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CRITICAL LARGE DEVIATIONS FOR GAUSSIAN FIELDS IN THE PHASE TRANSITION REGIME, I[1]

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Dedicated to the memory of Frank Spitzer

We investigate large deviations for the empirical distribution functional of a Gaussian random field on $\mathbb{R}^{\mathbb{Z}^d}$, $d \geq 3$, in the phase transition regime. We first prove that the specific entropy governs an N^d volume order large deviation principle outside the Gibbsian class. Within the Gibbsian class we derive an N^{d-2} capacity order large deviation principle with exact rate function, and we apply this result to the asymptotics of microcanonical ensembles. We also give a spins' profile description of the field and show that *smooth* profiles obey N^{d-2} order large deviations, whereas *discontinuous* profiles obey N^{d-1} surface order large deviations.

0. Introduction. Let $\mathbb{Z}^d$ be the d-dimensional square lattice, where we always assume that $d \geq 3$. Next let $\Omega = \mathbb{R}^{\mathbb{Z}^d}$. We give Ω the product topology and use $\mathscr{B}_\Omega$ to denote the associated Borel field. Let $\mathscr{M}_1(\Omega)$ be the space of probability measures on $(\Omega, \mathscr{B}_\Omega)$ and set

$$\langle f, \mu \rangle = \int_\Omega f \, d\mu, \quad \text{for } f \in L^1(\mu) \text{ and } \mu \in \mathscr{M}_1(\Omega).$$

On $\mathscr{M}_1(\Omega)$ we consider the weak topology generated by the bounded continuous functions on Ω. Let $\mathscr{M}_1^{\mathbf{S}}(\Omega)$ be the subset of measures $\nu \in \mathscr{M}_1(\Omega)$ which are invariant with respect to the shift transformation $\{\theta^{\mathbf{k}}, \mathbf{k} \in \mathbb{Z}^d\}$ determined by

$$(\theta^{\mathbf{k}}\omega)_{\mathbf{j}} = \omega_{\mathbf{k}+\mathbf{j}}, \quad \text{for all } \mathbf{k}, \mathbf{j} \in \mathbb{Z}^d \text{ and } \omega \in \Omega.$$

$\mathscr{M}_1^{\mathbf{E}}(\Omega)$ will denote the set of ergodic $\nu \in \mathscr{M}_1^{\mathbf{S}}(\Omega)$.

Consider a shift invariant symmetric transition function $\mathbf{Q}$: $\mathbb{Z}^d \times \mathbb{Z}^d \to [0,1]$ with $Q(\mathbf{k}, \mathbf{j}) = Q(\mathbf{j}, \mathbf{k}) = Q(\mathbf{k} - \mathbf{j}, \mathbf{0})$,

$$\sum_{\mathbf{k} \in \mathbb{Z}^d} Q(\mathbf{0}, \mathbf{k}) = 1, \qquad Q(\mathbf{0}, \mathbf{0}) = 0 \quad \text{and} \quad Q(\mathbf{0}, \mathbf{k}) = 0 \quad \text{if } |\mathbf{k}| \geq R,$$

for some $R \in \mathbb{Z}^+$. We will assume that $\mathbf{Q}$ is *irreducible*. Since $d \geq 3$, $\mathbf{Q}$ is *transient*. We let $\mathbf{G} = \{G(\mathbf{k}, \mathbf{j})\text{: } \mathbf{k}, \mathbf{j} \in \mathbb{Z}^d\}$ denote the corresponding Green

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function:

$$\mathbf{G} = \sum_{n=0}^{\infty} \mathbf{Q}^n = (-\Delta_{\mathbf{Q}})^{-1},$$

where $\mathbf{Q}^n$ is the nth product of the matrix $\mathbf{Q}$ and $\Delta_{\mathbf{Q}} = \mathbf{Q} - \mathbf{I}$, where $\mathbf{I}$ is the identity.

Let $P \in \mathscr{M}_1^{\mathbf{S}}(\Omega)$ be the Gaussian field with

$$\langle \omega_{\mathbf{k}}, P\rangle = 0 \quad \text{and} \quad \mathrm{cov}_P(\omega_{\mathbf{k}}, \omega_{\mathbf{j}}) = G(\mathbf{k},\mathbf{j}), \qquad \mathbf{j},\mathbf{k} \in \mathbb{Z}^d.$$

The classical example is the simple random walk for $d \geq 3$ with

$$Q(\mathbf{k},\mathbf{j}) = \begin{cases} \dfrac{1}{2d}, & \text{if } |\mathbf{k}-\mathbf{j}| = 1, \\ 0, & \text{otherwise}, \end{cases} \tag{0.1}$$

for which $\Delta_{\mathbf{Q}}$ is the usual discrete Laplacian on $\mathbb{Z}^d$. In this case the associated Gaussian field is sometimes called the (discrete) *massless free field* or the *harmonic oscillator*. It will be helpful to introduce a *Gibbsian* description of P. Let $N(m,\sigma^2) \in \mathscr{M}_1(\mathbb{R})$ be the Gaussian distribution with mean $m \in \mathbb{R}$ and variance $\sigma^2 \in \mathbb{R}^+$ and consider the interaction potential $\mathscr{U} = \{U_F\colon \varnothing \neq F \subset\subset \mathbb{Z}^d\} \subseteq C(\Omega;\mathbb{R})$,

$$U_F(\omega) = \begin{cases} Q(\mathbf{k},\mathbf{j})\omega_{\mathbf{k}}\omega_{\mathbf{j}}, & F = \{\mathbf{k},\mathbf{j}\}, \\ 0, & |F| \neq 2. \end{cases}$$

Next let $\mathfrak{G}^{\mathbf{E}}(\mathbf{Q})$ denote the set of Gaussian $\{\gamma_m\colon m \in \mathbb{R}\} \subseteq \mathscr{M}_1^{\mathbf{S}}(\Omega)$ such that

$$\mathrm{cov}_{\gamma_m}(\omega_{\mathbf{k}}, \omega_{\mathbf{j}}) = G(\mathbf{k},\mathbf{j}) \quad \text{and} \quad \langle \omega_{\mathbf{k}}, \gamma_m\rangle = m, \qquad \mathbf{k} \in \mathbb{Z}^d,$$

and set

$$\mathfrak{G}^{\mathbf{S}}(\mathbf{Q}) \equiv \left\{\gamma = \int_V \gamma_{\phi(x)}\, dx\colon \phi \in L^2(V)\right\},$$

where $V = [0,1]^d$ is the unit cube in $\mathbb{R}^d$. Note that $P = \gamma_0$. $\mathfrak{G}^{\mathbf{E}}(\mathbf{Q})$ and $\mathfrak{G}^{\mathbf{S}}(\mathbf{Q})$ are precisely the set of extremal shift invariant Gibbs states L^2-finite (respectively, shift invariant Gibbs states) for the potential $\mathscr{U}$ with a priori measure $N(0,1)$ (cf. [15]). This means that if, for each $\mathbf{k} \in \mathbb{Z}^d$ and $\mu \in \mathscr{M}_1^{\mathbf{S}}(\Omega)$, $\mu_{\mathbf{k}}(\cdot|\omega)$ denotes the conditional distribution of $\omega_{\mathbf{k}}$ given $\{\omega_{\mathbf{j}}, \mathbf{j} \neq \mathbf{k}\}$, then

$$\gamma \in \mathfrak{G}^{\mathbf{S}}(\mathbf{Q}) \quad \text{if and only if} \quad \gamma_{\mathbf{k}}(\cdot|\omega) = N\Big(\sum_{\mathbf{j}\neq\mathbf{k}} Q(\mathbf{k},\mathbf{j})\omega_{\mathbf{j}}, \mathbf{1}\Big), \qquad \mathbf{k} \in \mathbb{Z}^d.$$

The aim of this paper is to investigate the ergodic property of the law of the *empirical distribution functional*

$$\omega \in \Omega \to \mathbf{R}_N(\omega) \equiv \frac{1}{|V_N|} \sum_{\mathbf{k}\in V_N} \delta_{\theta^{\mathbf{k}}\omega_N} \in \mathscr{M}_1^{\mathbf{S}}(\Omega),$$

under P, where V_N is the cube $V_N = [0, N-1]^d \cap \mathbb{Z}^d$ and ω_N is the configuration obtained by extending $\omega|_{V_N}$ periodically to $\mathbb{Z}^d$. Since P is ergodic, we know that with respect to weak convergence on $\mathscr{M}_1^{\mathbf{S}}(\Omega)$ we have

$$\lim_{N\to\infty} \mathbf{R}_N = P, \quad P \text{ almost surely}.$$

In particular, if

$$\mathbf{L}_N(\omega) \equiv \frac{1}{|V_N|} \sum_{\mathbf{k}\in V_N} \delta_{\omega_{\mathbf{k}}} \in \mathscr{M}_1(\mathbb{R}),$$

then

$$\lim_{N\to\infty} \mathbf{L}_N = N(0, \sigma^2), \quad P \text{ almost surely},$$

where $\sigma^2 \equiv G(\mathbf{0}, \mathbf{0})$.

Thus if $\Gamma \subseteq \mathscr{M}_1^{\mathbf{S}}(\Omega)$ is an open set containing P, then

$$\lim_{N\to\infty} P(\mathbf{R}_N \notin \Gamma) = 0. \tag{0.2}$$

Our main task will be to investigate at which exponential rate the preceding convergence occurs. Let us introduce some useful notation: If $(a_N)_{N\in\mathbb{N}}$ is a sequence of positive real numbers tending to ∞, and if U_N is a sequence of random elements with values in a topological space X, we say that U_N satisfies an (a_N)-large deviation principle, (a_N)-LDP for short, with a rate function J: $X \to [0,\infty]$, if J is lower semicontinuous with compact level sets and if, for any Borel set $A \subseteq X$,

$$\begin{aligned} -\inf_{\mathrm{int}(A)} J &\le \liminf_{N\to\infty} a_N^{-1} \log P(U_N \in A) \\ &\le \limsup_{N\to\infty} a_N^{-1} \log P(U_N \in A) \le -\inf_{\mathrm{cl}(A)} J. \end{aligned}$$

We also say that U_N is (a_N)-exponentially tight if

$$\inf_{\mathbf{K}\subset\subset\mathbf{X}} \liminf_{N\to\infty} a_N^{-1} \log P(U_N \notin K) = -\infty.$$

If the upper bound holds only for A compact, we say that U_N satisfies a weak LDP.

Our first result (cf. Theorem 1.4) shows that *outside of the Gibbsian class* $\mathfrak{G}^{\mathbf{S}}(\mathbf{Q})$ the convergence in (0.2) occurs at an exponential *volume rate* as $N \to \infty$. More precisely, let $\mathbf{h}(\cdot|P)$: $\mathscr{M}_1^{\mathbf{S}}(\Omega) \to [0,\infty]$ be the *specific entropy relative to* P:

$$\mathbf{h}(\mu|P) = \lim_{N\to\infty} \frac{1}{|V_N|} \mathbf{H}_N(\mu|P), \qquad \mu \in \mathscr{M}_1^{\mathbf{S}}(\Omega),$$

where $\mathbf{H}_N(\mu|P)$ is the relative entropy of μ with respect to P restricted to the cube V_N; $\mathbf{h}(\cdot|P)$ is well defined and satisfies

$$\mathbf{h}(\mu|P) = 0 \quad \text{if and only if} \quad \mu \in \mathfrak{G}^{\mathbf{S}}(\mathbf{Q}). \tag{0.3}$$

Then $\mathbf{h}(\cdot|P)$ is the rate function of a *weak* N^d*-large deviation principle* for $\mathbf{R}_N$.

The restriction to compact sets in the upper bound can be explained by the fact that the rate function $\mathbf{h}(\cdot|P)$ does not have compact level sets on the whole of $\mathscr{M}_1^{\mathrm{S}}(\Omega)$ (cf. Remark 1.16).

In view of the characterization (0.3) of $\mathfrak{G}^{\mathrm{S}}(\mathbf{Q})$, the preceding discussion does not give any information on the large deviations within the Gibbsian class $\mathfrak{G}^{\mathrm{S}}(\mathbf{Q})$. The main objective of this paper is to study the precise asymptotics within this class. The exact asymptotic behavior of $\mathbf{G}$ plays a crucial role. Let A be the symmetric $d \times d$ matrix associated with the covariances of $\mathbf{Q}$:

$$|y|_A^2 \equiv y \cdot Ay = \sum_{\mathbf{k} \in \mathbb{Z}^d} (y \cdot \mathbf{k})^2 Q(\mathbf{k}, \mathbf{0}), \qquad y \in \mathbb{R}^d, \tag{0.4}$$

and denote by $|A|$ the determinant of A. As $\mathbf{Q}$ is assumed to be irreducible, A is nonsingular. A classical result of Spitzer (cf. [26]) states that

$$\lim_{|\mathbf{k}| \to \infty} \frac{G(\mathbf{k}, \mathbf{0})}{g(\mathbf{k})} = 1, \tag{0.5}$$

where $g\colon \mathbb{R}^d \setminus \{0\} \to \mathbb{R}^+$ given by

$$g(x) = \frac{\Gamma(d/2)}{(d-2)\pi^{d/2}} |A|^{-1/2} \frac{1}{(x \cdot A^{-1}x)^{(d-2)/2}} \tag{0.6}$$

is the Green function associated with the operator

$$\frac{1}{2}\Delta_A = \frac{1}{2} \sum_{i,j=1}^{d} A_{i,j} \frac{\partial^2}{\partial x_i \, \partial x_j}.$$

We will actually need a stronger result than (0.5):

$$|G(\mathbf{k}, \mathbf{0}) - g(\mathbf{k})| = O(|\mathbf{k}|^{-d+1}). \tag{0.7}$$

This comes as a consequence of the local central limit theorem with error bounds. (See [19], Theorem 1.5.4 for the nearest-neighbor case. The general case is similar.)

From this it is easy to guess that the exponential rate of the volume order $|V_N| = N^d$ must be replaced by a capacity order N^{d-2}. Namely, introduce the empirical mean or magnetization $\mathbf{M}_N\colon \Omega \to \mathbb{R}$ of the box V_N,

$$\mathbf{M}_N(\omega) = \frac{1}{|V_N|} \sum_{\mathbf{k} \in V_n} \omega_{\mathbf{k}} = \langle \omega_0, \mathbf{R}_N(\omega) \rangle, \qquad N \in \mathbb{Z}^+.$$

Then, since $\mathbf{M}_N$ is a Gaussian random variable, a simple computation of the variance shows that, for each $m > 0$,

$$\lim_{N \to \infty} \frac{1}{N^{d-2}} \log P(\mathbf{M}_N \geq m) = -\frac{m^2}{2\langle 1_V, K_V 1_V \rangle_V}, \tag{0.8}$$

where

$$\langle \phi, K_V \psi \rangle_V \equiv \iint_{V \times V} \phi(x) g(x-y) \psi(y)\, dx\, dy, \qquad \phi, \psi \in L^2(V)$$

(cf. [6]). Note that this does not contradict Theorem 1.4 since in (0.8) we are looking at a large deviation within $\mathfrak{G}^{\mathbf{S}}(\mathbf{Q})$ as

$$\{\mu \in \mathscr{M}_1^{\mathbf{S}}(\Omega) : \langle \omega_0, \mu \rangle \geq m\} \cap \mathfrak{G}^{\mathbf{S}}(\mathbf{Q}) \neq \varnothing.$$

Another approach is to look at the behavior of the relative entropy $\mathbf{H}_N(\gamma|P)$ on the box V_N for some $\gamma \in \mathfrak{G}^{\mathbf{S}}(\mathbf{Q})$. We will prove that the *capacity specific entropy* $\mathbf{c}(\cdot|P)$: $\mathscr{M}_1^{\mathbf{S}}(\omega) \to [0, \infty]$ relative to P,

$$\mathbf{c}(\mu|P) \equiv \lim_{N \to \infty} \frac{1}{N^{d-2}} \mathbf{H}_N(\mu|P),$$

is well defined and satisfies

$$\mathbf{c}(\mu|P) = \begin{cases} \dfrac{\|\phi\|_V^2}{2} \mathscr{E}_V(1_V), & \text{if } \mu = \int_V \gamma_{\phi(x)}\, dx \in \mathfrak{G}^{\mathbf{S}}(\mathbf{Q}), \\ \infty, & \text{otherwise}, \end{cases} \tag{0.9}$$

where $\mathscr{E}_V$: $L^2(V) \to [0, \infty]$,

$$\mathscr{E}_V(\phi) = \inf\{\tfrac{1}{2}\langle |\nabla h|_A^2 \rangle_{\mathbb{R}^d} : h \in H^1(\mathbb{R}^d),\ h = \phi \text{ a.e. on } V\}.$$

[Here $H^1(\mathbb{R}^d)$ is the usual Sobolev space of once-differentiable functions $\mathbb{R}^d \to \mathbb{R}$.] $\mathscr{E}_V$ can also be viewed as the Dirichlet form of the diffusion embedded in the cube V generated by $\frac{1}{2}\Delta_A$. In particular, $\mathscr{E}_V(1_V) = \operatorname{cap}_A(V)$, where $\operatorname{cap}_A(V)$ is the corresponding capacity of V, and therefore

$$\mathbf{c}(\gamma_m|P) = \frac{m^2}{2} \operatorname{cap}_A(V), \qquad \gamma_m \in \mathfrak{G}^{\mathbf{E}}(\mathbf{Q}).$$

The first guess might be that $\mathbf{c}(\cdot|P)$ should govern an N^{d-2}-large deviation principle. It turns out that the order is correct but the rate function $\mathbf{c}(\cdot|P)$ is wrong. This can be seen from (0.8) and (0.9) since an alternative expression of the capacity shows that

$$\frac{\langle \phi, 1_V \rangle_V^2}{2\langle 1_V, K_V 1_V \rangle_V} < \frac{\|\phi\|_V^2}{2} \operatorname{cap}_A(V),$$

for any $\phi \in L^2(V)$. The correct rate function is given in the following strong large deviation principle, which is also our main result.

THEOREM 0.10. $\mathbf{R}_N$ *satisfies an* N^{d-2}*-LDP with rate function* $\mathscr{C}(\cdot|P)$: $\mathscr{M}_1^{\mathbf{S}}(\Omega) \to [0, \infty]$ *given by*

$$\mathscr{C}(\mu|P) \equiv \inf\left\{\tfrac{1}{2}\mathscr{E}_V(\phi) : \phi \in L^2(V) \text{ with } \mu = \int_V \gamma_{\phi(x)}\, dx\right\}$$

(*as usual* $\inf \varnothing = \infty$).

Note that $\mathscr{E}(\gamma_m|P) = \mathbf{c}(\gamma_m|P)$ for $\gamma_m \in \mathfrak{G}^{\mathbf{E}}(\mathbf{Q})$ and, in general,

$$\mathbf{c}(\gamma|P) \leq \frac{1}{\lambda_1 \operatorname{cap}_A(V)} \mathscr{E}(\gamma|P),$$

where $\lambda_1 \in \mathbb{R}^+$ is the first eigenvalue associated with the integral operator K_V on $L^2(V)$. However, the behavior of the two functions can be dramatically different: $\mathbf{c}(\cdot|P)$ is affine but $\mathscr{E}(\cdot|P)$ is not even convex. For example, since $\mathscr{E}_V(\phi) = \infty$ for discontinuous ϕ inside V, $\mathscr{E}(\gamma|P) = \infty$ for γ's which are discrete mixtures of extremals, for example,

$$\gamma = \sum_{i=1}^{n} p_i \gamma_{m_i}, \quad \text{with } 0 < p_i < 1 \text{ and } \sum_{i=1}^{n} p_i = 1.$$

The best way to understand why $\mathscr{E}(\cdot|P)$ is the correct rate function in Theorem 0.10 is to use the following "profile" description. Let $\mathscr{M}(V)$ and $\mathscr{M}_1(V \times \mathbb{R})$ be the space of finite signed measures on the cube V, respectively, of probability measures on $\mathbb{R} \times V$. Consider the random measures $\mathbf{X}_N\colon \Omega \to \mathscr{M}(V)$ and $\mathbf{Y}_N\colon \Omega \to \mathscr{M}_1(V \times \mathbb{R})$:

$$\mathbf{X}_N(\omega)(dx) = \frac{1}{|V_N|} \sum_{\mathbf{k} \in V_N} \delta_{\mathbf{k}/N}(dx) \cdot \omega_{\mathbf{k}},$$

$$\mathbf{Y}_N(\omega)(dx, dt) = \frac{1}{|V_N|} \sum_{\mathbf{k} \in V_N} \delta_{\mathbf{k}/N}(dx) \otimes \delta_{\omega_{\mathbf{k}}}(dt).$$

Note that we recover both $\mathbf{X}_N(\omega)$ and $\mathbf{L}_N(\omega)$ by projections from $\mathbf{Y}_N(\omega)$:

$$\langle \cdot \otimes \omega_0, \mathbf{Y}_N(\omega) \rangle_{\mathbb{R}} = \int_{\mathbb{R}} t \mathbf{Y}_N(\omega)(dt) = \mathbf{X}_N(\omega),$$

$$\langle 1_V \otimes \cdot, \mathbf{Y}_N(\omega) \rangle_V = \int_V \mathbf{Y}_N(\omega)(dx) = \mathbf{L}_N(\omega).$$

Next, define the rate functions $J\colon \mathscr{M}(V) \to [0, \infty]$ and $\bar{J}\colon \mathscr{M}_1(V \times \mathbb{R}) \to [0, \infty]$:

$$J(\mu) = \begin{cases} \dfrac{1}{2} \mathscr{E}_V(\phi), & \dfrac{d\mu}{dx} = \phi \in L^2(V), \\ \infty, & \text{otherwise}, \end{cases}$$

and

$$\bar{J}(\tilde{\mu}) = \begin{cases} \dfrac{1}{2} \mathscr{E}_V(\phi), & \tilde{\mu}(dx, dt) = dx \otimes N(\phi(x), \sigma^2)(dt), \\ & \qquad \phi \in L^2(V), \\ \infty, & \text{otherwise}. \end{cases} \tag{0.11}$$

In Section 3 we prove that both $\mathbf{X}_N$ and $\mathbf{Y}_N$ satisfy an N^{d-2}-large deviation principle with rate functions J and $\bar{J}$. $\mathbf{X}_N$ is a Gaussian field, so the proof of the LDP involves only covariance calculations.

The proof of the LDP for $\mathbf{Y}_N$ is more delicate. It turns out that the empirical field locally looks like an extremal Gibbs field, at least up to events which have probabilities too small to influence the LDP. This is the content of Proposition 3.10, which is proved using a conditioning argument on a wider lattice, a technique reminiscent of the hydrodynamical limits (cf. [16]). In some sense $\mathbf{X}_N$ describes the profile of spins $\omega_{\mathbf{k}}$ within the box V_N. We can relate these two critical large deviation principles with Theorem 0.10 in the following way: Take $\gamma \in \mathfrak{G}^{\mathrm{S}}(\mathbf{Q})$ and choose $\phi \in L^2(V)$ such that $\gamma = \int_V \gamma_{\phi(x)}\, dx$ minimizes $\mathscr{E}_V(\phi)$. Next, define the Gaussian field $\gamma^{\phi_N} \in \mathscr{M}_1(\Omega)$ with

$$\text{(0.12)} \quad \operatorname{cov}_{\gamma^{\phi_N}}(\omega_{\mathbf{k}}, \omega_{\mathbf{j}}) = G(\mathbf{k},\mathbf{j}), \quad \langle \omega_{\mathbf{k}}, \gamma^{\phi_N} \rangle = \phi_N(\mathbf{k}) = \phi\left(\frac{\mathbf{k}}{N}\right), \quad \mathbf{k}, \mathbf{j} \in \mathbb{Z}^d.$$

Then, for any open $\Gamma \subseteq \mathscr{M}_1(\Omega)$ containing γ, one can check that

$$\lim_{N\to\infty} \gamma^{\phi_N}(\mathbf{R}_N \in \Gamma) = 1$$

and that

$$\lim_{N\to\infty} \frac{1}{N^{d-2}} \mathbf{H}_N(\gamma^{\phi_N}|P) = \frac{1}{2}\mathscr{E}_V(\phi).$$

In other words, the best way for the empirical field $\mathbf{R}_N$ to realize a large deviation γ under P is to adopt the profile $\phi(\mathbf{k}/N)$ for the spins $\omega_{\mathbf{k}}$, $\mathbf{k} \in V_N$, and look at the appropriate microscopic scale like one of the extremal Gibbs $\gamma_{\phi(\mathbf{k}/N)}$. This will be, roughly speaking, the argument of the proof of the lower bound of Theorem 0.10 (cf. Theorem 2.9).

The exact rate function also gives us the answer to some microcanonical ensembles based on P. To be more precise, let us consider the empirical mean $\mathbf{M}_N$ and second moment $\mathbf{S}_N^2$ of the box V_N:

$$\mathbf{S}_N^2(\omega) \equiv \frac{1}{|V_N|} \sum_{\mathbf{k}\in V_N} \omega_{\mathbf{k}}^2 = \langle \omega_0^2, \mathbf{R}_N(\omega) \rangle, \qquad N \in \mathbb{Z}^+.$$

For $m \in \mathbb{R}$ and $s^2, \delta \in \mathbb{R}^+$ we would like to know the limiting distributions of the conditional law of P given $|\mathbf{M}_N - m| \le \delta$ and $|\mathbf{S}_N^2 - s^2| \le \delta$:

$$P\big(\cdot \,|\, |\mathbf{M}_N - m| \le \delta\big) \quad \text{and} \quad P\big(\cdot \,|\, |\mathbf{S}_N^2 - s^2| \le \delta\big),$$

when first $N \to \infty$ and then $\delta \searrow 0$. As a consequence of our large deviation principle we get the following.

THEOREM 0.13. *Let $m \in \mathbb{R}$ and $s^2 > \sigma^2$. Then, with respect to the weak convergence on $\mathscr{M}_1(\Omega)$,*

$$\lim_{\delta\searrow 0} \lim_{N\to\infty} P\big(\cdot \,|\, |\mathbf{M}_N - m| \le \delta\big) = \int_V \gamma_{\phi^*(x)}\, dx,$$

$$\lim_{\delta\searrow 0} \lim_{N\to\infty} P\big(\cdot \,|\, |\mathbf{S}_N^2 - s^2| \le \delta\big) = \int_V \gamma_{\psi^*(x)}\, dx,$$

where

$$\phi^*(x) = \frac{m}{\langle 1_V, K_V 1_V \rangle_V} K_V 1_V(x), \qquad \psi^*(x) = (s^2 - \sigma^2)^{1/2} e_1(x), \qquad x \in V;$$

$e_1 \in L^2(V)$ *is the normalized eigenfunction associated with* λ_1, *the first eigenvalue of* K_V. *However, if* $0 < s^2 < \sigma^2$, *then*

$$\lim_{\delta \searrow 0} \lim_{N \to \infty} P\big(\cdot \mid |\mathbf{S}_N^2 - s^2| \le \delta\big) = P^{\varepsilon^*},$$

where P^{ε^*} *is the centered Gaussian field with covariance* $\mathbf{G}^{\varepsilon^*} = (\varepsilon^* \mathbf{I} - \Delta_Q)^{-1}$, $\varepsilon^* > 0$ *being chosen such that* $G^{\varepsilon^*}(\mathbf{0}, \mathbf{0}) = s^2$.

The rest of the paper is divided into five sections. In Section 1 we show the noncritical large deviation principle. In Section 2 we show the existence of the capacity entropy and prove the lower bound in Theorem 0.10. The proof of the upper bound is given in Section 3 together with the large deviation principles for $\mathbf{X}_N$ and $\mathbf{Y}_N$. Section 4 deals with discontinuous profiles where surface phenomena of the order N^{d-1} occur. Finally, in the Appendix we derive a few covariance estimates which are used in the proofs.

Before concluding this introduction, let us have a look at the literature: The Gibbsian description of a Gaussian random field was initiated by Rozanov [23] and Dobrushin [9]. The relation between the covariance $\mathbf{G}$ and the transition function $\mathbf{Q}$ comes from Spitzer [25]. Although we discuss only Gaussian fields whose covariances can be expressed as Green functions of some nonnegative transition function $\mathbf{Q}$, we believe that our result could be extended to any situation with slowly decaying covariances. In a further paper we will investigate the lower lattice dimensions $d = 1, 2$ with a transient matrix $\mathbf{Q}$ of the form

$$\lim_{|\mathbf{k}| \to \infty} \frac{Q(\mathbf{0}, \mathbf{k})}{|\mathbf{k}|^{d+\alpha}} > 0 \quad \text{for some } \alpha \in (0, d \wedge 2).$$

In this case the capacity order N^{d-2} in Theorem 0.10 must be replaced by $N^{d-\alpha}$; cf. [2]. The characterization (0.3) of the specific entropy has been derived by Künsch in [17]. Large deviations for Gibbsian random fields at the volume order have been investigated by several authors (cf. [13], [20], [5] and [8]). However, none of these results apply to our situation since we are dealing here with an unbounded interaction. Donsker and Varadhan [12] treated the one-dimensional Gaussian case under rapidly decaying covariances where no phase transition occurs; see also [18] for a variational formula for a continuous version of P^ε in Theorem 0.13.

The understanding of critical large deviations for Gibbsian random fields within the phase transition regime is still at a very preliminary stage. Some results at the level of the empirical mean for Gaussian processes can be found in [6]. For discrete spins Föllmer and Ort [14] showed that the relative entropy of two Gibbs states within the box V_N grows like the surface N^{d-1}, and they derived a lower bound at this order for the empirical field. The best-studied model is the d-dimensional *Ising model*. In this case Schonmann [24] showed upper and lower bounds at the surface order N^{d-1} for the large deviation of the empirical mean $\mathbf{M}_N$. Dobrushin, Kotecky and Shlosman derive a large deviation principle for the spins' profile $\mathbf{X}_N$ of the two-dimensional Ising model, giving an explicit form for the exact rate function. The analogue of the canonical ensemble in Theorem 0.13 is related to the Wulff construction of the droplet (cf. [10] and [21]). It is interesting to compare our result with [10]: In their situation, surface phenomena at the order N^{d-1} occur, whereas in our situation we have capacity order N^{d-2}. The capacity order comes from the fact that the $\mathbb{R}$-valued spins have a *continuous symmetry*. We have an N^{d-2} order as long as we are dealing with "smooth" profiles in $H_1(\mathbb{R}^d)$. On the other hand, discontinuous profiles yield a surface order N^{d-1} with a rate function similar to the one in [10] (cf. Section 4).

1. Noncritical large deviations. In this section we show that the specific entropy governs the large deviations of the empirical field at the volume order.

Let us fix some notation: $c, c_1, c_2, \dots$ are generic constants greater than 0, not necessarily the same at different occurrences.

We start with a definition and characterization of the specific entropy. Given a nonempty $\Lambda \subseteq \mathbb{Z}^d$ and $\omega \in \Omega$, ω_Λ denotes the element of $\mathbb{R}^\Lambda$ obtained by restricting ω to Λ, and $\mathscr{B}_\Lambda$ is the σ-algebra over Ω generated by the projection map $\omega \in \Omega \to \omega_\Lambda \in \mathbb{R}^\Lambda$. For any $\mu \in \mathscr{M}_1(\Omega)$ and $N \in \mathbb{Z}^+$, let μ_{V_N} denote the marginal distribution of $\omega \in \Omega \to \omega_{V_N} \in \mathbb{R}^{V_N}$ under μ. Next, define the entropy $\mathbf{H}_N(\mu|P)$ of $\mu \in \mathscr{M}_1^S(\Omega)$ relative to P on V_N by

$$\mathbf{H}_N(\mu|P) = \begin{cases} \displaystyle\int_{\mathbb{R}^{v_N}} \log f_{V_N}\, d\mu_{V_N}, & \text{if } \mu_{V_N} \ll P_{V_N} \text{ and } f_{V_N} = \dfrac{d\mu_{V_N}}{dP_{V_N}}, \\ \infty, & \text{otherwise,} \end{cases}$$

and set

$$\bar{h}(\mu|P) \equiv \limsup_{N\to\infty} \frac{1}{|V_N|}\mathbf{H}_N(\mu|P) \quad \text{and} \quad \underline{h}(\nu|P) \equiv \liminf_{N\to\infty} \frac{1}{|V_N|}\mathbf{H}_N(\mu|P).$$

If they coincide, we call

$$\mathbf{h}(\mu|P) = \bar{\mathbf{h}}(\mu|P) = \underline{\mathbf{h}}(\mu|P)$$

the *specific entropy* of μ relative to P. For $L \in \mathbb{Z}^+$, set

$$\mathscr{K}_L = \{\nu \in \mathscr{M}_1^{\mathbf{S}}(\Omega) : \langle \omega_0^2, \nu\rangle \le L\}.$$

Finally, for each $\varepsilon > 0$, let $P^\varepsilon \in \mathscr{M}_1^{\mathbf{S}}(\Omega)$ be the centered Gaussian field with covariances

$$\mathbf{G}^\varepsilon = (\varepsilon \mathbf{I} - \Delta_Q)^{-1},$$

or, equivalently, with spectral density $\hat{g}^\varepsilon = \hat{g}/(\varepsilon\hat{g} + 1)$, where $\hat{g}$ is the spectral density of $\mathbf{G}$. Note that

$$\mathbf{G}^{\varepsilon} = (1+\varepsilon)^{-1} \sum_{n=0}^{\infty} (1+\varepsilon)^{-n} \mathbf{Q}^{n}, \tag{1.1}$$

that is, $\mathbf{G}^{\varepsilon}$ is the Green function of a random walk with a killing rate $\varepsilon/(1+\varepsilon)$, with killing starting at time 0. P^{ε} is usually referred to in the literature as the (discrete) *free field with positive mass* ε and can also be viewed as the unique Gibbs state with potential $\mathscr{T}^{\varepsilon} = \{J_F^{\varepsilon},\ F \Subset \mathbb{Z}^d\}$,

$$J_F^{\varepsilon}(\omega) = \begin{cases} -\dfrac{\varepsilon}{2}\omega_{\mathbf{k}}^2, & F = \{\mathbf{k}\}, \\ Q(\mathbf{k},\mathbf{j})\omega_{\mathbf{k}}\omega_{\mathbf{j}}, & F = \{\mathbf{k},\mathbf{j}\}, \\ 0, & |F| \geq 3, \end{cases}$$

and a priori measure $N(0,1)$. It follows from (1.1) that P^{ε} is hypercontractive (cf. [18], [4]) and has exponentially decaying covariances. This implies that $\mathbf{h}(\cdot|P^{\varepsilon})$ is well defined and has compact level sets in $\mathscr{M}_1^{\mathbf{S}}(\Omega)$ (cf. [18]).

Furthermore, $\mathbf{R}_N$ satisfies an N^d-large deviation principle under P^{ε} with rate function $\mathbf{h}(\cdot|P^{\varepsilon})$. For a proof, see [18] and [8].

The following gives a characterization of the specific entropy relative to P.

PROPOSITION 1.2. *The specific entropy* $\mathbf{h}(\cdot|P)$ *is well defined on* $\mathscr{M}_1^{\mathbf{S}}(\Omega)$ *and satisfies* $\mathbf{h}(\nu|P) = \infty$ *for* $\nu \notin \mathscr{K}_{\infty} \equiv \bigcup_L \mathscr{K}_L$ *and*

$$\mathbf{h}(\nu|P) = 0 \quad \text{if and only if} \quad \nu \in \mathfrak{G}^{\mathbf{S}}(\mathbf{Q}).$$

Moreover, for each $L \in \mathbb{Z}^+$, $\mathbf{h}(\cdot|P)$ *has compact level sets in* $\mathscr{K}_L$.

PROOF. Let us first check that $\mathbf{h}(\nu|P) = \infty$ for $\nu \notin \mathscr{K}_{\infty}$: Simply note that

$$\mathbf{H}_N(\nu|P) \geq \mathbf{H}_1(\nu|P) \geq \frac{1}{4\sigma^2} E^{\nu}\left[\omega_0^2\right] + \frac{1}{2}\log\frac{1}{2}$$

(cf. [7]). The existence and characterization of the specific entropy follows from [17]. Moreover, for each $\varepsilon > 0$ and $\nu \in \mathscr{K}_{\infty}$, we have

$$\mathbf{h}(\nu|P) = \mathbf{h}(\nu|P^{\varepsilon}) - \frac{\varepsilon}{2}\langle\omega_0^2\rangle_{\nu} - \phi(\varepsilon), \tag{1.3}$$

where

$$\phi(\varepsilon) = -\frac{1}{2}\frac{1}{(2\pi)^d}\int_{(-\pi,\pi]^d} \log(\varepsilon\hat{g}(x) + 1)\,dx$$

(cf. [17]), but this implies the lower semicontinuity of $\mathbf{h}(\cdot|P)$ on each $\mathscr{K}_L$ and concludes the proof. $\square$

Our main result in this section is the following weak large deviation principle.

THEOREM 1.4. *For each closed* $F \subseteq \mathscr{M}_1^{\mathbf{S}}(\Omega)$ *and* $L \in \mathbb{Z}^+$,

$$\limsup_{N\to\infty} \frac{1}{|V_N|} \log P(\mathbf{R}_N \in F \cap \mathscr{K}_L) \le -\inf_{F\cap\mathscr{K}_L} \mathbf{h}(\cdot|P),$$

and, for each open $G \in \mathscr{M}_1^{\mathbf{S}}(\Omega)$,

$$\liminf_{N\to\infty} \frac{1}{|V_N|} \log P(\mathbf{R}_N \in G) \ge -\inf_G \mathbf{h}(\cdot|P).$$

PROOF. We will only give a sketch of the proof of the lower bound, since the argument is identical to the one used by Donsker and Varadhan in [11]; see also [4]. We need to show that, for any $\nu \in \Gamma$ such that $\mathbf{h}(\nu|P) < \infty$,

$$\liminf_{N\to\infty} \frac{1}{|V_N|} \log P(\mathbf{R}_N \in G) \ge -\mathbf{h}(\nu|P). \tag{1.5}$$

Actually it is enough to consider an open *convex* Γ and $\nu \in \mathscr{M}_1^{\mathbf{S}}(\Omega)$ of the form

$$\nu = \sum_{i=1}^n p_i \nu^i, \quad \text{where } \nu^i \in \mathscr{M}_1^{\mathbf{E}}(\Omega) \cap \mathscr{K}_L \text{ and } \sum_{i=1}^n p_i = 1,$$

for some $L \in \mathbb{Z}^+$ (cf. [7]). Let $\mathscr{P}(V)$ be the set of finite partitions of V into disjoint cubes $\{V^1, \dots, V^n\}$. Choose a partition $\{V^1, \dots, V^n\} \in \mathscr{P}(V)$ such that $|V^i| = p_i$ and set

$$V_N^i = \left\{\mathbf{k} \in \mathbb{Z}^d: \frac{\mathbf{k}}{N} \in V^i\right\}.$$

Next, define the measure $\nu^{(N)} \in \mathscr{M}_1(\Omega)$,

$$\nu^{(N)} = \prod_{i=1}^{n+1} \nu^i_{V_N^i},$$

with $V_N^{n+1} = \mathbb{Z}^d \setminus V_N$ and $\nu^{n+1} = P$. Then by the ergodic theorem and the convexity of Γ we have

$$\lim_{N\to\infty} \nu^{(N)}(\mathbf{R}_N \in \Gamma) = 1. \tag{1.6}$$

Repeating the argument of Künsch in [17], we see that

$$\mathbf{H}_N(\nu^{(N)}|P) \le \sum_{i=1}^n \mathbf{H}_{N_i}(\nu^i|P) + C(n)\cdot LN^{d-1},$$

where $N_i = p_i N$ and $C(n) \in (0,\infty)$ is some constant depending on n; therefore,

$$\limsup_{N\to\infty} \frac{1}{|V_N|} \mathbf{H}_N(\nu^N|P) \le \sum_{i=1}^n p_i \mathbf{h}(\nu^i|P) = \mathbf{h}(\nu|P). \tag{1.7}$$

Now (1.6) together with (1.7) implies (1.5) (cf. [7]).

The proof of the upper bound is more delicate and uses explicitly the set $\mathscr{K}_L$. Fix $L \in \mathbb{Z}^+$ and set $\Gamma_L = \Gamma \cap \mathscr{K}_L$ for some closed $\Gamma \subseteq \mathscr{M}_1^{\mathbf{S}}(\Omega)$. Next, for $\varepsilon > 0$ define $P^{N,\varepsilon} \in \mathscr{M}_1(\Omega)$:

$$\frac{P^{N,\varepsilon}(d\omega)}{P(d\omega)} = F^{N,\varepsilon}(\omega) \equiv \exp\left[-\frac{\varepsilon}{2}|V_N|\langle \omega^2(\mathbf{0}), \mathbf{R}_N(\omega)\rangle\right]\frac{1}{Z_N^\varepsilon},$$

where

$$(1.8)\quad Z_N^\varepsilon = E^P\left[\exp\left[-\frac{\varepsilon}{2}|V_N|\langle \omega^2(\mathbf{0}), \mathbf{R}_N(\omega)\rangle\right]\right] = \det(\varepsilon\mathbf{G}_N + \mathbf{I}_N)^{-1/2},$$

where $\mathbf{G}_N$ and $\mathbf{I}_N$ are the restrictions of $\mathbf{G}$ and the identity, respectively, to the box V_N. Then

$$(1.9)\qquad \lim_{N\to\infty}\frac{1}{|V_N|}\log Z_N^\varepsilon = \phi(\varepsilon)$$

[cf. (1.3)].

The covariance operator $\mathbf{G}^{N,\varepsilon}$ of $P^{N,\varepsilon}$ can be written as

$$(1.10)\qquad \mathbf{G}^{N,\varepsilon} = \left(\mathbf{I} - \frac{\varepsilon}{1+\varepsilon}\mathbf{I}_N\right)\sum_{n=0}^{\infty}\left(\mathbf{Q} - \frac{\varepsilon}{1+\varepsilon}\mathbf{I}_N\mathbf{Q}\right)^n,$$

that is, killing occurs only in V_N. Using (1.9), we get

$$(1.11)\qquad \begin{aligned}&\limsup_{N\to\infty}\frac{1}{|V_N|}\log P(\mathbf{R}_N \in \Gamma_L)\\ &\quad\le \limsup_{N\to\infty}\frac{1}{|V_N|}\log E^{P^{N,\varepsilon}}\left[\frac{1}{F_{N,\varepsilon}}; \mathbf{R}_N \in \Gamma_L\right]\\ &\quad\le \frac{\varepsilon L}{2} + \phi(\varepsilon) + \limsup_{N\to\infty}\frac{1}{|V_N|}\log P^{N,\varepsilon}(\mathbf{R}_N \in \Gamma_L).\end{aligned}$$

We will show that

$$(1.12)\qquad \limsup_{N\to\infty}\frac{1}{|V_N|}\log P^{N,\varepsilon}(\mathbf{R}_N \in \Gamma_L) \le -\inf_{\Gamma_L}\mathbf{h}(\cdot|P^\varepsilon).$$

Combining this with (1.3) and (1.11), we get

$$\limsup_{N\to\infty}\frac{1}{|V_N|}\log P(\mathbf{R}_N \in \Gamma_L) \le \frac{\varepsilon L}{2} - \inf_{\Gamma_L}\mathbf{h}(\cdot|P);$$

as $\varepsilon > 0$ is arbitrary, the upper bound follows.

Let $M \in \mathbb{Z}^+$ be fixed. We may assume that

$$\Gamma_L \subseteq \mathscr{M}_1(\mathbb{R}^{V_M}) \equiv \mathscr{M}_1(\Omega_M).$$

Let d_M be the Prohorov metric on $\mathscr{M}_1(\Omega_M)$, which is dominated by the total variation norm. If $0 < \tau < 1$, consider

$$\mathbf{R}_{N,\tau} = \frac{1}{|V_{N,\tau}|} \sum_{\mathbf{k} \in V_{N,\tau}} \delta_{\theta_k \omega_N},$$

where

$$V_{N,\tau} \equiv \big([\tau N, (1-\tau)N] \cap \mathbb{Z}\big)^d.$$

Given $a > 0$, we can choose $\tau > 0$ and N large enough such that

$$d_M(\mathbf{R}_N, \mathbf{R}_{N,\tau}) \le a.$$

Let Γ_L^a denote the closed a-neighborhood of Γ_L. If N is large enough, then $\mathbf{R}_{N,\tau}$ is $\mathscr{B}_{V_{N,\tau/2}}$-measurable. Let $\phi_N \equiv (dP^{N,\varepsilon}|_{\mathrm{B}_{V_{N,\tau/2}}})/(dP^{\varepsilon}|_{\mathscr{B}_{V_{N,\tau/2}}})$. Then for any $1 < p < \infty$, $q \equiv p/(p-1)$,

$$\begin{aligned}
&\limsup_{N\to\infty} \frac{1}{|V_N|} \log P^{N,\varepsilon}(\mathbf{R}_N \in \Gamma_L) \\
&\quad\le \limsup_{N\to\infty} \frac{1}{|V_N|} \log P^{N,\varepsilon}(\mathbf{R}_{N,\tau} \in \Gamma_L^a) \\
&\quad\le \frac{1}{p} \limsup_{N\to\infty} \frac{1}{|V_N|} \log E^{P^\varepsilon}[\phi_N^p] \\
&\qquad + \frac{1}{q} \limsup_{N\to\infty} \frac{1}{|V_N|} \log P^{\varepsilon}\big(\mathbf{R}_N \in \Gamma_L^{2a}\big).
\end{aligned} \tag{1.13}$$

We first claim that

$$\limsup_{N\to\infty} \frac{1}{|V_N|} \log E^{P^\varepsilon}[\phi_N^p] = 0. \tag{1.14}$$

Let $\mathbf{H}$ and $\mathbf{K}$ be the covariance matrices $\mathbf{G}^{\varepsilon}$ and $\mathbf{G}^{N,\varepsilon}$, respectively, restricted to $V_{N,\tau/2}$. From (1.1) and (1.10), it easily follows that

$$\sup_{\mathbf{i},\mathbf{k} \in V_{N,\tau/2}} |H(\mathbf{i},\mathbf{k}) - K(\mathbf{i},\mathbf{k})| = O(e^{-CN^d}), \tag{1.15}$$

where $C = c(\varepsilon, \tau) > 0$, for $\varepsilon, \tau > 0$. Thus

$$E^{P^\varepsilon}(\phi_N^p) = \det(\mathbf{H}^{-1}\mathbf{K})^{1/2} \det\big[p(\mathbf{H}^{-1}\mathbf{K} - \mathbf{I}') + \mathbf{I}'\big]^{1/2},$$

where $\mathbf{I}' = \mathbf{I}_{V_{N,\tau/2}}$. Using (1.15), one gets

$$\lim_{N\to\infty} E^{P^\varepsilon}[\phi_N^p] = 1,$$

which is much more than what is needed for (1.14).

Next, using the upper bound in the N^d-LDP for $\mathbf{R}_N$ under P^{ε}, we see that

$$\frac{1}{q} \limsup_{N\to\infty} \frac{1}{|V_N|} \log P^{\varepsilon}\big(\mathbf{R}_N \in \Gamma_L^{2a}\big) \le -\frac{1}{q} \inf_{\Gamma_L^{2a}} \mathbf{h}(\cdot|P^{\varepsilon}).$$

Since a and $p > 1$ are arbitrary, we get (1.12). □

REMARK 1.16. The restriction of $\mathbf{h}(\cdot|P)$ to $\mathscr{K}_L$ is not artificial, but necessary—namely, the specific entropy is not lower semicontinuous on the whole of $\mathscr{M}_1^{\mathbf{S}}(\Omega)$: Choose $\{\phi_n: n \in \mathbb{Z}^+\} \subseteq L^2(V)$ such that ϕ_n converges in $L^1(V)$ to ϕ with $\|\phi\|_{L^2(V)} = \infty$ and set

$$\gamma^n = \int_V \gamma_{\phi_n(x)}\, dx, \qquad \gamma = \int_V \gamma_{\phi(x)}\, dx.$$

Then γ^n converges weakly to γ but $\mathbf{h}(\gamma^n|P) = 0$, $n \in \mathbb{Z}^+$, and $\mathbf{h}(\gamma|P) = \infty$. Also we do not have exponential tightness at the volume order $|V_N|$. As we will see in Section 3, exponential tightness holds at the capacity order N^{d-2}. More precisely, for each $L > \sigma^2$,

$$\lim_{N\to\infty} \frac{1}{N^{d-2}} \log P(\mathbf{R}_N \notin \mathscr{K}_L) = -\frac{L-\sigma^2}{2\lambda_1},$$

where λ_1 is the first eigenvalue of K_V.

We conclude this section with an application of Theorem 1.4 in the setting of microcanonical distributions. Let $\mathbf{S}_N^2(\omega) = \langle \omega^2(\mathbf{0}), \mathbf{R}_N(\omega)\rangle$ be the empirical second moment of the box V_N.

PROPOSITION 1.17. *Let $0 < s^2 < \sigma^2$. Then with respect to weak convergence on $\mathscr{M}_1(\Omega)$,*

$$\lim_{\delta\searrow 0}\lim_{N\to\infty} P\big(\cdot\,\big|\,|\mathbf{S}_N^2 - s^2| \le \delta\big) = P^{\varepsilon^*},$$

where $\varepsilon^* > 0$ is chosen such that

$$E^{P^{\varepsilon^*}}\big[\omega^2(\mathbf{0})\big] = G^{\varepsilon^*}(\mathbf{0},\mathbf{0}) = \frac{1}{(2\pi)^d}\int_{(-\pi,\pi]^d} \frac{\hat{g}(x)}{\varepsilon^*\hat{g}(x)+1}\, dx = s^2.$$

PROOF. From (1.9) we know that the law of $\mathbf{S}_N^2$ under P satisfies a weak large deviation principle with rate function J: $[0,\infty) \to [0,\infty]$,

$$J(s^2) = \begin{cases} \sup_{\varepsilon>0}\Big\{-\dfrac{\varepsilon}{2}s^2 - \phi(\varepsilon)\Big\}, & s^2 \in [0,\sigma^2),\\ 0, & s^2 \in [\sigma^2,\infty).\end{cases}$$

The supremum over $\varepsilon > 0$ is obtained at ε^* such that

$$s^2 = -2\phi'(\varepsilon^*) = \frac{1}{(2\pi)^d}\int_{(-\pi,\pi]^d} \frac{\hat{g}(x)}{\varepsilon^*\hat{g}(x)+1}\, dx. \tag{1.18}$$

In order to show the above convergence, we simply have to verify that, for each $s^2 \in (0,\sigma^2)$, P^{ε^*} is the unique solution of the variational problem

$$\inf\big\{h(\nu|P): \nu \in \mathscr{M}_1^{\mathbf{S}}(\Omega) \text{ with } E^{\nu}\big[\omega^2(\mathbf{0})\big] = s^2\big\} = J(s^2).$$

However, this follows from (1.3) and the fact that $\mathbf{h}(\nu|P^{\varepsilon}) = 0$ if and only if $\nu = P^{\varepsilon}$. $\square$

Proposition 1.17 will not give the microcanonical distribution for an $s^2 > \sigma^2$. This will be answered in the next section.

2. Capacity entropy and the lower bound. In this section we prove the existence and give a characterization of the capacity entropy. We also give a proof of the lower bound for the critical large deviations.

Our analysis is based on the exact asymptotic behavior of the covariances $\mathbf{Q}$. Let g: $\mathbb{R}^d \setminus \{0\}$ be the Green function of the diffusion associated with $\frac{1}{2}\Delta_A$ [cf. (0.6)]. Define the integral operator K_V: $L^2(V) \to L^2(V)$,

$$K_V\phi(x) = \int_V g(x-y)\phi(y)\,dy, \qquad x \in V.$$

K_V is a compact, positive define operator. We let $\{\lambda_n, \lambda_n \geq \lambda_{n+1}, n \in \mathbb{Z}^+\} \subseteq \mathbb{R}^+$ and $\{e_n, n \in \mathbb{Z}^+\} \subset L^2(V)$ denote the corresponding positive eigenvalues and eigenfunctions. Let $\mathscr{E}_V$: $L^2(V) \to [0, \infty]$,

$$\mathscr{E}_V(\phi) \equiv \sum_{n=1}^{\infty} \frac{1}{\lambda_n} \langle \phi, e_n \rangle_V^2.$$

The following gives an alternative expression for $\mathscr{E}_V$.

LEMMA 2.1. *For each* $\phi \in L^2(V)$,

$$\begin{aligned}\mathscr{E}_V(\phi) &= \sup\left\{\langle f, \phi\rangle_V - \tfrac{1}{2}\langle f, K_V f\rangle_V \colon f \in L^2(V)\right\}\\ &= \inf\left\{\tfrac{1}{2}\|\,|\nabla h|_A\|^2_{L^2(\mathbb{R}^d)} \colon h \in H^1(\mathbb{R}^d),\ h = \phi \text{ a.e. on } V\right\}.\end{aligned}$$

In particular, $\mathscr{E}_V(1_V) = \mathrm{cap}_A(V)$.

PROOF. The first equality is a trivial consequence of the spectral decomposition theorem. Set

$$\tilde{\mathscr{E}}_V(\phi) \equiv \inf\left\{\tfrac{1}{2}\|\,|\nabla h|_A\|^2_{L^2(\mathbb{R}^d)} \colon h \in H^1(\mathbb{R}^d),\ h = \phi \text{ a.e. on } V\right\}.$$

Assume that $\tilde{\mathscr{E}}_V(\phi) < \infty$. If Ω is an open domain in $\mathbb{R}^d$, and $f \in H^1(\Omega)$, we write

$$\|f\|^2_{A,\Omega} \equiv \int_\Omega |\nabla f|_A^2\,dx.$$

Let $\mathring{V} = \mathrm{int}(V)$. Evidently, for any $f \in L^2(V)$ and any $h \in H^1(\mathbb{R}^d)$,

$$\langle f, h\rangle - \tfrac{1}{2}\langle f, Kf\rangle \leq \tfrac{1}{2}\|h\|_{A,\mathbb{R}^d},$$

and so $\mathscr{E}_V(\phi) \leq \tilde{\mathscr{E}}_V(\phi)$. We prove the other direction.

Let $\phi \in H_1(\mathring{V})$. We choose a sequence $\phi_n \in C^\infty(V)$ which satisfies $\|\phi_n - \phi\|_{A,\mathring{V}} \to 0$ (see [1], Theorem 3.18). We first restrict ϕ_n to a smaller cube $C_n = [\delta_n, 1 - \delta_n]^d$ and then take the harmonic extension of this restriction to

$\mathbb{R}^d$, which is the unique function $\psi_n \in C_0(\mathbb{R}^d)$ satisfying $\Delta_A \psi_n = 0$ outside C_n and $\psi_n = \phi_n$ on C_n. By choosing $\delta_n \searrow 0$ appropriately, we may assume that $\|\phi_n - \psi_n\|_{A,\mathring{V}} \to 0$. We now approximate ψ_n by functions $\tau_n \in C_0^\infty(\mathbb{R}^d)$ which agree with ψ_n outside V and satisfy $\|\tau_n - \psi_n\|_{A,\mathring{V}} \to 0$. Then we put $f_n \equiv \Delta\tau_n$, which is 0 outside V. Therefore,

$$\langle f_n, \tau_n\rangle_V - \tfrac{1}{2}\langle f_n, K_V f_n\rangle_V = \langle f_n, \tau_n\rangle - \tfrac{1}{2}\langle f_n, K f_n\rangle = \tfrac{1}{2}\|\tau_n\|_{A,V}^2.$$

Furthermore,

$$|\langle f_n, \phi - \tau_n\rangle_V| = |\langle \nabla_A \tau_n, \nabla_A(\phi - \tau_n)\rangle| \le \|\tau_n\|_{A,\mathbb{R}^d}\|\phi - \tau_n\|_{A,\mathbb{R}^d} \to 0,$$

for $n \to \infty$. Therefore, we get

$$\limsup_{n\to\infty} \langle f_n, \phi\rangle_V - \tfrac{1}{2}\langle f_n, K_V f_n\rangle_V \ge \limsup_{n\to\infty} \tfrac{1}{2}\|\tau_n\|_{A,\mathbb{R}^d}^2,$$

that is,

$$\mathscr{E}_V(\phi) \ge \limsup_{n\to\infty} \tfrac{1}{2}\|\tau_n\|_{A,\mathbb{R}^d}^2.$$

By the Calderon extension theorem (see [1], Theorem 4.32), there exists a continuous extension operator Λ: $H^1(\mathring{V}) \to H^1(\mathbb{R}^d)$. Then

$$h_n \equiv \tau_n + \Lambda\big((\phi - \tau_n)|_{\mathring{V}}\big)$$

is an extension of ϕ which satisfies

$$\|h_n\|_{A,R^d} \le \|\tau_n\|_{A,\mathbb{R}^d} + C\|\phi - \tau_n\|_{A,\mathring{V}}.$$

Therefore

$$\mathscr{E}_V(\phi) \ge \limsup_{n\to\infty} \|h_n\|_{A,\mathbb{R}^d} \ge \tilde{\mathscr{E}}(\phi). \qquad \square$$

The following lemma will play an important role in our proofs.

LEMMA 2.2. *Let $\phi \in C(V;\mathbb{R})$ and set $\phi_N(\mathbf{k}) = \phi(\mathbf{k}/N)$, $\mathbf{k} \in V_N$. Then*

$$\lim_{N\to\infty} \frac{1}{N^{d+2}}\langle \phi_N, \mathbf{G}_N \phi_N\rangle_{V_N} = \langle \phi, K_V\phi\rangle_V. \tag{2.3}$$

Let $\phi \in C^1(V;\mathbb{R})$ and let $\mathbf{G}_N^{-1}$ be the inverse of the matrix $\mathbf{G}_N$. Then

$$\lim_{N\to\infty} \frac{1}{N^{d-2}}\langle \phi_N, \mathbf{G}_N^{-1}\phi_N\rangle_{V_N} = \mathscr{E}_V(\phi). \tag{2.4}$$

In particular, if γ^{ϕ_N} denotes the Gaussian field on $\mathbb{R}^{V_N}$ with

$$E^{\gamma^{\phi_N}}[\omega_k] = \phi_N(\mathbf{k}) = \phi\left(\frac{\mathbf{k}}{N}\right), \qquad \mathrm{cov}_{\gamma^{\phi_N}}(\omega_{\mathbf{k}}, \omega_{\mathbf{j}}) = G(\mathbf{k},\mathbf{j}), \qquad \mathbf{k},\mathbf{j} \in \mathbb{Z}^d,$$

then

$$\lim_{N\to\infty} \frac{1}{N^{d-2}}\mathbf{H}_N(\gamma^{\phi_N}|P) = \frac{1}{2}\mathscr{E}_V(\phi). \tag{2.5}$$

PROOF. In view of Spitzer's result (0.5), we see that, for each $M > 1$,

$$\begin{aligned}
&\lim_{N\to\infty} \frac{1}{N^{d+2}} \langle \phi_N, \mathbf{G}_N \phi N \rangle_{V_N} \\
&\quad = \lim_{N\to\infty} \frac{1}{N^{2d}} \langle \phi_N, N^{d-2} \mathbf{G}_N \phi N \rangle_{V_N} \\
&\quad = \lim_{N\to\infty} \frac{1}{N^{2d}} \sum_{k, j \in V_N, |\mathbf{k}-\mathbf{j}|>M} \phi\left(\frac{\mathbf{k}}{N}\right) g\left(\frac{\mathbf{k}}{N} - \frac{\mathbf{j}}{N}\right) \phi\left(\frac{\mathbf{j}}{N}\right) \\
&\quad = \langle \phi, K_{V^\phi} \rangle_V.
\end{aligned}$$

This shows (2.3). Next, take any $h \in H^1(\mathbb{R}^d)$ such that $h = \phi$ on V and write $h_N \equiv h(\cdot/N)$. Then

$$\begin{aligned}
\langle \phi_N, \mathbf{G}_N^{-1} \phi_N \rangle_{V_N} &= 2 \sup\left\{ \langle \phi_N, f \rangle_{V_N} - \tfrac{1}{2} \langle f, \mathbf{G}_N f \rangle_{V_N} \colon f \in L^2(V_N) \right\} \\
&= 2 \sup\left\{ \langle h_N, f \rangle_{\mathbb{Z}^d} - \tfrac{1}{2} \langle f, \mathbf{G} f \rangle_{\mathbb{Z}^d} \colon f \in L^2(V_N) \right\} \\
&\le 2 \sup\left\{ \langle h_N, f \rangle_{\mathbb{Z}^d} - \tfrac{1}{2} \langle f, \mathbf{G} f \rangle_{\mathbb{Z}^d} \colon f \in L^2(\mathbb{Z}^d) \right\} \\
&= \langle h_N, \mathbf{G}^{-1} h_N \rangle_{\mathbb{Z}^d} = \tfrac{1}{2} \langle |\nabla_Q h_N|^2 \rangle_{\mathbb{Z}^d},
\end{aligned}$$

where

$$|\nabla_Q h_N|^2(\mathbf{k}) = \sum_{\mathbf{j} \in \mathbb{Z}^d} Q(\mathbf{j}, \mathbf{k}) \big(h_N(\mathbf{j}) - h_N(\mathbf{k})\big)^2.$$

Thus, using the mean value theorem, we get

$$\limsup_{N\to\infty} \frac{1}{N^{d-2}} \langle \phi_N, \mathbf{G}_N^{-1} \phi_N \rangle_{V_N} \le \frac{1}{2} \limsup_{N\to\infty} \frac{1}{N^{d-2}} \langle |\nabla_Q h_N|^2 \rangle_{\mathbb{Z}^d} = \frac{1}{2} \| \, |\nabla h|_A \|_{\mathbb{R}^d}^2.$$

On the other hand, for any $f \in C^1(V; \mathbb{R})$,

$$\frac{1}{N^{d-2}} \langle \phi_N, \mathbf{G}_N^{-1} \phi_N \rangle_{V_N} \ge 2 \left(\frac{1}{N^d} \langle f_N, \phi_N \rangle_{V_N} - \frac{1}{2} \frac{1}{N^{d+2}} \langle f_N, \mathbf{G}_N f_N \rangle_{V_N} \right);$$

thus, in view of (2.3),

$$\liminf_{N\to\infty} \frac{1}{N^{d-2}} \langle \phi_N, \mathbf{G}_N^{-1} \phi_N \rangle_{V_N} \ge 2 \left(\langle f, \phi \rangle_V - \frac{1}{2} \langle f, K_V f \rangle_V \right).$$

Now (2.4) follows from Lemma 2.1 and the fact that

$$\mathscr{E}_V(\phi) = 2 \sup\left\{ \langle f, \phi \rangle_V - \tfrac{1}{2} \langle f, K_V f \rangle_V \colon f \in C^1(V; \mathbb{R}) \right\},$$

since $e_n \in C^1(V; \mathbb{R})$, $n \in \mathbb{Z}^+$ (cf. [22]). Finally, since γ^{ϕ_N} and P_{V_N} are both Gaussian with the same covariance $\mathbf{G}_N$,

$$\mathbf{H}_N(\gamma^{\phi_N} | P) = \tfrac{1}{2} \langle \phi_N, \mathbf{G}_N^{-1} \phi_N \rangle_{V_N}$$

and we get (2.5) by (2.4). □

As we saw in the previous section, the specific entropy vanishes on $\mathfrak{G}^{\mathbf{S}}(\mathbf{Q})$. We will prove that a relative entropy $\mathbf{H}_N(\gamma|P)$ of $\gamma \in \mathfrak{G}^{\mathbf{S}}(\mathbf{Q})$ with respect to P on the box V_N has a growth of the capacity order N^{d-2} as $N \to \infty$. More precisely, define the capacity entropy $c(\cdot|P)$: $\mathscr{M}_1^{\mathbf{S}}(\Omega) \to [0, \infty]$,

$$\mathbf{c}(\nu|P) \equiv \lim_{N\to\infty} \frac{1}{N^{d-2}} \mathbf{H}_N(\nu|P). \tag{2.6}$$

Then we have the following.

THEOREM 2.7. *The capacity entropy is well defined on $\mathscr{M}_1^{\mathbf{S}}(\Omega)$ and satisfies*

$$\mathbf{c}(\nu|P) = \begin{cases} \dfrac{\|\phi\|_V^2}{2} \operatorname{cap}_A(V), & \nu = \int_V \gamma_{\phi(x)}\,dx \in \mathfrak{G}^{\mathbf{S}}(\mathbf{Q}), \\ \infty, & \nu \notin \mathfrak{G}^{\mathbf{S}}(\mathbf{Q}). \end{cases}$$

PROOF. Set

$$\bar{\mathbf{c}}(\nu|P) = \limsup_{N\to\infty} \frac{1}{N^{d-2}} \mathbf{H}_N(\nu|P), \qquad \underline{\mathbf{c}}(\nu|P) = \liminf_{N\to\infty} \frac{1}{N^{d-2}} \mathbf{H}_N(\nu|P).$$

Then, in view of Proposition 1.2, $\underline{\mathbf{c}}(\nu|P) = \bar{\mathbf{c}}(\nu|P) = \infty$ for $\nu \notin \mathfrak{G}^{\mathbf{S}}(\mathbf{Q})$. Also as a direct consequence of Lemma 2.2,

$$\underline{\mathbf{c}}(\gamma_m|P) = \bar{\mathbf{c}}(\gamma_m|P) = \frac{m^2}{2} \operatorname{cap}_A(V), \qquad \gamma_m \in \mathfrak{G}^{\mathbf{E}}(\mathbf{Q}),$$

since $\mathscr{E}_V(\phi) = m^2 \operatorname{cap}_A(V)$ for $\phi = m1_V$. Next let $\gamma = \int_V \gamma_{\phi(x)}\,dx \in \mathfrak{G}^{\mathbf{S}}(\mathbf{Q})$. Since

$$\mathbf{H}_N(\gamma|P) \le \int_V \mathbf{H}_N\big(\gamma_{\phi(x)}|P\big)\,dx = \|\phi\|_V^2 \mathbf{H}_N(\gamma_1|P),$$

we see that

$$\bar{\mathbf{c}}(\gamma|P) \le \frac{\|\phi\|_V^2}{2} \operatorname{cap}_A(V).$$

Take a finite partition $\{V^1, \dots, V^n\} \in \mathscr{P}(V)$. Then

$$\gamma = \sum_{i=1}^n \gamma^i |V^i|, \quad \text{where } \gamma^i = \frac{1}{|V^i|} \int_{V^i} \gamma_{\phi(x)}\,dx,$$

and therefore

$$\underline{\mathbf{c}}(\gamma|P) \ge \sum_{i=1}^n \mathbf{c}(\gamma^i|P)|V^i| \tag{2.8}$$

(cf. [7], Exercise 4.4.41). Let

$$f_{m,N} = \frac{d(\gamma_m)_{V_N}}{dP_{V_N}} \quad \text{and} \quad f_N^i = \frac{d\gamma_{V_N}^i}{dP_{V_N}} = \frac{1}{|V^i|} \int_{V^i} f_{\phi(x),N}\,dx.$$

Then, using Jensen's inequality, we see that

$$\mathbf{H}_N(\gamma^i|P) = E^{\gamma^i}\big[\log f_N^i\big] \geq E^{\gamma^i}\left[\frac{1}{|V^i|}\int_{V^i}\log f_{\phi(x),N}\,dx\right]$$
$$= \frac{1}{|V^i|}\int_{V^i} E^{\gamma^i}\left[-\frac{1}{2}\phi^2(x)\langle \mathbf{1}, \mathbf{G}_N^{-1}\mathbf{1}\rangle_{V_N} + \phi(x)\langle \mathbf{1}, \mathbf{G}_N^{-1}\omega\rangle_{V_N}\right]dx$$
$$= \frac{\langle \mathbf{1}, \mathbf{G}_N^{-1}\mathbf{1}\rangle_{V_N}}{|V^i|^2}\int_{V^i}\int_{V^i}\left\{-\frac{1}{2}\phi(x)^2 + \phi(x)\phi(y)\right\}dx\,dy.$$

Since $\lim_{N\to\infty}(1/N^{d-2})\langle \mathbf{1}, \mathbf{G}_N^{-1}\mathbf{1}\rangle_{V_N} = \operatorname{cap}_A(V)$ (cf. Lemma 2.2), we get

$$\underline{\mathbf{c}}(\gamma|P) \geq \frac{\|\phi\|_V^2}{2}\operatorname{cap}_A(V) - \operatorname{cap}_A(V)\sum_{i=1}^n \frac{1}{|V^i|}\int_{V^i}\int_{V^i}\{\phi(x) - \phi(y)\}^2\,dx\,dy.$$

Now the result follows from this, (2.8) and

$$\inf\left\{\sum_{i=1}^n \frac{1}{|V^i|}\int_{V^i}\int_{V^i}\{\phi(x)-\phi(y)\}^2\,dx\,dy : \{V^1,\dots,V^n\} \in \mathscr{P}(V)\right\} = 0,$$
$$\phi \in L^2(V). \qquad \square$$

Our next step will be the proof of the lower bound.

THEOREM 2.9. *Let Γ be an open set of $\mathscr{M}_1^{\mathbf{S}}(\Omega)$. Then*

$$\liminf_{N\to\infty}\frac{1}{N^{d-2}}\log P(\mathbf{R}_N \in \Gamma) \geq -\inf_{\Gamma}\mathscr{E}(\cdot|P),$$

where

$$\mathscr{E}(\nu|P) = \inf\left\{\tfrac{1}{2}\mathscr{E}_V(\phi) : \phi \in L^2(V),\ \nu = \int_V \gamma_{\phi(x)}\,dx\right\}.$$

Moreover $\mathscr{E}(\cdot|P)$ has compact level sets in $\mathscr{M}_1(\Omega)$.

PROOF. The last statement follows from

$$\mathscr{E}(\gamma|P) \geq \frac{\|\phi\|_V^2}{2\lambda_1}.$$

In order to show the lower bound it is enough to prove that

$$\liminf_{N\to\infty}\frac{1}{N^{d-2}}\log P(\mathbf{R}_N \in \Gamma) \geq -\frac{1}{2}\mathscr{E}_V(\psi),$$

for each $\psi \in L^2(V)$ such that $\gamma = \int_V \gamma_{\psi(x)}\,dx \in \Gamma$. We may assume that $\mathscr{E}_V(\psi) < \infty$ and, using the eigenfunction expansion of ψ, choose for each $\varepsilon > 0$ a $\phi \in C^1(V;\mathbb{R})$ such that

$$\gamma' = \int_V \gamma_{\phi(x)}\,dx \in \Gamma \quad\text{and}\quad \mathscr{E}_V(\phi) \leq \mathscr{E}_V(\psi) + \varepsilon.$$

Let γ^{ϕ_N} be the Gaussian field on $\mathbb{R}^{V_N}$ as in Lemma 2.2. We will show that

$$\lim_{N\to\infty} \gamma^{\phi_N}(\mathbf{R}_N \in \Gamma) = 1. \tag{2.10}$$

This, together with (2.5), implies that

$$\liminf_{N\to\infty} \frac{1}{N^{d-2}} \log P(\mathbf{R}_N \in \Gamma) \geq -\frac{1}{2}\mathscr{E}_V(\phi) \geq -\frac{1}{2}\mathscr{E}_V(\psi) - \varepsilon$$

(cf. Lemma 5.4.21 of [7]), and concludes the proof. Equation (2.10) would be trivial for simple functions of the form

$$\phi'(x) = \sum_{i=1}^{n} m(i) 1_{V^i}(x), \quad \text{for some } \{V^1, \dots, V^n\} \in \mathscr{P}(V).$$

However, in this case $\mathscr{E}_V(\phi') = \infty$. For given $\delta > 0$ choose a simple function ϕ' such that

$$\limsup_{N\to\infty} \frac{1}{|V_N|} \sum_{\mathbf{k}\in V_N} |\phi'_N(\mathbf{k}) - \phi_N(\mathbf{k})| \leq \delta.$$

If d_L denotes the *Wasserstein* metric on $\mathscr{M}_1(\mathbb{R}^{V_L})$, then, for each $\omega, \omega' \in \Omega$,

$$d_L(\mathbf{R}_N(\omega), \mathbf{R}_N(\omega')) \leq \frac{|V_L|}{|V_N|} \sum_{\mathbf{k}\in V_N} |\omega_{\mathbf{k}} - \omega'_{\mathbf{k}}|.$$

Also, since

$$\gamma^{\Phi_N}(\mathbf{R}_N(\cdot) \in \Gamma) = \gamma^{\phi'_N}(\mathbf{R}_N(\cdot + \phi_N - \phi'_N) \in \Gamma),$$

we see that we can choose $\delta > 0$ small enough that

$$\gamma^{\phi_N}(\mathbf{R}_N \in \Gamma) \geq \gamma^{\phi'_N}(\mathbf{R}_N \in \Gamma'),$$

for some open Γ' containing $\gamma' = \int_V \gamma_{\phi'(x)}\, dx$; therefore,

$$\liminf_{N\to\infty} \gamma^{\phi_N}(\mathbf{R}_N \in \Gamma) \geq \liminf_{N\to\infty} \gamma^{\phi'_N}(\mathbf{R}_N \in \Gamma') = 1. \qquad \square$$

Let $\gamma \in \mathfrak{G}^{\mathbf{S}}(\mathbf{Q})$. Then there exists a unique $\rho \in \mathscr{M}_1(\mathbb{R})$ such that

$$\gamma = \int_{\mathbb{R}} \gamma_m \rho(dm). \tag{2.11}$$

If F_ρ: $\mathbb{R} \to [0,1]$ denotes the distribution function of ρ: $F_\rho(t) \equiv \rho((t,\infty))$, then any $\phi \in L^2(V)$ such that $\gamma = \int_V \gamma_{\phi(x)}\, dx$ must satisfy

$$|\{x \in V: \phi(x) > t\}| = F_\rho(t), \qquad t \in \mathbb{R}. \tag{2.12}$$

In general it is quite difficult to find the minimizing ϕ in Theorem 2.9 and to evaluate $\mathscr{E}(\gamma|P)$. For certain distributions concentrated on the positive (or negative) half-line, this problem can be explicitly solved via the rearrangement theorem (cf. [7], Section 4.3). More precisely, we assume that $\rho \in \mathscr{M}_1([0,\infty))$

with

$$\frac{d}{dt}F_\rho(t) < 0, \quad \text{for } t \in \operatorname{supp}(\rho) \setminus \{0\}. \tag{2.13}$$

Moreover, we suppose that $\rho(\{0\}) > 0$ is large enough that

$$B\left(0, \left(\frac{F_\rho(0)}{\kappa_d}\right)^{1/d}\right) \subseteq \bar{A}^{-1/2}\left(V - \frac{\mathbf{1}}{\mathbf{2}}\right), \tag{2.14}$$

where $\frac{\mathbf{1}}{\mathbf{2}} = (\frac{1}{2}, \dots, \frac{1}{2})$, $B(0, r) \equiv \{y \in \mathbb{R}^d\colon\ |y| < r\}$, $\kappa_d = |B(0,1)| = \pi^{d/2}/\Gamma(d/2 + 1)$ and $\bar{A} = 1/|A|^{1/d}$; here A is the $d \times d$ matrix of the covariances of $\mathbf{Q}$ [cf. (0.4)].

PROPOSITION 2.15. *Let* $\gamma \in \mathfrak{G}^{\mathbf{S}}(\mathbf{Q})$ *be of the form* (2.11), *where* ρ *satisfies* (2.13) *and* (2.14), *and set* $\tilde{\phi}\colon \mathbb{R} \to [0, \infty)$,

$$\tilde{\phi}(t) = \begin{cases} F_\rho^{-1}(t), & t \in \big(0, F_\rho(0)\big), \\ 0, & t \notin \big(0, F_\rho(0)\big). \end{cases}$$

Then

$$\mathscr{E}(\gamma|P) = \frac{1}{4}\|\,|\nabla\phi|_A\|^2_{L^2(\mathbb{R}^d)} = \frac{|A|^{1/d}d^2\kappa_d^{2/d}}{4}\int_0^{F_\rho(0)} y^{2-2/d}\tilde{\phi}'(y)^2\,dy, \tag{2.16}$$

with

$$\phi\left(x + \frac{\mathbf{1}}{\mathbf{2}}\right) \equiv \tilde{\phi}\big(\kappa_d \cdot |\bar{A}^{-1/2}x|^d\big), \qquad x \in \mathbb{R}^d.$$

PROOF. Let $\psi(x + \frac{\mathbf{1}}{\mathbf{2}}) \equiv \phi(\bar{A}^{1/2}(x + \frac{\mathbf{1}}{\mathbf{2}})) = \tilde{\phi}(\kappa_d \cdot |x|^d)$, $x \in \mathbb{R}^d$. Then (2.14) implies that ϕ has compact support in V, respectively, ψ in $\bar{A}^{-1/2}V$. Next let us verify (2.12). If $t < 0$, then

$$|\{x \in V\colon \phi(x) > t\}| = |V| = 1 = F_\rho(t).$$

For $t \geq 0$ we have

$$\begin{aligned} |\{x \in V\colon \phi(x) > t\}| &= \left|\bar{A}^{1/2}\{x \in \bar{A}^{-1/2}V\colon \psi(x) > t\}\right| \\ &= \left|\left\{x \in \bar{A}^{-1/2}\left(V - \frac{\mathbf{1}}{\mathbf{2}}\right)\colon |x|^d < \frac{F_\rho(t)}{\kappa_d}\right\}\right| = F_\rho(t). \end{aligned}$$

The rearrangement theorem also shows that such a ϕ minimizes

$$\|\,|\nabla\phi|_A\|^2_{L^2(\mathbb{R}^d)} = |A|^{1/d}\|\,|\nabla\phi|_{\bar{A}}\|^2_{L^2(\mathbb{R}^d)} = |A|^{1/d}\|\,|\nabla\psi|\,\|^2_{L^2(\mathbb{R}^d)},$$

and (2.16) follows from this and substitution. □

From (2.16) we get the following information about the tail of ρ.

COROLLARY 2.17. *Let* $\gamma = \int_{\mathbb{R}} \gamma_m \rho(dm) \in \mathfrak{G}^{\mathbf{S}}(\mathbf{Q})$ *satisfy* $\mathscr{E}(\gamma|P) < \infty$. *Then*

$$F_\rho(t) = o(t^{-2d/(d-2)}) \quad \text{as } t \to \infty$$

and

$$1 - F_\rho(t) = o(|t|^{-2d/(d-2)}) \quad \text{as } t \to -\infty;$$

in particular, $\langle |m|^p \rangle_\rho < \infty$ *for* $2 < p < 2d/(d-2)$.

EXAMPLE 2.18. Let $0 < p < 1$ be such that $B(0, (p/\kappa_d)^{1/d}) \subseteq \bar{A}^{-1/2}(V - \frac{1}{2})$; for example, in the isotropic case, where $\bar{A} = I$, one needs $p < \kappa_d / 2^d$. Next let $0 < q < p$ and $M > 0$ be given and set

$$\mathfrak{R}(p, q, M) \equiv \left\{ \gamma = \int_{\mathbb{R}} \gamma_m \rho(dm) \colon \operatorname{supp}(\rho) \subseteq [0, \infty), \right.$$
$$\left. \rho(\{0\}) = 1 - p,\ \rho([M, \infty)) \geq q \right\}.$$

Thus $\mathfrak{R}(p, q, M)$ defines the set of mixtures concentrated on the positive half-line, with $(1-p)$-mass at 0 and at least q-mass on $[M, \infty)$. Then the preceding considerations show that

$$\inf\{\mathscr{E}(\gamma|P) \colon \gamma \in \mathfrak{R}(p, q, M)\} = \frac{1}{4} \| \, |\nabla \phi|_A \|^2_{L^2(\mathbb{R}^d)} = \frac{|A|^{1/d} d(d-2) \kappa_d^{2/d} \cdot M^2}{4(q_d - p_d)},$$

where $q_d = q^{2/d - 1}$, $p_d = p^{2/d-1}$ and

$$\phi\left(x + \frac{1}{2}\right) = \begin{cases} M, & \kappa_d \cdot |\bar{A}^{-1/2} x|^d \leq q, \\ \dfrac{M}{q_d - p_d} \cdot \left[\kappa_d^{2/d-1} |\bar{A}^{-1/2} x|^{2-d} - p_d\right], & q < \kappa_d \cdot |\bar{A}^{-1/2} x|^d < p, \\ 0, & \kappa_d \cdot |\bar{A}^{-1/2} x|^d \geq p. \end{cases}$$

Thus the corresponding minimizing ρ has a distribution of the form

$$F_\rho(t) = \begin{cases} 1, & t < 0, \\ \left(\dfrac{t}{M}(q_d - p_d) + p_d\right)^{-d/(d-2)}, & 0 \leq t < M, \\ 0, & t \geq M. \end{cases}$$

□

We conclude this section with the proof of Theorem 0.13.

PROOF OF THEOREM 0.13. Assuming Theorem 0.10, we simply have to show that $\gamma^{(m)} = \int_V \gamma_{\phi^*(x)}\, dx$, respectively, $\gamma^{(s^2)} = \int_V \gamma_{\psi^*(x)}\, dx$, are solutions of the variational problems

$$\begin{aligned}&\inf\{\mathscr{E}(\gamma|P):\gamma\in\mathscr{M}_1^{\mathbf{S}}(\Omega),\langle\omega_0\rangle_\gamma=m\}\\&\qquad=\inf\{\tfrac{1}{2}\mathscr{E}_V(\phi):\phi\in L^2(V),\langle\phi,1_V\rangle_V=m\}\end{aligned}$$

and

$$\begin{aligned}&\inf\{\mathscr{E}(\gamma|P):\gamma\in\mathscr{M}_1^{\mathbf{S}}(\Omega),\langle\omega_0^2\rangle_\gamma=s^2\}\\&\qquad=\inf\{\tfrac{1}{2}\mathscr{E}_V(\phi):\phi\in L^2(V),\langle\phi,\phi\rangle_V=s^2-\sigma^2\},\end{aligned}$$

where in both cases $\gamma=\int_V\gamma_{\phi(x)}\,dx$. We can reformulate these two problems in terms of the eigenfunctions:

$$\inf\left\{\sum_{n=1}^{\infty}\frac{1}{\lambda_n}\langle\phi,e_n\rangle_V^2:\ \sum_{n=1}^{\infty}\langle\phi,e_n\rangle_V\langle 1_V,e_n\rangle_V=m\right\}$$

and

$$\inf\left\{\sum_{n=1}^{\infty}\frac{1}{\lambda_n}\langle\phi,e_n\rangle_V^2:\ \sum_{n=1}^{\infty}\langle\phi,e_n\rangle_V^2=s^2-\sigma^2\right\}.$$

Since $\lambda_1>\lambda_n$, $n\ge 2$, we see immediately that the solution to the second problem is simply ψ^*. As for the first problem, Lagrange multipliers yield

$$\langle\phi,e_n\rangle_V=\mu\cdot\lambda_n\langle 1_V,e_n\rangle_V$$

for some constant $\mu\in\mathbb{R}$. However, this means that $\phi=\mu\cdot K_V1_V$ and shows the result. □

3. Large deviation principles for $\mathbf{X}_N$ and $\mathbf{Y}_N$. In this section we prove the large deviation principle for $\mathbf{X}_N$ and $\mathbf{Y}_N$, and we prove the upper bound in Theorem 0.10.

If $r>0$, let $\mathscr{M}_r(V)$ be the set of signed measures on V with total variation less than or equal to r. We equip $\mathscr{M}_r(V)$ with the weak topology with which it becomes a compact space.

The set of all signed measures

$$\mathscr{M}(V)\equiv\bigcup_r\mathscr{M}_r(V)$$

is equipped with the inductive topology: A set $A\subset\mathscr{M}(V)$ is open if and only if $A\cap\mathscr{M}_r(V)$ is open in $\mathscr{M}_r(V)$. The empirical measure

$$\mathbf{X}_N(\omega)\equiv\frac{1}{|V_N|}\sum_{\mathbf{k}\in V_N}\delta_{\mathbf{k}/N}\cdot\omega_{\mathbf{k}}$$

is an $\mathscr{M}(V)$-valued random variable. Let $J\colon\mathscr{M}(V)\to[0,\infty]$,

$$J(\nu)=\tfrac{1}{2}\sup_{f\in C(V)}\{\langle\nu,f\rangle_V-\tfrac{1}{2}\langle f,K_Vf\rangle_V\};$$

J is convex and lower semicontinuous on $\mathscr{M}(V)$.

LEMMA 3.1. (a) *If* $\nu(dx) = \phi(x)\,dx$, $\phi \in H^1(\mathring{V})$, *then* $J(\nu) = \frac{1}{2}\mathscr{E}_V(\phi)$, *and* $J(\nu) = \infty$ *otherwise.*

(b) J *has compact level sets.*

PROOF. (b) follows from (a) and from the fact that the embedding $\phi \in H^1(\mathring{V}) \to \phi \in L^1(V)$ is compact and the mapping $\phi \in L^1(V) \to \phi(x)\,dx \in \mathscr{M}(V)$ is continuous. Therefore, we prove (a). If $\nu(dx) = \phi(x)\,dx$, $\phi \in H^1(\mathring{V})$, then

$$\sup_{f \in C(V)} \{\langle \nu, f\rangle_V - \tfrac{1}{2}\langle f, K_V f\rangle_V\} = \sup_{f\in L^2(V)} \{\langle \phi, f\rangle_V - \tfrac{1}{2}\langle f, K_V f\rangle_V\};$$

so $J(\nu) = \frac{1}{2}\mathscr{E}_V(\phi)$ follows from Lemma 2.1. It remains to show that $J(\nu) = \infty$ otherwise.

For any compact subset $W \in \mathbb{R}^d$ and finite signed measure τ with topological support in W, we put

$$J_W(\tau) \equiv \tfrac{1}{2} \sup_{f \in C(W)} \{\langle \tau, f\rangle_W - \tfrac{1}{2}\langle f, K_W f\rangle_W\}.$$

Then, for $\operatorname{supp}(\tau) \subset W_1 \subset W_2$,

$$J_{W_1}(\tau) = J_{W_2}(\tau).$$

Let h be a smooth probability density with support in the unit ball and $h_\varepsilon(x) = \varepsilon^{-d}h(x/\varepsilon)$. If $V' \equiv [-1,2]^d$ and $x \in \mathbb{R}^d$, $|x| \le 1$, then $J_{V'}(\theta_x \nu) = J_V(\nu)$, where $\theta_x\nu(A) = \nu(A - x)$. Using the convexity of $J_{V'}$, we get for $|\varepsilon| < 1$

$$J_{V'}(\nu_\varepsilon) = J_{V'}\left(\int dx\, h_\varepsilon(x)\theta_x\nu\right) \le \int dx\, h_\varepsilon(x) J_{V'}(\theta_x\nu) = J(\nu).$$

ν_ε has a smooth density $\psi_\varepsilon(x) = \int h_\varepsilon(x-y)\nu(dy)$ with support in the interior of V', if $|\varepsilon| < 1$, and therefore

$$J_{V'}(\nu_\varepsilon) = \tfrac{1}{2}\|\nabla\psi_\varepsilon\|_A^2.$$

The right-hand side converges to ∞ for $\varepsilon \to 0$ if ν does not have a density in $H^1(\mathring{V})$. □

PROPOSITION 3.2. $\mathbf{X}_N$ *satisfies an* N^{d-2}*-LDP with rate function* J.

PROOF. The lower bound follows from Section 2 and Lemma 3.1. To prove exponential tightness, it suffices to show that

$$\lim_{r\to\infty} \limsup_{N\to\infty} N^{-d+2} \log P(\mathbf{X}_N \notin \mathscr{M}_r) = -\infty. \tag{3.3}$$

To prove this, we use Lemma A.1 from the Appendix. Using

$$\sup_{\mathbf{i}\in V_N} \sum_{\mathbf{j}\in V_N} G(\mathbf{i},\mathbf{j}) = O(N^2), \qquad \operatorname{tr}(\mathbf{G}_N) = N^d\sigma^2, \qquad \operatorname{tr}(\mathbf{G}_N^2) = O(N^{d+1}),$$

where $\mathbf{G}_N$ is the restriction of $\mathbf{G}$ to V_N, and (A.3) from the Appendix, we see that, for each $a \in (0, 1/\lambda_1)$, where λ_1 is the maximal eigenvalue of the covariance operator K_V,

$$\limsup_{N\to\infty} \frac{1}{N^{d-2}} \log E^P\left[\exp\left(\frac{N^{d-2}}{2} a\langle \omega^2(\mathbf{0}), \mathbf{R}_N\rangle\right)\right] \le \frac{\sigma^2 a}{2}.$$

This shows that

$$\begin{aligned}\limsup_{N\to\infty} \frac{1}{N^{d-2}} \log P(\mathbf{X}_N \in \mathscr{M}_r) &\le \limsup_{N\to\infty} \frac{1}{N^{d-2}} \log P\left(\frac{1}{|V_N|}\sum_{\mathbf{j}\in V_N} |\omega_{\mathbf{j}}| > r\right) \\ &\le \limsup_{N\to\infty} \frac{1}{N^{d-2}} \log P\big(\langle \omega^2(\mathbf{0}), \mathbf{R}_N\rangle > r^2\big) \\ &\le -\frac{r^2-\sigma^2}{2\lambda_1}.\end{aligned}$$

The upper bound for Proposition 3.2 is now straightforward: If $f \in C(V)$, then $\langle f, \mathbf{X}_N\rangle$ is Gaussian with limiting variance

$$\lim_{N\to\infty} N^{d-2} E^P\big[\langle f, \mathbf{X}_N\rangle^2\big] = \langle f, K_V f\rangle_V$$

[cf. (23.)]. Therefore,

$$\lim_{N\to\infty} N^{-d+2} \log E^P\big[\exp\big(N^{d-2}\langle f, \mathbf{X}_N\rangle\big)\big] = \tfrac{1}{2}\langle f, K_V f\rangle_V.$$

Together with (3.3), this proves the proposition in the standard way. □

Let $\mathscr{M}_1(V\times\mathbb{R})$ be the set of probability measures on $V\times\mathbb{R}$. If ϕ is a continuous function, defined on V or on $\mathbb{R}$, we write

$$\|\phi\|_{BL} \equiv \|\phi\|_\infty + \sup_{x\ne y} \frac{|\phi(x)-\phi(y)|}{|x-y|}.$$

We use the following metric on $\mathscr{M}_1(V\times\mathbb{R})$: If $\mu,\ \nu \in \mathscr{M}_1(V\times\mathbb{R})$,

$$\begin{aligned} d_*(\mu,\nu) = \sup\{|\langle \phi\otimes\psi, \nu-\mu\rangle| : \phi \in C(V), \|\phi\|_{BL} \le 1, \\ \psi\in C(\mathbb{R}), \|\psi\|_{BL}\le 1\};\end{aligned}$$

d_* metrizes the weak topology on $\mathscr{M}_1(V\times\mathbb{R})$. This follows from the well-known fact that the Wasserstein metric [i.e., where the supremum over $\phi\otimes\psi$ is replaced by the supremum over $\tau\in C(V\times\mathbb{R})$ satisfying $\|\tau\|_{BL}\le 1$] induces the weak topology and the fact that the functions $\phi\otimes\psi$ appearing above separate $\mathscr{M}_1(V\times\mathbb{R})$.

We define a continuous mapping Φ: $L_1(V) \to \mathscr{M}_1(V\times\mathbb{R})$ by

$$\Phi(\phi)(dx,dt) \equiv dx \otimes N(\phi(x),\sigma^2)(dt),$$

where $N(a,\sigma^2)$ is the the normal distribution on $\mathbb{R}$ with mean $a\in\mathbb{R}$ and variance $\sigma^2 = G(\mathbf{0},\mathbf{0})$. It is obvious that Φ is a one-to-one continuous mapping. We define $\bar{J}$: $\mathscr{M}_1(V\times\mathbb{R}) \to [0,\infty]$ as in (0.10).

As the embedding $H^1(\mathring{V}) \to L_1(V)$ is compact, it follows that $\bar{J}$ has compact level sets and therefore is also lower semicontinuous. The main result of this section is the following.

THEOREM 3.4. *$\mathbf{Y}_N(\omega) \equiv |V_N|^{-1}\Sigma_{j \in V_N}\delta_{\mathbf{j}/N} \otimes \delta_{\omega_{\mathbf{j}}}$ satisfies an N^{d-2}-LDP with rate function $\bar{J}$.*

The basic idea of the proof is a conditioning argument, conditioning the field $(\omega_{\mathbf{k}})_{\mathbf{k} \in \mathbb{Z}^d}$ on the subfield $(\omega_{\mathbf{i}L})_{\mathbf{i} \in \mathbb{Z}^d}$ for suitable $L \in \mathbb{N}$. As the fields are Gaussian, we can decompose the field as

$$\omega_{\mathbf{i}} = y_{\mathbf{i}} + \xi_{\mathbf{i}}, \qquad \mathbf{i} \in \mathbb{Z}^d, \tag{3.5}$$

where $\xi_{\mathbf{i}} = E^P[\omega_{\mathbf{i}} | \mathscr{B}_{L\mathbb{Z}^d}]$ and the field $(y_{\mathbf{i}})_{\mathbf{i} \in \mathbb{Z}^d}$ is independent of $(\omega_{\mathbf{i}L})_{\mathbf{i} \in \mathbb{Z}^d}$. Furthermore, ξ_h has a linear representation,

$$\xi_{\mathbf{i}} = \sum_{\mathbf{k}} q_{\mathbf{i}}(\mathbf{k})\omega_{\mathbf{k}L}. \tag{3.6}$$

The $q_{\mathbf{i}}(\mathbf{k})$ are expressible in terms of the $\mathbf{Q}$-random walk on $\mathbb{Z}^d$, as explained in the Appendix. If we denote by $\mathbf{G}^L$ the covariance matrix of the $(y_{\mathbf{i}})_{\mathbf{i} \in \mathbb{Z}^d}$-field, that is,

$$G^L(\mathbf{i},\mathbf{j}) = E^P[y_{\mathbf{i}}y_{\mathbf{j}}],$$

then we have

$$\sum_{\mathbf{k}} q_{\mathbf{i}}(\mathbf{k})G(\mathbf{j},\mathbf{k}L) = G(\mathbf{j},\mathbf{i}) - G^L(\mathbf{i},\mathbf{j}). \tag{3.7}$$

It is crucial that $G^L(\mathbf{i},\mathbf{j})$ is rapidly decaying. More precisely, we have the following.

LEMMA 3.8. *There exist constants $c_1, c_2 > 0$ such that, for all $\mathbf{i},\mathbf{j} \in \mathbb{Z}^d$ and $L \in \mathbb{N}$,*

$$G^L(\mathbf{i},\mathbf{j}) \le c_1 \exp\left(-c_2|\mathbf{i} - \mathbf{j}|/L^{d/2}\right).$$

The proof will be given in the Appendix.

We let L depend on N: $L = L_N \equiv [\log N]$ and define $q_{\mathbf{i}}(\mathbf{k})$, $\xi_{\mathbf{i}}$ and $y_{\mathbf{i}}$ in terms of this L. Let $\mathbf{Z}_N$: $\Omega \to \mathscr{M}_1(V \times \mathbb{R})$ be given by

$$\mathbf{Z}_N(\omega) \equiv \frac{1}{|V_N|}\sum_{\mathbf{j} \in V_N} \delta_{\mathbf{j}/N} \otimes N\left(\xi_{\mathbf{j}}(\omega), \sigma^2\right).$$

The proof of Theorem 3.4 will then be a direct consequence of the following two propositions:

PROPOSITION 3.9. *$\mathbf{Z}_N$ satisfies an N^{d-2}-LDP with rate function $\bar{J}$.*

PROPOSITION 3.10. *For any $a > 0$, $\delta > 0$ and any bounded Lipschitz-continuous ϕ: $V \to \mathbb{R}$ and ψ: $\mathbb{R} \to \mathbb{R}$, we have*

$$\lim_{N \to \infty} N^{-d+\delta} \log P\left(|\langle \phi \otimes \psi, \mathbf{Y}_N - \mathbf{Z}_N\rangle| \ge a\right) = -\infty.$$

The proof of the first proposition is given in three lemmas. Let h be a smooth symmetric probability density on $\mathbb{R}^d$ with support in the unit ball, and for $\varepsilon > 0$ set $h_\varepsilon(x) \equiv \varepsilon^{-d} h(x/\varepsilon)$. If $\mu \in \mathscr{M}(V)$, let $\mu_\varepsilon \in C(V)$ be given by $\mu_\varepsilon(x) = \int_V h_\varepsilon(x-y)\mu(dy)$. We then define $\Phi_\varepsilon \colon \mathscr{M}(V) \to \mathscr{M}_1(V \times \mathbb{R})$ by

$$\Phi_\varepsilon(\mu) \equiv \Phi(\mu_\varepsilon).$$

Φ_ε is a continuous mapping and therefore the contraction principle applies, yielding Lemma 3.11.

LEMMA 3.11. *For each $\varepsilon > 0$, $\Phi_\varepsilon(\mathbf{X}_N)$ satisfies an N^{d-2}-LDP with rate function*

$$\bar{J}_\varepsilon(\nu) \equiv \inf\{J(\mu) \colon \Phi_\varepsilon(\mu) = \nu\}.$$

The crucial step in the proof of Proposition 3.9 is Lemma 3.12.

LEMMA 3.12. *For any $a > 0$,*

$$\lim_{\varepsilon \searrow 0} \limsup_{N \to \infty} N^{-d+2} \log P\big(d_*(\Phi_\varepsilon(\mathbf{X}_N), \mathbf{Z}_N) \ge a\big) = -\infty.$$

PROOF. For any measurable $\psi \colon \mathbb{R} \to \mathbb{R}$, satisfying $\|\psi\|_\infty \le 1$,

$$\bar{\psi}(x) \equiv \int_{\mathbb{R}} \psi(y) N(x, \sigma^2)(dy)$$

satisfies $\|\bar{\psi}\|_\infty \le 1$ and $\|\bar{\psi}'\|_\infty \le \sqrt{2} / \sqrt{\pi}\,\sigma$. Using this, one easily obtains

$$d_*(\Phi_\varepsilon(\mathbf{X}_N), \mathbf{Z}_N) \le \frac{c}{|V_N|} \left\{ \sum_{\mathbf{j} \in V_N} |\eta^\varepsilon_{\mathbf{j}}| + \frac{\varepsilon^{-d-1}}{N} \sum_{\mathbf{j} \in V_N} |\omega_{\mathbf{j}}| \right\}, \tag{3.13}$$

where

$$\eta^\varepsilon_{\mathbf{j}}(\omega) \equiv \xi_{\mathbf{j}}(\omega) - \frac{1}{|V_N|} \sum_{\mathbf{k} \in V_N} h_\varepsilon\left(\frac{\mathbf{j}}{N} - \frac{\mathbf{k}}{N}\right)\omega_{\mathbf{k}}.$$

For any $a, \varepsilon > 0$,

$$\limsup_{N \to \infty} N^{-d+2} \log P\left(\frac{\varepsilon^{-d-1}}{N|V_N|} \sum_{\mathbf{j} \in V_N} |\omega_{\mathbf{j}}| \ge a \right) = -\infty; \tag{3.14}$$

this follows from Lemma A.1 [see the proof of (3.3)]. We also use this lemma to treat the first summand on the right-hand side of (3.13).

Let $\Gamma^\varepsilon_N(\mathbf{i}, \mathbf{j}) \equiv E^P[\eta^\varepsilon_{\mathbf{i}} \eta^\varepsilon_{\mathbf{j}}]$, $\mathbf{i}, \mathbf{j} \in V_N$. It is easy to prove

$$\operatorname{tr}\big((\Gamma^\varepsilon_N)^2\big) = O(N^{d+1}). \tag{3.15}$$

[Actually, for $d \ge 5$, it is only $O(N^d)$, but this is of no importance.] We claim that

$$\lim_{N \to \infty} |V_N|^{-1} \operatorname{tr}(\Gamma^\varepsilon_N) = 0, \tag{3.16}$$

for all $\varepsilon > 0$, and

$$\lim_{\varepsilon \searrow 0} \limsup_{N\to\infty} N^{-2} \sup_{\mathbf{i}} \sum_{\mathbf{j}} |\Gamma_N^\varepsilon(\mathbf{i},\mathbf{j})| = 0. \tag{3.17}$$

Equations (3.15)–(3.17) and Lemma A.1 imply that

$$\lim_{\varepsilon \searrow 0} \limsup_{N\to\infty} N^{-d+2} \log P\left(\frac{1}{|V_N|} \sum_{\mathbf{j}\in V_N} |\eta_{\mathbf{j}}^\varepsilon| \ge a \right) = -\infty$$

for all $a > 0$. This, together with (3.13) and (3.14), proves Lemma 3.12. It remains to prove (3.16) and (3.17). We can estimate $\operatorname{tr}(\Gamma_N^\varepsilon) = \sum_{\mathbf{j}\in V_N} \operatorname{var}(\eta_{\mathbf{j}}^\varepsilon)$ quite crudely:

$$\operatorname{var}(\eta_{\mathbf{j}}^\varepsilon) \le 2E^P[\xi_{\mathbf{j}}^2] + 2E^P\left[\left(\frac{1}{|V_N|} \sum_{\mathbf{k}} h_\varepsilon\left(\frac{\mathbf{j}}{N} - \frac{\mathbf{k}}{N} \right) \omega_{\mathbf{k}} \right)^2 \right].$$

The second summand on the right-hand side is $O(N^{-d+2})$ for any fixed $\varepsilon > 0$, uniformly in $\mathbf{j} \in V_N$, and, by Lemma A.14,

$$\lim_{N\to\infty} |V_N|^{-1} \sum_{\mathbf{j}\in V_N} E^P[\xi_{\mathbf{j}}^2] = \lim_{N\to\infty} |V_N|^{-1} \sum_{\mathbf{j}\in V_N} \{G(\mathbf{j},\mathbf{j}) - G^L(\mathbf{j},\mathbf{j})\} = 0.$$

This proves (3.16). To prove (3.17), we write

$$\hat{\xi}_{\mathbf{j}} \equiv \xi_{\mathbf{j}} - \frac{1}{|V_N|} \sum_{\mathbf{k}\in V_N} h_\varepsilon\left(\frac{\mathbf{j}}{N} - \frac{\mathbf{k}}{N} \right) \xi_{\mathbf{k}}.$$

Using the independence of the ξ- and y-fields, we get

$$\Gamma_N^\varepsilon = \hat{\Gamma}_N^\varepsilon + \Lambda_N^\varepsilon,$$

where $\hat{\Gamma}_N^\varepsilon$ is the covariance matrix of the $\hat{\xi}$-field and Λ_N^ε that of

$$\frac{1}{|V_N|} \sum_{\mathbf{k}\in V_N} h_\varepsilon\left(\frac{\mathbf{j}}{N} - \frac{\mathbf{k}}{N} \right) y_{\mathbf{k}}, \qquad \mathbf{j} \in V_N.$$

Using the fast decay of the covariances of the y-field (cf. Lemma 3.8), we get

$$\limsup_{N\to\infty} N^{-\delta} \sup_{\mathbf{i}} \sum_{\mathbf{j}\in V_N} \Lambda_N^\varepsilon(\mathbf{i},\mathbf{j}) = 0,$$

for every $\delta > 0$, $\varepsilon > 0$.

Let

$$G^\xi(\mathbf{i},\mathbf{j}) \equiv E^P[\xi_{\mathbf{i}}\xi_{\mathbf{j}}] = G(\mathbf{i},\mathbf{j}) - G^L(\mathbf{i},\mathbf{j}).$$

Using $G(\mathbf{i},\mathbf{j}) = g(\mathbf{i}-\mathbf{j}) + O(|\mathbf{i}-\mathbf{j}|^{-d+1})$, $\mathbf{i} \ne \mathbf{j}$ [cf. (0.7)] and Lemma 3.8, and putting

$$R(\mathbf{i},\mathbf{j}) = G^\xi(\mathbf{i},\mathbf{j}) - g(\mathbf{i}-\mathbf{j}),$$

we get

$$|R(\mathbf{i},\mathbf{j})| \le c_1|\mathbf{i}-\mathbf{j}|^{-d+1} + \exp\left(-c_2\frac{|\mathbf{i}-\mathbf{j}|}{(\log N)^{d/2}}\right).$$

Therefore, the contribution $R(\mathbf{i},\mathbf{j})$ to $\Sigma_{\mathbf{j}\in V_N}|\hat{\Gamma}_N^{\varepsilon}(\mathbf{i},\mathbf{j})|$ is again $O(N^{\delta+1})$ for any $\delta, \varepsilon > 0$; it remains to prove

$$\lim_{\varepsilon\searrow 0}\limsup_{N\to\infty} N^{-2}\sup_{\mathbf{i}}\sum_{\mathbf{j}\in V_N}|\alpha_N^{\varepsilon}(\mathbf{i},\mathbf{j})| = 0, \tag{3.18}$$

where

$$\alpha_N^{\varepsilon}(\mathbf{i},\mathbf{j}) \equiv \bar{g}(\mathbf{i}-\mathbf{j}) - 2\sum_{\mathbf{k}\in V_N} p_{\varepsilon}(\mathbf{i},\mathbf{k})\bar{g}(\mathbf{k}-\mathbf{j})$$
$$+ \sum_{\mathbf{e},\mathbf{k}\in V_N} p_{\varepsilon}(\mathbf{i},\mathbf{k})p_{\varepsilon}(\mathbf{j},\mathbf{e})\bar{g}(\mathbf{k}-\mathbf{e})$$

with $p_{\varepsilon}(\mathbf{i},\mathbf{j}) \equiv |V_N|^{-1}h_{\varepsilon}(\mathbf{i}/N - \mathbf{j}/N)$ and $\bar{g}(\mathbf{i}-\mathbf{j}) \equiv g(\mathbf{i}-\mathbf{j})1_{\{\mathbf{i}\neq\mathbf{j}\}}$.

An easy approximation leads to

$$\lim_{N\to\infty} N^2 \sup_{\mathbf{i}}\sum_{\mathbf{j}\in V_N}|\alpha_N^{\varepsilon}(\mathbf{i},\mathbf{j})|$$
$$= \sup_x \int_V dy\,|g(x-y) - 2h_{\varepsilon} * g(x-y) + h_{\varepsilon} * h_{\varepsilon} * g(x-y)|.$$

The right-hand side converges to 0 as $\varepsilon \searrow 0$ by Lebesgue dominated convergence. This proves (3.18) and, therefore, Lemma 3.12. □

Lemma 3.12, together with Lemma 3.11, proves that $\mathbf{Z}_N$ satisfies an N^{d-2}-LDP with rate function

$$\tilde{J}(\mu) \equiv \liminf_{\substack{\varepsilon\searrow 0\\ \nu\to\mu}} \bar{J}_{\varepsilon}(\nu). \tag{3.19}$$

Proposition 3.9 therefore follows from the next lemma.

LEMMA 3.20.

$$\tilde{J} = \bar{J}.$$

PROOF. If $\bar{J}(\nu) < \infty$, then $\nu = \Phi(f)$ for some $f \in H^1(\mathring{V})$ and $\Phi_{\varepsilon}(f\,dx) \to \nu$ as $\varepsilon \searrow 0$. Therefore $\tilde{J}(\nu) \le \bar{J}(\nu)$.

To prove the converse, assume $\tilde{J}(\nu) < \infty$, and let $\mu^{(\varepsilon)}$, $\varepsilon > 0$, be elements in $\mathscr{M}(V)$ with $J(\mu^{(\varepsilon)}) \to \tilde{J}(\nu)$, $\Phi_{\varepsilon}(\mu(\varepsilon)) \to \nu$, as $\varepsilon \searrow 0$. As $J(\mu^{(\varepsilon)}) < \infty$, we have $\mu^{(\varepsilon)}(dx) = f^{(\varepsilon)}(x)\,dx$, with $f^{(\varepsilon)} \in H^1(\mathring{V})$; in fact, $f^{(\varepsilon)}$ is relatively compact in $L^1(V)$, as $\varepsilon \searrow 0$. Passing to a subsequence $\varepsilon_n \to 0$, we may assume that $f^{(\varepsilon)}$ is convergent as $\varepsilon \searrow 0$, $f^{(\varepsilon)} \to f$, say. Then, however, also $\mu_{\varepsilon}^{(\varepsilon)} \to f$ and therefore $\Phi(f) = \nu$. Therefore, $\bar{J}(\nu) = \frac{1}{2}\mathscr{E}_V(f) \le \lim_{\varepsilon\searrow 0} J(\mu^{(\varepsilon)}) = \tilde{J}(\nu)$. □

The proof of Theorem 3.4 is completed with the following.

PROOF OF PROPOSITION 3.10. First note that

$$\langle \phi \otimes \psi, \mathbf{Y}_N - \mathbf{Z}_N \rangle \le |V_N|^{-1} \left| \sum_{\mathbf{j} \in V_N} \phi\left(\frac{\mathbf{j}}{N}\right) \left[\psi(\omega_{\mathbf{j}}) - \left\langle \psi, N\left(\xi_{\mathbf{j}}, \sigma^2\right) \right\rangle \right] \right| + O\left(\frac{1}{N}\right).$$

We write $\omega_{\mathbf{j}} = \xi_{\mathbf{j}} + y_{\mathbf{j}}$ and, using the independence of the ξ- and y-fields, we get

$$P\left(|V_N|^{-1} \left| \sum_{\mathbf{j} \in V_N} \phi\left(\frac{\mathbf{j}}{N}\right) \left[\psi(\omega_{\mathbf{j}}) - \left\langle \psi, N\left(\xi_{\mathbf{j}}, \sigma^2\right) \right\rangle \right] \right| \ge a \right)$$
$$\le \sup_{\mathbf{b}} P\left(|V_N|^{-1} \left| \sum_{\mathbf{j} \in V_N} \phi\left(\frac{\mathbf{j}}{N}\right) \left(\psi(b_{\mathbf{j}} + y_{\mathbf{j}}) - \left\langle \psi, N\left(b_{\mathbf{j}}, \sigma^2\right) \right\rangle \right) \right| \ge a \right),$$

where the supremum is taken over $b = (b_{\mathbf{j}}) \in \mathbb{R}^{V_N}$. For any $b \in \mathbb{R}$,

$$\left| \left\langle \psi, N(b, \sigma^2) \right\rangle - \left\langle \psi, N(b, \gamma^2) \right\rangle \right| \le |\sigma - \gamma| \, \|\psi\|_{BL}.$$

Putting $\tau_{\mathbf{j}}^2 \equiv E^P[y_{\mathbf{j}}^2]$, we have, because $L \to \infty$ as $N \to \infty$,

$$N^{-d} \sum_{\mathbf{j} \in V_N} |\tau_{\mathbf{j}} - \sigma| \to 0$$

(cf. Lemma A.14). So we see that it suffices to prove that, uniformly in $\mathbf{b}$,

$$P\left(|V_N|^{-1} \left| \sum_{\mathbf{j}} \phi\left(\frac{\mathbf{j}}{N}\right) \left[\psi(b_{\mathbf{j}} + y_{\mathbf{j}}) - \left\langle \psi, N\left(b_{\mathbf{j}}, \tau_{\mathbf{j}}^2\right) \right\rangle \right] \right| \ge a \right) \le \exp(-cN^{d-\delta}), \tag{3.21}$$

for large enough N.

By the inequality (A.5),

$$E^P\left[\exp\left(\pm |V_N|^{-1} \sum_{\mathbf{j} \in V_N} \phi\left(\frac{\mathbf{j}}{N}\right) \left[\psi(b_{\mathbf{j}} + y_{\mathbf{j}}) - \left\langle \psi, N\left(b_{\mathbf{j}}, \tau_{\mathbf{j}}^2\right) \right\rangle \right] \right) \right]$$
$$\le \exp\left(\|\phi\|_\infty \|\psi\|_{BL} |V_N|^{-1} \sum_{\mathbf{j}, \mathbf{k} \in V_N} E^P[y_{\mathbf{j}} y_{\mathbf{k}}] \right).$$

Using this, together with Lemma 3.8, (3.21) follows in a standard way. □

As a direct consequence of Theorem 3.4 we get the following.

PROOF OF THE UPPER BOUND OF THEOREM 0.10. In view of Theorem 1.4, all we need to show is that, for each $\gamma \in \mathfrak{G}^{\mathbf{S}}(\mathbf{Q})$ and $M \in \mathbb{Z}^+$,

$$\lim_{\varepsilon \searrow 0} \limsup_{N \to \infty} \frac{1}{N^{d-2}} \log P(\mathbf{R}_N \in B_M(\gamma, \varepsilon)) \le -\mathscr{E}(\gamma | P), \tag{3.22}$$

where $B_M(\gamma, \varepsilon) \subseteq \mathscr{M}_1(\Omega_M)$ is the Prohorov ball of radius $\varepsilon > 0$ around γ. Assume $\gamma = \int_V \gamma_{\phi(x)}\, dx$ for some $\phi \in L^2(V)$. Then there is an $\varepsilon' > 0$ such that

$$\{\mathbf{R}_N \in B_M(\gamma, \varepsilon)\} \subseteq \left\{\mathbf{L}_N \in B_1\left(\int_V N(\phi(x), \sigma^2)\, dx, \varepsilon'\right)\right\}$$

$$= \left\{\langle \mathbf{Y}_N, \cdot \otimes 1\rangle_V \in B_1\left(\int_V N(\phi(x), \sigma^2)\, dx, \varepsilon'\right)\right\},$$

and (3.22) follows from the contraction principle and Theorem 3.4. □

4. Surface entropy. In this section we introduce a surface entropy for discontinuous profiles and derive the corresponding N^{d-1}-large deviation principle.

In Section 2 we have seen that for smooth profiles $\phi \in C^1(V; \mathbb{R})$, the relative entropy $\mathbf{H}_N(\gamma^{\phi_N}|P)$ grows at a capacity order N^{d-2}. We will show that for discontinuous ϕ, the entropy grows at a surface order N^{d-1}. More precisely, take ϕ of the form

$$\phi(x) = \sum_{i=1}^{n} \phi^i(x) 1_{V^i}(x), \qquad x \in V, \tag{4.1}$$

where $\{V^1, \dots, V^n\}$ is a partition of V into open sets V^i with piecewise smooth boundary and $\phi^i \in C^1(\overline{V^i}; \mathbb{R})$. Next let

$$W_{i,j} \equiv \overline{V^i} \cap \overline{V^j}, \qquad i \neq j$$

and let $\sigma_{i,j}$ be the surface measure on $W_{i,j}$. For $x \in W_{i,j}$, let $n_{i,j}(x) \in \mathbf{S}^{d-1}$ be the outer normal vector to ∂V_i at x and set

$$\nabla_{i,j}\phi(x) \equiv \lim_{t \searrow 0} \phi(x + t n_{i,j}(x)) - \phi(x - t n_{i,j}(x)).$$

Finally, write

$$\|y\|_{\mathbf{Q}} \equiv \sum_{\mathbf{k} \in \mathbb{Z}^d} (\mathbf{k} \cdot y)^+ Q(\mathbf{k}, \mathbf{0}), \qquad y \in \mathbb{R}^d.$$

PROPOSITION 4.2. *Take ϕ as in (4.1) and let γ^{ϕ_N} be the Gaussian field as in Lemma 2.2. Then*

$$\lim_{N \to \infty} \frac{1}{N^{d-1}} \mathbf{H}_N(\gamma^{\phi_N}|P) = \frac{1}{2} \mathscr{S}(\phi), \tag{4.3}$$

where

$$\mathscr{S}(\phi) = \tfrac{1}{2} \sum_{i \neq j} \int_{W_{i,j}} \left|\nabla_{i,j}\phi(x)\right|^2 \|n_{i,j}(x)\|_{\mathbf{Q}} \sigma_{i,j}(dx).$$

PROOF. In view of Lemma 2.2, we have

$$2\frac{1}{N^{d-1}}\mathbf{H}_N(\gamma^{\phi_N}|P) = \frac{1}{N^{d-1}}\sum_{\mathbf{k}\in V_N,\,\mathbf{l}\in V_N}\phi\left(\frac{\mathbf{k}}{N}\right)G_N^{-1}(\mathbf{k},\mathbf{l})\phi\left(\frac{\mathbf{l}}{N}\right)$$
$$= \frac{1}{N^{d-1}}\langle \phi_N, \mathbf{G}_N^{-1}\phi_N\rangle_{V_N}.$$

The summations over the exterior boundary terms are of the order N^{d-2} and can be neglected. We may therefore assume that ϕ has compact support in V and get

$$\lim_{N\to\infty}\frac{1}{N^{d-1}}\langle \phi_N, \mathbf{G}_N^{-1}\phi_N\rangle_{V_N}$$
$$= \lim_{N\to\infty}\frac{1}{N^{d-1}}\frac{1}{2}\langle |\nabla_{\mathbf{Q}}\phi_N|^2\rangle_{\mathbb{Z}^d}$$
$$= \lim_{N\to\infty}\frac{1}{2}\frac{1}{N^{d-1}}\sum_{\mathbf{k}\in\mathbb{Z}^d}\sum_{\mathbf{l}\in\mathbb{Z}^d}Q(\mathbf{k},\mathbf{l})\left(\phi\left(\frac{\mathbf{k}}{N}\right)-\phi\left(\frac{\mathbf{l}}{N}\right)\right)^2$$
$$= \frac{1}{2}\sum_{i,j=1}^{n}\lim_{N\to\infty}\frac{1}{N^{d-1}}\sum_{\mathbf{k}\in V_N^i}\sum_{\mathbf{l}\in V_N^j}Q(\mathbf{k},\mathbf{l})\left(\phi\left(\frac{\mathbf{k}}{N}\right)-\phi\left(\frac{\mathbf{l}}{N}\right)\right)^2,$$

where $V_N^i = \{k\in\mathbb{Z}^d\colon \mathbf{k}/N\in V^i\}$. Using the smoothness of ϕ^i inside V^i, we see that

$$\lim_{N\to\infty}\frac{1}{N^{d-1}}\sum_{\mathbf{k}\in V_N^i}\sum_{\mathbf{l}\in V_N^i}Q(\mathbf{k},\mathbf{l})\left(\phi\left(\frac{\mathbf{k}}{N}\right)-\phi\left(\frac{\mathbf{l}}{N}\right)\right)^2 = 0,$$

and (4.3) follows from

$$\lim_{N\to\infty}\frac{1}{N^{d-1}}\sum_{\mathbf{k}\in V_N^i}\sum_{\mathbf{l}\in V_N^j}Q(\mathbf{k},\mathbf{l})\left(\phi\left(\frac{\mathbf{k}}{N}\right)-\phi\left(\frac{\mathbf{l}}{N}\right)\right)^2$$
$$(4.4)\qquad = \int_{W_{i,j}}|\nabla_{i,j}\phi(x)|^2\|n_{i,j}(x)\|_{\mathbf{Q}}\sigma_{i,j}(dx),\quad \text{for } i\neq j.$$

We will prove (4.4) assuming that $d = 3$ and $\mathbf{Q}$ is the transition function of the simple random walk [cf. (0.1)]. The general case can be shown along the same lines.

Since ϕ^i and ϕ^j are continuous on V^i and V^j, we can take $\phi^i(x)\equiv f^i$ and $\phi^j(x)\equiv f^j$ to be constant. Using the smoothness of the boundaries and a suitable triangularization, we may assume that $W_{i,j}$ is the triangle spanned between $\mathbf{a} = (a,0,0)$, $\mathbf{b} = (0,b,0)$ and $\mathbf{c} = (0,0,c)$:

$$W_{i,j} = \{x = (x_1,x_2,x_3) = \varepsilon_1\mathbf{a}+\varepsilon_2\mathbf{b}+\varepsilon_3\mathbf{c},\ 0\le\varepsilon_i\le 1,\ \varepsilon_1+\varepsilon_2+\varepsilon_3 = 1\}.$$

In this case the normal vector is constant with

$$n_{i,j} = (bc, ac, ab)'/\sqrt{b^2c^2 + a^2c^2 + a^2b^2}.$$

If $|W_{i,j}|$ denotes the surface of $W_{i,j}$ we get

$$\begin{aligned}\int_{W_{i,j}} |\nabla_{i,j}\phi(x)|^2 \|n_{i,j}(x)\|_{\mathbf{Q}} \sigma_{i,j}(dx) &= |f^i - f^j|^2 \|n_{i,j}\|_{\mathbf{Q}} |W_{i,j}| \\ &= \frac{1}{6}|f^i - f^j|^2\left(\frac{|bc|}{2} + \frac{|ac|}{2} + \frac{|ab|}{2}\right).\end{aligned}$$

Next, let $W_{i,j}(u)$ be the orthogonal projections of $W_{i,j}$ onto the planes $\{x_u = 0\}$, $u = 1, 2, 3$. Let $R(i, j; N)$ be the number of bonds $(\mathbf{k}, \mathbf{l}) \subseteq V^i \times V^j$ cut by the plane $N \cdot W_{i,j}$. Then, up to a boundary effect of order N, we have

$$\begin{aligned}R(i, j; N) &= N^2(|W_{i,j}(1)| + |W_{i,j}(2)| + |W_{i,j}(3)|) + O(N) \\ &= N^2\left(\frac{|bc|}{2} + \frac{|ac|}{2} + \frac{|ab|}{2}\right) + O(N).\end{aligned}$$

However, this implies

$$\begin{aligned}&\lim_{N\to\infty} \frac{1}{N^2} \sum_{\mathbf{k}\in V_N^i} \sum_{\mathbf{l}\in V_N^j} Q(\mathbf{k}, \mathbf{l})\left(\phi\left(\frac{\mathbf{k}}{N}\right) - \phi\left(\frac{\mathbf{l}}{N}\right)\right)^2 \\ &\quad= \lim_{N\to\infty} \frac{1}{N^2} \frac{1}{6} R(i, j; N)|f^i - f^j|^2 \\ &\quad= \frac{1}{6}\left(\frac{|bc|}{2} + \frac{|ac|}{2} + \frac{|ab|}{2}\right)|f^i - f^j|^2\end{aligned}$$

and shows (4.4). □

Note that we could consider the lower semicontinuous extension of $\mathscr{S}$ on $L^2(V)$. In particular, if $\psi \in L^2(V)$ is such that $\mathscr{E}(\psi) < \infty$, then we can find discontinuous ϕ_n, $n \in \mathbb{Z}^+$, converging to ψ such that $\mathscr{S}(\psi) \le \liminf_{n\to\infty} \mathscr{S}(\phi_n) = 0$.

Also, it is interesting to compare the solutions of the following variational problems. Let $d = 3$ and let $\mathbf{Q}$ be the transition matrix of the simple random walk [cf. (0.1)]. We want to look at profiles which minimize the surface, respectively, the capacity, rate function for a distribution concentrated on two points 0 and $M > 0$. Of course in the second case we have to consider a suitable approximation. More precisely, for $\delta \ge 0$ and $0 < p < \frac{1}{2}$, set

$$\begin{aligned}A_\delta(p) = \{\phi \in L^2(V): &|\{x: \phi(x) = 0\}| = 1 - p, \\ &|\{x: \phi(x) = M\}| = p - \delta\}.\end{aligned}$$

Then the solution of $\mathscr{S}(\phi^*) = \inf\{\mathscr{S}(\phi): \phi \in A_0(p)\}$ is achieved at

$$\phi^*(x) = M \cdot 1_{V(p)}(x), \quad \text{where } V(p) = \begin{cases} (0, p^{1/3})^3, & 0 < p < (\frac{2}{3})^6, \\ (0, p^{1/2})^2 \times (0,1), & (\frac{2}{3})^6 < p < \frac{1}{4}, \\ (0,p) \times (0,1)^2, & \frac{1}{4} < p < \frac{1}{2}, \end{cases}$$

whereas if we are looking at $\mathscr{E}(\psi_\delta^*) = \inf\{\mathscr{E}(\phi): \phi \in A_\delta(p)\}$, $\delta > 0$, and then let $\delta \searrow 0$, we get, by Example 2.16,

$$\psi^*(x) = \lim_{\delta \to 0} \psi_\delta^*(x) = M \cdot 1_W(x), \quad \text{where } W = \left\{x \in V: |x|^3 < \frac{p}{\kappa_3}\right\}.$$

This shows that approximate and exact canonical distributions can be quite different!

In view of Lemma 4.2 it is clear that we can get a *lower bound* of the large deviations of $\mathbf{X}_N$ for discontinuous profiles ϕ at the order N^{d-1}. However, with respect to the topology introduced in Section 3, any open set containing ϕ will also contain a smooth ψ for which $\mathscr{E}(\psi) < \infty$ and $\mathscr{S}(\psi) = 0$. So if we want to deal effectively with discontinuous profiles as before, we must find the suitable topology. This problem is not trivial. We illustrate this phenomenon only in the case where one looks for discontinuities of the one-dimensional profile obtained from arithmetic means along hyperplanes orthogonal to the first of the coordinate axes. Even that restricted situation requires some care. We hope that the approach will be extendable to cope with more general surface phenomena.

To keep things as simple as possible, we also restrict ourselves to the simple random walk [cf. (0.1)].

We introduce $\mathbf{W}_N: \Omega \to \mathscr{M}([0,1])$ defined by

$$\mathbf{W}_N(\omega) \equiv \sum_{j=1}^{N} \nabla_j S_N(\omega) \delta_{j/N},$$

where

$$\nabla_j S_N \equiv S_{N,j} - S_{N,j-1}, \qquad S_{N,j}(\omega) \equiv N^{-d+1} \sum_{(j_2,\dots,j_d) \in V_N^{(d-1)}} \omega_{(j,j_2,\dots,j_d)},$$

with $S_{N,0}(\omega) \equiv 0$ and $V_N^{(d-1)} \equiv [1,N]^{d-1} \cap \mathbb{Z}^{d-1}$.

Let E_0 be the set of finite linear combinations of Dirac measures in $[0,1]$. We introduce a Skorohod-type metric d_s on E_0: If $\alpha, \beta \in E_0$, $d_s(\alpha, \beta)$ is the infimum over numbers $\varepsilon > 0$ such that there exists a continuous, strictly increasing function $\lambda: [0,1] \to [0,1]$, satisfying $\lambda(0) = 0$, $\lambda(1) = 1$, $\max|\lambda(t) - t| \le \varepsilon$ and

$$\sup_{t \in [0,1]} \left|\alpha(\{\lambda^{-1}(t)\}) - \beta(\{t\})\right| \le \varepsilon.$$

We define a rate function I_0 on E_0,

$$I_0\left(\sum_{j=1}^{m} x_j \delta_{\xi_j}\right) \equiv \frac{1}{4d} \sum_{j=1}^{m} x_j^2.$$

It is obvious that I_0 is lower semicontinuous (but not continuous) on E_0.

Let (E, d_s) be the completion of (E_0, d_s), which is a Polish space, and let I: $E \to [0, \infty]$ be defined by

$$I(\alpha) \equiv \lim_{\substack{\beta \to \alpha \\ \beta \in E_0}} I_0(\beta).$$

LEMMA 4.5. *I has compact level sets.*

PROOF. It suffices to prove that any sequence $(\alpha_n)_{n \in \mathbb{N}}$, $\alpha_n \in E_0$, satisfying

$$\rho \equiv \sup_n I_0(\alpha_n) < \infty,$$

has a Cauchy subsequence. If $\alpha_n = \sum_{i=1}^{m(n)} x_i^{(n)} \delta_{\xi_i^{(n)}}$, then

$$\max_i |x_i^{(n)}| \le c\sqrt{\rho}.$$

Fix $k \in \mathbb{N}$ and let

$$\Lambda_n(k) = \left\{ j\colon 1 \le j \le m(n), |x_j^{(n)}| \ge \frac{1}{k} \right\}.$$

Obviously $|\Lambda_n(k)| \le c\rho k^2$.

We may pass to a subsequence of (n) along which $|\Lambda_n(k)|$ is constant (k fixed). We denote this subsequence again by (n). We may identify all the $\Lambda_n(k)$: $\Lambda_n(k) = \Lambda(k)$ and, passing again to a subsequence, we may assume that the $\xi_i^{(n)}$ and $x_i^{(n)}$, $i \in \Lambda(k)$, are convergent. Therefore, we get

$$\limsup_{n, m \to \infty} d_s(\alpha_n, \alpha_m) \le \frac{1}{k}.$$

As k is arbitrary, we may pass to a Cauchy subsequence. □

Our main result in this section is the following theorem.

THEOREM 4.6. $\mathbf{W}_N$ *satisfies an* N^{d-1}*-LDP with rate function I.*

Let $\alpha = \sum_{i=1}^{n} x_i \delta_{\xi_i} \in E_0$ with $0 = \xi_0 < \xi_1 < \cdots < \xi_n < \xi_{n+1} = 1$ and set

$$V^i = (\xi_{i-1}, \xi_i) \times (0,1)^{d-1} \subseteq \mathbb{R}^d, \qquad i = 1, \dots, n.$$

The profile $\phi \in L^2(V)$ corresponding to α is then given by

$$\phi(x) = \sum_{i=1}^{n} f^i 1_{V^i}(x), \quad \text{where } f^i = \sum_{j=1}^{i} x_j, \ i = 1, \dots, n,$$

and

$$\mathscr{S}(\phi) = I(\alpha) = \frac{1}{4d} \sum_{i=1}^{n} x_i^2.$$

From this and (4.3) it is not too difficult to show the lower bound in Theorem 4.6 by the usual entropy argument. The upper bound will be proved in several steps. We first need estimates of the covariances of the $\nabla_j S_N$.

LEMMA 4.7.

$$E^P\left[\nabla_i S_N \nabla_j S_N\right] = 2d\delta_{i,j}N^{-d+1} + O(N^{-d}\log N).$$

The proof will be given in the Appendix.

Let us first prove exponential tightness of $\mathbf{W}_N$. For two sequences $\rho = (\rho_n)$ and $\mathbf{l} = (l_n)$ of positive real numbers satisfying $\rho_n \searrow 0$ and $l_n \nearrow \infty$, let

$$K_0(\rho,\mathbf{l}) \equiv \bigcap_n G(\rho_n, l_n),$$

where

$$G(\rho, l) \equiv \left\{\sum x_i \delta_{\xi_i} \in E_0 \colon \sum_{i\colon |x_i| \geq \rho} |x_i| \leq l\right\}.$$

The same kind of argument as that used in Lemma 4.5 proves the following.

LEMMA 4.8. *The closure K of K_0 is compact.*

We now want to show that, for appropriately chosen sequences, $P(\mathbf{W}_N \notin K)$ is small; this is more precisely stated as follows.

LEMMA 4.9. *let $\boldsymbol{\rho} = (\rho_n)$ be given by $\rho_n = n^{-1/4}$ and $\mathbf{l}^{(r)} = (l_n^{(r)})$ by $l_n^{(r)} = r^2 n^2$, $r \in \mathbb{N}$. Then*

$$\lim_{r\to\infty} \limsup_{N\to\infty} N^{-d+1} \log P\left(\mathbf{W}_N \notin K(\boldsymbol{\rho}, \mathbf{l}^{(r)})\right) = -\infty.$$

PROOF.

$$P\left(\mathbf{W}_N \notin K(\boldsymbol{\rho}, \mathbf{l}^{(r)})\right) \leq \sum_{m=1}^{\infty} P\left(\mathbf{W}_N \notin G(m^{-1/4}, r^2 m^2)\right).$$

If $\rho > 0$, let $\nu_N(\rho)$ be the number of elements $i \in \{1,\dots,N\}$ for which $|\nabla_i S_N| \geq \rho$. Then

$$\begin{aligned}
&P\left(\mathbf{W}_N \notin G(m^{-1/4}, r^2 m^2)\right)\\
&\quad\leq \sum_{k=1}^{2rm} P\left(\mathbf{W}_N \notin G(m^{-1/4}, r^2m^2), \nu_N(m^{-1/4}) = k\right)\\
&\qquad + 1_{\{2rm \leq N\}} P\left(\nu_N(m^{-1/4}) > 2rm\right)\\
&\quad\leq 2rmNP\left(|\nabla_1 S_N| \geq \frac{rm}{2}\right) + 1_{\{2rm \leq N\}} P\left(\left|\sum_{j=1}^{rm} \nabla_{i_j} S_N\right| \geq rm^{3/4}\right)\\
&\quad= A_1(r,m,N) + A_2(r,m,N),
\end{aligned}$$

for some subset $\{i_1,\dots,i_{rm}\}$ of $\{1,\dots,N\}$.

By Lemma 4.7 we have

$$A_1(r,m,N) \leq c_1 rmN \exp\big(-c_2 r^2 m^2 N^{d-1}\big).$$

From this, we easily get

$$\lim_{r\to\infty} \limsup_{N\to\infty} N^{-d+1} \log \sum_{m=1}^{\infty} A_1(r,m,N) = -\infty.$$

Again by Lemma 4.7, we get

$$\operatorname{var}\left(\sum_{j=1}^{rm} \nabla_{i_j} S_N\right) \leq c\big(rmN^{-d+1} + r^2 m^2 N^{-d} \log N\big).$$

Therefore,

$$A_2(r,m,N) \leq c_1 N^{rm} \exp\left(-c_2 \frac{rm^{1/2} N^{d-1}}{1 + rmN^{-1}\log N}\right).$$

Note that, for $rm \leq N$,

$$rm \log N - c\frac{rm^{1/2} N^{d-1}}{1 + rmN^{-1}\log N} \leq -\frac{c}{2}\frac{rm^{1/2} N^{d-1}}{1 + rmN^{-1}\log N}$$

if N is larger than some fixed N_0. Therefore,

$$A_2(r,m,N) \leq c_1 1_{\{rm \leq N\}} \exp\left(-\frac{c_2}{2}\frac{rm^{1/2} N^{d-1}}{1 + rmN^{-1}\log N}\right).$$

This immediately gives

$$\lim_{r\to\infty} \limsup_{N\to\infty} N^{-d+1} \log \sum_{m=1}^{[N/r]} A_2(r,m,N) = -\infty$$

and proves the lemma. □

PROOF OF THE UPPER BOUND IN THEOREM 4.6. We introduce an approximating rate function: If $\delta > 0$, $\alpha = \sum_j x_j \delta_{\xi_j} \in E_0$, let

$$I_\delta(\alpha) = \frac{1}{4d} \sum_{i:\,|x_i|>\delta} x_i^2.$$

I_δ is lower semicontinuous and is extended to a lower semicontinuous function on E. Obviously, $I_\delta \leq I$ and it is straightforward that, for any compact set K,

$$\lim_{\delta\downarrow 0} \inf_{\alpha\in K} I_\delta(\alpha) = \inf_{\alpha\in K} I(\alpha). \tag{4.10}$$

Using this and Lemma 4.9, we see that it suffices to prove that, for any δ, $\lambda > 0$ and $\alpha \in E_0$, we can find $\varepsilon > 0$ such that

$$\limsup_{N\to\infty} N^{-d+1} \log P\big(\mathbf{W}_N \in B_{d_s}(\alpha,\varepsilon)\big) \leq -I_\delta(\alpha) + \lambda. \tag{4.11}$$

Let $\alpha = \sum_{j=1}^{n} x_j \delta_{\xi_j}$, $I_\delta(\alpha) = k$. Let $M \equiv [4\,dk/\delta^2] + 1$. Then there is a subset of indices $\{j_1, \dots, j_L\} \subset \{1, \dots, n\}$ with $L \le M$, $|x_{j_i}| > \delta$ and $(1/4d)\sum_{i=1}^{L} x_{j_i}^2 \ge k$. We choose now $\varepsilon = \varepsilon(\delta, k, \lambda)$ such that, for any $\beta \in B_{d_s}(\alpha, \varepsilon) \cap E_0$ of the form $\beta = \sum_{j=1}^{n'} y_j \delta_{\eta_j}$, there exist L indices with $(1/4d)\sum_{i=1}^{L} y_{j_i}^2 \ge k - \lambda$. Therefore,

$$P\big(\mathbf{W}_N \in B_{d_s}(\alpha, \varepsilon)\big) \le \binom{N}{M} \max_{j_1, \dots, j_M} P\left(\frac{1}{4d} \sum_{i=1}^{m} (\nabla_{j_i} S_N)^2 \ge k - \lambda\right)$$

and (4.11) follows from Lemma 4.7, and the upper bound is proved. □

APPENDIX

In this Appendix we prove a few useful covariance inequalities.

We start with a quite general inequality:

LEMMA A.1. *Let $P \in \mathscr{M}_1(\mathbb{R}^n)$ be the centered Gaussian law with covariance matrix $\Gamma \in \mathbb{R}^{n\times n}$. Then, for $t \ge 0$,*

$$P\left(\sum_{i=1}^{n} x_i^2 \ge \operatorname{tr}(\Gamma) + t\right) \le \exp\left[-\frac{1}{8}\min\left(\frac{t}{\overline{\Gamma}}, \frac{t^2}{\operatorname{tr}(\Gamma^2)}\right)\right], \tag{A.2}$$

where $\overline{\Gamma} \equiv \max_i \sum_{j=1}^{n} |\Gamma_{ij}|$ and $\operatorname{tr}(\ )$ denotes the trace of a matrix.

PROOF. Let $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n \ge 0$ be the eigenvalues of Γ. Note that $\lambda_1 \le \overline{\Gamma}$. If $0 < a < 1/\lambda_1$, then

$$\begin{aligned}
\log E^P\left[\exp\left(\frac{1}{2}a \sum_{j=1}^{n} x_j^2\right)\right] &= \log \prod_{j=1}^{n} (1 - a\lambda_j)^{-1/2} = -\frac{1}{2}\sum_{j=1}^{n} \log(1 - a\lambda_j) \\
&= \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{\infty} \frac{1}{k}(a\lambda_j)^k \\
&\le \frac{a}{2}\sum_{j=1}^{n} \lambda_j + \frac{1}{2}\sum_{j=1}^{n}\sum_{k=0}^{\infty} \frac{1}{k+2}(a\lambda_j)^2 (a\lambda_1)^k \\
&\le \frac{1}{2}\operatorname{tr}(\Gamma) \\
&\quad + \frac{a^2}{2}\operatorname{tr}(\Gamma^2)\frac{1}{(a\lambda_1)^2}(-\log(1 - a\lambda_1) - a\lambda_1).
\end{aligned} \tag{A.3}$$

Therefore, if $a\lambda_1 \le \frac{1}{2}$,

$$\begin{aligned}
P\left(\sum_{j=1}^{n} x_j^2 \ge \operatorname{tr}(\Gamma) + x\right) &= P\left(\exp\left[\frac{a}{2}\sum_{j=1}^{n} x_j^2\right] \ge \exp\left[\frac{a}{2}(\operatorname{tr}(\Gamma) + t)\right]\right) \\
&\le \exp\left(-\frac{a}{2}t + \frac{a^2}{2}\operatorname{tr}(\Gamma^2)\right).
\end{aligned}$$

If $\operatorname{tr}(\Gamma^2) \le t\lambda_1$, then

$$\sup_{a \le 1/2\lambda_1} \left(\frac{at}{2} - \frac{a^2}{2} \operatorname{tr}(\Gamma^2) \right) \ge \frac{t}{4\lambda_1} - \frac{1}{8\lambda_1^2} \operatorname{tr}(\Gamma^2) \ge \frac{t}{8\lambda_1} \ge \frac{t}{8\bar{\Gamma}}.$$

If $\operatorname{tr}(\Gamma^2) \ge t\lambda_1$, then $t/(2\operatorname{tr}(\Gamma^2)) \le 1/(2\lambda_1)$; therefore,

$$\sup_{a < 1/(2\lambda_1)} \left(\frac{at}{2} - \frac{a^2}{2} \operatorname{tr}(\Gamma^2) \right) = \frac{t^2}{8\operatorname{tr}(\Gamma^2)}. \qquad \square$$

Let $\mathscr{L}(\mathbb{R}^n)$ be the set of Lipschitz continuous $f\colon \mathbb{R}^n \to \mathbb{R}$ and write

$$\delta_k(f) \equiv \sup\left\{ \frac{|f(x) - f(y)|}{|x_k - y_k|} : x_j = y_j \text{ for } j \ne k \right\}.$$

Let $P \in \mathscr{M}_1(\mathbb{R}^n)$ be as in Lemma A.1.

LEMMA A.4. *For any $f \in \mathscr{L}(\mathbb{R}^n)$, we have*

$$E^P[\exp[f - \langle f \rangle_P]] \le \exp\left[\tfrac{1}{2} \sum_{k,j=1}^{n} \delta_k(f) |\Gamma_{k,j}| \delta_j(f) \right]. \tag{A.5}$$

PROOF. Using a suitable approximation, we may assume that $f \in C^1(\mathbb{R}^n; \mathbb{R})$. In this case,

$$\delta_k(f) = \left\| \frac{\partial}{\partial x_k} f \right\|_\infty.$$

Let $A \equiv \Gamma^{1/2}$ and set

$$\Psi(x) \equiv f(A \cdot x), \qquad x \in \mathbb{R}^n.$$

Let $\mathscr{W}$ be the Wiener measure on $C([0,1]; \mathbb{R}^n)$ and let $\{P_t,\ t > 0\}$ be the corresponding semigroup. Then by Itô's formula we have

$$\begin{aligned}\Psi(W_1) &= E^{\mathscr{W}}[\Psi(W_1)] + \int_0^1 \nabla P_{1-t}\Psi(W_t) \cdot dW_t \\ &= E^{\mathscr{W}}[\Psi(W_1)] + \int_0^1 P_{1-t}(\nabla\Psi)(W_t) \cdot dW_t.\end{aligned}$$

Therefore,

$$\begin{aligned}1 &= E^{\mathscr{W}}\left[\exp\left[\int_0^1 P_{1-t}(\nabla\Psi)(W_t) \cdot dW_t - \tfrac{1}{2}\int_0^1 |P_{1-t}(\nabla\Psi)(W_t)|^2\,dt\right]\right] \\ &\ge E^{\mathscr{W}}\left[\exp\left[\Psi(W_1) - E^{\mathscr{W}}[\Psi(W_1)] - \tfrac{1}{2}\int_0^1 P_{1-t}(|\nabla\Psi|^2)(W_t)\,dt\right]\right].\end{aligned}$$

This yields

$$E^P\big[\exp[f - E^P[f]]\big] = E^{\mathscr{W}}\big[\exp[\Psi(W_1) - E^{\mathscr{W}}[\Psi(W_1)]]\big]$$
$$\le \exp\bigg[\tfrac{1}{2}\sup_{x\in\mathbb{R}^n} |\nabla\Psi|^2(x)\bigg] = \exp\bigg[\tfrac{1}{2}\sup_{x\in\mathbb{R}^n} \nabla f(x)\cdot\Gamma\,\nabla f(x)\bigg]$$
$$\le \exp\bigg[\tfrac{1}{2}\sum_{i,j=1}^{n} \delta_i(f)|\Gamma_{i,j}|\delta_j(f)\bigg]. \qquad \square$$

We write $\mathbb{P}_{\mathbf{i}}$, $\mathbf{i}\in\mathbb{Z}^d$, for the law of a discrete-time random walk $\{\eta_t: t\in\mathbb{N}_0\}$ ($\mathbb{N}_0 \equiv \mathbb{N}\cup\{0\}$) on $\mathbb{Z}^d$ with transition probabilities $Q(\mathbf{i},\mathbf{j})$ starting in $\mathbf{i}$. Let $\tau_L \equiv \inf\{t\in\mathbb{N}_0: \eta_t \in L\mathbb{Z}^d\}$. Obviously, $\tau_L < \infty$ $\mathbb{P}_{\mathbf{i}}$-a.s. for all $\mathbf{i}\in\mathbb{Z}^d$. If $\mathbf{i}$, $\mathbf{k}\in\mathbb{Z}^d$, we define

$$q_{\mathbf{i}}(\mathbf{k}) \equiv \mathbb{P}_{\mathbf{i}}(\eta_{\tau_L} = \mathbf{k}L).$$

Clearly, $q_{\mathbf{i}L}(\mathbf{k}) = \delta_{\mathbf{i},\mathbf{k}}$ for $\mathbf{i}\in\mathbb{Z}^d$.

LEMMA A.6.

$$\sum_{\mathbf{k}\in\mathbb{Z}^d} q_{\mathbf{i}}(\mathbf{k})G(\mathbf{j},\mathbf{k}L) = G(\mathbf{i},\mathbf{j}) - G^L(\mathbf{i},\mathbf{j}), \quad \textit{where } G^L(\mathbf{i},\mathbf{j}) \equiv \mathbb{E}_{\mathbf{i}}\Bigg[\sum_{t=0}^{\tau_L-1} 1_{\{\eta_t=\mathbf{j}\}}\Bigg].$$

PROOF.

$$\sum_{\mathbf{k}\in\mathbb{Z}^d} q_{\mathbf{i}}(\mathbf{k})G(\mathbf{j},\mathbf{k}L) = \sum_{\mathbf{k}\in\mathbb{Z}^d} \mathbb{P}_{\mathbf{i}}(\eta_{\tau_L} = \mathbf{k}L)\mathbb{E}_{\mathbf{k}L}\Bigg[\sum_{t=0}^{\infty} 1_{\{\eta_t=\mathbf{j}\}}\Bigg]$$
$$= \mathbb{E}_{\mathbf{i}}\Bigg[\sum_{t=\tau_L}^{\infty} 1_{\{\eta_t=\mathbf{j}\}}\Bigg] = G(\mathbf{i},\mathbf{j}) - G^L(\mathbf{i},\mathbf{j}). \qquad \square$$

Note that $G^L(\mathbf{i},\mathbf{j}) = 0$ if $\mathbf{i},\mathbf{j}\in L\mathbb{Z}^d$. It is essential that $G^L(\mathbf{i},\mathbf{j})$ is rapidly decaying in $|\mathbf{i}-\mathbf{j}|$; this is more precisely stated as follows.

LEMMA A.7. *There exist* $c_1, c_2 > 0$ *such that*

$$G^L(\mathbf{i},\mathbf{j}) \le c_1\exp(-c_2|\mathbf{i}-\mathbf{j}|/L^{d/2}),$$

for all $\mathbf{i},\mathbf{j}\in\mathbb{Z}^d$, $L\in\mathbb{N}$.

PROOF. If $\mathbf{j}\in\mathbb{Z}^d$, we write $\tau_{\mathbf{j}}$ for the first entrance time into $\mathbf{j}$:

$$G^L(\mathbf{i},\mathbf{j}) = E_{\mathbf{i}}\Bigg[\sum_{t=\tau_{\mathbf{j}}}^{\tau_L-1} 1_{\{\eta_t=\mathbf{j}\}}; \tau_{\mathbf{j}} < \tau_L\Bigg] = \mathbb{P}_{\mathbf{i}}(\tau_{\mathbf{j}} < \tau_L)\mathbb{E}_{\mathbf{j}}\Bigg[\sum_{t=0}^{\tau_L-1} 1_{\{\eta_t=\mathbf{j}\}}\Bigg]$$
$$\le G(\mathbf{0},\mathbf{0})\mathbb{P}_{\mathbf{i}}(\tau_{\mathbf{j}} < \tau_L) \le G(\mathbf{0},\mathbf{0})\big[\mathbb{P}_{\mathbf{i}}(\tau_{\mathbf{j}} < T) + \mathbb{P}_{\mathbf{i}}(\tau_L > T)\big],$$

for any $T > 0$. We claim that

$$\mathbb{P}_{\mathbf{i}}(\tau_L > T) \le \exp(-cT/L^d), \tag{A.8}$$

$$\mathbb{P}_{\mathbf{i}}(\tau_{\mathbf{j}} < T) \le c_1 \exp\bigl(-c_2|\mathbf{i}-\mathbf{j}|^2/T\bigr), \tag{A.9}$$

the last estimate for $T \ge |\mathbf{i}-\mathbf{j}|$.

The desired estimate for Lemma A.7 follows from these estimates by choosing $T \equiv |\mathbf{i}-\mathbf{j}|L^{d/2}$.

PROOF OF (A.8). Instead of investigating τ_L for η_t, we can change to a random walk $\eta_t^{(L)}$ on the finite torus $\{0,\dots,L-1\}^d$ (identifying L with 0) and investigate the first entrance time into $\mathbf{0}$. It is well known that, for $s \ge L^2$, the distribution of $\eta_s^{(L)}$ is close to the uniform distribution if $\mathbf{Q}$ is aperiodic; the periodic case needs a slight modification, which we omit. More precisely, there exists $c > 0$ such that, for $s \ge L^2$,

$$\mathbb{P}_{\mathbf{i}}\bigl(\eta_s^{(L)} = \mathbf{0}\bigr) = \mathbb{P}_{\mathbf{i}}\bigl(\eta_s \in L\mathbb{Z}^d\bigr) \ge cL^{-d}. \tag{A.10}$$

This follows, for example, immediately from a local central limit theorem with error rate (see [26], Section 26):

$$\begin{aligned}\mathbb{P}_{\mathbf{i}}(\eta_s = \mathbf{j}) &= (2\pi s)^{-d/2}|A|^{-1/2}\exp\biggl(-\frac{1}{2s}\langle \mathbf{j}-\mathbf{i}, A^{-1}(\mathbf{j}-\mathbf{i})\rangle\biggr)\\ &\quad + O(s^{-(d+1)/2}),\end{aligned} \tag{A.11}$$

where A is the $d \times d$ covariance matrix of the distribution $\mathbf{Q}$ [cf. (0.4)]. For small s, $s \le 2L^2$, we have an estimate from below:

$$\mathbb{P}_{\mathbf{i}}\bigl(\eta_s \in L\mathbb{Z}^d\bigr) \ge c\min(s^{-d/2}, 1). \tag{A.12}$$

Equation (A.10) is fairly immediate from (A.11), too.

Using (A.10) and (A.12) (and $d \ge 3$), we get

$$\begin{aligned} cL^{-d+2} &\le \sum_{t=L^2}^{2L^2} \mathbb{P}_i\bigl(\eta_t \in L\mathbb{Z}^d\bigr) \le \mathbb{E}_{\mathbf{i}}\Biggl[\sum_{t=0}^{2L^2} 1_{\{\eta_t \in L\mathbb{Z}^d\}}\Biggr]\\ &\le \mathbb{P}_{\mathbf{i}}\bigl(\tau_L \le 2L^2\bigr)\mathbb{E}_{\mathbf{0}}\Biggl[\sum_{t=0}^{2L^2} 1_{\{\eta_t \in L\mathbb{Z}^d\}}\Biggr] \le c'\mathbb{P}_{\mathbf{i}}\bigl(\tau_L \le 2L^2\bigr), \end{aligned}$$

uniformly in $\mathbf{i}$, that is,

$$\mathbb{P}_{\mathbf{i}}\bigl(\tau_L \le 2L^2\bigr) \ge cL^{-d+2}.$$

By the usual renewal argument,

$$\mathbb{P}_{\mathbf{i}}\bigl(\tau_L > 2kL^2\bigr) \le \bigl(1 - cL^{-d+2}\bigr)^k \le \exp(-ckL^{-d+2}),$$

which proves (A.8). □

PROOF OF (A.9).

$$\mathbb{P}_{\mathbf{i}}(\tau_{\mathbf{j}} < T) \le \mathbb{P}_{\mathbf{0}}\Big(\sup_{0\le s\le T} |\eta_s| \ge |\mathbf{i}-\mathbf{j}|\Big) \le 2 \sup_{0\le s\le T} \mathbb{P}_{\mathbf{0}}\big(|\eta_s| \ge \tfrac{1}{2}|\mathbf{i}-\mathbf{j}|\big),$$

by a maximal inequality. The right-hand side is now estimated by the usual large deviation estimates for sums of i.i.d. vectors, which lead to (A.6) for $|\mathbf{i}-\mathbf{j}| \ge T$. This proves our lemma. □

We return to the original setting of the paper; in particular, $P \in \mathscr{M}_1^{\mathbf{S}}(\Omega)$ is the centered Gaussian field with covariances $\mathbf{G}$. Let

$$\xi_{\mathbf{i}}(\omega) \equiv \sum_{\mathbf{k}\in\mathbb{Z}^d} q_{\mathbf{j}}(\mathbf{k})\omega_{\mathbf{k}L} \quad\text{and}\quad y_i(\omega) \equiv \omega_{\mathbf{i}} - \xi_{\mathbf{i}}(\omega), \quad \mathbf{i}\in\mathbb{Z}^d.$$

LEMMA A.13. (a) $E^P[y_{\mathbf{i}}y_{\mathbf{j}}] = G^L(\mathbf{i},\mathbf{j})$.
(b) *The fields* $(y_{\mathbf{i}})_{\mathbf{i}\in\mathbb{Z}^d}$ *and* $(\xi_{\mathbf{i}})_{\mathbf{i}\in\mathbb{Z}^d}$ *are independent.*

PROOF. For (b), it suffices to prove that $y_{\mathbf{i}}$ and $\xi_{\mathbf{k}L}$ are uncorrelated for any $\mathbf{i},\mathbf{k}\in\mathbb{Z}^d$:

$$\begin{aligned} E^P[y_{\mathbf{i}}\xi_{\mathbf{k}L}] &= G(\mathbf{i},\mathbf{k}L) - \sum_{\mathbf{j}\in\mathbb{Z}^d} q_{\mathbf{i}}(\mathbf{j})G(\mathbf{j}L,\mathbf{k}L) \\ &= G(\mathbf{i},\mathbf{k}L) - G(\mathbf{i},\mathbf{k}L) + G^L(\mathbf{i}L,\mathbf{k}L) \\ &= 0, \end{aligned}$$

by using Lemma A.6. Part (a) is immediate, too. □

LEMMA A.14. *Let* $L = L(N) = \log N$ *and* $\sigma^2 = G(\mathbf{0},\mathbf{0})$. *Then*

$$\lim_{N\to\infty} N^{-d}\sum_{\mathbf{i}\in V_N}\Big|\sigma - \sqrt{G^L(\mathbf{i},\mathbf{i})}\Big| = 0 \quad\textit{and}\quad \lim_{N\to\infty} N^{-d}\sum_{\mathbf{i}\in V_N} E^P\big[\xi_{\mathbf{i}}^2\big] = 0.$$

PROOF. Simply note that, by Lemma A.6,

$$E^P\big[\xi_{\mathbf{i}}^2\big] = G(\mathbf{i},\mathbf{i}) - G^L(\mathbf{i},\mathbf{i}) = \mathbb{E}_{\mathbf{i}}\Bigg[\sum_{t=\tau_L}^{\infty} 1_{\{\eta_t=\mathbf{i}\}}\Bigg];$$

this shows the lemma since $L(N)\to\infty$ as $N\to\infty$. □

Let $\mathbf{Q}$ be the transition matrix of the simple random walk [cf. (0.1)]. In this case (cf. [19]), page 32), (0.7) can actually be sharpened to

$$G(\mathbf{i},\mathbf{0}) = g(\mathbf{i}) + O(|\mathbf{i}|^{-d}). \tag{A.15}$$

For $j = 1,\dots,N$, let

$$\begin{aligned} \nabla_j S_N(\omega) &\equiv S_{N,j}(\omega) - S_{N,j-1}(\omega), \\ S_{N,j}(\omega) &\equiv N^{-d+1}\sum_{(j_2,\dots,j_d)\in V_N^{(d-1)}} \omega_{(j,j_2,\dots,j_d)}, \end{aligned}$$

with $S_{N,0}(\omega)\equiv 0$ and $V_N^{(d-1)} \equiv [1,N]^{d-1}\cap\mathbb{Z}^{d-1}$.

LEMMA A.16.

$$E^P[\nabla_i S_N \nabla_j S_N] = 2d\delta_{i,j}N^{-d+1} + O(N^{-d}\log N).$$

PROOF.

$$E^P[\nabla_i S_N \nabla_j S_N] = N^{-2d+2} \sum_{\mathbf{k},\mathbf{l}\in V_N^{(d-1)}} [2\gamma(i-j,\mathbf{k}-\mathbf{l}) - \gamma(i-j+1,\mathbf{k}-\mathbf{l}) - \gamma(i-j-1,\mathbf{k}-\mathbf{l})],$$

where, for $r \in \mathbb{Z}$, $\mathbf{s} \in \mathbb{Z}^{d-1}$ and $(r,\mathbf{s}) \in \mathbb{Z}^d$, we write

$$\gamma(r,\mathbf{s}) \equiv G((r,\mathbf{s}),(0,\mathbf{0})).$$

For $(r,\mathbf{s}) \in \mathbb{Z} \times \mathbb{Z}^{d-1}$, we have

$$2\gamma(r,\mathbf{s}) - \gamma(r+1,\mathbf{s}) - \gamma(r-1,\mathbf{s}) = \Delta_{d-1}\gamma(r,\mathbf{s}) + 2d\delta_{((r,\mathbf{s}),(0,\mathbf{0}))},$$

where

$$\Delta_{d-1}\gamma(r,\mathbf{s}) \equiv \sum_{\mathbf{l}:\, |\mathbf{l}-\mathbf{s}|=1} (\gamma(r,\mathbf{s}+\mathbf{l}) - \gamma(r,\mathbf{s})).$$

Therefore,

$$E^P[\nabla_i S_N \nabla_j S_N] = 2d\delta_{i,j}N^{-d+1} + \sum_{\mathbf{k},\mathbf{l}\in V_N^{(d-1)}} N^{-2d+2}\Delta_{d-1}\gamma(i-j,\mathbf{k}-\mathbf{l}). \tag{A.17}$$

Let $\rho(\mathbf{k}) \equiv (\Delta_{d-1}1_{V_N^{(d-1)}}(\mathbf{k})$. Then $\rho(\mathbf{k}) = 0$ except when $\mathbf{k}$ belongs either to $\partial^i V_N^{(d-1)}$, the boundary of $V_N^{(d-1)}$, or to $\partial^\circ V_N^{(d-1)}$, the boundary of the complement $V_N^{(d-1)}$. Fixing $\mathbf{l}$ and performing the summation over $\mathbf{k}$, we get, by (A.15),

$$\begin{aligned}\sum_{\mathbf{k}\in V_N^{(d-1)}} \Delta_{d-1}\gamma(i-j,\mathbf{k}-\mathbf{l}) &= \sum_{\mathbf{k}\in\mathbb{Z}^{d-1}} \rho(\mathbf{k})\gamma(i-j,\mathbf{k}-\mathbf{l})\\ &= c\sum_{\mathbf{k}}\rho(\mathbf{k})\Big[|(i,\mathbf{k})-(j,\mathbf{l})|^{-d+2}\wedge 1\Big]\\ &\quad + O\Big(\sum_{\mathbf{k}}|\rho(\mathbf{k})|\Big[|(i,\mathbf{k})-(j,\mathbf{l})|^{-d}\wedge 1\Big]\Big).\end{aligned} \tag{A.18}$$

Set $\partial V_N^{(d-1)} = \partial^i V_N^{(d-1)} \cup \partial^\circ V_N^{(d-1)}$. The $O(\ )$ summand is easy to handle. A straightforward calculation, using $\sum_{n\in\mathbb{Z}^{d-1}}(1\wedge|n|^{-d}) < \infty$, yields

$$\sum_{\mathbf{l}\in V_N^{(d-1)}} \sum_{k\in\partial V_N^{(d-1)}} \left(1\wedge|(i,\mathbf{k})-(i,\mathbf{l})|^{-d}\right) = O(N^{d-2}), \tag{A.19}$$

uniformly in i,j.

As to the first summand, we must have a closer look at $\rho(\mathbf{k})$. If $\mathbf{k} \in \partial^i V_N^{(d-1)}$, then $\rho(\mathbf{k})$ is minus the number of neighbors of $\mathbf{k}$ which do not belong to $V_N^{(d-1)}$ and, for $\mathbf{k} \in \partial^\circ V_N^{(d-1)}$, vice versa with the other sign. Performing for each

$\mathbf{k}' \in \partial^i V_N^{(d-1)}$ the summation over $\mathbf{k}$ being neighbors of $\mathbf{k}'$ in the first summand on the right-hand side of (A.18), this partial sum can be bounded by

$$\mathbf{c}\Big[\big|(i,\mathbf{k}') - (j,\mathbf{l})\big|^{-d+1} \wedge 1\Big].$$

Using

$$\sum_{\substack{n \in \mathbb{Z}^{d-1} \\ |n| \le N}} \left(1 \wedge |n|^{-d+1}\right) = O(\log N),$$

we easily get

$$\sum_{\mathbf{k}' \in \partial^i V_N^{(d-1)}} \big|(i,\mathbf{k}') - (j,\mathbf{l})\big|^{-d+1} \wedge 1 = O(N^{d-2} \log N).$$

Combining with (A.17)–(A.19), this proves the lemma. □

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