mathbb{C}}$. Then

$$\|D_t(v,\cdot)\|^2 = \langle D_t(v,\cdot), D_t(v,\cdot)\rangle = P_0(J(v)) = 1 \quad \text{[see (2.23)]}$$

and

$$\|D_t(v,\cdot)\|^2 = \sum_{k=0}^{|t|} w_k(v)\|e_t^{(k)}\|^2 = 1$$

for $t=0$. Then $1 = w_0(v)\|e_0^{(0)}\|^2 = w_0(v)$. Thus,

$$|P_{|t|}(J(v))|^2 + \sum_{k=1}^{|t|} w_k(v)\|e_t^{(k)}\|^2 = 1$$

and then we have

$$\lim_{|v|\to q^{\pm 1/2}} \sum_{k=1}^{|t|} w_k(v)\|e_t^{(k)}\|^2 = 0.$$

Thus

$$R_v(b) = \sum_{k=1}^{|t|} \sqrt{w_k(v)}\, e_t^{(k)}(b) \to 0 \qquad \text{as } |v| \to q^{\pm 1/2},$$

since

$$R_v \in V_{|t|} \text{ and } R_v = \sum_{s\in S_n} R_v^s 1_{\Omega_s}$$

$$\Rightarrow \int_B |R_v(b)|^2 \nu_0(db) = \Pi_{|t|}^{-1} \sum_{s\in S_{|t|}} |R_v^s|^2. \qquad \square$$

THEOREM 3.4. *The function Δ_t is continuous on K.*

PROOF. The restriction of Δ_t to $]-1,+1[\times B$ is continuous from the definition of Δ_t. We shall now use (3.4) and (3.5) to show that Δ_t is continuous at ± 1. Let us prove continuity at 1. If $|t| = n$, we number by $1, 2, \ldots, \Pi_n$ the Π_n vertices of S_n and by $\Omega_1, \ldots, \Omega_{\Pi_n}$ the Π_n corresponding open sets of B. Let us denote by Δ'_t the restriction of Δ_t to $]-1,+1[\times B$. We shall show that if S is an open set of $\mathbb{C}$ containing 1, then $\Delta_t^{-1}(S) = \Delta_t'^{-1}(S) \cup \{1\}$ is an open subset of K.

For $1 \le k \le \Pi_n$, let us define $U_k = \Delta_t^{-1}(S) \cap\,]-1,+1[\times \Omega_k$, which is an open subset of $]-1,+1[\times\Omega_k$. $\Delta'_t(z,b)$ is independent of b when $b\in\Omega_k$. This remark together with the result (3.4) allows us to conclude that there exists

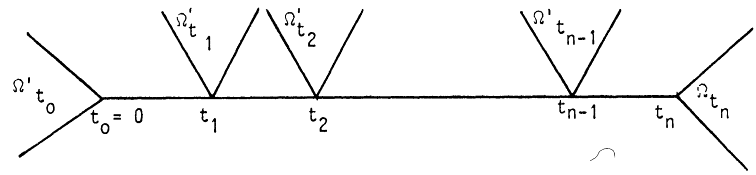

FIG. 3.

$\alpha_k < 1$ such that $]\alpha_k, 1[\times\Omega_k \subset U_k$, and then $\Delta_t'^{-1}(S) = \bigcup_{k=1}^{\Pi_n} U_k$ is an open set of $]-1, +1[\times B$ which contains $]\alpha, 1[\times B$, with $\alpha = \max_{1\le k\le \Pi_n} \alpha_k < 1$. Since $(]\alpha, 1[\times B) \cup \{1\} \subset \Delta_t^{-1}(S) = \Delta_t'^{-1}(S) \cup \{1\}$, we conclude that $\Delta_t^{-1}(S)$ is an open set of K. Continuity in -1 is similar. □

With the following theorem, we shall show that the linear space E spanned by the family $(\Delta_t)_{t\in T}$ is a dense subset of the linear space $\mathscr{C}(K)$ of continuous complex functions on K.

THEOREM 3.5.

(i) *Let η be a positive measure on the compact set B and let z be a fixed element of $]-1, +1[$ and denote*

$$\hat{\eta}_t^{(z)} = \int_B \Delta_t(z,b)\eta(db) \quad \text{for } t \in T. \tag{3.6}$$

Then $(\hat{\eta}_t^{(z)})_{t\in T}$ determines η.

(ii) *Let α be a probability measure in the Cantor sphere K and denote $\hat{\alpha}_t = \int_K \Delta_t(\xi)\alpha(d\xi)$. Then $(\hat{\alpha}_t)_{t\in T}$ determines α.*

PROOF. Let $[t_0 = 0, t_1, \dots, t_n, \dots]$ be an infinite chain of vertices of T. We have $\delta_b(0, t_n) = 2k - n$ if $b \in \Omega'_{t_k}$, and $0 \le k \le n$, $\delta_b(0, t_n) = n$ if $b \in \Omega_{t_n}$. (See Figure 3.) Let z be a fixed element of $]-1, +1[$. We shall denote by $\Delta'_{t_m,k}$ the constant value taken by $\Delta_{t_m}(z,b)$ when $b \in \Omega'_{t_k}$, $k \le m-1$. We shall denote by $\Delta_{t_m,m}$ the constant value taken by $\Delta_{t_m}(z,b)$ when $b \in \Omega_{t_m}$. Let us calculate

$$\begin{aligned}\hat{\eta}_{t_m}^{(z)} &= \int_B \Delta_{t_m}(z,b)\eta(db)\\ &= \sum_{k=0}^{m-1} \eta(\Omega'_{t_k})\,\Delta'_{t_m,k} + \eta(\Omega_{t_m})\,\Delta_{t_m,m}.\end{aligned}$$

Suppose there exist two measures η and η' on B such that $\hat{\eta}_t^{(z)} = \hat{\eta}_t'^{(z)}$ for each $t \in T$. By considering $\beta = \eta - \eta'$, we obtain

(0) $\quad 0 = \beta(B)$ by using $\hat{\eta}_{t_0}^{(z)} = \hat{\eta}_{t_0}'^{(z)}$,

(1) $\quad 0 = \Delta'_{t_1,0}\beta(\Omega'_{t_0}) + \Delta_{t_1,1}\beta(\Omega_{t_1})$ by using $\hat{\eta}_{t_1}^{(z)} = \hat{\eta}'_{t_1}(z)$,

(2) $\quad 0 = \Delta'_{t_2,0}\beta(\Omega'_{t_0}) + \Delta'_{t_2,1}\beta(\Omega'_{t_1}) + \Delta_{t_2,2}\beta(\Omega_{t_2})$,

$\vdots$

$$(n) \qquad \begin{aligned} 0 = \Delta'_{t_n,0}\beta(\Omega'_{t_0}) + \Delta'_{t_0,1}\beta(\Omega'_{t_1}) + \cdots + \Delta'_{t_n,n-1}\beta(\Omega'_{t_{n-1}}) \\ + \Delta_{t_n,n}\beta(\Omega_{t_n}), \end{aligned}$$

Since $\beta(B) = 0$, we then have $\beta(\Omega_{t_m}) = -\beta(\Omega'_{t_0}) \dots -\beta(\Omega'_{t_{m-1}})$, and the equations $(1), \dots, (n)$ become

(1′) $\quad 0 = \beta(\Omega'_{t_0})(\Delta'_{t_1,0} - \Delta_{t_1,1})$,

(2′) $\quad 0 = \beta(\Omega'_{t_0})(\Delta'_{t_2,0} - \Delta_{t_2,2}) + \beta(\Omega'_{t_1})(\Delta'_{t_2,1} - \Delta_{t_2,2})$,

$\vdots$

$$(n') \qquad 0 = \sum_{k=0}^{n-1} \beta(\Omega'_{t_k})(\Delta'_{t_n,k} - \Delta_{t_n,n}).$$

We shall show that $\Delta_{t_m,m} \neq \Delta'_{t_m,m-1}$ for a fixed $z \in\,]-1,+1[$. If $|z| \leq \rho_q$, then $z = J(v)$, $v \in E_{\mathbb{C}}$, and

$$\Delta'_{t_m,m-1} - \Delta_{t_m,m} = \left(\sqrt{q}\,v\right)^{m-2}(1 - qv^2) \neq 0, \qquad m \geq 1.$$

If $\rho_q < |z| < 1$, $z = J(v)$, $v \in \mathring{E}_{\mathbb{R}}$,

$$\Delta'_{t_m,m-1} = \sum_{k=0}^{m} \sqrt{w_k(v)}\, e_{t_m}^{(k)}(b_{m-1}), \qquad b_{m-1} \in \Omega'_{t_{m-1}},$$

$$\Delta_{t_m,m} = \sum_{k=0}^{m} \sqrt{w_k(v)}\, e_{t_m}^{(k)}(b_m), \qquad b_m \in \Omega_{t_m}.$$

Thus,

$$\Delta'_{t_m,m-1} - \Delta_{t_m,m} = \sqrt{w_m(v)}\left(e_{t_m}^{(m)}(b_{m-1}) - e_{t_m}^{(m)}(b_m)\right) = \sqrt{w_m(v)}\,\alpha_m \cdot q \neq 0.$$

Then, by induction, we conclude that $\beta(\Omega'_{t_n}) = 0$, $\forall n$, and since the sequence $(t_n)_{n \geq 0}$ was arbitrary, it follows that $\beta = 0$ and $\eta = \eta'$.

For the proof of (2) we choose a probability measure α on the compact set K. Define by α'' the probability measure on $[-1,+1] \times B$ such that for each Borel set U of $]-1,+1[\times B$ we have $\alpha''(U) = \alpha(U)$ and such that for each Borel set Ω of B we have

$$\alpha''(\{1\} \times \Omega) = \alpha(\{1\})\nu_0(\Omega),$$
$$\alpha''(\{-1\} \times \Omega) = \alpha(\{-1\})\nu_0(\Omega).$$

We can also define, for $t \in T$, Δ''_t: $[-1, +1] \times B \to \mathbb{C}$ by

$$\Delta''_t = \Delta_t \text{ on }]-1, +1[\times B,$$
$$\Delta''_t(-1, b) = (-1)^{|t|},$$
$$\Delta''_t(1, b) = 1.$$

For each polynomial P the product $P \cdot \Delta_t$: $K \to \mathbb{C}$ is well defined by $(P \cdot \Delta_t)(\xi) = P(z)\,\Delta_t(z, b)$ if $\xi = (z, b) \in]-1, +1[\times B$ and by $P(\pm 1)\,\Delta_t(\pm 1)$ if $\xi = \pm 1$. It is then evident that

$$L(P) = \int_K (P \cdot \Delta_t)(\xi)\alpha(d\xi) = \iint_{[-1,+1]\times B} P(z)\,\Delta''_t(z, b)\alpha''(dz, db).$$

Our first step is to show that L is determined by $(\hat{a}_s)_{s \in T}$.

Let E'' be the subspace of $\mathscr{C}([-1, +1] \times B)$ spanned by the Δ''_t. The main point is that if $(z, b) \to f(z, b)$ is an element of E'' and $z \to P(z)$ is a polynomial, then $(z, b) \to f(z, b)P(z)$ is an element of E''. Indeed, the Dunau polynomials form a basis of the polynomials and it then suffices to show that

$$\Delta''_t(z, b)P_n(z) = \Pi_n^{-1} \sum_{d(s,t)=n} \Delta''_s(z, b). \tag{3.7}$$

We shall show that (3.7) is true with $z \in\ ]-1, +1[$; in this case it is defined by the following identity:

$$D_t(v, b)P_n(J(v)) = \Pi_n^{-1} \sum_{d(s,t)=n} D_s(v, b), \qquad v \in \mathring{E}_{\mathbb{R}} \cup E_{\mathbb{C}}. \tag{3.8}$$

If $v \in E_{\mathbb{C}}$, $D_t(v, b) = e_t(b)$ and we deduce (3.8) from Propositions 2.7 and 2.6. If $v \in \mathring{E}_{\mathbb{R}}$, it suffices to examine the projections on W_k of the two members of equality (3.8):

$$\mathrm{Proj}_{W_k} D_t \cdot P_n = P_n(J(v))\sqrt{w_k(v)}\,e_t^{(k)}(v, \cdot)$$

and

$$\mathrm{Proj}_{W_k} \Pi_n^{-1} \sum_{d(s,t)=n} D_s = \Pi_n^{-1}\sqrt{w_k(v)} \sum_{d(s,t)=n} e_s^{(k)}.$$

The problem is then to show that

$$e_t^{(k)} \cdot P_n = \Pi_n^{-1} \sum_{d(s,t)=n} e_s^{(k)},$$

which is immediate by projection on W_k of the two members of equality (2.12) applied to $e_\cdot(b)$ in t; we deduce (3.7) from (3.8) by continuity at the points $z = \pm 1$.

The next step is to write the probability measure α'' as follows: $\alpha''(dz, db) = m(dz)\eta(z, db)$, where η is a probability transition kernel with

$$m(1) = \alpha(1), \qquad m(-1) = \alpha(-1),$$
$$\eta(1, db) = \eta(-1, db) = \nu_0(db).$$

Then

$$L(P) = \iint_{[-1,+1]\times B} \Delta_t''(z,b)P(z)\alpha''(dz,db)$$
$$= \int_{[-1,+1]} P(dz)\left\{\int_B \Delta_t''(z,b)\eta(z,db)\right\}m(dz).$$

We then use the theorem of uniqueness of moments on a compact interval to conclude that if we are given the $L(P)$, that is, if we are given the $\hat{\alpha}_s$, $s \in T$, we can determine the measures $M_t(dz) = \int_B \Delta_t''(z,b)\eta(z,db)m(dz)$.

We have

$$M_t(+1) = \alpha(1), \qquad M_t(-1) = (-1)^{|t|}\alpha(-1)$$

and for $z \in]-1,+1[$, $M_t(dz) = \hat{\eta}_t(z)m(dz)$ since $\hat{\eta}_0(z) = \int_B \eta(z,db) = 1$. Thus, $\hat{\eta}_t(z)$ is determined m-almost everywhere and this implies from the first part (1) that $\eta(z,db)$ is determined m-almost everywhere and then α is completely determined. □

COROLLARY 3.6. *The linear space spanned by the* $(\Delta_t)_{t\in T}$ *is dense in* $\mathscr{C}_{\mathbb{C}}(K)$.

PROOF. It is an immediate consequence of the Radon–Riesz theorem; see, for example, [14]. □

We have arrived at last to the main theorem of this paper.

THEOREM 3.7. *Let* $(X_t)_{t\in T}$ *be a stationary process on a homogeneous tree* T, *with real spectral measure* μ. *Let* $H(X)$ *be the linear space spanned by the* $(X_t)_{t\in T}$ *and let* m *be the positive measure on the Cantor sphere defined by* $\mu \times \nu_0$ *on* $]-1,+1[\times B$ *and such that* $m(\{-1\}) = \mu(\{-1\})$ *and* $m(\{+1\}) = \mu(\{+1\})$. *If* ψ: $H(X) \to L^2(K,m)$ *denotes the linear functional defined by* $\psi(X_t) = \Delta_t$, *then* ψ *is a surjective isometry.*

PROOF. It can be seen that ψ is an isometry since

$$E(\overline{X}_s X_t) = \int_{-1}^{+1} P_{d(s,t)}(z)\mu(dz)$$
$$= \iint_{]-1,+1[\times B} \Delta_t(z,b)\,\overline{\Delta_s(z,b)}(\mu\times\nu_0)(dz,db)$$
$$+ P_{d(s,t)}(-1)\mu(-1) + P_{d(s,t)}(1)\mu(1)$$
$$= \int_K \Delta_t(\xi)\,\overline{\Delta}_s(\xi)m(d\xi),$$

this last identity being due to the equality

$$(-1)^{d(s,t)} = (-1)^{|t|+|s|}.$$

To prove the surjectivity, it suffices to apply Corollary 3.6 to deduce that the linear space spanned by the $(\Delta_t)_{t\in T}$ is dense in $\mathscr{C}_{\mathbb{C}}(K)$, and since the continuous functions are dense in L^2, we have then proved the result. □

4. Conclusion. We are henceforth in possession of a powerful tool to approach two of the great problems concerning second-order processes: prediction (in a wide sense) and filtering.

By the problem of prediction we mean the following: Let T' be a subset of the vertices T and denote by $H(X)$ and $H'(X)$ the closed subsets of $L^2(\Omega, \mathscr{A}, P)$ spanned respectively by the stationary processes $(X_t)_{t\in T}$ and $(X_t)_{t\in T'}$.

If Y is an element of $H(X)$, to predict Y is to find its projection PY on $H'(X)$. It is the case, for example, if $Y = X_t$ with $t \notin T'$. The computation of $\|Y - PY\|^2$ gives us the quality of this prediction. We can remark that in the case $q = 1$, the term "prediction" is generally reserved for the case where T' is the half-line of integers $]-\infty, 0[$; the case $T' = \mathbb{Z} \setminus \{0\}$ is usually called the problem of interpolation. Different generalizations of the idea of half-lines to a homogeneous tree seem possible: T' might be a connected component of T from which we take off a vertex or an edge. Benveniste in [4] considers the case $T' = \{t, \delta_\infty(0, t) \le 0\}$, when 0 and ∞ are a fixed vertex and end, and δ_∞ is defined in (2.5).

Exciting problems can be approached as the analogs of the Hardy space H^2, or the Szegö condition $\int_0^{2\pi} \log f dx > -\infty$, which is essential in prediction when $q = 1$ and $T' =]-\infty, 0]$. The isometry studied in this paper led us to study these Hardy spaces on the Cantor sphere introduced in Definition 3.1. We can classify under the same topic the search of analogs of ARMA and ARIMA processes already outlined in Benveniste.

The second problem concerns filtering ([8] and [9]). The idea is that when trying to make observations of a stationary process $(X_t)_{t\in T}$, we may be unable to observe the variables X_t themselves but only $Y_t = X_t + Z_t$ where $(Z_t)_{t\in T}$ represents "random noise." It is natural to assume that $(Z_t)_{t\in T}$ is also stationary and that it is orthogonal to $(X_t)_{t\in T}$. The correlation functions of both $(X_t)_{t\in T}$ and $(Z_t)_{t\in T}$ are supposed to be known. Let $H'(Y)$ be the closed subspace spanned by $\{Y_t, t \in T'\}$ with $T' \subset T$; then $H'(Y)$ represents what is actually observed. The best linear approximation of X represents the best attempt to sort out the "signal" X_t from the accompanying "noise" Z_t using the observed data $H'(Y)$ which consist of signal plus noise. Thus, to approximate X_t linearly, we seek the projection $\hat{X}_t$ of X_t onto $H'(Y)$.

REFERENCES

[1] ARNAUD, J. P. (1980). Fonctions sphériques et fonctions définies positives sur l'arbre homogène. *C. R. Acad. Sci. Paris Sér. I Math.* **290** 99–101.

[2] ARNAUD, J. P., DUNAU, J. L. and LETAC, G. (1983). Arbres homogènes et couples de Gelfand. Publication 02, Laboratory of Statistics and Probability, Univ. Paul Sabatier, Toulouse, France.

[3] ARNAUD, J. P. and LETAC, G. (1985). La formule de représentation spectrale d'un processus gaussien stationnaire sur un arbre homogène. Publication 01, Laboratory of Statistics and Probability, Univ. Paul Sabatier, Toulouse, France.
[4] BENVENISTE, A. and BASSEVILLE, M. (1988). Modèles statistiques temps échelle en traitement du signal. Internal publication 446, IRISA Rennes.
[5] CARTIER, P. (1971–1972). Géométrie et analyse sur les arbres. *Sem. Bourbaki*, 24th year, lecture 407.
[6] CARTIER, P. (1976). Harmonic analysis on trees. *Proc. Symp. Pure Math.* **26** 419–424.
[7] DUNAU, J. L. (1976). Etude d'une classe de marches aléatoires sur l'arbre homogène. Publication 04, Laboratory of Statistics and Probability, Univ. Paul Sabatier, Toulouse, France. Reprinted in [2].
[8] DYM, H. and MCKEAN, H. P. (1975). *Gaussian Processes, Complex Function Theory and the Inverse Spectral Method*. Academic, New York.
[9] LAMPERTI, J. (1977). *Stochastic Processes: A Survey of the Mathematical Theory*. Springer, New York.
[10] LASSER, R. (1983). Orthogonal polynomials and hypergroups. *Rend. Mat.* **3** 185–209.
[11] LASSER, R. and LEITNER, M. (1990). On the estimation of the mean of weakly stationary and polynomial weakly stationary sequences. *J. Multivariate Anal.* **35** 31–47.
[12] LETAC, G. (1981). Problèmes classiques de probabilité sur un couple de Gelfand. *Analytic Methods in Probability Theory. Lecture Notes in Math.* **361** 93–127. Springer, New York.
[13] NEHARI, Z. (1952). *Conformal Mapping*. McGraw-Hill, New York.
[14] ROYDEN, H. (1968). *Real Analysis*. Macmillan, New York.
[15] SAWYER, S. (1978). Isotropic random walks in a tree. *Z. Wahrsch. Verw. Gebiete* **42** 279–292.
[16] SZEGÖ, G. (1939). *Orthogonal Polynomials*. Amer. Math. Soc., Providence, RI.

UNIVERSITÉ PAUL SABATIER
LABORATOIRE DE STATISTIQUE ET PROBABILITÉS
U.R.A. C.N.R.S. D0745
118, ROUTE DE NARBONNE
31062 TOULOUSE CEDEX
FRANCE