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STRONG LARGE DEVIATION AND LOCAL LIMIT THEOREMS[1]

By Narasinga Rao Chaganty[2] and Jayaram Sethuraman[3]

Old Dominion University and Florida State University

Most large deviation results give asymptotic expressions for log $P(Y_n \geq y_n)$, where the event $\{Y_n \geq y_n\}$ is a large deviation event, that is, $P(Y_n \geq y_n)$ goes to 0 exponentially fast. We refer to such results as weak large deviation results. In this paper we obtain strong large deviation results for arbitrary random variables $\{Y_n\}$, that is, we obtain asymptotic expressions for $P(Y_n \geq y_n)$, where $\{Y_n \geq y_n\}$ is a large deviation event. These strong large deviation results are obtained for lattice valued and nonlattice valued random variables and require some conditions on their moment generating functions. These results strengthen existing results which apply mainly to sums of independent and identically distributed random variables.

Since Y_n may not possess a probability density function, we consider the function $q_n(y; b_n, S) = [(b_n/\mu(S))P(b_n(Y_n - y) \in S)]$, where $b_n \to \infty$, μ is the Lebesgue measure on R, and S is a measurable subset of R such that $0 < \mu(S) < \infty$. The function $q_n(y; b_n, S)$ is the p.d.f. of $Y_n + Z_n$, where Z_n is uniform on $-S/b_n$, and will be called the pseudodensity function of Y_n. By a local limit theorem we mean the convergence of $q_n(y_n; b_n, S)$ as $n \to \infty$ and $y_n \to y^*$. In this paper we obtain local limit theorems for arbitrary random variables based on easily verifiable conditions on their characteristic functions. These local limit theorems play a major role in the proofs of the strong large deviation results of this paper. We illustrate these results with two typical applications.

1. Introduction. The establishment of a limit distribution for a sequence of random variables $\{Y_n,\ n \geq 1\}$ provides an approximation to $P(Y_n \geq y)$. However, there are other aspects relating to the distribution of Y_n for which one often desires an approximation. This could be $P(Y_n \geq y_n)$, known in the literature as a large deviation, especially when it tends to 0 exponentially fast. Another example is $f_n(y_n)$, the probability density function (p.d.f.) of Y_n at y_n. The term, a large deviation local limit result for Y_n, is used when an asymptotic expression is established for $f_n(y_n)$ and y_n is in the range of a large deviation for Y_n. Still another example is the pseudodensity function of Y_n, $q_n(y; b_n, S)$, which is available even when Y_n does not possess a p.d.f. and is defined as the p.d.f. of $Y_n + Z_n$, where Z_n is uniform on $-S/b_n$. Such a result will be referred to as a local limit result for Y_n. This paper will deal with strong large deviation and local limit theorems for arbitrary random variables.

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The theory of large deviations for sums of independent and identically distributed (i.i.d.) random variables and its many generalizations has a long history [see, e.g., Cramér (1938), Chernoff (1952), Ellis (1984) and Varadhan (1984)]. However, most of these results give asymptotic expressions for $\log P(Y_n \geq y_n)$ and so we choose to call them weak large deviation results. For arbitrary random variables T_n and $Y_n = T_n/a_n$ for some sequence $a_n \to \infty$, this paper gives asymptotic expressions for $P(Y_n \geq y_n)$, which we call strong large deviation results. These results are found in Theorems 3.3 and 3.5 which impose conditions on the moment generating function (m.g.f.) of T_n. These extend the well-known strong large deviation results for sums of i.i.d. random variables due to Bahadur and Ranga Rao (1960).

The proofs of our strong large deviation theorems depend on the local limit results for Y_n. These are established in this paper in Theorems 2.1, 2.2, 2.3 and 2.9 and they are in the spirit of Feller (1967) wherein can be found some of the first local limit results for sums of i.i.d. random variables. Local limit results for extreme values are established in de Haan and Resnick (1982). Local limit results for sums of triangular arrays of i.i.d. random variables can be found in Jain and Pruitt (1987). The local limit results in this paper apply to arbitrary random variables Y_n and require some easily verifiable boundedness conditions on their characteristic functions.

We illustrate our general results with two applications in Section 4. The first application is a local limit result for sums of dependent random variables given by a general model considered in Chaganty and Sethuraman (1987). The second application is a strong large deviation result for the Wilcoxon signed-rank statistic under the null hypothesis.

We do not study large deviation local limit results in this paper. We have obtained such results for arbitrary random variables in Chaganty and Sethuraman (1985) for one-dimensional random variables and in Chaganty and Sethuraman (1986) for multidimensional random variables.

2. Local limit theorems. Let $\{Y_n,\ n \geq 1\}$ be a sequence of real valued random variables which converge weakly to a random variable Y. Then $\hat{f}_n(t) \to \hat{f}(t)$ for each t, where $\hat{f}_n(t)$ and $\hat{f}(t)$ are the characteristic functions (c.f.) of Y_n and Y, respectively. In this section we show that if $\hat{f}_n(t)$ satisfies some boundedness conditions, then the p.d.f. of Y_n, or more generally the pseudodensity function of Y_n converges uniformly to the p.d.f. of Y. Both the cases where Y_n is nonlattice valued and lattice valued will be considered. To motivate the boundedness conditions used in the main Theorems 2.3 and 2.9 of this section, we begin with two theorems, using bounds on $\hat{f}_n(t)$ over the whole real line, which are straightforward and must be well known.

THEOREM 2.1. *Let $\{Y_n,\ n \geq 1\}$ be a sequence of real valued random variables which converge weakly to a random variable Y. Let $\hat{f}_n(t)$ and $\hat{f}(t)$ be the* c.f.'s *of Y_n and Y, respectively. Suppose that there exists an integrable function*

$f^*(t)$ such that

$$\sup_n \left|\hat{f}_n(t)\right| \le f^*(t) \tag{2.1}$$

for all t. Then Y_n possesses a bounded and continuous p.d.f. f_n, Y also possesses a bounded and continuous p.d.f. f, and $f_n(y_n)$ converges to $f(y^*)$ if $y_n \to y^*$.

Proof. Condition (2.1) implies that the c.f.'s $\hat{f}_n$ and $\hat{f}$ are integrable. Hence both Y_n and Y possess bounded and continuous p.d.f.'s. The inversion formula and the dominated convergence theorem show that $f_n(y_n)$ converges to $f(y^*)$ if $y_n \to y^*$ as $n \to \infty$. □

A random variable X is said to be lattice valued with span p and displacement c if $P(X \in L) = 1$, where $L = \{c + kp$, for some $k = 0, \pm 1, \pm 2, \dots\}$ and $0 \le c < p$ and p is the largest such number. Some authors say that X is lattice valued only when $c = 0$; in this paper we do not make this distinction. A lattice valued random variable cannot satisfy condition (2.1) since the modulus of its c.f. is periodic. Hence Theorem 2.1 is not applicable to lattice valued random variables. The following theorem will apply to lattice valued random variables.

Theorem 2.2. *Let Y_n be lattice valued random variables with span h_n converging to 0. Let Y_n converge weakly to Y. Assume that there exists an integrable function f^* such that*

$$\sup_n \left|\hat{f}_n(t)\right| I(|t| \le \pi/h_n) \le f^*(t) \tag{2.2}$$

for each t. Then Y possesses a bounded and continuous p.d.f. f, and there exists a constant $M < \infty$ such that

$$\sup_n \sup_y \left[\frac{1}{h_n} P(Y_n = y)\right] \le M. \tag{2.3}$$

Further, if y_n is in the range of Y_n, and y_n converges to y^* then

$$\frac{1}{h_n} P(Y_n = y_n) \to f(y^*) \tag{2.4}$$

as $n \to \infty$.

Proof. Let y_n be in the range of Y_n. Then an application of the inversion formula yields

$$\frac{1}{h_n} P(Y_n = y_n) = \frac{1}{2\pi} \int_{-\pi/h_n}^{\pi/h_n} \exp(-ity_n)\, \hat{f}_n(t)\, dt. \tag{2.5}$$

The assertions (2.3) and (2.4) now follow from condition (2.2) and the dominated convergence theorem. □

The conditions (2.1) in the nonlattice case and (2.2) in the lattice case are too strong to be useful in most situations. We show in Theorems 2.3 and 2.9 that appropriate bounds on the c.f. $\hat{f}_n(t)$ on increasing sequences of bounded intervals are sufficient to obtain results similar to those of Theorems 2.1 and 2.2.

Since Y_n may not possess a p.d.f., we will define its pseudodensity function at y as the p.d.f. of $Y_n + Z_n$, where Z_n is independent of Y_n and uniformly distributed on $-S/b_n$, where $b_n \to \infty$ and S is a set such that $0 < \mu(S^0) = \mu(\bar{S}) < \infty$, where μ is the Lebesgue measure on R. More directly, the pseudodensity $q_n(y; b_n, S)$ of Y_n is defined by

$$q_n(y; b_n, S) = \frac{b_n}{\mu(S)} P(b_n(Y_n - y) \in S). \tag{2.6}$$

We will keep the set S fixed throughout this section when dealing with pseudodensities. Let $\{y_n\}$ be a sequence of real numbers such that $y_n \to y^*$. The convergence of $q_n(y_n; b_n, S)$ to the p.d.f. of Y at y^* will be referred to as a local limit theorem in this paper. This is the spirit under which local limit theorems have been studied for normalized sums of i.i.d. random variables by Feller (1967), for normalized extreme values in de Haan and Resnick (1982) and for normalized triangular arrays of i.i.d. random variables in Jain and Pruitt (1987).

THEOREM 2.3. *Let $\{Y_n,\ n \geq 1\}$ be a sequence of real valued random variables which converge weakly to a random variable Y. Let $\hat{f}_n(t)$ and $\hat{f}(t)$ be the c.f.'s of Y_n and Y, respectively. Suppose that there exist an integrable function $f^*(t)$ and sequences $\{\beta_n\}$ and $\{b_n\}$ with $\beta_n \to \infty$, $b_n \to \infty$, such that*

$$\sup_n |\hat{f}_n(t)| I(|t| \leq \beta_n) \leq f^*(t) \tag{2.7}$$

for each t, and

$$\theta_n(\lambda) =_{\text{def}} \sup_{\beta_n < |t| \leq \lambda b_n} |\hat{f}_n(t)| = o\left(\frac{1}{b_n}\right) \tag{2.8}$$

for each $\lambda > 0$, where the above supremum is defined to be 0 if $\{t\colon \beta_n < |t| \leq \lambda b_n\}$ is empty. Then the random variable Y possesses a bounded and continuous p.d.f. f. Let $q_n(y; b_n, S)$ be the pseudodensity function of Y_n as defined in (2.6). Then there exists a finite constant M and an integer n_s, which may depend on S, such that

$$\sup_y [q_n(y; b_n, S)] \leq M \tag{2.9}$$

for $n \geq n_s$. Furthermore, if $y_n \to y^*$, then

$$q_n(y_n; b_n, S) \to f(y^*) \tag{2.10}$$

as $n \to \infty$.

PROOF. Since $\hat{f}_n(t) \to \hat{f}(t)$ pointwise and $\beta_n \to \infty$, condition (2.7) implies that $\hat{f}$ is bounded by f^*. Hence Y possesses a bounded and continuous p.d.f. f. Suppose β_n/b_n is bounded. Since $b_n\theta_n(\lambda) \to 0$, for each $\lambda > 0$, we can find a sequence $\{\lambda_n\}$ satisfying

$$\lambda_n \to \infty \quad \text{and} \quad \lambda_n b_n \theta_n \to 0 \tag{2.11}$$

as $n \to \infty$, where $\theta_n =_{\text{def}} \theta_n(\lambda_n)$. Now, suppose that $\beta_n/b_n \to \infty$. In this case we let $\lambda_n = \beta_n/b_n$ and (2.11) is satisfied because $\theta_n =_{\text{def}} \theta_n(\lambda_n) = 0$. Let U_n be the uniform distribution on the set $-S/b_n$ and $u_n, \hat{u}_n$ be the p.d.f. and c.f. corresponding to U_n. We also introduce another distribution function (d.f.) V_n with p.d.f. v_n and c.f. $\hat{v}_n$ which vanishes outside of $[-\lambda_n b_n, \lambda_n b_n]$, to obtain the important identity (2.14):

$$v_n(x) = \frac{\lambda_n b_n}{2\pi}\left[\frac{\sin(\lambda_n b_n x/2)}{(\lambda_n b_n x/2)}\right]^2, \qquad -\infty < x < \infty, \tag{2.12}$$

$$\hat{v}_n(t) = \begin{cases} 1 - \dfrac{|t|}{\lambda_n b_n}, & \text{if } |t| \leq \lambda_n b_n, \\ 0, & \text{otherwise.} \end{cases} \tag{2.13}$$

Let F_n be the d.f. of Y_n, and let $Q_n = F_n * U_n$, $M_n = Q_n * V_n$, where $*$ denotes the convolution operation. Notice that $q_n(y; b_n, S)$ defined in (2.6) is the p.d.f. of Q_n. Let $m_n(y)$ be the p.d.f. of M_n. The c.f. $\hat{m}_n(t)$ of M_n, which is equal to $\hat{f}_n(t)\hat{u}_n(t)\hat{v}_n(t)$, vanishes outside the interval $[-\lambda_n b_n, \lambda_n b_n]$. The inversion theorem yields the following identity:

$$\begin{aligned} m_n(y) &= \int_{-\infty}^{\infty} q_n(y - x; b_n, S) v_n(x)\, dx \\ &= \frac{b_n}{\mu(S)} \int_{-\infty}^{\infty} P(b_n(Y_n - y + x) \in S) v_n(x)\, dx \\ &= \frac{1}{2\pi} \int_{-\lambda_n b_n}^{\lambda_n b_n} \exp(-ity)\hat{m}_n(t)\, dt. \end{aligned} \tag{2.14}$$

Relation (2.14) is the starting point of the main part of this proof and it relates $q_n(y; b_n, S)$ to the integrable c.f. $\hat{m}_n(t)$. We first show that $m_n(y_n)$ converges to $f(y^*)$ and then obtain lower and upper bounds for $m_n(y_n)$ which depend on $q_n(y_n; b_n, S)$. This will then establish (2.9) and (2.10). Notice that from (2.11) we get

$$\left|\frac{1}{2\pi}\int_{\beta_n < |t| \leq \lambda_n b_n} \exp(-ity_n)\hat{m}_n(t)\, dt\right| \leq \frac{\lambda_n b_n \theta_n}{\pi} \to 0. \tag{2.15}$$

From condition (2.7), the dominated convergence theorem and the inversion formula we get

$$\frac{1}{2\pi}\int_{-\beta_n}^{\beta_n} \exp(-ity_n)\hat{m}_n(t)\,dt \to \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(-ity^*)\hat{f}(t)\,dt = f(y^*). \tag{2.16}$$

This shows that

$$m_n(y_n) \to f(y^*). \tag{2.17}$$

Let $\eta > 0$. Let $s(x, \eta)$ be a closed interval centered at x, that is, $s(x, \eta) = \{y: |y - x| \le \eta\}$. Let $S_\eta = \{x: s(x, \eta) \subset S\}$ and $S^\eta = \{y: |y - x| \le \eta$, for some $x \in S\}$. Since we have assumed that $\mu(S^0) = \mu(\bar{S})$ we can find $\eta(= \eta_s) > 0$ such that

$$\mu(S_\eta) > 0 \quad \text{and} \quad [\mu(S^\eta)/\mu(S)] \le 2. \tag{2.18}$$

Note that $y \in S_\eta$ implies that $y + x \in s(y, \eta) \subset S$ if $|x| \le \eta$. From this, we get a lower bound for $m_n(y)$ as follows:

$$\begin{aligned} m_n(y) &\ge \frac{b_n}{\mu(S)}\int_{|x|\le\eta/b_n} P(b_n(Y_n - y + x) \in S)v_n(x)\,dx \\ &\ge \frac{b_n}{\mu(S)}P\big(b_n(Y_n - y) \in S_\eta\big)\int_{|x|\le\eta/b_n} v_n(x)\,dx \\ &\ge \frac{b_n}{\mu(S)}P\big(b_n(Y_n - y) \in S_\eta\big)\left[1 - \frac{4}{\pi\lambda_n\eta}\right]. \end{aligned} \tag{2.19}$$

Using (2.14), (2.15), (2.19) and condition (2.7) we get

$$\begin{aligned} &\frac{b_n}{\mu(S)}P\big(b_n(Y_n - y) \in S_\eta\big)\left[1 - \frac{4}{\pi\lambda_n\eta}\right] \\ &\quad\le m_n(y) \le \frac{\lambda_n b_n\theta_n}{\pi} + \frac{1}{2\pi}\int_{-\infty}^{\infty} f^*(t)\,dt \\ &\quad\le \frac{1}{\pi}\int_{-\infty}^{\infty} f^*(t)\,dt \end{aligned} \tag{2.20}$$

for sufficiently large n. By replacing S by S^η and using (2.18) and the fact that $S \subset (S^\eta)_\eta$, we get

$$\begin{aligned} &\frac{b_n}{\mu(S)}P\big(b_n(Y_n - y) \in S\big)\left[1 - \frac{4}{\pi\lambda_n\eta}\right] \\ &\quad\le \frac{\mu(S^\eta)}{\mu(S)}\frac{1}{\pi}\int_{-\infty}^{\infty} f^*(t)\,dt \le \frac{2}{\pi}\int_{-\infty}^{\infty} f^*(t)\,dt. \end{aligned} \tag{2.21}$$

Since $\lambda_n \to \infty$ as $n \to \infty$ we can find an integer n_s so that

$$\sup_y\left[\frac{b_n}{\mu(S)}P\big(b_n(Y_n - y) \in S\big)\right] \le M \tag{2.22}$$

for $n \geq n_s$, where

$$M = \frac{3}{\pi}\int_{-\infty}^{\infty} f^*(t)\,dt. \tag{2.23}$$

This proves assertion (2.9). Note that $y \in S$ implies that $y - x \in S^\eta$ for $|x| \leq \eta$. Therefore for $n \geq n_s$ an upper bound for $m_n(y)$ is given by

$$\begin{aligned} m_n(y) &= \frac{b_n}{\mu(S)}\int_{-\infty}^{\infty} P(b_n(Y_n - y + x) \in S)v_n(x)\,dx \\ &\leq \frac{b_n}{\mu(S)}P(b_n(Y_n - y) \in S^\eta)\int_{|x|\leq \eta/b_n} v_n(x)\,dx \\ &\quad + M\int_{|x|>\eta/b_n} v_n(x)\,dx \\ &\leq \frac{b_n}{\mu(S)}P(b_n(Y_n - y) \in S^\eta) + \frac{4M}{\pi\lambda_n\eta}. \end{aligned} \tag{2.24}$$

Thus, from (2.17), (2.19) and (2.24) we get that

$$\begin{aligned} &\limsup_n \frac{b_n}{\mu(S)}P\big(b_n(Y_n - y_n) \in S_\eta\big) \\ &\quad \leq f(y^*) \leq \liminf_n \frac{b_n}{\mu(S)}P(b_n(Y_n - y_n) \in S^\eta). \end{aligned} \tag{2.25}$$

By replacing S by S^η in the l.h.s. and S by S_η in the r.h.s. and using the relations $S \subset (S^\eta)_\eta$ and $(S_\eta)^\eta \subset S$ we get that

$$\begin{aligned} &\limsup_n \frac{b_n}{\mu(S^\eta)}P(b_n(Y_n - y_n) \in S) \\ &\quad \leq f(y^*) \leq \liminf_n \frac{b_n}{\mu(S_\eta)}P(b_n(Y_n - y_n) \in S). \end{aligned} \tag{2.26}$$

Letting $\eta \to 0$ and using the fact $\mu(S^0) = \mu(\bar{S})$ we get the assertion (2.10). □

COROLLARY 2.4. *Suppose that the* c.f.'s *of the sequence* Y_n *satisfy condition* (2.1), *which is stronger than condition* (2.7). *Then* (2.9) *and* (2.10) *hold for any sequence* $b_n \to \infty$.

REMARK 2.5. The conclusions of Theorem 2.3 hold if we replace condition (2.8) by

$$\int_{\beta_n<|t|\leq\lambda b_n} \big|\hat{f}_n(t)\big|\,dt \to 0 \quad \text{as } n \to \infty, \tag{2.27}$$

for each $\lambda > 0$. Notice that condition (2.8) is needed to obtain a sequence $\{\lambda_n\}$ satisfying (2.11) and to show that $m_n(y_n) \to f(y^*)$. If (2.27) holds for each

$\lambda > 0$, we can find a sequence of real numbers $\{\lambda_n\}$, such that $\lambda_n \to \infty$ and

$$\int_{\beta_n < |t| \le \lambda_n b_n} |\hat{f}_n(t)|\, dt \to 0 \quad \text{as } n \to \infty. \tag{2.28}$$

These are enough to show that $m_n(y_n) \to f(y^*)$, following the proof of Theorem 2.3.

The next theorem provides a convenient way to verify condition (2.7) of Theorem 2.3. In Lemma 3.3 of Section 3 we will use Theorem 2.3 and this method of verification of condition (2.7).

THEOREM 2.6. *Let $\{Y_n, n \ge 1\}$ be a sequence of random variables with* c.f.'s $\{\hat{f}_n(t)\}$. *Let $\{d_n\}$ be a sequence of real numbers such that $d_n \to \infty$. Assume that there exists $\delta > 0$ such that $g_n(t) = d_n^{-2} \log|\hat{f}_n(d_n t)|$ is finite and twice differentiable in the interval $[-\delta, \delta]$, for all $n \ge 1$. Suppose that there exists $\alpha > 0$ such that for $|t| \le \delta$,*

$$-g_n''(t) \ge \alpha \tag{2.29}$$

for all $n \ge 1$. Then condition (2.7) *of Theorem* 2.3 *is satisfied with $\beta_n = \delta d_n$.*

PROOF. By expanding $g_n(t)$ around 0 by Taylor's theorem, we find that, for $|t| \le \delta$,

$$\begin{aligned} g_n(t) &= g_n(0) + t g_n'(0) + \frac{t^2}{2} g_n''(r_n) \\ &= \frac{t^2}{2} g_n''(r_n) \\ &\le -\frac{\alpha t^2}{2}, \end{aligned} \tag{2.30}$$

where r_n is such that $|r_n| < |t| \le \delta$. Let $\beta_n = \delta d_n$. Thus for $|t| \le \beta_n$, we have for all $n \ge 1$,

$$\begin{aligned} |\hat{f}_n(t)| &= \exp(d_n^2(g_n(t/d_n))) \\ &\le \exp(-\alpha t^2/2), \end{aligned} \tag{2.31}$$

which is an integrable function. This completes the proof of the theorem. □

The next theorem obtains the limit of a function related to the Laplace transform of the positive part of Y_n when (2.9) and (2.10) hold. It plays an important role in the proofs of the strong large deviation theorems of Section 3.

THEOREM 2.7. *Let $\{Y_n, n \ge 1\}$ be a sequence of random variables converging weakly to Y, which possess a p.d.f. f. Let $\{b_n\}$ be a sequence of real numbers such that $b_n \to \infty$. Let $q_n(y; b_n, S)$ be as defined in* (2.6). *Assume that*

$q_n(y; b_n, S)$ satisfies (2.9) *and* (2.10). *Then*

$$b_n E[\exp(-b_n Y_n) I(Y_n \geq 0)] \to f(0) \tag{2.32}$$

as $n \to \infty$.

PROOF. Let $h > 0$. Consider

$$\begin{aligned} I_n &= E[\exp(-b_n Y_n) I(Y_n \geq 0)] \\ &= \sum_{k=1}^{\infty} E\left[\exp(-b_n Y_n) I\left(\frac{(k-1)h}{b_n} \leq Y_n < \frac{kh}{b_n}\right)\right] \\ &= \sum_{k=1}^{\infty} E\left[\exp(-b_n Y_n) I\left(-\frac{h}{2b_n} \leq Y_n - y_{nk} < \frac{h}{2b_n}\right)\right], \end{aligned} \tag{2.33}$$

where $y_{nk} = (2k-1)h/2b_n$. Let $k_h = [1/h^2]$ and $S_h = [-h/2, h/2)$. We now get lower and upper bounds for I_n as follows:

$$\begin{aligned} I_n &\geq \sum_{k=1}^{k_h} \exp(-kh) P\left(-\frac{h}{2b_n} \leq Y_n - y_{nk} < \frac{h}{2b_n}\right) \\ &= \frac{h}{b_n} \sum_{k=1}^{k_h} \exp(-kh) q_n(y_{nk}; b_n, S_h) \end{aligned} \tag{2.34}$$

and

$$\begin{aligned} I_n &\leq \sum_{k=1}^{\infty} \exp(-(k-1)h) P\left(-\frac{h}{2b_n} \leq Y_n - y_{nk} < \frac{h}{2b_n}\right) \\ &= \frac{h}{b_n} \sum_{k=1}^{k_h} \exp(-(k-1)h) q_n(y_{nk}; b_n, S_h) \\ &\quad + \frac{h}{b_n} \sum_{k=k_h+1}^{\infty} \exp(-(k-1)h) q_n(y_{nk}; b_n, S_h). \end{aligned} \tag{2.35}$$

Using (2.9) and (2.10) after noting that $y_{nk} \to 0$ as $n \to \infty$ for each k, we get

$$\begin{aligned} \liminf_n (b_n I_n) &\geq f(0) h \sum_{k=1}^{k_h} \exp(-kh) \\ &= f(0) \frac{h(\exp(-h) - \exp(-(k_h+1)h))}{1 - \exp(-h)} \end{aligned} \tag{2.36}$$

and

$$\begin{aligned} \limsup_n (b_n I_n) &\leq f(0) h \sum_{k=1}^{k_h} \exp(-(k-1)h) \\ &\quad + Mh \sum_{k=k_h+1}^{\infty} \exp(-(k-1)h) \\ &= f(0) \frac{h(1 - \exp(-k_h h))}{1 - \exp(-h)} + Mh \frac{\exp(-k_h h)}{1 - \exp(-h)}. \end{aligned} \tag{2.37}$$

Letting $h \to 0$ in (2.36) and (2.37) we get

$$\lim_n (b_n I_n) = f(0). \tag{2.38}$$

This completes the proof of the lemma. □

REMARK 2.8. The conditions used in Theorem 2.3 can be satisfied by both nonlattice and lattice random variables. For a lattice valued random variable Y_n with span h_n, condition (2.7) can be satisfied only if h_n converges to 0 and condition (2.8) can be satisfied only if $b_n h_n$ converges to 0. We will now extend Theorems 2.3 and 2.7 to the case where $0 < \liminf_n b_n h_n < \infty$. Thus we might as well take $b_n = 1/h_n$. We can also take $S = (-1/2, 1/2)$ and notice that

$$q_n(y; b_n, S) = \frac{1}{h_n} P(Y_n = y), \tag{2.39}$$

where y is in the range of Y_n.

THEOREM 2.9. *Let $\{Y_n, n \geq 1\}$ be lattice valued random variables with span h_n converging to 0. Let Y_n converge weakly to Y. Let $\hat{f}_n(t)$ and $\hat{f}(t)$ be the c.f.'s of Y_n and Y, respectively. Assume that there exists an integrable function f^* such that*

$$\sup_n |\hat{f}_n(t)| I(|t| \leq \beta_n) \leq f^*(t) \tag{2.40}$$

for each t, and

$$\theta_n^* =_{\text{def}} \sup_{\beta_n < |t| \leq \pi/h_n} |\hat{f}_n(t)| = o(h_n), \tag{2.41}$$

for some sequence of real numbers $\{\beta_n\}$ such that $\beta_n \to \infty$ and $\beta_n < \pi/h_n$ for all $n \geq 1$. Then Y possesses a bounded and continuous p.d.f. f, and there exists a constant M such that (2.3) *holds. If y_n is in the range of Y_n and y_n converges to y^* as $n \to \infty$, then* (2.4) *holds.*

PROOF. Let y_n be a possible value of Y_n. Then an application of the inversion formula yields

$$\begin{aligned} \frac{1}{h_n} P(Y_n = y_n) &= \frac{1}{2\pi} \int_{-\pi/h_n}^{\pi/h_n} \exp(-ity_n) \hat{f}_n(t)\, dt \\ &= \frac{1}{2\pi} \int_{|t| \leq \beta_n} \exp(-ity_n) \hat{f}_n(t)\, dt \\ &\quad + \frac{1}{2\pi} \int_{\beta_n < |t| \leq \pi/h_n} \exp(-ity_n) \hat{f}_n(t)\, dt \\ &= I_{n1} + I_{n2}, \quad \text{say}. \end{aligned} \tag{2.42}$$

It is easy to check that condition (2.40) and the dominated convergence theorem imply that I_{n1} converges to $(1/2\pi)\int \exp(-ity^*)\hat{f}(t)\,dt = f(y^*)$. Next,

$$\begin{aligned} |I_{n2}| &\le \frac{1}{h_n} \sup_{\beta_n < |t| \le \pi/h_n} |\hat{f}_n(t)| \\ &= \frac{\theta_n^*}{h_n}, \end{aligned} \tag{2.43}$$

which converges to 0 by condition (2.41), as $n \to \infty$. This completes the proof of (2.4). Next, from (2.42) and (2.43) we get

$$\begin{aligned} \sup_y \left[\frac{1}{h_n} P(Y_n = y)\right] &\le \frac{1}{2\pi}\int_{|t| \le \beta_n} |\hat{f}_n(t)|\,dt + \frac{\theta_n^*}{h_n} \\ &\le \frac{1}{2\pi}\int_{-\infty}^{\infty} f^*(t)\,dt + \frac{\theta_n^*}{h_n} \\ &\le \frac{1}{\pi}\int_{-\infty}^{\infty} f^*(t)\,dt = M \end{aligned} \tag{2.44}$$

for $n \ge n_0$. Hence (2.3) holds. This completes the proof of the theorem. □

THEOREM 2.10. *Let Y_n be a lattice valued random variable taking values in the lattice $\{kh_n\colon k = 0, \pm 1, \pm 2, \dots\}$, where $h_n > 0$ for $n \ge 1$. Assume that the span h_n of Y_n converges to 0 as $n \to \infty$. Let Y_n converge in distribution to Y. Let $\{b_n\}$ be a sequence of real numbers such that $0 < \liminf_n (b_n h_n) = b < \infty$. Suppose that Y possesses a p.d.f. f and Y_n satisfies the conditions* (2.3) *and* (2.4). *Then*

$$\frac{(1 - \exp(-b_n h_n))}{h_n} E[\exp(-b_n Y_n) I(Y_n \ge 0)] \to f(0) \tag{2.45}$$

as $n \to \infty$.

PROOF. Consider

$$\begin{aligned} I_n &= E(\exp(-b_n Y_n) I(Y_n \ge 0)) \\ &= \sum_{k=0}^{\infty} \exp(-kb_n h_n) P(Y_n = kh_n). \end{aligned} \tag{2.46}$$

Let $N > 1$ be fixed. A lower bound for I_n is given by

$$\sum_{k=0}^{N-1} \exp(-kb_n h_n) P(Y_n = kh_n) \tag{2.47}$$

and an upper bound is given by

$$\sum_{k=0}^{N-1} \exp(-kb_n h_n) P(Y_n = kh_n) + Mh_n \sum_{k=N}^{\infty} \exp(-kb_n h_n), \tag{2.48}$$

wherein we have used (2.3). Combining (2.46), (2.47), (2.48) and using (2.4) we get

$$\liminf_n \left[\frac{(1 - \exp(-b_n h_n))}{h_n} I_n \right] \geq f(0) \liminf_n (1 - \exp(-N b_n h_n)) \tag{2.49}$$
$$= f(0)(1 - \exp(-Nb))$$

and

$$\limsup_n \left[\frac{(1 - \exp(-b_n h_n))}{h_n} I_n \right] \leq f(0) + \limsup_n (M \exp(-N b_n h_n)) \tag{2.50}$$
$$= f(0) + M \exp(-Nb),$$

where $b = \liminf_n (b_n h_n)$. We let $N \to \infty$ in (2.49) and (2.50) and conclude that

$$\lim_n \left[\frac{(1 - \exp(-b_n h_n))}{h_n} I_n \right] = f(0). \tag{2.51}$$

This completes the proof of the theorem. □

3. Strong large deviation theorems. Let $\{T_n, n \geq 1\}$ be a sequence of random variables. Let $\{a_n\}$ be a sequence of real numbers and $\{m_n\}$ be a bounded sequence of real numbers. Weak large deviation results give asymptotic expressions for $\log P(T_n/a_n \geq m_n)$ where the event $\{T_n/a_n \geq m_n\}$ represents a large deviation. A number of authors, including Sievers (1969), Steinebach (1978) and Ellis (1984) have obtained such results under suitable conditions on the m.g.f. of T_n. Strong large deviation results give asymptotic expressions for $P(T_n/a_n \geq m_n)$. One of the earliest strong large deviation theorems was obtained by Bahadur and Ranga Rao (1960) when T_n is the sum of i.i.d. random variables. In Theorems 3.3 and 3.5 of this section we obtain strong large deviation limit theorems for arbitrary sequences of random variables $\{T_n, n \geq 1\}$, under some conditions on the m.g.f.'s of T_n's. In Remark 3.6 we demonstrate that Theorems 3.3. and 3.5 are sufficient to establish the strong large deviation results for sums of i.i.d. random variables obtained by Bahadur and Ranga Rao (1960). This shows that the generalization to general random variables in Theorems 3.3 and 3.5 has not been obtained by introducing unnecessary or restrictive conditions. The proofs of our strong large deviation results depend heavily on the local limit theorems of Section 2. We use the notation $A_n \sim B_n$, if $A_n/B_n \to 1$. We shall develop some more notation before stating the main theorem.

Let $\{T_n, n \geq 1\}$ be a sequence of random variables with m.g.f. $\phi_n(z) = E[\exp(zT_n)]$, which is nonvanishing and analytic in the region $\Omega = \{z \in \mathscr{C}: |z| < a\}$, where $a > 0$ and $\mathscr{C}$ is the set of all complex numbers. Let $\{a_n\}$ be a sequence of real numbers. Let

$$\psi_n(z) = a_n^{-1} \log \phi_n(z), \quad \text{for } z \in \Omega, \tag{3.1}$$

$$\gamma_n(u) = \sup_{|s|<a,\, s\in R_1} \left[us - \psi_n(s) \right], \quad \text{for } u \in R_1. \tag{3.2}$$

Note that ψ_n is a convex function on $(-a, a)$. Let $\{m_n\}$ be a bounded sequence of real numbers such that there exists a sequence $\{\tau_n\}$ satisfying

$$\psi_n'(\tau_n) = m_n \quad \text{and} \quad 0 < \tau_n < a_0 < a, \quad \text{for all } n \geq 1. \tag{3.3}$$

Under these conditions we can see that $\gamma_n(m_n) = m_n\tau_n - \psi_n(\tau_n)$. Let K_n be the d.f. of T_n. We will use the left continuous version of the distribution function which will enable us to write the identities in (3.5). Let

$$H_n(y) = \int_{-\infty<u<y} \exp(u\tau_n - a_n\psi_n(\tau_n))\, dK_n(u) \tag{3.4}$$

and let T_n^* be a random variable with d.f. $H_n(y)$. Let $T_n' = T_n^* - a_n m_n$, $Y_n = T_n'/d_n$, $d_n = \sqrt{a_n\psi_n''(\tau_n)}$ and $b_n = \tau_n d_n$. Note that if T_n is a lattice valued random variable with span $p_n > 0$, we can assume, without loss of generality, when considering $P(T_n/a_n \geq m_n)$, that m_n is in the range of T_n/a_n. In this case we can easily verify that Y_n is a lattice valued random variable with span $h_n = p_n/d_n$ and displacement 0. The study of the strong large deviation properties of T_n is carried out by relating them to local limit results for the random variables Y_n previously defined and using the identity (3.5) which follows. This identity is easily verified:

$$\begin{aligned} P\left(\frac{T_n}{a_n} \geq m_n\right) &= \int_{a_n m_n}^{\infty} dK_n(y) \\ &= \int_{a_n m_n}^{\infty} \exp(-y\tau_n + a_n\psi_n(\tau_n))\, dH_n(y) \\ &= \exp(a_n\psi_n(\tau_n)) E(\exp(-\tau_n T_n^*) I(T_n^* \geq a_n m_n)) \\ &= \exp(-a_n\gamma_n(m_n)) E(\exp(-\tau_n T_n') I(T_n' \geq 0)) \\ &= \exp(-a_n\gamma_n(m_n)) E(\exp(-b_n Y_n) I(Y_n \geq 0)). \end{aligned} \tag{3.5}$$

This is usually called the use of the Esscher transformation, and is the starting point of most investigations in large deviations.

LEMMA 3.1. *Let $\{T_n, n \geq 1\}$ be a sequence of random variables. Let $\{m_n\}$ be a bounded sequence of real numbers such that there exists a sequence $\{\tau_n\}$ satisfying* (3.3). *Let $d_n = \sqrt{a_n\psi_n''(\tau_n)}$ and let $a_n \to \infty$. Let the random variable Y_n be as defined following identity* (3.4). *Let $\hat{f}_n(t)$ be the c.f. of Y_n. Assume the following conditions for T_n:*

(a) *There exists $\beta < \infty$ such that $|\psi_n(z)| < \beta$ for all $n \geq 1$, $z \in \Omega$.*
(b) *There exists $\alpha > 0$ such that $\psi_n''(\tau_n) \geq \alpha$ for all $n \geq 1$.*

Then Y_n converges in distribution to the standard normal and there exists $\delta > 0$ such that

$$\sup_n |\hat{f}_n(t)| I(|t| \le \delta d_n) \le \exp(-\alpha t^2/2). \tag{3.6}$$

PROOF. The c.f. of Y_n is given by

$$\hat{f}(t) = \exp\left(-\frac{ita_n m_n}{d_n}\right)\frac{\phi_n(\tau_n + it/d_n)}{\phi_n(\tau_n)}. \tag{3.7}$$

Since $\psi_n(z) = a_n^{-1}\log\phi_n(z)$ is a finite and analytic function in Ω, and $0 < \tau_n < a_0$, using a Taylor series expansion, we can write, for $|t| < (a - a_0)/2$,

$$\psi_n(\tau_n + it) = \psi_n(\tau_n) + it\psi_n'(\tau_n) - (t^2/2)\psi_n''(\tau_n) + R_n(\tau_n + it), \tag{3.8}$$

where the remainder term R_n satisfies

$$|R_n(\tau_n + it)| \le \frac{\beta|t|^3}{((a - a_0)/2)^3}, \tag{3.9}$$

from condition (a) and Cauchy's inequality [see Apostol (1974), page 451]. From (3.7), (3.8), (3.9) and condition (b) we obtain, for any fixed t, that

$$\begin{aligned} \log\hat{f}_n(t) &= -(ita_n m_n)/d_n + a_n[\psi_n(\tau_n + it/d_n) - \psi_n(\tau_n)] \\ &= -(ita_n m_n)/d_n + a_n\big[it\psi_n'(\tau_n)/d_n \\ &\qquad -\big(t^2\psi_n''(\tau_n)\big)/\big(2d_n^2\big) + R_n(\tau_n + it/d_n)\big] \\ &= -t^2/2 + a_n R_n(\tau_n + it/d_n) \end{aligned} \tag{3.10}$$

and

$$\big|a_n R_n(\tau_n + it/d_n)\big| \le \frac{\beta|t|^3}{\alpha d_n((a - a_0)/2)^3} \to 0, \quad \text{as } n \to \infty. \tag{3.11}$$

Hence Y_n converges weakly to the standard normal random variable. The conclusion (3.6) will follow from Theorem 2.6, if we verify that Y_n satisfies condition (2.29). Let

$$\begin{aligned} g_n(t) &= d_n^{-2}\log|\hat{f}_n(d_n t)| \\ &= \frac{1}{\psi_n''(\tau_n)}\big[\mathrm{Real}(\psi_n(\tau_n + it) - \psi_n(\tau_n))\big]. \end{aligned} \tag{3.12}$$

Thus

$$\begin{aligned} g_n''(t) &= \frac{-\mathrm{Real}(\psi_n''(\tau_n + it))}{\psi_n''(\tau_n)} \\ &= \frac{-\mathrm{Real}(\psi_n''(\tau_n) + it\xi_n)}{\psi_n''(\tau_n)} \\ &= -1 - \mathrm{Real}(it\xi_n/\psi_n''(\tau_n)) \\ &\le -1 + |t|\frac{|\xi_n|}{\alpha}, \end{aligned} \tag{3.13}$$

where ξ_n is an appropriate complex number depending on the third derivative of ψ_n. From condition (a) and Cauchy's inequality for derivatives we get

$$|\xi_n| \le \frac{3!\beta}{((a - a_0)/2)^3} \quad \text{for } n \ge 1. \tag{3.14}$$

Therefore we can find $\delta > 0$ such that $\delta < \delta_0$ and for $|t| \le \delta$,

$$g_n''(t) \le -(1/2) \quad \text{for all } n \ge 1. \tag{3.15}$$

This verifies condition (2.29) of Theorem 2.6 and the proof is complete. □

LEMMA 3.2. *Let $\{T_n, n \ge 1\}$ be a sequence of random variables. Let $\{m_n\}$ be a bounded sequence of real numbers such that there exists a sequence $\{\tau_n\}$ satisfying* (3.3). *Let $d_n = \sqrt{a_n \psi_n''(\tau_n)}$ and $b_n = \tau_n d_n$ for $n \ge 1$. Suppose that $a_n \to \infty$ such that $\tau_n \sqrt{a_n} \to \infty$. Let the random variable Y_n be as defined following identity* (3.4). *Suppose that T_n satisfies conditions* (a) *and* (b) *and the following condition* (c):

(c) *There exists $\delta_0 > 0$ such that*

$$\sup_{\delta < |t| \le \lambda\tau_n} \left| \frac{\phi_n(\tau_n + it)}{\phi_n(\tau_n)} \right| = o\left(\frac{1}{\sqrt{a_n}} \right) \tag{3.16}$$

for any given δ and λ such that $0 < \delta < \delta_0 < \lambda$. Then

$$b_n E(\exp(-b_n Y_n) I(Y_n \ge 0)) \to \frac{1}{\sqrt{2\pi}}. \tag{3.17}$$

PROOF. Lemma 3.1 shows that Y_n converges weakly to the standard normal and there exists a $\delta > 0$ such that (2.7) holds with $\beta_n = \delta d_n$. Using condition (c) we get that for fixed $\lambda > 0$,

$$\begin{aligned} \sup_{\delta d_n < |t| \le \lambda b_n} |\hat{f}_n(t)| &= \sup_{\delta < |t| \le \lambda\tau_n} \left| \frac{\phi_n(\tau_n + it)}{\phi_n(\tau_n)} \right| \\ &= o\left(\frac{1}{\sqrt{a_n}} \right) \\ &= o\left(\frac{1}{b_n} \right). \end{aligned} \tag{3.18}$$

This verifies condition (2.8). The assertion (3.17) now follows from Theorem 2.3 and Theorem 2.7. □

We are now in a position to state the main theorem of this section.

THEOREM 3.3. *Let $\{T_n, n \ge 1\}$ be a sequence of random variables. Let $\{m_n\}$ be a bounded sequence of real numbers and $\{\tau_n\}$ be a sequence satisfying* (3.3).

Suppose that $a_n \to \infty$ *such that* $\tau_n\sqrt{a_n} \to \infty$. *Assume that* T_n *satisfies conditions* (a), (b) *and* (c). *Then*

$$P\left(\frac{T_n}{a_n} \geq m_n\right) \sim \frac{1}{\tau_n\sqrt{2\pi a_n\psi_n''(\tau_n)}} \exp(-a_n\gamma_n(m_n)). \tag{3.19}$$

PROOF. The conclusion (3.19) follows from Lemma 3.2 and the identity (3.5). □

REMARK 3.4. We will now examine the class of lattice valued random variables $\{T_n\}$ satisfying conditions (a), (b) and (c). Let T_n be a lattice valued random variable with span p_n and m_n be in the range of T_n/a_n. Then the random variable Y_n is lattice valued with span $h_n = p_n/d_n$ and displacement 0. The c.f. of Y_n is periodic and its absolute value achieves the value 1 at multiples of $2\pi/h_n$. When (a) and (b) hold, Lemma 3.1 shows that (3.6) holds, which implies that p_n is bounded. Again, when (c) holds, $p_n\tau_n \to 0$ as $n \to \infty$. Theorem 3.5 considers the case where the p_n's and τ_n's are bounded below by a positive number, and obtains the strong large deviation conclusion (3.20) by replacing condition (c) with condition (c′). When $p_n\tau_n \to 0$ condition (c′) implies condition (c) and the conclusion (3.20) of Theorem 3.5 agrees with the conclusion (3.19) of Theorem 3.3.

THEOREM 3.5. *Let* T_n *be a lattice valued random variable with span* p_n. *Let* m_n *be in the range of* T_n/a_n *and let* $\{\tau_n\}$ *be a sequence satisfying* (3.3). *Assume that* $0 < p^* < p_n < p^{**}$ *and* $0 < d < \tau_n < a_0 < a$ *for all* $n \geq 1$. *Let* T_n *satisfy conditions* (a) *and* (b) *of Lemma* 3.1 *and the following condition* (c′):

(c′) *There exists* $\delta_1 > 0$, *such that for* $0 < \delta < \delta_1$,

$$\sup_{\delta < |t| \leq \pi/p_n} \left|\frac{\phi_n(\tau_n + it)}{\phi_n(\tau_n)}\right| = o\left(\frac{1}{\sqrt{a_n}}\right).$$

Then

$$P\left(\frac{T_n}{a_n} \geq m_n\right) \sim \frac{p_n}{\sqrt{2\pi a_n\psi_n''(\tau_n)}} \frac{\exp(-a_n\gamma_n(m_n))}{(1 - \exp(-\tau_n p_n))}. \tag{3.20}$$

PROOF. Let Y_n be as defined following identity (3.4). Then Y_n is a lattice valued random variable with span $h_n = p_n/d_n$ and displacement 0. Let $b_n = \tau_n d_n$. Since p_n and τ_n are bounded above and below by positive numbers we have $0 < \liminf_n b_n h_n < \infty$. Lemma 3.1 shows that Y_n converges weakly to the standard normal and there exists a $\delta > 0$ such that (2.40) holds with $\beta_n = \delta d_n$. Using condition (c′) we can easily verify that Y_n satisfies (2.41). The conclusion (3.20) now follows from Theorems 2.9 and 2.10. □

REMARK 3.6. Bahadur and Ranga Rao (1960) obtained a strong large deviation result for sums of i.i.d. random variables, which included a result of

Blackwell and Hodges (1959) in the lattice case. We will now show that the result of Bahadur and Ranga Rao (1960) can be obtained from our Theorems 3.3 and 3.5. Let $X_1, X_2, \dots$ be i.i.d. nondegenerate random variables with m.g.f. $\phi(z)$ and let $\psi(z) = \log(\phi(z))$, be finite for $|z| < a$. Let $T_n = X_1 + \cdots + X_n$. The m.g.f. of T_n is given by $\phi_n(z) = \phi^n(z)$. Let m be a real number such that there exists $0 < \tau < a$ satisfying $\psi'(\tau) = m$. Let $m_n = m$ and $a_n = n$ for all $n \geq 1$. Conditions (a) and (b) are trivially satisfied since $\psi_n \equiv \psi$ and $\tau_n = \tau$ for all $n \geq 1$.

If X_1 is nonlattice valued, then $\phi(\tau + it)/\phi(\tau)$ is the c.f. of a nonlattice valued random variable and hence its supremum on an compact interval not containing 0 is less than 1. This verifies condition (c). Let $\gamma(m) = \sup_s[ms - \psi(s)]$. The strong large deviation result

$$P\left(\frac{T_n}{n} \geq m\right) \sim \frac{1}{\tau\sqrt{2\pi n \psi''(\tau)}} \exp(-n\gamma(m)) \tag{3.21}$$

follows from Theorem 3.3. Bahadur and Ranga Rao (1960) established (3.21) by considering separately the two cases where X_1 satisfies Cramér's condition and where X_1 is nonlattice and does not satisfy Cramér's condition. The above application of Theorem 3.3 shows that it is not necessary to separate these two cases.

If X_1 is lattice valued random variable with span p, then $\phi(\tau + it)/\phi(\tau)$ is the c.f. of a random variable with span p. It follows that

$$\sup_{\delta < |t| \leq \pi/p} \left| \frac{\phi(\tau + it)}{\phi(\tau)} \right| < 1,$$

for any $\delta > 0$. This verifies condition (c′). The strong large deviation result

$$P\left(\frac{T_n}{n} \geq m\right) \sim \frac{p}{\sqrt{2\pi n \psi''(\tau)}} \frac{\exp(-n\gamma(m))}{(1 - \exp(-\tau p))} \tag{3.22}$$

follows from Theorem 3.5. Thus we have verified that all cases of Theorem 1 of Bahadur and Ranga Rao (1960) follow from our Theorems 3.3 and 3.5.

4. Applications. In this section we give two typical applications to illustrate the large deviation limit theorems and strong large deviation limit theorems of the previous sections. The first example is a local limit result and illustrates Theorem 2.1. The second example is a strong large deviation result for a lattice valued random variable and illustrates Theorem 3.3.

EXAMPLE 4.1. This example applies to a general class of sums of dependent random variables considered in Chaganty and Sethuraman (1987). Though it was proved in that paper that the limit distribution could be both normal and nonnormal, our example applies only to the case where the limit distribution is normal. We first present a particular application and then state a more general application referring to conditions found in Chaganty and Sethuraman (1987).

Let $\{X_1^{(n)}, X_2^{(n)}, \dots, X_n^{(n)}\}$ be a triangular array of random variables with joint density function

$$(4.1)\qquad dQ_n^*(\mathbf{x}) = z_n^{-1}(2\pi)^{-n/2}\left[\cosh\left(\frac{s_n}{\beta n}\right)\right]^n \exp\left(-\sum_{j=1}^n \frac{x_j^2}{2}\right) d\mathbf{x},$$

where $\mathbf{x} = (x_1, \dots, x_n)$, $s_n = x_1 + \cdots + x_n$, $\beta > 1$ and z_n is a normalizing constant. Such dependent random variables arise in generalized Curie–Weiss models used to describe ferromagnets. Using Theorem 3.7 of Chaganty and Sethuraman (1987) or using (4.3) we can show that $Y_n = (X_1^{(n)} + \cdots + X_n^{(n)})/\sqrt{n}$ converges in distribution to a normal distribution with mean 0 and variance $\sigma^2 = \beta^2/(\beta^2 - 1)$ [Example 4.4 of Chaganty and Sethuraman (1987) considered the case $\beta = 1$ and obtained a nonnormal distribution under a different normalization]. We will now show that Theorem 2.1 applies to Y_n. Since

$$(4.2)\qquad (\cosh\omega)^n = \sum_{y\in C_n} \exp(\omega y)\lambda_n(y)$$

with $\lambda_n(y) = \binom{n}{(n+y)/2}2^{-n}$ and $C_n = \{-n, -n+2, \dots, n\}$, the c.f. of Y_n is given by

$$(4.3)\qquad \begin{aligned} \hat{f}_n(t) &= E(\exp(itY_n)) \\ &= z_n^{-1}\sum_{y\in C_n}\left[\frac{1}{(2\pi)^{n/2}}\int \exp\left(\frac{its_n}{\sqrt{n}} + \frac{ys_n}{\beta n} - \sum_{j=1}^n \frac{x_j^2}{2}\right)d\mathbf{x}\right]\lambda_n(y) \\ &= \exp\left(-\frac{t^2}{2}\right) z_n^{-1}\sum_{y\in C_n}\exp\left(\frac{ity}{\beta\sqrt{n}} + \frac{y^2}{2\beta^2 n}\right)\lambda_n(y). \end{aligned}$$

Since $\hat{f}_n(0) = 1$, we have

$$(4.4)\qquad |\hat{f}_n(t)| \le \exp(-t^2/2) \quad \text{for all } n \text{ and } t.$$

Thus from Theorem 2.1 we get, if $y_n \to y$,

$$(4.5)\qquad f_n(y_n) \to \frac{1}{\sqrt{2\pi}\,\sigma}\exp\left(-\frac{y^2}{2\sigma^2}\right),$$

where $f_n(y)$ is the p.d.f. of Y_n and $\sigma^2 = \beta^2/(\beta^2 - 1)$.

From the preceding discussion and from a full use of Theorem 3.7 of Chaganty and Sethuraman (1987) we have the following application which we state without proof.

Let $\{X_1^{(n)}, X_2^{(n)}, \dots, X_n^{(n)}\}$ be a triangular array of random variables whose joint distribution is as given in (3.13) of Theorem 3.7 of Chaganty and Sethuraman (1987). We will impose conditions on the probability measure P and the index r appearing in that theorem. Let P be the standard normal distribution and let $r = 1$. Under these conditions, Theorem 3.7 of Chaganty and Sethuraman (1987) shows that there is a sequence of constants $\{m_n\}$ such that

$$Y_n = \left(\sum_{j=1}^{n} X_i^{(n)} - nm_n\right)/\sqrt{n} \tag{4.6}$$

has a limiting normal distribution with mean 0 and variance σ^2. Let $\hat{f}_n(t)$ be the c.f. of Y_n. For this case, if we proceed as in the previous application, we can establish (4.4) for all n and t. This shows that (4.5) is true with the appropriate σ.

EXAMPLE 4.2. We now obtain a strong large deviation result for the Wilcoxon signed-rank statistic under the null hypothesis. This strengthens the well-known weak large deviation results for this statistic [see Klotz (1965)].

Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. continuous random variables symmetric about their median m. Arrange $|X_1|, |X_2|, \dots, |X_n|$ in increasing order of magnitude and assign ranks $1, 2, \dots, n$. The Wilcoxon signed-rank statistic W_n is defined as the sum of the ranks of positive X_i's. The statistic W_n is used to test the null hypothesis H_0: $m = 0$ versus H_1: $m \neq 0$. Let $T_n = U_n/n$. The random variable T_n is a lattice random variable with span $p_n = 1/n$. The m.g.f. of T_n under the null hypothesis H_0 is given by

$$\phi_n(z) = \prod_{k=1}^{n} [(\exp(kz/n) + 1)/2], \qquad z \in \mathscr{C}. \tag{4.7}$$

It is easy to check that $\phi_n(z)$ is analytic and nonvanishing in the region $\Omega = \{z \in \mathscr{C}: |z| < \pi/2\}$. Let

$$\psi_n(z) = n^{-1} \log \phi_n(z). \tag{4.8}$$

It is easy to check that there exists $\beta > 0$ such that $|\psi_n(z)| < \beta$ for $|z| < \pi/2$. Straightforward calculations show that $\psi_n''(\tau)$ is bounded below by a positive number α for real τ such that $|\tau| < \pi/2$. Thus T_n satisfies conditions (a) and (b). Next we first note that $\psi_n'(s) \to \int_0^1(x)/(1 + \exp(-sx))\,dx$ and that the range of $\psi_n'(s)$ for real s contains the open interval $(0, 1/2)$ for all $n \geq 1$. Thus if $\{m_n\}$ is a sequence of real numbers such that $1/4 < m_n < \overline{m} < \int_0^1(x)/(1 + \exp(-\pi x/2))\,dx$, then we can find a positive number a_0 and a sequence $\{\tau_n\}$ satisfying $0 < \tau_n < a_0 < \pi/2$ and $\psi_n'(\tau_n) = m_n$, for all $n \geq 1$. From the analysis in Example 3.1 of Chaganty and Sethuraman (1985) it can be seen that there exists n_0 and $\delta_1 > 0$ such that for $0 < \delta < \delta_1$,

$$\sup_{\delta < |t| \leq \pi/p_n} \left|\frac{\phi_n(\tau_n + it)}{\phi_n(\tau_n)}\right| \leq \exp\left(-\frac{n\alpha\delta^2}{4}\right) \tag{4.9}$$

for $n \geq n_0$. Since $p_n \to 0$ this verifies condition (c). Therefore Theorem 3.3 shows that the conclusion (3.19) holds.

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REFERENCES

APOSTOL, T. M. (1974). *Mathematical Analysis*. Addison-Wesley, Reading, MA.

BAHADUR, R. R. and RANGA RAO, R. (1960). On deviations of the sample mean. *Ann. Math. Statist.* **31** 1015–1027.

BLACKWELL, D. and HODGES, J. L. (1959). The probability in the extreme tail of a convolution. *Ann. Math. Statist.* **30** 1113–1120.

CHAGANTY, N. R. and SETHURAMAN, J. (1985). Large deviation local limit theorems for arbitrary sequences of random variables. *Ann. Probab.* **13** 97–114.

CHAGANTY, N. R. and SETHURAMAN, J. (1986). Multi-dimensional large deviation local limit theorems. *J. Multivariate Anal.* **20** 190–204.

CHAGANTY, N. R. and SETHURAMAN, J. (1987). Limit theorems in the area of large deviations for some dependent random variables. *Ann. Probab.* **15** 628–645.

CHERNOFF, H. (1952). A measure of asymptotic efficiency for tests of an hypothesis based on the sum of observations. *Ann. Math. Statist.* **23** 493–507.

CRAMÉR, H. (1938). Sur un nouveau théorème-limite de la théorie des probabilitès. *Actualites Sci. Ind.* **736** 5–23.

DE HAAN, L. and RESNICK, S. I. (1982). Local limit theorems for sample extremes. *Ann. Probab.* **10** 396–413.

ELLIS, R. S. (1984). Large deviations for a general class of dependent random vectors. *Ann. Probab.* **12** 1–12.

FELLER, W. (1967). On regular variation and local limit theorems. *Proc. Fifth Berkeley Symp. Math. Statist. Probab.* **2** 373–388. Univ. California Press, Berkeley.

JAIN, N. C. and PRUITT, W. E. (1987). Lower tail probability estimates for subordinators and nondecreasing random walks. *Ann. Probab.* **15** 75–101.

KLOTZ, J. (1965). Alternative efficiencies for signed rank tests. *Ann. Math. Statist.* **36** 1759–1766.

SIEVERS, G. L. (1969). On the probability of large deviations. *Ann. Math. Statist.* **40** 1908–1921.

STEINEBACH, J. (1978). Convergence rates of large deviation probabilities in the multidimensional case. *Ann. Probab.* **6** 751–759.

VARADHAN, S. R. S. (1984). *Large Deviations and Applications*. SIAM, Philadelphia.

DEPARTMENT OF MATHEMATICS AND STATISTICS
OLD DOMINION UNIVERSITY
NORFOLK, VIRGINIA 23529

DEPARTMENT OF STATISTICS
FLORIDA STATE UNIVERSITY
TALLAHASSEE, FLORIDA 32306