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ON THE AVERAGE DIFFERENCE BETWEEN CONCOMITANTS AND ORDER STATISTICS[1]

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For a sequence of bivariate pairs (X_i, Y_i), the concomitant $Y_{[i]}$ of the ith largest x-value $X_{(i)}$ equals that value of Y paired with $X_{(i)}$. In assessing the quality of a file-merging or file-matching procedure, the penalty for incorrect matching may often be expressed as the average value of a function of the difference $Y_{[i]} - Y_{(i)}$. We establish strong laws and central limit theorems for such quantities. Our proof is based on the observation that if $G_x(\cdot)$ denotes the distribution function of Y given $X = x$, then $G_X(Y)$ is stochastically independent of X, even though $G_x(\cdot)$ depends numerically on x.

1. Introduction. An abbreviated account of the problem of micro data file-merging or file-matching may be given as follows. [The reader is referred to Office of Federal Statistical Policy and Standards (1980), DeGroot (1987) and Goel and Ramalingam (1989) for more detail.] Suppose observations are collected in the form of independent pairs (X_i, Y_i), $1 \le i \le n$, but that before the data are recorded the linkage between the X and Y values is broken. Thus, only the two marginal data files $\mathbf{X} = \{X_i,\ 1 \le i \le n\}$ and $\mathbf{Y} = \{Y_i,\ 1 \le i \le n\}$ are available, and we must endeavor to reconstruct the original paired data by *merging* the $\mathbf{X}$ file with the $\mathbf{Y}$ file by *matching* X's and Y's. The reconstructed bivariate file has the form $(X_{(i)}, Y_{(\varphi(i))})$, where $X_{(1)} \le \cdots \le X_{(n)}$ and $Y_{(1)} \le \cdots \le Y_{(n)}$ denote the ordered marginal data, and φ is some permutation of the integers $1, \dots, n$.

Practical examples of the use of file-matching techniques have been discussed by Goel and Ramalingam (1989). They include two classical exact file-matching problems—the Framingham heart study [Dawber, Kannel and Lyell (1963)] and the study of Japanese A-bomb survivors [Beebe (1979)]—and a file-merging problem that results from incomplete U.S. Treasury records. In the latter case, file-merging methods are the key to solving the practically important problems of estimating the correlation between X and Y variables.

DeGroot, Feder and Goel (1971) have shown that for a large class of models for the joint distribution of (X, Y), the maximum likelihood solution to the file-merging problem is to pair the ith largest X with the ith largest value of $\delta(Y)$, where δ is a known function. If the parent density h of (X, Y) has a monotone (increasing) likelihood ratio, meaning that $h(x_1, y_1)h(x_2, y_2) \ge$

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$h(x_1, y_2)h(x_2, y_1)$ whenever $x_1 \le x_2$ and $y_1 \le y_2$, then we may take δ to be the identity function. For this case, the maximum likelihood solution is to take $\varphi(i) = i$, $1 \le i \le n$ [Chew (1973)], that is, to pair the x and y observations in natural increasing order. If the likelihood ratio is monotone decreasing, then it is optimal to take $\varphi(i) = n - i + 1$, that is, to pair the x and y observations in reverse order, or equivalently to take $\varphi(i) = i$ when the pairs are changed to $(X_i, -Y_i)$. Examples with monotone increasing or decreasing likelihood ratios include bivariate normal distributions with correlation $\rho > 0$ or $\rho < 0$, respectively.

Thus, the optimal solution to a wide range of micro data file-merging problems is to pair $X_{(i)}$ with $Y_{(i)}$. The "true" pairing corresponds to the actual observation in which $X_{(i)}$ is paired with $Y_{[i]}$, the latter being the *concomitant* of $X_{(i)}$ in the Y-sample, that is,

$$Y_{[i]} = \sum_{j=1}^{n} Y_j I_{\{\text{rank}(X_j)=i\}}, \tag{1.1}$$

where $I_{\{A\}}$ denotes the indicator function of the set A. The concomitants of order statistics are also called *induced order statistics*. The general theory of concomitants of order statistics is discussed in David (1973; 1981, pages 109 and 110, 1982), David and Galambos (1979), and David, O'Connell and Yang (1977). Work on the ordered multivariate data has been surveyed by Barnett (1976), Bhattacharya (1984) and Galambos (1975, 1985).

The limiting behavior of sample paths of normalized sums of $Y_{[i]}$, $i = 1, 2, \dots, n$, as n increases to ∞, is discussed in Bhattacharya (1974, 1976). However, if (X, Y) has a nondegenerate bivariate distribution, then it will often be the case that $Y_{(i)}$ and $Y_{[i]}$ assume different values. Thus, there is a cost or penalty associated with mismatching. If the cost is measured in terms of the distance of $Y_{(i)}$ from $Y_{[i]}$, then the total cost of mismatches is

$$S = \sum_{i=1}^{n} \mathscr{a}(Y_{(i)} - Y_{[i]}), \tag{1.2}$$

where $\mathscr{a}(\cdot)$ is some specified function. Our aim in this paper is to derive a strong law and central limit theorem for S.

Earlier results of this type are largely confined to the case where $\mathscr{a}(0) = 0$ and $\mathscr{a}(x) = 1$ for $x \ne 0$. Here, there is unit cost associated with anything other than perfect matching and S denotes the number of incorrect matches. It has been shown that, in this context, $n - S$ has an asymptotic Poisson distribution [Goel and Ramalingam (1989), page 58]. For a slightly more general cost function, in which $\mathscr{a}(Y_{(i)} - Y_{[i]}) = 1$ if $|Y_{(i)} - Y_{[i]}| \ge \varepsilon$ and $= 0$ otherwise, weak laws of large numbers have been derived by Yahav (1982). The strong laws of large numbers for this cost function are derived in Goel and Ramalingam [(1989), page 45]. However, in cases of practical importance, $\mathscr{a}$ is often a continuous function on $(0, \infty)$. For example, a majority of loss functions in decision theory are of this form. The absolute value function is a very common measure of distance [see, e.g., Goel and Ramalingam (1989), page 8].

Large sample theory for such functions is highly important in the context of file-merging problems. We shall generalize the results of other authors to a class of functions $\mathscr{a}$, which include examples such as these, by developing strong laws of large numbers. We shall also develop a central limit theory for these cost functions. In addition, we shall present a new proof for the central limit theorem for order statistics from a sequence of multivariate random variables. This result was proved under stronger assumptions by Babu and Rao (1988), and it is needed in the proof of our central limit theorem for S.

The new method of proof is based on the observation that if $G_x(\cdot)$ denotes the distribution function of Y given $X = x$, then $G_X(Y)$ is stochastically independent of X, even though $G_x(\cdot)$ depends numerically on x. This fact may be used to prove a range of other results, including simpler proofs of limit theorems for linear combinations of concomitants.

2. Main results and summary. In Section 2.1 we present a result in joint limit theory for quantiles from bivariate sequences, proved by Babu and Rao (1988) under stronger assumptions. In particular, this result states that the correlation coefficient for the asymptotic distribution of quantiles from different marginals depends only on the bivariate distribution function, not on the density. The correlation is proportional to the signed measure of association for the 2×2 contingency table generated by the pair of marginal population quantiles. This result is needed for the proof of Theorem 2.3.

Sections 2.2 and 2.3 present respectively a strong law and a central limit theorem for the series S, defined in (1.2), when the function $\mathscr{a}$ is bounded. Those results take the form $n^{-1}S \to \mu$ with probability 1, and $n^{-1/2}(S - n\mu) \to N(0, \sigma^2)$ in distribution, for appropriate constants μ and σ^2. Remarks following those results discuss the regularity conditions imposed. The proofs of these results are given in Section 3. In each case it is possible to relax the assumption that $\mathscr{a}$ be bounded, at the expense of more stringent conditions on the "tails" of the distribution of (X, Y). However, a general account of such results involves cumbersome regularity conditions, and particularly lengthy proofs, and so is omitted.

2.1. *Asymptotic theory for bivariate quantiles.* Let (X_i, Y_i), $i \ge 1$, denote independent and identically distributed random vectors, with absolutely continuous distribution function. Write $X_{(1)} \le \cdots \le X_{(n)}$ and $Y_{(1)} \le \cdots \le Y_{(n)}$ for the ordered marginals of the first n pairs (X_i, Y_i). Let (X, Y) denote a generic (X_i, Y_i), and define $\pi_1(x, y) = P(X \le x,\ Y \le y)$, $\pi_2(x, y) = P(X \le x,\ Y > y)$, $\pi_3(x, y) = P(X > x,\ Y \le y)$, $\pi_4(x, y) = P(X > x,\ Y > y)$, representing the probabilities in the four quadrants defined by the point (x, y). Furthermore, let $F(x)$ and $G(y)$ [$f(x)$ and $g(y)$] denote the marginal cdf's (densities) of X and Y, respectively. For $0 < \alpha, \beta < 1$, let $\xi_\alpha = F^{-1}(\alpha)$, $\eta_\beta = G^{-1}(\beta)$ denote the quantiles of the two marginals, and let (r, s) be integers satisfying $r = \alpha n + o(n^{1/2})$, $s = \beta n + o(n^{1/2})$, and finally let $p_i = \pi_i(\xi_\alpha, \eta_\beta)$, $i = 1, \dots, 4$. The following result was proved in Babu and Rao (1988) by using Bahadur's representation of sample quantiles. However, a stronger assumption, in that the marginals F and G be continuously twice differentiable in neighborhoods of ξ_α and η_β, was required for their proof.

THEOREM 2.1. *Assume that both of the first partial derivatives of each function $\pi_i(x, y)$ exist in neighborhoods of (ξ_α, η_β) and are continuous at (ξ_α, η_β) and that $f(\xi_\alpha)$, $g(\eta_\beta)$ are nonzero. Then $\{n^{1/2}(X_{(r)} - \xi_\alpha), n^{1/2}(Y_{(s)} - \eta_\beta)\}$ has an asymptotic joint normal distribution with respective variances $\alpha(1-\alpha)f(\xi_\alpha)^{-2}$, $\beta(1-\beta)g(\eta_\beta)^{-2}$ and correlation coefficient*

$$\rho = (p_1 p_4 - p_2 p_3)\{\alpha(1-\alpha)\beta(1-\beta)\}^{-1/2}.$$

REMARK 2.1. The existence of $f(x)$, $g(y)$ for x, y in neighborhoods of ξ_α, η_β, respectively, follows from the existence of the first derivatives of the function π_i.

REMARK 2.2. The correlation coefficient ρ is proportional to the signed root of the "χ^2-statistic" for a 2×2 contingency table with cell numbers proportional to $p_1, \dots, p_4$.

REMARK 2.3. Babu and Rao (1988) state that the correlation ρ is equal to $(p_1 - \alpha\beta)\{\alpha(1-\alpha)\beta(1-\beta)\}^{-1/2}$. However, it is easy to check that the expression for ρ in Theorem 2.1 is equal to this one as the first factor simplifies to $(p_1 - \alpha\beta)$.

REMARK 2.4. Theorem 2.1 implies that $n^{1/2}(X_{(r)} - \xi_\alpha)$ and $n^{1/2}(Y_{(s)} - \eta_\beta)$ are asymptotically independent if and only if

$$P(X \le \xi_\alpha, Y \le \eta_\beta)P(X > \xi_\alpha, Y > \eta_\beta)$$
$$= P(X \le \xi_\alpha, Y > \eta_\beta)P(X > \xi_\alpha, Y \le \eta_\beta),$$

or equivalently that, for fixed α and β,

$$P(X \le \xi_\alpha, Y \le \eta_\beta) = P(X \le \xi_\alpha)P(Y \le \eta_\beta).$$

This is, of course, a weaker condition than the condition that X, Y be independent.

REMARK 2.5. Asymptotic normality for sample quantiles formed from the marginals of independent m-vectors may be deduced by an argument similar to that which produces Theorem 2.1. The covariance matrix in this more general case, given in Babu and Rao (1988), may be written down immediately from its counterpart for $m = 2$.

2.2. *Strong law for S.* Adopt the notation of Section 2.1 and assume that (X, Y) has a continuous distribution, with uniquely defined quantiles on both marginals. Let $G_x(y) = P(Y \le y | X = x)$, and let $\mathscr{a}: \mathbb{R} \to \mathbb{R}$ be a bounded function and let S be defined by (1.2). Assume that G_x^{-1} is continuous in x, in the sense that for each $\lambda > 0$,

$$\sup_{|y| \le \varepsilon} \sup_{0<u<1,\, |x| \le \lambda} \left| G_{x+y}^{-1}(u) - G_x^{-1}(u) \right| \to 0 \quad \text{as } \varepsilon \to 0. \tag{2.1}$$

In addition, suppose that:

(2.2) either a is continuous, or a has a finite number of jump discontinuities and the density of Y given $X = x$ is bounded uniformly in $|x| \le \lambda$, for each $\lambda > 0$.

Let U follow the uniform distribution on the interval (0, 1) and let

$$\mu = \int_0^1 E\left[a\left\{ G_{\xi_t}^{-1}(U) - \eta_t \right\} \right] dt. \tag{2.3}$$

THEOREM 2.2. *Under the above conditions,* $n^{-1}S \to \mu$ *with probability* 1.

REMARK 2.6. In cases of practical interest, condition (2.1) is a consequence of the fact that $D_x(u) = (\partial/\partial x)G_x^{-1}(u)$ is bounded uniformly in x and u. This condition is easily checked in many examples where X, Y have uniformly distributed marginals. It also holds when (X, Y) has a bivariate normal distribution, as follows immediately from the fact that Y given $X = x$ is normal $N(c_1 x, c_2)$ where c_1, c_2 are constants. Thus, $D_x(u) \equiv c_1$.

2.3. *Central limit theorem for* S. Using the notation introduced in Sections 2.1 and 2.2, we introduce the following regularity conditions on the function a and on the distribution of (X, Y):

(2.4) a, a' exist and are bounded and uniformly continuous on $(-\infty, \infty)$;

(2.5a) for both $H = F$ and $H = G$, $\min\{H(x), 1 - H(x)\} \le \text{const.}(1 + |x|)H'(x)$ and $\int \{\min(H, 1 - H)\}^{1/2} < \infty$, and

(2.5b) $h = H'$ is continuous on its support;

(2.6) both first partial derivatives of each function $\pi_i(x, y)$ exist and are continuous for $-\infty < x, y < \infty$;

(2.7) $\sup_{u,x} |D_x(u)| < \infty$ and D_x is uniformly continuous in x.

Again, let U denote a random variable with the uniform distribution on the interval (0, 1), and let μ be as defined in (2.3). Given $0 < t, u, t_1, t_2 < 1$, define

$$K(u, t) = G_{\xi_t}^{-1}(u) - \eta_t, \tag{2.8}$$

(2.9)

$$c_{1t} = f(\xi_t)^{-1} E\left[D_{\xi_t}(U) a'\{K(U, t)\} \right],$$

$$c_{2t} = g(\eta_t)^{-1} E\left[a'\{K(U, t)\} \right],$$

$$\omega_i(t_1, t_2) = \pi_i(\xi_{t_1}, \eta_{t_2}), \qquad i = 1, \dots, 4, \tag{2.10}$$

and

$$\sigma_1^2 = \int_0^1 \operatorname{var}[\mathscr{a}\{K(U,t)\}]\,dt. \tag{2.11}$$

Let ζ denote a normal $N(0, \sigma_1^2)$ random variable and let $(\zeta_1, \zeta_2) = \{(\zeta_{1t_1}, \zeta_{2t_2}),$ $0 < t_1, t_2 < 1\}$ denote a zero-mean bivariate Gaussian process, constructed to have a joint normal distribution with covariances

$$E(\zeta_{it_1}\zeta_{it_2}) = (t_1 \wedge t_2)\{1 - (t_1 \vee t_2)\}, \qquad i = 1, 2, \tag{2.12}$$

$$E(\zeta_{1t_1}\zeta_{2t_2}) = \frac{(\omega_1\omega_4 - \omega_2\omega_3)}{\{(\omega_1 + \omega_2)(\omega_3 + \omega_4)(\omega_1 + \omega_3)(\omega_2 + \omega_4)\}^{1/2}}, \tag{2.13}$$

where $\omega_i = \omega_i(t_1, t_2)$ are defined in (2.10). Furthermore, for all $0 < t, t_1, t_2 < 1$, let the covariances of ζ and (ζ_1, ζ_2) be defined by $E(\zeta\zeta_{1t}) = 0$ and

$$\begin{aligned} E(\zeta\zeta_{2t}) = -\int_0^1 \bigg\{ &\int_{-\infty}^{\eta_t} \mathscr{a}(y - \eta_u)\, dG_{\xi_u}(y) \\ &- G_{\xi_u}(\eta_t) \int_{-\infty}^{\infty} \mathscr{a}(y - \eta_u)\, dG_{\xi_u}(y) \bigg\}\, du. \end{aligned} \tag{2.14}$$

The existence of the joint distribution of ζ, ζ_1, ζ_2 is demonstrated in the proof of Theorem 2.3, where it is shown that these processes arise as weak limits of other processes. Now for (c_{1t}, c_{2t}) defined in (2.9), consider the zero-mean normal random variable defined by

$$\zeta + \int_0^1 (c_{1t}\zeta_{1t} - c_{2t}\zeta_{2t})\, dt. \tag{2.15}$$

Let σ^2 denote the variance of this random variable.

THEOREM 2.3. *Under conditions* (2.4)–(2.7), $n^{-1/2}(S - n\mu)$ *has an asymptotic normal* $N(0, \sigma^2)$ *distribution.*

REMARK 2.7. It may be proved by an argument similar to that used to derive Theorem 2.3 that, under the same conditions, $(S - ES)/\{\operatorname{Var}(S)\}^{1/2}$ is asymptotically normal $N(0, 1)$.

REMARK 2.8. Condition (2.5a) implies that $E(X^2) + E(Y^2) < \infty$. Conversely, the constraint $\int\{\min(H, 1 - H)\}^{1/2} < \infty$ holds for $H = F, G$ if for some $\varepsilon > 0$, $E(|X|^{2+\varepsilon}) + E(|Y|^{2+\varepsilon}) < \infty$. To appreciate why, observe that if Z has distribution function H and $\int\{\min(H, 1 - H)\}^{1/2} < \infty$, then $u^{1/2}|H^{-1}(u)| \to 0$ as $u \to 0$, so that

$$\begin{aligned} &\int_{-\infty}^{H^{-1}(1/2)} u^2\, dH(u) \\ &\quad = \int_0^{1/2} \{H^{-1}(u)\}^2\, du = \int_0^{1/2} \{u^{1/2}|H^{-1}(u)|\} u^{-1/2} |H^{-1}(u)|\, du \\ &\quad \le c_1 \int_0^{1/2} u^{-1/2} |H^{-1}(u)|\, du \le c_2 \int_{-\infty}^{H^{-1}(1/2)} \{H(u)\}^{1/2}\, du < \infty, \end{aligned}$$

and similarly $\int_{u > H^{-1}(1/2)} u^2\, dH(u) < \infty$. Furthermore, if $E(|Z|^{2+\varepsilon}) < \infty$, then it follows that $\min\{H(u), 1 - H(u)\} \le c_3(1 + |u|)^{-(2+\varepsilon)}$, whence $\int\{\min(H, 1 - H)\}^{1/2} < \infty$.

REMARK 2.9. The remaining part of condition (2.5a), that is, $\min\{H(x), 1 - H(x)\} \le c(1 + |x|)H'(x)$, holds for most continuous distributions—for example, for beta distributions (including the uniform), for the exponential, gamma, normal, lognormal and Student's t distributions and for more general distributions whose densities decrease like $|x|^{-\alpha}$ as $|x| \to \infty$, for $\alpha > 0$.

The reasonableness of condition (2.7) was noted in Remark 2.6.

3. Proofs of the main results. In this section we shall give a new proof of Theorem 2.1, under assumptions weaker than those in Babu and Rao (1988). We shall also prove the remaining two theorems described in Section 2 as well as state and prove two results concerning the limiting values of the linear combinations of differences of sample and population quantiles, which are needed in the proof of Theorem 2.3.

3.1. *Proof of Theorem* 2.1. Define $A(i, j, k) = n!\{k!(i - k)!(j - k)!(n - i - j + k)!\}^{-1}$, which is to be interpreted as 0 if any one of k, $i - k$, $j - k$, $n - i - j + k$ is negative. Fix real numbers u and v and put

$$x = \xi_\alpha + n^{-1/2} f(\xi_\alpha)^{-1} u, \qquad y = \eta_\beta + n^{-1/2} g(\eta_\beta)^{-1} v,$$

$$U = n^{-1/2}(X_{(r)} - \xi_\alpha) f(\xi_\alpha), \qquad V = n^{-1/2}(Y_{(s)} - \eta_\beta) g(\eta_\beta).$$

Let π_i denote $\pi_i(x, y)$ evaluated at these particular values of x, y, and set $\delta_i = n^{1/2}(\pi_i - p_i)$. Define t_1, t_2, t_3 by $i = n(p_1 + p_2) + n^{1/2} t_1$, $j = n(p_1 + p_3) + n^{1/2} t_2$, $k = np_1 + n^{1/2} t_3$. Then

$$P(U \le u, V \le v) = \sum_{i=r}^{n} \sum_{j=s}^{n} \sum_{k=0}^{n} A(i, j, k) \pi_1^k \pi_2^{i-k} \pi_3^{j-k} \pi_4^{n-i-j+k}.$$

Now for i, j, k chosen such that t_1, t_2, t_3 are bounded and using Stirling's formula, we get

$$A(i, j, k) \pi_1^k \pi_2^{i-k} \pi_3^{j-k} \pi_4^{n-i-j+k}$$
$$= (2\pi n)^{-3/2} (p_1 p_2 p_3 p_4)^{-1/2} \exp\{-(1/2)(Pt_3^2 - 2Qt_3 + R) + o(1)\},$$

where

$$P = \sum p_i^{-1},$$

$$Q = p_1^{-1}\delta_1 + p_2^{-1}(t_1 - \delta_2) + p_3^{-1}(t_2 - \delta_3) + p_4^{-1}(t_1 + t_2 + \delta_4),$$

$$R = p_i^{-1}\delta_1^2 + p_2^{-1}(t_1 - \delta_2)^2 + p_3^{-1}(t_2 - \delta_3)^2 + p_4^{-1}(t_1 + t_2 + \delta_4)^2.$$

Thus,

$$\begin{aligned}P(U \le u, V \le v) &\sim \int_{t_1 \ge 0,\, t_2 \ge 0,\, -\infty < t_3 < \infty} (2\pi)^{-3/2}(p_1p_2p_3p_4)^{-1/2} \\ &\quad \times \exp\{-(1/2)(Pt_3^2 - 2Qt_3 + R)\}\, dt_1\, dt_2\, dt_3 \\ &= (2\pi)^{-1}(p_1p_2p_3p_4)^{-1/2}P^{-1/2} \\ &\quad \times \int_{t_1 \ge 0,\, t_2 \ge 0} \exp\{-(1/2)(R - P^{-1}Q)^2\}\, dt_1\, dt_2 \\ &= I \quad \text{(say)}.\end{aligned}$$

Some of the steps above, which require tedious algebra, have been omitted. To simplify the formula for I, change the variables from (t_1, t_2) to (w, z) in the last integral, where $w = \delta_1 + \delta_2 - t_1$, $z = \delta_1 + \delta_3 - t_2$. This gives

$$\begin{aligned}I &= (2\pi)^{-1}(p_1p_2p_3p_4)^{-1/2}P^{-1/2} \\ &\quad \times \int_{w \le \delta_1 + \delta_2,\, z \le \delta_1 + \delta_3} \exp\{-(1/2)P^{-1}(A_1w^2 + A_2z^2 + 2A_3wz)\}\, dw\, dz,\end{aligned}$$

where $A_1 = (p_1^{-1} + p_3^{-1})(p_2^{-1} + p_4^{-1})$, $A_2 = (p_1^{-1} + p_2^{-1})(p_3^{-1} + p_4^{-1})$, $A_3 = p_1^{-1}p_4^{-1} - p_2^{-1}p_3^{-1}$. It follows from the definition of the δ_i's that $\delta_1 + \delta_2 \to u$ and $\delta_1 + \delta_3 \to v$ as $n \to \infty$. Therefore, the limit of I equals the integral of a nonnegative function of (w, z) over $w \le u, z \le v$. This function must be the density of the asymptotic distribution of (U, V), whence follows the theorem. □

3.2. *Proof of Theorem* 2.2. The key ingredient of our proofs of Theorem 2.2 and 2.3 is the observation that we may write $Y_{[i]} = G_{X_{(i)}}^{-1}(U_i)$, $1 \le i \le n$, where $U_1, \dots, U_n$ are independent, have the uniform distribution on the interval $(0, 1)$ and are independent of $X_1, \dots, X_n$. Since for each $0 < \varepsilon < \frac{1}{2}$,

$$\sup_{\varepsilon n \le i \le (1-\varepsilon)n} \left(|X_{(i)} - \xi_{i/n}| + |Y_{(i)} - \eta_{i/n}|\right) \to 0,$$

with probability 1, then, in view of (2.1),

$$\sup_{\varepsilon n \le i \le (1-\varepsilon)n} \left| Y_{[i]} - Y_{(i)} - \left\{K\left(U_i, \frac{i}{n}\right)\right\}\right| \to 0,$$

where $K(u, t)$ is defined by (2.8). It now follows from (2.2) that with probability 1,

$$\sum_{i=\varepsilon n}^{(1-\varepsilon)n} \mathscr{a}(Y_{[i]} - Y_{(i)}) = \sum_{i=\varepsilon n}^{(1-\varepsilon)n} \mathscr{a}\left\{K\left(U_i, \frac{i}{n}\right)\right\} + o(n). \tag{3.1}$$

Write S_ε for the series on the right-hand side. Since the U_i's are independent and $\mathscr{a}$ is bounded, it follows that $E[(S_\varepsilon - ES_\varepsilon)^4] = O(n^2)$. Therefore, by Markov's inequality, $P(|S_\varepsilon - ES_\varepsilon| > \delta n) = O(n^{-2})$ for each $\delta > 0$. The

Borel–Cantelli lemma now implies that $n^{-1}(S_\varepsilon - ES_\varepsilon) \to 0$ with probability 1. The theorem follows from this result, (3.1) and the fact that

$$\limsup_{n\to\infty} |n^{-1}E(S_\varepsilon) - \mu| \to 0 \quad \text{as } \varepsilon \to 0. \qquad \square$$

3.3. *Proof of Theorem* 2.3. The proof of the theorem requires the following two lemmas, whose proofs are deferred to the end of the section.

LEMMA 3.1. *Let $Z_1, Z_2, \dots$ be independent and identically distributed random variables with absolutely continuous distribution function H, satisfying* (2.5a). *Let $\{M_{ni},\ 1 \le i \le n < \infty\}$ denote nonnegative random variables with the properties $\sup_{n,i} M_{ni} \le C_2 < \infty$, and for each $0 < \varepsilon < \frac{1}{2}$, $\max_{\varepsilon n \le i \le (1-\varepsilon)n} M_{ni} \to 0$ in probability. Write $Z_{(i)}$ for the ith largest of $Z_1, \dots, Z_n$ and put $\zeta_{i/n} = H^{-1}(i/n)$ for $1 \le i \le n-1$, $\zeta_{n/n} = \zeta_{(n-1)/n}$. Then, as $n \to \infty$,*

$$n^{-1/2} \sum_{i=1}^{n} |Z_{(i)} - \zeta_{i/n}| M_{ni} \to 0 \quad \textit{in probability}.$$

LEMMA 3.2. *Let $Z_1, Z_2, \dots$ be independent and identically distributed random variables with absolutely continuous distribution function H and density h, satisfying* (2.5a) *and* (2.5b). *Using Rényi's representation* [*David* (1981), *page* 21], *write*

$$Z_{(i)} = H^{-1}\left\{\exp\left(-\sum_{j=i}^{n} j^{-1} e_j\right)\right\},$$

where $e_1, \dots, e_n$ are independent exponential random variables. Let B_{ni}, $1 \le i \le n < \infty$, be random variables satisfying $\sup_{n,i}|B_{ni}| \le C_2 < \infty$. Then

$$\sum_{i=1}^{n} (Z_{(i)} - \zeta_{i/n}) B_{ni} = -\sum_{j=1}^{n} (e_j - 1)(jn)^{-1} \sum_{i=1}^{j} i\big(h(\zeta_{i/n})\big)^{-1} B_{ni} + o_p(n^{1/2}).$$

REMARK 3.1. Our proof of Lemma 3.2 involves truncation arguments, which also yield the following result. Given $0 < \varepsilon < \frac{1}{2}$, write $\sum_i^{(\varepsilon, j)}$ for summation over i such that $\varepsilon n \le i \le \min\{j, (1-\varepsilon)n\}$. (The sum is null if $j < \varepsilon n$.) Put

$$R_\varepsilon = -\sum_{j=1}^{n} (e_j - 1)(jn)^{-1} \sum_{i}^{(\varepsilon, j)} i\big(h(\zeta_{i/n})\big)^{-1} B_{ni}.$$

Then, under the conditions of Lemma 3.2 and for each $\delta > 0$,

$$\lim_{\varepsilon\to 0} \limsup_{n\to\infty} P\left\{\left|\sum_{i=1}^{n} (Z_{(i)} - \zeta_{i/n}) B_{ni} - R_\varepsilon\right| > n^{1/2}\delta\right\} = 0.$$

To begin the proof of Theorem 2.3, let $U_1, \dots, U_n$ be as defined in Section 3.2. With a slight abuse of notation, define $\xi_{n/n} = \xi_{(n-1)/n}$ and $\eta_{n/n} = \eta_{(n-1)/n}$. In view of (2.7) and (2.8), we can write

$$Y_{[i]} - Y_{(i)} = K\left(U_i, \frac{i}{n}\right) + (X_{(i)} - \xi_{i/n})D_{\xi_{i/n}}(U_i) - (Y_{(i)} - \eta_{i/n}) + (X_{(i)} - \xi_{i/n})s_{ni},$$

where $\sup_{n,i}|s_{ni}| \le \sup_{u,x}|D_x(u)| < \infty$, and for each $0 < \varepsilon < \frac{1}{2}$,

$$\sup_{\varepsilon n \le i \le (r\varepsilon)n} |s_{ni}| \to 0 \quad \text{in probability.}$$

Therefore, by (2.4) and a Taylor series expansion, we can write

$$\begin{aligned} a(Y_{[i]} - Y_{(i)}) = a\left\{K\left(U_i, \frac{i}{n}\right)\right\} &+ \left\{(X_{(i)} - \xi_{i/n})D_{\xi_{i/n}}(U_i) - (Y_{(i)} - \eta_{i/n})\right\} a'\left\{K\left(U_i, \frac{i}{n}\right)\right\} \\ &+ \left(|X_{(r)} - \xi_{i/n}| + |Y_{(i)} - \eta_{i/n}|\right)t_{ni}, \end{aligned}$$

where

$$\sup_{n,i}|t_{ni}| \le C < \infty, \qquad \max_{\varepsilon n \le i \le (ri)n} |t_{ni}| \to 0.$$

However, it follows from Lemma 3.1 that

$$n^{-1/2} \sum_{i=1}^{n} \left(|X_{(i)} - \xi_{i/n}| + |Y_{(i)} - \eta_{i/n}|\right)|t_{ni}| \to 0,$$

whence

$$S = \sum_{i=1}^{n} a(Y_{[i]} - Y_{(i)}) = S_0 + S_1 - S_2 + o_p(n^{1/2}), \tag{3.2}$$

where

$$\begin{aligned} S_0 &= \sum_{i=1}^{n} a\left\{K\left(U_i, \frac{i}{n}\right)\right\}, \\ S_1 &= \sum_{i=1}^{n} (X_{(i)} - \xi_{i/n})D_{\xi_{i/n}}(U_i) a'\left\{K\left(U_i, \frac{i}{n}\right)\right\}, \\ S_2 &= \sum_{i=1}^{n} (Y_{(i)} - \eta_{i/n}) a'\left\{K\left(U_i, \frac{i}{n}\right)\right\}. \end{aligned} \tag{3.3}$$

Both of the series S_1, S_2 may be expressed in the form

$$S_l = \sum_{i=1}^{n} (Z_{(i)} - \zeta_{i/n})b_{ni}(U_i),$$

where $Z_{(i)} - \zeta_{i/n} = X_{(i)} - \xi_{i/n}$ when $l = 1$, and equals $Y_{(i)} - \eta_{i/n}$ when $l = 2$; b_{ni} is a bounded function. We claim that the weights $b_{ni}(U_i)$ can in effect be replaced by their expected values, that is,

$$S_l = S_l^* + o_p(n^{1/2}), \tag{3.4}$$

where

$$S_l^* = \sum_{i=1}^{n} (Z_{(i)} - \zeta_{i/n}) E\{b_{ni}(U_i)\}, \qquad l = 1, 2. \tag{3.4a}$$

Here, Lemma 3.2 forms the first step in a proof of (3.4). In the case $l = 1$ the U_i's are independent of the e_i's, and therefore

$$E\left[\sum_{j=1}^{n} (e_j - 1)(jn)^{-1} \sum_{i=1}^{j} i\big(h(\zeta_{i/n})\big)^{-1}\{b_{ni}(U_i) - Eb_{ni}(U_i)\}\right]^2$$
$$= \sum_{j=1}^{n} (jn)^{-2} \sum_{i=1}^{j} i^2\big(h(\zeta_{i/n})\big)^{-2} E\{b_{ni}(U_i) - Eb_{ni}(U_i)\}^2 \tag{3.5}$$
$$= O\left\{\sum_{j=1}^{n} (jn)^{-2} \sum_{i=1}^{j} i^2\big(h(\zeta_{i/n})\big)^{-2}\right\} = o(n).$$

Hence,

$$\sum_{j=1}^{n} (e_j - 1)(jn)^{-1} \sum_{i=1}^{j} ih(\zeta_{i/n})^{-1} b_{ni}(U_i)$$
$$= \sum_{j=1}^{n} (e_j - 1)(jn)^{-1} \sum_{i=1}^{j} ih(\zeta_{i/n})^{-1} E\{b_{ni}(U_i)\} + o_p(n^{1/2}).$$

From this result and two applications of Lemma 3.2, the first with $B_{ni} = b_{ni}(U_i)$ and the second with $B_{ni} = E\{b_{ni}(U_i)\}$, and both having $Z_{(i)} - \zeta_{i/n} = X_{(i)} - \xi_{i/n}$, we deduce (3.4) for $l = 1$. The case $l = 2$ is similar, except that a more complicated argument is necessary in place of (3.5), with the expected value of the square of the sum over j being written as a double sum over j_1 and j_2, and expectation being taken term-by-term in the latter.

We may now write (3.2) in the form

$$S = S_0 + S_1^* - S_2^* + o_p(n^{1/2}), \tag{3.6}$$

where S_0 is as defined in (3.3) and

$$S_l^* = n^{-1} \sum_{i=1}^{n} \hat{\zeta}_{l(i/n)} c_{l(i/n)}, \qquad l = 1, 2, \tag{3.6a}$$

where $\hat{\zeta}_{1(i/n)} = n^{1/2}(X_{(i)} - \xi_{i/n}) f(\xi_{i/n})$, $\hat{\zeta}_{2(i/n)} = n^{1/2}(Y_{(i)} - \eta_{i/n}) g(\eta_{i/n})$ and (c_{1t}, c_{2t}), $0 < t < 1$, are defined by (2.9). The stochastic process $\hat{\zeta}_{l(i/n)}$ is readily extended by interpolation to the continuous-time process $\hat{\zeta}_{lt}$, $0 \le t \le 1$.

Define $\hat{\zeta} = n^{-1/2}(S_0 - ES_0)$. We claim that the process $\hat{\zeta} \times (\hat{\zeta}_1, \hat{\zeta}_2)$ converges weakly to the Gaussian process $\zeta \times (\zeta_1, \zeta_2)$, the latter defined by (2.12) to (2.14), just prior to the statement of Theorem 2.3.

Tightness of $(\hat{\zeta}_1, \hat{\zeta}_2)$ on $C(0,1)^2$ is easily proved. Therefore, to establish the claim, we need only check convergence of finite-dimensional distributions. That is, we must prove that for $0 < t_1, \dots, t_M, s_1, \dots, s_N < 1$,

$$(3.7) \quad \left(\hat{\zeta}, \hat{\zeta}_{1t_1}, \dots, \hat{\zeta}_{1t_M}, \hat{\zeta}_{2s_1}, \dots, \hat{\zeta}_{2s_N}\right) \to_d \left(\zeta, \zeta_{1t_1}, \dots, \zeta_{1t_M}, \zeta_{2s_1}, \dots, \zeta_{2s_N}\right).$$

We present the proof only in outline, which gives explicitly the derivation of the convergence for each *pair* of $\hat{\zeta}$'s and sketches the derivation of (3.7) in full generality.

For $i = 1, 2$ and any fixed L, the convergence

$$(3.8) \qquad \left(\hat{\zeta}_{it_1}, \dots, \hat{\zeta}_{it_L}\right) \to_d \left(\zeta_{it_1}, \dots, \zeta_{it_L}\right)$$

follows from the well-known central limit theorem for order statistics from the same marginal [see, e.g., David (1981), page 255]. Convergence of $(\hat{\zeta}_{1t_1}, \hat{\zeta}_{2t_2})$ to $(\zeta_{1t_1}, \zeta_{2t_2})$ follows from Theorem 2.1. It is straightforward to combine the methods used there with the methods employed to establish (3.8), to show that

$$(3.9) \quad \left(\hat{\zeta}_{1t_1}, \dots, \hat{\zeta}_{1t_M}, \hat{\zeta}_{2s_1}, \dots, \hat{\zeta}_{2s_N}\right) \to_d \left(\zeta_{1t_1}, \dots, \zeta_{1t_M}, \zeta_{2s_1}, \dots, \zeta_{2s_N}\right).$$

We shall conclude by outlining a proof that

$$(3.10) \qquad \left(\hat{\zeta}, \hat{\zeta}_{2t}\right) \to_d \left(\zeta, \zeta_{2t}\right).$$

Those methods, together with the techniques leading to (3.10) and the observation that $\hat{\zeta}$ is stochastically independent of $\hat{\zeta}_{1t}$ (since the U_i's and X_i's are totally independent) allows us to establish (3.7).

To prove (3.10), let r be an integer with the property $r = nt + o(n^{1/2})$ as $n \to \infty$ and note that

$$(3.11) \quad \begin{aligned} Y_{(r)} - \eta_{r/n} &= (U'_{(r)} - t) g(\eta_t)^{-1} + o_p(n^{-1}) \\ &= \left\{ t - n^{-1} \sum_{i=1}^{n} I(Y_i \le \eta_t) \right\} g(\eta_t)^{-1} + o_p(n^{-1/2}). \end{aligned}$$

Define $\mu_i = E[\alpha\{K(U_i, i/n)\}]$ and observe that, for all x,

$$E\left[\alpha\left\{ K\left(U_i, \frac{i}{n}\right)\right\} \middle| X_{(i)} = x \right] = \mu_i.$$

Therefore, conditional on $X_1, \dots, X_n$, the variable

$$S_0^* = \sum_i \left[\alpha\{K(U_i, i/n)\} - \mu_i \right]$$

has zero mean. Put $Y_i' = Y_{[i]}$,

$$Q_1 = \sum_{i=1}^{n} \left\{ I(Y_i' \le \eta_t) - G_{X_{(i)}}(\eta_t) \right\} \quad \text{and} \quad Q_2 = \sum_{i=1}^{n} G_{X_{(i)}}(\eta_t) - nt.$$

In view of (3.11),

$$Y_{(r)} - \eta_{r/n} = -n^{-1}(Q_1 + Q_2)g(\eta_t)^{-1} + o_p(n^{-1/2}).$$

Let $-\infty < x_1 < \cdots < x_n < \infty$ and condition on $X_1, \dots, X_n$ and on $X_{(i)} = x_i$, $1 \le i \le n$; write $|\mathbf{X}$ to indicate this conditioning. Note that, conditional on $\mathbf{X}$, $Y_1', \dots, Y_n'$ are independent, and Y_i' has distribution function G_{x_i}. Therefore, conditionally on $\mathbf{X}$, S_0^* and Q_1 are both sums of independent random variables. Likewise, $Q_2 = \Sigma G_{X_{(i)}}(\eta_t) - nt = \Sigma\{G_{x_i}(\eta_t) - t\}$ is a sum of independent random variables. Arguing thus, we may prove that, unconditionally, $(n^{-1/2}S_0^*, n^{-1/2}(Q_1 + Q_2))$ has an asymptotic normal distribution with covariance given by the limit of

$$\begin{aligned} &n^{-1}E\{S_0^*(Q_1 + Q_2)\} \\ &\quad = n^{-1}E[E\{S_0^*(Q_1 + Q_2)|\mathbf{X}\}] = n^{-1}E\{E(S_0^* Q_1|\mathbf{X})\} \\ &\quad = n^{-1}E\left\{\sum_{i=1}^{n}\left(\int_{-\infty}^{\eta_t} a\left[G_{\xi_{i/n}}^{-1}\{G_{x_i}(y)\} - \eta_{i/n}\right] dG_{x_i}(y) - \mu_i G_{x_i}(\eta_t)\right)\right\} \\ &\quad \to \int_0^1\left[\int_{-\infty}^{\eta_t} a(y - \eta_u)\, dG_{\xi_u}(y) - \left\{\int_{-\infty}^{\infty} a(y - \eta_u)\, dG_{\xi_u}(y)\right\}G_{\xi_u}(\eta_t)\right] du \\ &\quad = \gamma, \quad \text{say.} \end{aligned}$$

Since $S_0^* = S_0 - n\mu + o(n^{1/2})$, it follows that $\hat{\zeta} = -n^{-1/2}S_0^* + o(1)$; and since $Y_{(r)} - \eta_{r/n} = -n^{-1}(Q_1 + Q_2)g(\eta_t)^{-1} + o_p(n^{-1/2})$, it follows that $\hat{\zeta}_{2t} = -n^{-1/2}(Q_1 + Q_2) + o_p(1)$. Therefore, $(\hat{\zeta}, \hat{\zeta}_{2t})$ has an asymptotic normal distribution with covariance $-\gamma$; the variances are those of the limits of the marginals $\hat{\zeta}$ and $\hat{\zeta}_{2t}$, respectively, and equal σ_1^2 and $t(1 - t)$.

The convergence in distribution of

$$n^{-1/2}(S - n\mu) = \hat{\zeta} + n^{-1}\sum_{i=1}^{n}\left(\hat{\zeta}_{1(i/n)}c_{1(i/n)} - \hat{\zeta}_{2(i/n)}c_{2(i/n)}\right) + o_p(1)$$

[see (3.6a)] to

$$\zeta + \int_0^1 (\zeta_{1t}c_{1t} - \zeta_{2t}c_{2t})\, dt$$

follows from the weak convergence of $\hat{\zeta} \times (\hat{\zeta}_1, \hat{\zeta}_2)$ to $\zeta \times (\zeta_1, \zeta_2)$, after a little classical analysis of the type used in the proof of Lemma 3.1. □

We conclude the section with the proofs of Lemmas 3.1 and 3.2.

PROOF OF LEMMA 3.1. Let C_j, $j \ge 3$, denote positive constants not depending on n. We write $W_j = W_j(n)$ for nonnegative random variables satisfying

$$\lim_{\lambda \to \infty} \limsup_{n \to \infty} P(W_j > \lambda) = 0, \qquad j \ge 1. \tag{3.12}$$

Put $U_{(i)}' = H(Z_{(i)})$, representing the ith largest order statistic of an n-sample from the uniform distribution on $(0, 1)$. It follows from the results in David

[(1981), page 36], that

$$\max_{1\le i\le n} E\left(U'_{(i)} - \frac{i}{n}\right)^4 \le C_3 n^{-4}\{\min(i, n-i+1)\}^2. \tag{3.13}$$

Put $h = H'$, let $0 \le \theta_i \le 1$ and define $a_i = H^{-1}\{i/n + \theta_i(U'_{(i)} - i/n)\}$. Then, for an appropriate choice of θ_i and $1 \le i \le n-1$,

$$\begin{aligned} |Z_{(i)} - \zeta_{i/n}| &= \left|H^{-1}\left(\frac{i}{n} + U'_{(i)} - \frac{i}{n}\right) - H^{-1}\left(\frac{i}{n}\right)\right| = \left|U'_{(i)} - \frac{i}{n}\right| h\{H^{-1}(a_i)\}^{-1} \\ &\le C_1\left|U'_{(i)} - \frac{i}{n}\right|\{1 + |H^{-1}(a_i)|\}\min(a_i, 1-a_i). \end{aligned}$$

(The case $i = n$ is similar.) For large values of $|x|$, the function $\psi(x) = (1+|x|)[\min\{H(x), 1-H(x)\}]^{-1}$ is increasing for $x > 0$, and decreasing for $x < 0$. Also, ψ is bounded away from 0. It follows that for some $C_4 \ge 1$, $\psi(x_2) \le C_4\{\psi(x_1) + \psi(x_3)\}$ whenever $x_1 \le x_2 \le x_3$. Therefore,

$$\begin{aligned} \psi\{H^{-1}(a_i)\} &\le C_4\left[\psi\{H^{-1}(i/n)\} + \psi\{H^{-1}(U'_{(i)})\}\right] = C_4\{\psi(\zeta_{i/n}) + \psi(Z_{(i)})\} \\ &\le C_5\left(1 + |\zeta_{i/n}| + |Z_{(i)}|\right) \\ &\quad \times\left[\left\{\min\left(\frac{i}{n}, 1 - \frac{i-1}{n}\right)\right\}^{-1} + \left\{\min(U'_{(i)}, 1 - U'_{(i)})\right\}^{-1}\right] \\ &\le W_1\left(1 + |\zeta_{i/n}| + |Z_{(i)}|\right) n \max\left\{i^{-1}, (n-i+1)^{-1}\right\}. \end{aligned}$$

Hence, for arbitrary fixed $m \ge 1$,

$$\begin{aligned} &\sum_{i=m}^{n/2} |Z_{(i)} - \zeta_{i/n}| M_{ni} \\ &\quad \le C_1 W_1 \sum_{i=m}^{n/2}\left|U'_{(i)} - \frac{i}{n}\right| n i^{-1}\left(1 + |\zeta_{i/n}| + |Z_{(i)}|\right) M_{ni}. \end{aligned} \tag{3.14}$$

Define Δ_m as follows:

$$\Delta_m = \max_{m\le i\le n/2}\left|U'_{(i)} - \frac{i}{n}\right| n i^{-1}.$$

In view of (3.13),

$$\begin{aligned} P(\Delta_m > \varepsilon) &\le \sum_{i=m}^{n/2} P\left(\left|U'_{(i)} - \frac{i}{n}\right| n i^{-1} > \varepsilon\right) \\ &\le \sum_{i=m}^{n/2}\left(\frac{i}{n}\varepsilon\right)^{-4} C_3 n^{-4} i^2 = C_3 \varepsilon^{-4}\sum_{i=m}^{\infty} i^{-2}. \end{aligned} \tag{3.15}$$

Given $\delta > 0$, choose $C_6 > 0$ so large that $P(W_1 > C_1^{-1}C_6) \le \delta/2$ for all n.

By (3.15), we may choose m so large that $P(\Delta_m > \frac{1}{2}C_6^{-1}) \leq \delta/2$ for all n. Noting that (3.14) implies that

$$\sum_{i=m}^{n/2} |Z_{(i)} - \zeta_{i/n}| M_{ni}$$
$$\leq C_1 W_1 \sum_{i=m}^{n/2} \left| U'_{(i)} - \frac{i}{n} \right| ni^{-1}\left(1 + 2|\zeta_{i/n}| + |Z_{(i)} - \zeta_{i/n}|\right) M_{ni},$$

we see that with probability at least $1 - \delta$,

$$\sum_{i=m}^{n/2} |Z_{(i)} - \zeta_{i/n}| M_{ni} \leq C_1 W_1 \sum_{i=m}^{n/2} \left| U'_{(i)} - \frac{i}{n} \right| ni^{-1}\left(1 + 2|\zeta_{i/n}|\right) M_{ni}$$
$$+ C_1 \cdot C_1^{-1} C_6 \cdot \tfrac{1}{2} C_6^{-1} \sum_{i=m}^{n/2} |Z_{(i)} - \zeta_{i/n}| M_{ni}.$$

Hence,

$$\sum_{i=m}^{n/2} |Z_{(i)} - \zeta_{i/n}| M_{ni} \leq 2C_1 W_1 \sum_{i=m}^{n/2} \left| U'_{(i)} - \frac{i}{n} \right| ni^{-1}\left(1 + 2|\zeta_{i/n}|\right) M_{ni}. \tag{3.16}$$

Now, using (3.13),

$$E\left| U'_{(i)} - \frac{i}{n} \right| ni^{-1} \leq \left\{ E\left(U'_{(i)} - \frac{i}{n} \right)^4 \right\}^{1/4} ni^{-1} \leq C_3^{1/4} i^{-1/2}.$$

It follows that for each $0 < \varepsilon < \frac{1}{2}$,

$$n^{-1/2} \sum_{i=\varepsilon n+1}^{n/2} \left| U'_{(i)} - \frac{i}{n} \right| ni^{-1}\left(1 + 2|\zeta_{i/n}|\right) M_{ni} \to_p 0$$

and

$$E\left\{ n^{-1/2} \sum_{i=m}^{\varepsilon n} \left| U'_{(i)} - \frac{i}{n} \right| ni^{-1}\left(1 + 2|\zeta_{i/n}|\right) M_{ni} \right\}$$
$$\leq C_2 C_3^{1/4} n^{-1} \sum_{i=m}^{\varepsilon n} (n/i)^{1/2} \{1 + 2|H^{-1}(i/n)|\}$$
$$\leq C_7 \int_0^{\varepsilon} u^{-1/2} |H^{-1}(u)|\, du = 2C_7 \int_0^{\varepsilon} |H^{-1}(u)|\, d(u^{1/2})$$
$$= 2C_7 \left\{ \varepsilon^{1/2} |H^{-1}(\varepsilon)| + \int_{-\infty}^{H^{-1}(\varepsilon)} H(x)^{1/2}\, dx \right\} \to 0$$

as $\varepsilon \to 0$. Combining the results from (3.16) down, we conclude that, for each $\delta' > 0$,

$$\limsup_{n \to \infty} P\left(n^{-1/2} \sum_{i=m}^{n/2} |Z_{(i)} - \zeta_{i/n}| M_{ni} > \delta' \right) \leq \delta. \tag{3.17}$$

The condition $\int\{\min(H, 1 - H)\}^{1/2} < \infty$ implies that $E(Z^2) < \infty$, whence it follows that

$$n^{-1/2}\Big\{\max_{1\le i\le n}|Z_i| + \max_{1\le i\le n-1}|H^{-1}(i/n)|\Big\} \to_p 0. \tag{3.18}$$

Therefore, for each fixed m,

$$n^{-1/2}\sum_{i=1}^{m}|Z_{(i)} - \zeta_{i/n}|M_{ni} \to_p 0,$$

and so by (3.17),

$$n^{-1/2}\sum_{i=1}^{m/2}|Z_{(i)} - \zeta_{i/n}|M_{ni} \to_p 0. \tag{3.19}$$

Similarly, it may be proven that

$$\sum_{i=n/2+1}^{n}|Z_{(i)} - \zeta_{i/n}|M_{ni} \to_p 0,$$

which together with (3.19) yields the desired result. □

PROOF OF LEMMA 3.2. As in the proof of Lemma 3.1, define $U'_{(i)} = H(Z_{(i)})$ and $a_i = H^{-1}\{n^{-1}i + \theta_i(U'_{(i)} - n^{-1}i)\}$, where $0 \le \theta_i \le 1$. For an appropriate choice of θ_i,

$$\sum_{i=1}^{n}|Z_{(i)} - \zeta_{i/n}|B_{ni} = \sum_{i=1}^{n}\left(U'_{(i)} - \frac{i}{n}\right)h\{H^{-1}(a_i)\}^{-1}B_{ni} = T_1 + T_2, \tag{3.20}$$

where

$$T_1 = \sum_{i=1}^{n}\left(U'_{(i)} - \frac{i}{n}\right)h(\zeta_{i/n})^{-1}B_{ni},$$

$$T_2 = \sum_{i=1}^{n}\left(U'_{(i)} - \frac{i}{n}\right)\left[h\{H^{-1}(a_i)\}^{-1} - h(\zeta_{i/n})^{-1}\right]B_{ni}.$$

The first step is to prove that $T_2 = o_p(n^{1/2})$, of which the first part involves showing that the tails of the series defining T_2 are negligibly small. With C_1, C_4 and ψ as in the proof of Lemma 3.1, we have

$$\begin{aligned} h\{H^{-1}(a_i)\}^{-1} + h(\zeta_{i/n})^{-1} &\le C_1\left[\psi\{H^{-1}(a_i)\} + \psi(\zeta_{i/n})\right] \\ &\le 2C_1C_4\{\psi(Z_{(i)}) + \psi(\zeta_{i/n})\} \\ &\le W_2(1 + |\zeta_{i/n}| + |Z_{(i)}|)n\max\{i^{-1}, (n-i+1)^{-1}\}, \end{aligned}$$

where W_2 satisfies (3.12). Thus,

$$\sum_{i=1}^{\varepsilon n}\left|U'_{(i)}-\frac{i}{n}\right|\left[h\{H^{-1}(a_i)\}^{-1}+h(\zeta_{i/n})^{-1}\right]$$
$$\leq W_2\sum_{i=1}^{\varepsilon n}\left|U'_{(i)}-\frac{i}{n}\right|ni^{-1}\left(1+2|\zeta_{i/n}|+|Z_{(i)}-\zeta_{i/n}|\right).$$

The argument in the proof of Lemma 3.1 following (3.15) may now be used to show that, for any $\delta>0$,

$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty}P\left\{\sum_{i=1}^{\varepsilon n}\left|U'_{(i)}-\frac{i}{n}\right|\times ni^{-1}\left(1+2|\zeta_{i/n}|+|Z_{(i)}-\zeta_{i/n}|\right)>\delta n^{1/2}\right\}=0. \tag{3.21}$$

Therefore,

$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty}P\left[\sum_{i=1}^{\varepsilon n}\left|U'_{(i)}-\frac{i}{n}\right|\left|h\{H^{-1}(a_i)\}^{-1}-h(\zeta_{i/n})^{-1}\right|>\delta n^{1/2}\right]=0.$$

Similarly,

$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty}P\left[\sum_{i=(1-\varepsilon)n}^{n}\left|U'_{(i)}-\frac{i}{n}\right|\times\left|h\{H^{-1}(a_i)\}^{-1}-h(\zeta_{i/n})^{-1}\right|>\delta n^{1/2}\right]=0. \tag{3.22}$$

In view of the continuity of h, we have for each $\varepsilon>0$,

$$\max_{\varepsilon n\leq i\leq(1-\varepsilon)n}|h\{H^{-1}(a_i)\}^{-1}-h(\zeta_{i/n})^{-1}|\to_p 0.$$

Furthermore,

$$E\left(\sum_{i=1}^{n}\left|U'_{(i)}-\frac{i}{n}\right|\right)=O(n^{1/2}),$$

and therefore

$$\sum_{i=\varepsilon n}^{(1-\varepsilon)n}\left|U'_{(i)}-\frac{i}{n}\right|\left|h\{H^{-1}(a_i)\}^{-1}-h(\zeta_{i/n})^{-1}\right|=o_p(n^{1/2}). \tag{3.23}$$

Combining (3.21)–(3.23), we deduce that $T_2=o_p(n^{1/2})$.

Now, put $C_{ni} = \exp(-\sum_{j=i}^{n} j^{-1})$ and observe that

$$U'_{(i)} = C_{ni} \exp\left\{ -\sum_{j=i}^{n} j^{-1}(e_j - 1) \right\}$$
$$= C_{ni}\left\{ 1 - \sum_{j=i}^{n} j^{-1}(e_j - 1) + i^{-1}V_i \right\},$$

where $\sup_{i \le n}|V_i| = O_p(1)$. Now, $|C_{ni} - i/n| \le C_8 n^{-1}$, and so

$$T_1 = -\sum_{i=1}^{n} \frac{i}{n}\left\{ \sum_{j=i}^{n} j^{-1}(e_j - 1) \right\} h(\zeta_{i/n})^{-1} B_{ni} + O_p\left\{ n^{-1} \sum_{i=1}^{n} h(\zeta_{i/n})^{-1} \right\}. \tag{3.24}$$

Furthermore,

$$h(\zeta_{ni})^{-1} \le C_1\psi(\zeta_{i/n}) \le C_9(1 + |\zeta_{i/n}|)n \max\{i^{-1},(n - i + 1)^{-1}\}.$$

Therefore, since $n^{-1/2}\max_{i \le n-1}|H^{-1}(i/n)| \to 0$ [see (3.18)],

$$\begin{aligned} n^{-3/2}\sum_{i=1}^{n/2} h(\zeta_{i/n})^{-1} &\le C_{10} n^{-3/2} \sum_{i=1}^{n/2} (i/n)^{-1}|H^{-1}(i/n)| \\ &= C_{10} n^{-1} \sum_{i=1}^{n/2} (i/n)^{-1/2} i^{-1/2} |H^{-1}(i/n)| \\ &= o\left\{ n^{-1} \sum_{i=1}^{n/2} (i/n)^{-1/2} |H^{-1}(i/n)| \right\} \\ &= o\left\{ \int_0^{1/2} u^{-1/2}|H^{-1}(u)|\, du \right\} = o(1), \end{aligned} \tag{3.25}$$

and similarly,

$$n^{-3/2} \sum_{i=n/2}^{n} h(\zeta_{i/n})^{-1} = o(1). \tag{3.26}$$

Combining (3.24)–(3.26) and changing the order of summation in the double series on the right-hand side of (3.24), we deduce that

$$T_1 = -\sum_{j=i}^{n} (e_j - 1)(jn)^{-1} \sum_{i=1}^{j} i h(\zeta_{i/n})^{-1} B_{ni} + o_p(n^{1/2}). \tag{3.27}$$

Lemma 3.2 follows on combining (3.27) with (3.15) and noting that $T_2 = o_p(n^{1/2})$. □

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REFERENCES

BABU, G. J. and RAO, C. R. (1988). Joint asymptotic distribution of marginal quantiles and quantile functions in samples from a multivariate population. In *Multivariate Statistics and Probability—Essays in Honor of Paruchuri R. Krishnaiah* (C. R. Rao and M. M.Rao, eds.) 15–23. Academic, New York.

BARNETT, V. (1976). The ordering of multivariate data (with discussion). *J. Roy. Statist. Soc. Ser. A* **139** 318–354.

BEEBE, G. W. (1979). Reflections on the work of the atomic bomb casualty commission in Japan. *Epidemiol. Rev.* **1** 184–210.

BHATTACHARYA, P. K. (1974). Convergence of sample paths of normalized sums of induced order statistics. *Ann. Statist.* **2** 1034–1039.

BHATTACHARYA, P. K. (1976). An invariance principle in regression analysis. *Ann. Statist.* **4** 621–624.

BHATTACHARYA, P. K. (1984). Induced order statistics: theory and applications. In *Handbook of Statistics. Nonparametric Methods* (P. R. Krishnaiah and P. K. Sen, eds.) **4** 383–403. North-Holland, Amsterdam.

CHEW, M. C. (1973). On pairing observations from a distribution with monotone likelihood ratio. *Ann. Statist.* **1** 433–445.

DAVID, H. A. (1973). Concomitants of order statistics. *Bull. Internat. Statist. Inst.* **45** 295–300.

DAVID, H. A. (1981). *Order Statistics*, 2nd ed. Wiley, New York.

DAVID, H. A. (1982). Concomitants of order statistics: theory and applications. In *Some Recent Advances in Statistics* (J. Tiago de Oliveira and B. Epstein eds.) Academic, New York.

DAVID, H. A. and GALAMBOS, J. (1974). The asymptotic theory of concomitants of order statistics. *J. Appl. Probab.* **11** 762–770.

DAVID, H. A., O'CONNELL, M. J. and YANG, S. S. (1977). Distribution and expected value of the rank of a concomitant of an order statistic. *Ann. Statist.* **5** 216–223.

DAWBER, T. R., KANNEL, W. B. and LYELL, L. P. (1963). An approach to the longitudinal studies in a community: the Framingham study. *Ann. New York Acad. Sci.* **107** 539–556.

DEGROOT, M. H. (1987). Record linkage and matching systems. *Encyclopedia of Statistical Sciences* **7** 649–654. Wiley, New York.

DEGROOT, M. H., FEDER, P. I. and GOEL, P. K. (1971). Matchmaking. *Ann. Math. Statist.* **42** 578–593.

GALAMBOS, J. (1975). Order statistics of samples from multivariate distributions. *J. Amer. Statist. Assoc.* **70** 674–680.

GALAMBOS, J. (1985). Multivariate order statistics. *Encyclopedia of Statistical Sciences* **6** 100–104. Wiley, New York.

GOEL, P. K. and RAMALINGAM, T. (1989). *The Matchmaking Methodology: Some Statistical Properties. Lecture Notes in Statist.* **52**. Springer, New York.

OFFICE OF FEDERAL STATIST. POLICY AND STANDARDS (1980). Report on exact and statistical matching techniques. Subcommittee on Matching Techniques (D. B. Radner, R. Allen, M. E. Gonzalez, T. B. Jabine, and H. J. Muller). Statistical Policy Working Paper 5, U.S. Dept. Commerce, Washington, DC.

YAHAV, J. A. (1982). On matchmaking. In *Statistical Decision Theory and Related Topics, III* (S. S. Gupta and J. O. Berger, eds.) **2** 497–504. Academic, New York.

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