The Annals of Probability
1993, Vol. 21, No. 2, 819–830

AN ASYMPTOTIC INDEPENDENT REPRESENTATION IN LIMIT THEOREMS FOR MAXIMA OF NONSTATIONARY RANDOM SEQUENCES

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Let $\{X_k\}_{k \in \mathbb{N}}$ be a *nonstationary* sequence of random variables. Sufficient conditions are found for the existence of an independent sequence $\{\tilde{X}_k\}_{k \in \mathbb{N}}$ such that $\sup_{x \in \mathbb{R}^1} |P(M_n \le x) - P(\tilde{M}_n \le x)| \to 0$ as $n \to \infty$, where M_n and $\tilde{M}_n$ are nth partial maxima for $\{X_k\}$ and $\{\tilde{X}_k\}$, respectively.

Let $\{X_k\}_{k \in \mathbb{N}}$ be a sequence of random variables. Define $M_{m:n} = \max_{m < k \le n} X_k$ for $m < n$, $M_{m:n} = -\infty$ for $m \ge n$ and $M_n = M_{0:n}$.

Suppose one can find a sequence $\{\tilde{X}_k\}_{k \in \mathbb{N}}$ of *independent* random variables such that

$$\sup_{x \in \mathbb{R}^1} \left| P(M_n \le x) - P\left(\tilde{M}_n \le x\right) \right| \to 0, \quad \text{as } n \to \infty, \tag{1}$$

where $\tilde{M}_n$ is the nth partial maximum of $\tilde{X}_k$'s. In what follows such a sequence $\{\tilde{X}_k\}_{k \in \mathbb{N}}$ is said to be an *asymptotic independent representation* (a.i.r.) for maxima of $\{X_k\}_{k \in \mathbb{N}}$.

Existence of an a.i.r. reduces many problems on asymptotic properties of laws of $\{M_n\}_{n \in \mathbb{N}}$ to the easily computable independent case. For example, possible limit laws for suitably centered and normalized M_n's can be identified with those found by Meizler (1956); see also Galambos (1978), Chapter 3.

In extreme value limit theory, the idea of a replacement of the "original" sequence by an independent one, equivalent from some point of view, takes its beginning in papers by Watson (1954) and Loynes (1965). In the latter paper the notion of the "associated" sequence for a *stationary* $\{X_k\}$ was introduced —an i.i.d. sequence $\{\hat{X}_k\}$ with the same one-dimensional marginals: $\mathscr{L}(X_k) = \mathscr{L}(\hat{X}_k)$. Leadbetter (1974) proved that in a wide class of stationary sequences the limit behaviour of all order statistics is the same for both $\{X_k\}$ and $\{\hat{X}_k\}$. Even if the correspondence between higher order statistics breaks down, the maxima of $\{X_k\}$ and $\{\hat{X}_k\}$ can remain closely related. This holds, for example, if the so-called extremal index of $\{X_k\}$ exists; see Leadbetter (1983) and also Leadbetter, Lindgren and Rootzén (1983), Chapter 3.

In the stationary case it is quite natural to require that $\{\tilde{X}_k\}$ in (1) is an independent *identically distributed* sequence. If G is the distribution function

Received August 1990; revised August 1991.
AMS 1991 *subject classifications*. 60G70, 60F99, 60J10.
Key words and phrases. Maxima, phantom distribution function, asymptotic independent representation.

of $\tilde{X}_1$, then (1) can be rewritten as

$$\sup_{x \in \mathbb{R}^1} |P(M_n \le x) - G^n(x)| \to 0 \quad \text{as } n \to \infty.$$

This relation defines a phantom distribution function for $\{X_k\}$—a notion introduced by O'Brien (1987). O'Brien gave widely applicable sufficient conditions for existence of such G; an improvement of his results obtained by Jakubowski (1991) states that a *stationary* sequence $\{X_k\}$ has a phantom distribution function G satisfying

$$(2) \qquad G(G_* -) = 1 \quad \text{and} \quad \frac{1 - G(x)}{1 - G(x-)} \to 1, \quad \text{as } x \nearrow G_*,$$

where $G_* = \sup\{u;\ G(u) < 1\}$, if and only if there is a sequence $\{v_n\}$ of numbers such that $P(M_n \le v_n) \to \alpha$ for some α, $0 < \alpha < 1$, and

$$\sup_{j, k \in \mathbb{N}} \left| P(M_{j+k} \le v_n) - P(M_j \le v_n) P(M_k \le v_n) \right| \to 0, \quad \text{as } n \to \infty.$$

In the present paper we aim at proving an analogous result in the general setting of nonstationary sequences and using asymptotic independent representations in place of phantom distribution functions.

Some examples in Hüsler (1986) show that we cannot directly adapt methods developed in the stationary case. Therefore we suggest studying the whole path

$$\mathbb{R}^+ \ni t \mapsto P(M_{[nt]} \le v_n)$$

and its limit behaviour.

Roughly speaking, we assume

$$(3) \qquad P(M_{[nt]} \le v_n) \to \alpha_t, \quad \text{as } n \to +\infty, t \in D,$$

for some dense subset $D \subset \mathbb{R}^+$ and we recover an a.i.r. from the limiting function α_t, provided the latter is of special form.

Note that α_t is nonincreasing and can be regularized to the right-continuous function

$$\tilde{\alpha}_t = \sup_{D \ni u > t} \alpha_u$$

for which

$$(4) \qquad P(M_{[nt]} \le v_n) \to \tilde{\alpha}_t, \quad \text{as } n \to +\infty,$$

at every point of continuity of $\tilde{\alpha}_t$.

Let F_j be the distribution function of X_j and let $(F_\infty)_* = \sup_j (F_j)_*$. It is easy to see that if $\alpha_{t_0} < 1$ for some $t_0 \in D$, then $v_n < (F_\infty)_*$ for all but finitely many n, and that $\sup_{t \in D} \alpha_t = 1$ implies $\liminf_n v_n \ge (F_\infty)_*$. In fact, instead of $\{v_n\}$ we may consider the nondecreasing sequence

$$v_n^* = \inf\{v_k \colon k \ge n\}.$$

LEMMA 1. *Suppose* (3) *holds on some subset* $D \subset \mathbb{R}^+$. *If* $\sup_{t \in D} \alpha_t = 1$ *and* $\alpha_{t_0} < 1$ *for some* $t_0 \in D$, *then* $v_n^* = v_{k_n}$ *for some nondecreasing sequence* $k_n \geq n$, $v_n^* < (F_\infty)_*$ *for all* $n \in \mathbb{N}$, $v_n^* \nearrow (F_\infty)_*$ *and*

$$P(M_{[nt]} \leq v_n^*) \to \alpha_t, \quad \text{as } n \to +\infty, t \in D.$$

PROOF. To prove the last statement, observe that $k_n \to \infty$, so we have for n large enough

$$\alpha_t - \varepsilon < P(M_{[k_n t]} \leq v_{k_n}) = P(M_{[k_n t]} \leq v_n^*)$$
$$\leq P(M_{[nt]} \leq v_n^*) \leq P(M_{[nt]} \leq v_n) < \alpha_t + \varepsilon,$$

that is, $P(M_{[nt]} \leq v_n^*) \to \alpha_t$. □

THEOREM 2. *Assume there is a sequence* $\{v_n\}$ *such that* (3) *holds for some dense subset* $D \subset \mathbb{R}^+ = (0, +\infty)$, *where the limiting function* α_t *possesses the properties*

$$\alpha_t > 0, \qquad t \in D, \tag{5}$$

$$\sup_{t \in D} \alpha_t = 1, \tag{6}$$

$$\inf_{t \in D} \alpha_t = 0. \tag{7}$$

Then the following statements (i)–(iv) *are equivalent.*

(i) $\{X_k\}$ *admits an asymptotic independent representation.*

(ii) $\{X_k\}$ *admits an asymptotic independent representation defined by marginal distribution functions*

$$\tilde{X}_k \sim F_k(x) = \begin{cases} 0, & \text{if } x < v_1^*, \\ \tilde{\alpha}_{k/n}/\tilde{\alpha}_{(k-1)/n}, & \text{if } v_n^* \leq x < v_{n+1}^*, \\ 1, & \text{if } x \geq \sup_k v_k^*, \end{cases} \tag{8}$$

where $v_n^* = \inf\{v_k: k \geq n\}$.

(iii) *For each* $u \geq 1$ *the function* $f_u(t) = \tilde{\alpha}_{ut}/\tilde{\alpha}_t$ *is nonincreasing on* $(0, \infty)$.

(iv) *The function* $g_\alpha = \log \circ \tilde{\alpha} \circ \exp$ *is concave.*

COROLLARY 3. *Assume, in addition to* (3)–(7), *that*

$$\sup_{k \leq l} |P(M_l \leq v_n) - P(M_k \leq v_n) P(M_{k:l} \leq v_n)| \to 0 \quad \text{as } n \to \infty. \tag{9}$$

Then $\{X_k\}$ *admits an asymptotic independent representation for maxima.*

COROLLARY 4. *Suppose* $\{X_k\}_{k \in \mathbb{N}}$ *are independent and* (3)–(7) *hold for some dense* $D \subset \mathbb{R}^+$. *Then* $\lim_{n \to \infty} P(M_{[nt]} \leq v_n) = \alpha_t$ *exists for each* $t > 0$ *and* $\alpha_{(\cdot)} = \exp(g_\alpha(\log(\cdot)))$ *for some concave* g_α.

Corollary 4 describes all possible "nondegenerate" limiting paths in the independent case. If we restrict our attention to i.i.d. sequences, then obviously the limit must be of the form $e^{-t\beta}$ for some $\beta > 0$. Note that in such a case formula (8) also gives an i.i.d. sequence:

COROLLARY 5. *Suppose that* (3) *is satisfied with* $\alpha_t = \exp(-t \cdot \beta)$, *where* $\beta > 0$. *Then* $\{X_k\}$ *admits a phantom distribution function* G *given by*

$$G(x) = \begin{cases} 0, & \text{if } x < v_1^*, \\ \exp(-\beta)^{1/n}, & \text{if } v_n^* \le x < v_{n+1}^*, \\ 1, & \text{if } x \ge \sup_k v_k, \end{cases} \tag{10}$$

where, as previously, $v_n^* = \inf\{v_k\colon k \ge n\}$.

We divide the proof of Theorem 1 into several steps. Let us begin with a simple analytic observation.

LEMMA 6. *Properties* (iii) *and* (iv) *are equivalent.*

PROOF. Take $u \ge 1$ and $t > s > 0$. Write $h = \log u$ (≥ 0), $h' = \log(t/s)$ (> 0) and $x = \log s$. Let $g_\alpha = \log \circ \tilde{\alpha} \circ \exp$. Then property (iii) can be rewritten as

$$g_\alpha(x + h) - g_\alpha(x) \ge g_\alpha(x + h + h') - g_\alpha(x + h')$$

for every $x \in \mathbb{R}^1$ and $h, h' \ge 0$. This is nothing but concavity of the function g_α. $\square$

LEMMA 7. *Suppose* $\{v_n\}$ *is* nondecreasing *and conditions* (3)–(7) *hold. If the limit function* $\tilde{\alpha}_{(\cdot)}$ *has property* (iii), *then* $\{X_k\}$ *admits an asymptotic independent representation for maxima.*

PROOF. Define distribution functions F_k of the independent sequence to be found by formula (8). For each k, F_k is a distribution function. Indeed, if $\lim_{n\to\infty} v_n < +\infty$, then $\lim_{x\to\infty} F_k(x) = 1$ trivially. If $\lim_{n\to\infty} v_n = +\infty$, then $\lim_{x\to\infty} F_k(x) = \lim_{n\to\infty} v_n < +\infty$, then $\lim_{t\to 0} \tilde{\alpha}_t / \lim_{t\to 0} \tilde{\alpha}_t = 1$ in this case also, and monotonicity of F_k follows by property (iii) of function $\tilde{\alpha}_{(\cdot)}$, when setting $u = (n+1)/n$, $s = (k-1)/n$ and $t = k/(n+1)$.

Let $\{\tilde{X}_k\}$ be independent with $\tilde{X}_k$ distributed according to F_k and let $\tilde{M}_n = \max_{k\le n} \tilde{X}_k$. We have for each $t > 0$,

$$P\left(\tilde{M}_{[nt]} \le v_n\right) = \prod_{k=1}^{[nt]} \frac{\tilde{\alpha}_{k/n}}{\tilde{\alpha}_{(k-1)/n}} = \tilde{\alpha}_{[nt]/n}.$$

Since (iv) provides continuity of $\tilde{\alpha}_{(\cdot)}$, we get (3) with $D = \mathbb{R}^+$ and $M_{[nt]}$ replaced by $\tilde{M}_{[nt]}$. Now monotonicity of paths and pointwise convergence to

$\tilde{\alpha}_{(\cdot)}$ on a dense set imply convergence in points of continuity of the limit. Since $\tilde{\alpha}_{(\cdot)}$ is continuous, we get *uniform* ([on $[0,\infty)$]) convergence to $\tilde{\alpha}_{(\cdot)}$ for both $P(M_{[n(\cdot)]} \le v_n)$ and $P(\tilde{M}_{[n(\cdot)]} \le v_n)$. In particular, for every sequence $m_n \to \infty$ of integers,

$$(11) \qquad P(M_n \le v_{m_n}) = P(M_{[m_n \cdot (n/m_n)]} \le v_{m_n}) = \tilde{\alpha}_{(n/m_n)} + o(1).$$

We have to prove (1) or, equivalently,

$$(12) \qquad P(M_n \le u_n) - P(\tilde{M}_n \le u_n) \to 0$$

for every sequence $\{u_n\}$ of reals. It is enough to consider the case $u_n < (F_\infty)_*$ and $u_n \to (F_\infty)_*$. Define integers m_n by the formula

$$m_n = \begin{cases} 1, & \text{if } u_n \le v_1, \\ k, & \text{if } v_k \le u_n < v_{k+1}. \end{cases}$$

Since $u_n \to (F_\infty)_*$, we have $m_n \to \infty$, which allows us to apply (11):

$$\begin{aligned} \tilde{\alpha}_{n/m_n} &= P(M_n \le v_{m_n}) + o(1) \\ &\le P(M_n \le u_n) + o(1) \\ &\le P(M_n \le v_{m_n+1}) + o(1) \\ &= \tilde{\alpha}_{n/(m_n+1)} + o(1), \end{aligned}$$

that is, $P(M_n \le u_n) = \tilde{\alpha}_{n/m_n} + o(1)$. Since also $P(\tilde{M}_n \le u_n) = \tilde{\alpha}_{n/m_n} + o(1)$, our claim follows. □

LEMMA 8. *Suppose $\{v_n\}$ is nondecreasing and conditions* (3) *and* (5) *hold. If for every* $0 < s < t$,

$$(13) \qquad P(M_{[nt]} \le v_n) - P(M_{[ns]} \le v_n) P(M_{[ns]:[nt]} \le v_n) \to 0,$$

then the limit function $\tilde{\alpha}_{(\cdot)}$ has property (iii).

PROOF. By (4) we have convergence of $P(M_{[n(\cdot)]} \le v_n)$ to $\tilde{\alpha}_{(\cdot)}$ at every point of continuity of the limit. We shall prove that

$$(14) \qquad \frac{\tilde{\alpha}_{us}}{\tilde{\alpha}_s} \ge \frac{\tilde{\alpha}_{ut}}{\tilde{\alpha}_t}$$

provided $\tilde{\alpha}_{(\cdot)}$ is continuous at s, us and t, and then we shall derive from (14) continuity of $\tilde{\alpha}_{(\cdot)}$ in the entire half-line. This will give us property (iii).

So let $u > 1$ and let $t > s > 0$ and us be continuity points of $\tilde{\alpha}_{(\cdot)}$. By (13), the fact that $\tilde{\alpha}_t > 0$, $t \ge 0$ and right continuity of $\tilde{\alpha}_{(\cdot)}$, it is enough to prove

$$\begin{aligned} \lim_{n\to\infty} P(M_{[ns:n(us)]} \le v_n) &= \tilde{\alpha}_{us}/\tilde{\alpha}_s \ge \tilde{\alpha}_{ut+\delta}/\tilde{\alpha}_t \\ &= \lim_{n\to\infty} P(M_{[nt]:[n(ut+\delta)]} \le v_n), \end{aligned}$$

where δ is such that $ut + \delta$ is a point of continuity of $\tilde{\alpha}_{(\cdot)}$. Let us observe that for large n,

$$P(M_{[ns]:[n(us)]} \le v_n) \ge P(M_{[(ns/t)t]:[(ns/t)ut]} \le v_{[ns/t]})$$
$$\ge P(M_{[[ns/t]t:[[ns/t](ut+\delta)]} \le v_{[ns/t]}).$$

The last expression approaches $\tilde{\alpha}_{ut+\delta}/\tilde{\alpha}_t$, while the first one approaches $\tilde{\alpha}_{us}/\tilde{\alpha}_s$, as desired.

We conclude the proof while showing continuity of $\tilde{\alpha}_{(\cdot)}$ on $(0,\infty)$. If $0 < s < t$ and $\tilde{\alpha}$ is continuous at t, let $s_n \nearrow s$ and $t_n \nearrow t$ be such that $s_n u_n = s$, $t_n u_n = t$ and s_n, t_n are continuity points of $\tilde{\alpha}$. Then $\tilde{\alpha}_{u_n t_n}/\tilde{\alpha}_{t_n} \nearrow 1$, so $\tilde{\alpha}_{u_n s_n}/\tilde{\alpha}_{s_n} \nearrow 1$ by (14). Therefore $\tilde{\alpha}$ is continuous at s. □

REMARK 9. Observe that in the Lemma 8 we may assume that (3) and (13) hold along a subsequence $\{n'\} \subset \mathbb{N}$ only.

Now we are ready to complete the proof of Theorem 2. First, by Lemma 1, we can assume $\{v_n\}$ is nondecreasing. Next, Lemma 7 gives us implications (iii) ⇒ (ii) ⇒ (i). Since (iii) ⇔ (iv) is proved in Lemma 6, the only remaining implication is (i) ⇒ (iii). Let $\{\tilde{X}_k\}$ be an a.i.r. for $\{X_k\}$. By (1), the $\tilde{M}_n$'s satisfy (3) and (5), and condition (13) is satisfied trivially for an independent sequence. Hence we can apply Lemma 8 in order to get property (iii) for $\tilde{\alpha}_{(\cdot)}$.

The proof of Corollary 3 is similar:

1. Reduction to nondecreasing $\{v_n\}$ by Lemma 1. Observe that condition (9) remains to be true with v_n replaced by $v_n^* = v_{k_n}$.
2. Application of Lemma 8. Condition (13) is implied by (9).
3. Construction of an a.i.r. by Lemma 7.

Corollary 4 is an obvious consequence of Theorem 2.

COMMENT 1. If (6) and (7) are not satisfied, the limit function may contain no information.

EXAMPLE 10. Let

$$F(x) = \begin{cases} 1 - x^{-\beta}, & \text{for } x \ge 1, \\ 0, & \text{otherwise.} \end{cases}$$

If $\{Y_k\}$ are i.i.d. with $\mathscr{L}(Y_k) \sim F$, define $X_k = k^{-1/\beta} Y_k$ and $v_n = \log^{1/\beta} n$. Then for *every* $t > 0$,

$$P(M_{[nt]} \le v_n) \to e^{-1}.$$

COMMENT 2. Our assumption (3) is much weaker than the convergence in distribution of suitably centered and normalized M_n's to a nondegenerate H.

EXAMPLE 11. Let $\{X_k\}$ be an i.i.d. sequence with marginal distribution function G. It is well known [see O'Brien (1974); also Leadbetter, Lindgren and Rootzén (1983), page 24] that there exists a sequence $\{v_n\}$ such that

$G^n(v_n) \to \alpha$ for some $0 < \alpha < 1$ if and only if G satisfies (2) (e.g., G is continuous). In such a case for each $t > 0$,

$$P(M_{[nt]} \le v_n) = G^{[nt]}(v_n) \to \alpha^t,$$

that is, conditions (3)–(7) hold. On the other hand, linearly transformed M_n's are weakly convergent if and only if G belongs to the domain of attraction of a max-stable distribution [see Leadbetter, Lindgren and Rootzén (1983), Theorem 1.4.1, page 16].

COMMENT 3. The way of checking condition (9) in Corollary 3 is standard. First, by (7) we can restrict our attention to maxima of length at most $[nT]$, say. Next, if the X_k's satisfy for each $T > 0$,

$$\max_{k \le nT} P(M_{[nT]} \ge v_n) \to 0, \quad \text{as } n \to \infty,$$

we can reduce the problem to proving

$$(15) \quad P(M_{k_n} \le v_n, M_{(k_n+r_n):l_n} \le v_n) - P(M_{k_n} \le v_n) P(M_{(k_n+r_n):l_n} \le v_n) \to 0,$$

for every $k_n, l_n \to \infty$, $k_n + l_n \le [nT]$, where $r_n \to \infty$ is such that

$$r_n \max_{k \le [nT]} P(X_k > v_n) \to 0.$$

The form of (15) is already "typical" for mixing conditions and similar to condition $\mathrm{AIM}(u_n)$ in O'Brien (1987).

COMMENT 4. Under stationarity and in the presence of mixing properties, our condition (3) is reduced to knowing a single sequence $\{P(M_n \le v_n)\}_{n \in \mathbb{N}}$ only, as shown in the following result.

PROPOSITION 12. *Suppose a stationary sequence $\{X_k\}$ has a phantom distribution function and for some sequence $\{v_n\}_{n \in \mathbb{N}}$,*

$$0 < \liminf_{n \to \infty} P(M_n \le v_n) \le \limsup_{n \to \infty} P(M_n \le v_n) < 1.$$

Then

$$\sup_{t \ge 0} \left| P(M_{[nt]} \le v_n) - P(M_n \le v_n)^t \right| \to 0, \quad \textit{as } n \to \infty.$$

PROOF. Apply Proposition 2.5 from Jakubowski (1991). □

COMMENT 5. It may happen that directly checking property (iii) is possible without explicitly invoking arguments of "mixing." For example, one can use a martingale approach (or a "successive conditioning approach").

Recall that $\{\mathscr{F}_k\}_{k \in \mathbb{N} \cup \{0\}}$ is a *filtration* if the $\mathscr{F}_k$'s form a nondecreasing sequence of σ-algebras and that the sequence $\{X_k\}$ is *adapted* to $\{\mathscr{F}_k\}$ if X_k is $\mathscr{F}_k$-measurable for each $k \in \mathbb{N}$.

We will follow the idea of the "principle of conditioning" due to Jakubowski (1991), this being a heuristic rule for derivation of limit theorems for dependent summands from results proved in the independent case only. We are going to show that this idea works in limit theorems for maxima as well.

The heart of what follows is a lemma that corresponds to Lemma 1.2 in Jakubowski (1986).

LEMMA 13. *Let $\{X_k\}$ be adapted to $\{\mathscr{F}_k\}$ and suppose that*

$$\prod_{k=1}^{k_n} P(X_k \le v_n | \mathscr{F}_{k-1}) \underset{\mathscr{P}}{\to} \alpha > 0, \tag{16}$$

where α is a constant. Then also

$$P(M_{k_n} \le v_n) \to \alpha.$$

PROOF. One can get this lemma immediately from Lemma 2 in Jakubowski and Słomiński [(1986), page 66]. The proof there can be difficult, however, for readers not familiar with the general theory of stochastic processes. So it seems to be instructive to give here a direct elementary proof.

Define

$$\tau_n = k_n \wedge \sup\left\{ m \ge 0;\ \prod_{k=1}^{m} P(X_k \le v_n | \mathscr{F}_{k-1}) \ge (1/2)\alpha \right\},$$

where $\prod_{k=1}^{0}(\cdot) \equiv 1$. For each n, τ_n is a stopping time and the sequence

$$Y_{n,m} = \prod_{k=1}^{m \wedge \tau_n} \frac{I(X_k \le v_n)}{P(X_k \le v_n | \mathscr{F}_{k-1})}, \qquad m = 1, 2, \dots, k_n,$$

is a martingale with respect to $\{\mathscr{F}_k\}$ bounded by $2\alpha^{-1}$. In particular $EY_{n,k_n} = 1$. Bearing this in mind, we can estimate

$$\begin{aligned}
&|P(M_n \le v_n) - \alpha| \\
&\quad = \left| E \prod_{k=1}^{k_n} I(X_k \le v_n) - \alpha \right| \\
&\quad \le \left| E \prod_{k=1}^{k_n} I(X_k \le v_n) - E \prod_{k=1}^{k_n \wedge \tau_n} I(X_k \le v_n) \right| \\
&\qquad + \left| EY_{n,k_n} \prod_{k=1}^{k_n \wedge \tau_n} P(X_k \le v_n | \mathscr{F}_{k-1}) - EY_{n,k_n}\alpha \right| \\
&\quad \le P(\tau_n < k_n) + 2\alpha^{-1} E\left| \prod_{k=1}^{k_n \wedge \tau_n} P(X_k \le v_n | \mathscr{F}_{k-1}) - \alpha \right| \\
&\quad \le (1 + 2\alpha^{-1}) P(\tau_n < k_n) + 2\alpha^{-1} E\left| \prod_{k=1}^{k_n} P(X_k \le v_n | \mathscr{F}_{k-1}) - \alpha \right|.
\end{aligned}$$

But (16) implies both terms in the last expression tend to zero. □

Here is an example of how to check assumption (16) of Lemma 13.

COROLLARY 14. *Using the property* $\log(1 - x) \sim -x$, *as* $x \to 0$, *we obtain that*

$$\max_{1 \le k \le k_n} P(X_k > v_n | \mathscr{F}_{k-1}) \underset{\mathscr{P}}{\to} 0, \tag{17}$$

$$\sum_{k=1}^{k_n} P(X_k > v_n | \mathscr{F}_{k-1}) \underset{\mathscr{P}}{\to} \beta \tag{18}$$

imply

$$\prod_{k=1}^{k_n} P(X_k \le v_n | \mathscr{F}_{k-1}) \underset{\mathscr{P}}{\to} e^{-\beta} \tag{19}$$

and so

$$P(M_{k_n} \le v_n) \to e^{-\beta}.$$

Now we are ready to state our criterion based on the martingale approach.

THEOREM 15. *Suppose that* $\{X_k\}$ *is adapted to* $\{\mathscr{F}_k\}$ *and the following two conditions hold for each* $t > 0$:

$$\max_{1 \le k \le [nt]} P(X_k > v_n | \mathscr{F}_{k-1}) \underset{\mathscr{P}}{\to} 0, \tag{20}$$

$$\sum_{k=1}^{[nt]} P(X_k > v_n | \mathscr{F}_{k-1}) \underset{\mathscr{P}}{\to} \beta_t, \tag{21}$$

where $\{\beta_t\}_{t \ge 0}$ *are finite constants and*

$$\lim_{t \to 0+} \beta_t = 0, \qquad \lim_{t \to +\infty} \beta_t = +\infty. \tag{22}$$

Then $\{X_k\}$ *admits an asymptotic independent representation for maxima.*

PROOF. By Corollary 14 we know that (3)–(7) hold with

$$\alpha_t = e^{-\beta t}.$$

Moreover, by the same corollary,

$$\prod_{k=1}^{[nt]} P(X_k \le v_n | \mathscr{F}_{k-1}) \underset{\mathscr{P}}{\to} \alpha_t \tag{23}$$

for each $t \ge 0$. By reasoning as in the proof of Lemma 1 we may assume that $v_n \lessdot (F_\infty)_*$ and is nondecreasing.

For every $k \in \mathbb{N}$ choose a version of the regular conditional distribution of X_k with respect to $\mathscr{F}_{k-1}$ and denote it by $\mu_k(A, \omega)$. Fix $\omega \in \Omega$ and let $X_1^{(\omega)}, X_2^{(\omega)}, X_3^{(\omega)}, \dots$ be independent and distributed according to

$\mu_1(\cdot, \omega), \mu_2(\cdot, \omega), \mu_3(\cdot, \omega), \ldots,$ respectively. Then

$$P(X_k \le v_n | \mathscr{F}_{k-1})(\omega) = P\big(X_k^{(\omega)} \le v_n\big) \quad \text{a.s.},$$

$$\prod_{k=1}^{[nt]} P(X_k \le v_n | \mathscr{F}_{k-1})(\omega) = P\Big(M_{[nt]}^{(\omega)} \le v_n\Big) \quad \text{a.s.}$$

By (20) and (23) there exists a subsequence $\{n'\} \subset \mathbb{N}$ such that

$$\max_{1 \le k \le [n't]} P\big(X_k^{(\omega)} > v_{n'}\big) \to 0 \quad \text{and} \quad P\Big(M_{[n't]}^{(\omega)} \le v_{n'}\Big) \to \alpha_t, \quad t \ge 0, t \text{ rational},$$

for every ω in a set Ω' of probability 1. Fix $\omega \in \Omega'$ and observe that along $\{n'\}$ condition (13) holds [since $P(M_{[nt]}^{(\omega)} \le v_n) = P(M_{[ns]}^{(\omega)} \le v_n) P(M_{[ns]:[nt]}^{(\omega)} \le v_n)$]. By Remark 9, $\tilde{\alpha}_s$ has property (iii) of Theorem 2 and by this theorem $\{X_k\}$ admits an a.i.r. □

THEOREM 16. *Suppose $\{Z_k\}$ is a homogeneous Markov chain with state space $(\mathscr{S}, \mathscr{B}_S)$, transition probabilities $P(x, A)$ and a unique stationary initial distribution ν. Let f: $(\mathscr{S}, \mathscr{B}_S) \to (\mathbb{R}^1, \mathscr{B}^1)$ be a measurable function such that for some sequence $\{v_n\}$ we have*

$$(24) \qquad nP(\cdot, f > v_n) \to U(\cdot) \quad \text{in } L^1(\mathscr{S}, \mathscr{B}_S, \nu).$$

If $E_\nu U \ne 0$, then $\{X_k = f \circ Z_k\}_{k \in \mathbb{N}}$ has a phantom distribution function.

PROOF. By Corollary 5 it is enough to check the assumptions of Theorem 15 with $\beta_t = t(E_\nu U)$. Set $\mathscr{F}_0 = \{\varnothing, \Omega\}$ and $\mathscr{F}_k = \sigma(Z_1, Z_2, \ldots, Z_k)$. Then for each $t > 0$,

$$E_\nu \max_{1 \le k \le [nt]} \big(P(X_k > v_n | \mathscr{F}_{k-1})\big)^2 \le \big(\nu(f > v_n)\big)^2 + \sum_{k=2}^{n} E_\nu \big(P(Z_{k-1}, f > v_n)\big)^2$$

$$\le nE_\nu \big(P(Z_1, f > v_n)\big)^2 \to 0,$$

since $n(P(Z_1, f > v_n))^2 \to 0$ in probability and is dominated by the uniformly integrable sequence $\{nP(Z_1, f > v_n)\}$. Checking (21) is a little bit more complicated. First, we may neglect the term $P(X_1 > v_n | \mathscr{F}_0) = \nu(f > v_n)$. Then, by (24),

$$E_\nu \left| \sum_{k=2}^{[nt]} P(X_k > v_n | \mathscr{F}_{k-1}) - \left(\frac{1}{n}\right) \sum_{k=1}^{[nt]-1} U(Z_k) \right|$$

$$\le \frac{[nt] - 1}{n} E_\nu | nP(Z_1, f > v_n) - U(Z_1) | \to 0.$$

However the ergodic theorem gives

$$(1/n) \sum_{k=1}^{[nt]-1} U(Z_k) \to tE_\nu(U(Z_1))$$

and our theorem follows. □

Using assumptions stronger than (24) we are able to work independently of whether a stationary initial distribution for $\{Z_k\}$ exists or not.

COROLLARY 17. *Let* $\{Z_k\}$ *be a homogeneous Markov chain on* $(\mathscr{S}, \mathscr{B}_S)$, *with transition probabilities* $P(x, A)$ *and initial distribution* ν.

If f *is such that*

$$\gamma_n = \sup_{x \in \mathscr{S}} |\beta - nP(x, f > v_n)| \to 0, \tag{25}$$

for some $\beta > 0$, *and*

$$\nu(f > v_n) \to 0, \tag{26}$$

then $\{X_k = f \circ Z_k\}_{k \in \mathbb{N}}$ *has a phantom distribution function.*

PROOF. Following notation from the proof of Theorem 16 we have

$$\begin{aligned}\max_{1 \le k \le [nt]} P(X_k > v_n | \mathscr{F}_{k-1}) &\le \nu(f > v_n) + \max_{2 \le k \le n} P(Z_{k-1}, f > v_n)\\ &\le \nu(f > v_n) + \beta/n + \gamma_n/n \to 0.\end{aligned}$$

Similarly

$$\begin{aligned}&\left| \sum_{k=2}^{[nt]} P(X_k > v_n | \mathscr{F}_{k-1}) - \frac{([nt] - 1)}{n}\beta \right|\\ &\quad = \left| \nu(f > v_n) + n^{-1} \sum_{k=2}^{[nt]} (nP(Z_{k-1}, f > v_n) - \beta) \right|\\ &\quad \le \nu(f > v_n) + n^{-1} n\gamma_n \to 0.\end{aligned}$$

Hence $\beta_t = \lim_{n \to \infty}([nt] - 1)\beta/n = t\beta$. □

REMARK 18. Our assumptions deal with transition probabilities only. If we know more about structure of the Markov chain (e.g., Harris recurrence), then it is often possible to find a phantom distribution function directly; see O'Brien (1987) and Rootzén (1988).

Acknowledgment. The author wishes to thank the referee for remarks that improved the article.

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