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SDEs WITH OBLIQUE REFLECTION ON NONSMOOTH DOMAINS

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In this paper we consider stochastic differential equations with reflecting boundary conditions for domains that might have corners and for which the allowed directions of reflection at a point on the boundary of the domain are possibly oblique. The main results are strong existence and uniqueness for solutions of such equations. A key ingredient is a family of relatively regular functions appropriate to the given domain and directions of reflection. Two cases are treated in the paper. In the first case the direction of reflection is single valued and varies smoothly, and the main new feature is that the boundary of the domain may be nonsmooth. In the second case the domain is taken to be the intersection of a finite number of domains with relatively smooth boundary, and at the resulting corner points more than one oblique direction is allowed.

1. Introduction. The primary goal of this paper is to solve stochastic differential equations (SDEs) with reflecting boundary conditions for domains that might have corners and for which the allowed directions of reflection at a point on the boundary of the domain are possibly oblique. Such equations are of increasing importance in stochastic systems theory. Our approach, which is inspired by (and borrows many of the ideas of) the approach used by Lions and Sznitman in [8], is a direct approach based on the *Skorokhod problem* (SP). In previous work [4] we have discussed at length the regularity properties of the solution mapping to the SP (to be defined below) as a function of the problem data in a more restrictive setting than the one we will use in the present paper. In the present work, the weakened assumptions on the domain of interest and on the directions of reflection at points on the boundary of the domain yield much weaker regularity properties of the solution mapping to the SP than in [4]. However, in the spirit of [8], we observe that these weaker properties are still sufficient if one is only interested in solving the corresponding stochastic differential equation with reflection (SDER).

The key ingredient we use to extend the results of [8] is a family of relatively regular functions appropriate to the given domain and directions of reflection. The properties and the use of these functions will be described and illustrated in the sections which follow. The existence of these functions was proved in [5] and [3], where the functions were used to prove existence and uniqueness of viscosity solutions (see [2]) to degenerate, fully nonlinear elliptic PDEs, with

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the same domain and derivative boundary conditions in the same directions as the directions of reflection used here. Indeed, a secondary goal of this paper is to indicate the close connection between the constructions needed in the viscosity solution approach to problems on such domains and those needed in the SP approach to constructing solutions to SDER for such domains. We treat two cases in this paper, which correspond to the two types of problems treated in [5] and [3]. In the first case the direction of reflection is single valued and varies smoothly, and the main new feature is that the boundary of the domain may be nonsmooth. In the second case the domain is taken to be the intersection of a finite number of domains with relatively smooth boundary and at the resulting corner points more than one oblique direction is allowed.

2. Definitions and notation. Throughout this paper we will use the following definitions and assumptions. The domain of interest will be called G and we will assume that G is a bounded open subset of $\mathbb{R}^N$. Let $S(a,b) = \{x \in \mathbb{R}^N: |x-a| = b\}$ and $B(a,b) = \{x \in \mathbb{R}^N: |x-a| \le b\}$. To each point $x \in \partial G$, we will associate a set $r(x) \subset S(0,1)$ called the directions of reflection. The set of continuous functions mapping $[0,T]$ into $\mathbb{R}^N$ will be denoted by $C[0,T]$. For a function of bounded variation η mapping $[0,T]$ to $\mathbb{R}^N$, we let $|\eta|(t)$ denote the total variation over the interval $[0,t]$.

Definition 2.1 (Skorokhod problem). Let $\psi \in C[0,T]$ with $\psi(0) \in \overline{G}$ be given. Then $(\phi,\eta) \in C[0,T]^2$ solves the SP for ψ (with respect to G and r) if: (i) $\phi = \psi + \eta$, $\phi(0) = \psi(0)$; (ii) $\phi(t) \in \overline{G}$ for $t \in [0,T]$; (iii) $|\eta|(T) < \infty$; (iv) $|\eta|(t) = \int_{(0,t]} I_{\{\phi(s) \in \partial G\}}\, d|\eta|(s)$; (v) there exists measurable γ: $[0,T] \to \mathbb{R}^N$ such that $\gamma(s) \in r(\phi(s))$ ($d|\eta|$ a.s.) and $\eta(t) = \int_{(0,t]} \gamma(s)\, d|\eta|(s)$.

Hence ϕ never leaves $\overline{G}$ and η changes only when $\phi \in \partial G$, in which case the change points in one of the directions $r(\phi)$.

Our present interest in the SP is in its use in defining solutions to SDER for the given domain and directions of reflection. A precise definition is as follows. Let $(\Omega, \mathscr{F}, P)$ be a complete probability space and let $\{\mathscr{F}_t, t \ge 0\}$ be a filtration satisfying the usual conditions, that is, $\mathscr{F}_t$ is right-continuous and $\mathscr{F}_0$ contains all sets of P measure zero. Let $\sigma_{ij}(x)$ and $b_i(x)$, $(i,j) \in \{1,\dots,N\} \times \{1,\dots,r\}$, be continuous functions on $\mathbb{R}^N$ and suppose that $\{w(t), t \ge 0\}$ is an r-dimensional $\mathscr{F}_t$-Brownian motion.

Definition 2.2 (SDER). A continuous $\mathscr{F}_t$-adapted process $X(t)$ is a solution to the SDER for the domain G, directions of reflection $r(\cdot)$, initial condition $x \in \overline{G}$ and Brownian motion $\{w(t), t \ge 0\}$, if $X(t) \in \overline{G}$ for all $t \ge 0$ (a.s.) and

$$X(t) = x + \int_0^t b(X(s))\, ds + \int_0^t \sigma(X(s))\, dw(s) + Z(t),$$

where

$$|Z|(t) = \int_{(0,t]} I_{\{X(s) \in \partial G\}}\, d|Z|(s) < \infty,$$

and where there exists measurable $\gamma(s) \in r(X(s))$ such that

$$Z(t) = \int_{(0,t]} \gamma(s)\, d|Z|(s)$$

(a.s.).

In other words, $(X(\cdot), Z(\cdot))$ should solve (on an a.s. pathwise basis) the SP for $\psi(\cdot) = x + \int_0^{\cdot} b(X(s))\, ds + \int_0^{\cdot} \sigma(X(s))\, dw(s)$.

REMARK 2.3. Other possibilities (e.g., time dependence of the coefficients or control dependence of the drift that is not of the feedback form) are also of interest and may be solved using the methods described below.

As stated in the Introduction, the purpose of this paper is to construct SDER for domains with corners and oblique directions of reflection. A key ingredient in the proofs will be families of test functions originally developed to prove existence and uniqueness results for PDEs on domains with corners and oblique derivative boundary conditions. The main steps in the proof are the following:

1. Prove the existence of solutions to the SP when ψ is smooth. This is accomplished in Lemma 4.5 using a penalty function method similar to one used in [8].
2. Obtain equicontinuity results for collections of solutions to the SP. This step makes important use of the test functions and is carried out in Lemma 4.7.
3. Using the equicontinuity result, we extend the existence result to cover all continuous paths with initial point in $\overline{G}$ in Theorem 4.8.
4. Steps (1)–(3) allow us to define the "reflected" version of a general class of semimartingales. Using the test functions again we obtain estimates on this mapping that are sufficient to prove the existence and uniqueness of a fixed point of a related mapping, which will actually be the solution to the SDER. The needed estimate is proved in Theorem 5.1.

The symbol C will be used for a constant satisfying $0 < C < \infty$, whose value may change with each equation.

3. Assumptions and properties of the test functions. We will consider two cases of nonsmooth domains and oblique directions of reflection. The cases correspond to the assumptions required for the study of degenerate elliptic PDE in [3] and [5].

CASE 1. G is an open bounded set. The set of directions of reflection is actually a singleton: $r(x) = \{\gamma(x)\}$, where

$$\gamma \in C^2(\mathbb{R}^N, \mathbb{R}^N). \tag{3.1}$$

There is $b \in (0,1)$ such that

$$\bigcup_{0 \le t \le b} B(x - t\gamma(x), tb) \subset G^c \quad \text{for } x \in \partial G. \tag{3.2}$$

CASE 2. G is an open bounded set with representation

$$G = \bigcap_{i \in I} G_i,$$

where I is a finite index set and each G_i is an open bounded set. Let $I(x) = \{i: x \notin G_i\}$ and assume that $I(\cdot)$ is upper semicontinuous in the sense that for each $x \in \partial G$, there is an open neighborhood V of x such that $y \in V \Rightarrow I(y) \subset I(x)$. Then we assume:

(3.3) For each $i \in I$ the boundary ∂G_i is of class C^1.

We assume that there are vector fields

$$\gamma_i \in C^{0,1}(\mathbb{R}^N, \mathbb{R}^N) \tag{3.4}$$

such that if $n_i(x)$ denotes the inward normal to ∂G_i at $x \in \partial G$, then $\langle \gamma_i(x), n_i(x) \rangle > 0$, and that for each $x \in \partial G$ the convex hull of $\{\gamma_i(x): i \in I(x)\}$ does not contain the origin. We then define the directions of reflection by

$$r(x) = \left\{ \sum_{i \in I(x)} a_i \gamma_i(x) : a_i \geq 0, \left| \sum_{i \in I(x)} a_i \gamma_i(x) \right| = 1 \right\}. \tag{3.5}$$

Thus the set $\{\alpha\gamma: \alpha \geq 0, \gamma \in r(x)\}$ is a closed convex cone and furthermore there is a hyperplane through the origin that intersects this cone only at the origin. We also assume that at each point $x \in \partial G$ there is $\gamma \in r(x)$ pointing into G. More precisely, we assume for each $x \in \partial G$ the existence of $a_i \geq 0$, $i \in I(x)$, such that

$$\left\langle \sum_{i \in I(x)} a_i \gamma_i(x), n_j(x) \right\rangle > 0 \quad \text{for } j \in I(x). \tag{3.6}$$

Our key assumption is the following. For each $x \in \partial G$ there is an open neighborhood W of x and a family $\{B(y): y \in W\}$ of compact convex subsets of $\mathbb{R}^N$ with $0 \in B(y)$ for all $y \in W$, such that the family is of class $C^{2,+}$ (defined immediately after the remark below) and such that for all $y \in W \cap \partial G$ and $p \in \partial B(y)$,

$$\langle \gamma_i(y), n \rangle \begin{cases} \geq 0, & \text{if } \langle p, n_i(y) \rangle \geq -1, \\ \leq 0, & \text{if } \langle p, n_i(y) \rangle \leq 1, \end{cases} \tag{3.7}$$

whenever n is an inward normal to $B(y)$ at p.

REMARK 3.1. The condition (3.7) occurs naturally in the study of the SP and we refer the reader to [4] for other examples of its use. A sufficient (but not necessary) condition for (3.6) and (3.7) that is simpler to verify in practice is as follows. For each $x \in \partial G$ we require the existence of scalars $b_i \geq 0$, $i \in I(x)$, such that

$$b_i \langle \gamma_i(x), n_i(x) \rangle > \sum_{j \in I(x) \setminus \{i\}} b_j \left| \langle \gamma_j(x), n_i(x) \rangle \right|. \tag{3.8}$$

It is obvious that (3.6) holds. For the proof that (3.7) holds, we refer the reader to [5], Section 5. This reference also proves the equivalence of (3.8) to the following algebraic characterization. Assume without loss of generality that $I(x) = \{1, \dots, m\}$ and that $\langle \gamma_i(x), n_i(x) \rangle = 1$ for $i \in I(x)$. Set $v_{ij} = |\langle \gamma_i(x), n_j(x) \rangle| - \delta_{ij}$, where δ_{ij} is 1 if $i = j$ and 0 if $i \neq j$. Then (3.8) holds if and only if $\sigma(V) < 1$, where $V = (v_{ij})$ and where $\sigma(V)$ denotes the spectral radius of V. Harrison and Reiman [6] were the first to recognize the usefulness of conditions of this type in connection with problems involving reflected processes and domains with corners.

DEFINITIONS. Let S^M denote the set of all symmetric $M \times M$ matrices. For a real valued function f defined on $U \subset \mathbb{R}^M$ and $x \in U$, the superdifferentials of first order at x are defined to be

$$D^+f(x) = \{p \in \mathbb{R}^M \colon f(x+h) \le f(x) + \langle p, h \rangle + o(|h|) \text{ for } x + h \in U \text{ and as } h \to 0\}.$$

The superdifferentials of second order at x are defined to be

$$D^{2,+}f(x) = \{(p, A) \in \mathbb{R}^M \times S^M \colon f(x+h) \le f(x) + \langle p, h \rangle + \tfrac{1}{2}\langle Ah, h \rangle + o(|h|^2) \text{ for } x + h \in U \text{ and as } h \to 0\}.$$

We will abuse notation and let I denote the identity matrix as well as the index set for Case 2. The intended usage will be clear from the context. We define $C^{2,+}(U)$ to be those real valued functions in $C^{0,1}(U)$ having the property that for each compact subset K of U, there is a constant $C < \infty$ such that if $x \in K$, then $(p, CI) \in D^{2,+}f(x)$ for some $p \in \mathbb{R}^M$. Thus $C^{2,+}(U)$ is the set of real, locally semiconcave functions on U. For any set B, let $d(x, B) = \inf\{|x - y| \colon y \in B\}$. A family of nonempty convex sets $\{B(x) \colon x \in U\}$ is said to be of class $C^{2,+}$ if the function

$$(x, y) \to d(y, B(x))^2$$

is in $C^{2,+}(U \times \mathbb{R}^M)$.

Under the assumptions listed at the beginning of this section, it is proved that families of functions having certain convenient properties exist.

THEOREM 3.2. *(Case 1.) Given $\theta \in (0,1)$, there exists a C^2 function $g(x, r)$ on $\overline{G} \times \mathbb{R}^N$ and a constant $C < \infty$ such that*

$$g(x, 0) = 1, \tag{3.9}$$

$$g(x, r) \ge |r|^2, \tag{3.10}$$

$$\langle D_r g(x, r), \gamma(x) \rangle \ge 0 \quad \text{if } \langle r, \gamma(x) \rangle \ge -\theta |r|, \tag{3.11}$$

$$\begin{gathered} |D_x g(x, r)| \vee \|D_x^2 g(x, r)\| \le C|r|^2, \\ |D_r g(x, r)| \vee \|D_x D_r g(x, r)\| \le C|r|, \\ \|D_r^2 g(x, r)\| \le C. \end{gathered} \tag{3.12}$$

Given $\theta \in (0,1)$, *there exists a family of* C^2 *functions* $\{f_\varepsilon(x,y)\colon \varepsilon > 0\}$ *defined on* $\overline{G} \times \overline{G}$ *such that if* $p = D_x f_\varepsilon(x,y)$ *and* $q = D_y f_\varepsilon(x,y)$, *then*

$$f_\varepsilon(x,y) \geq \frac{|x-y|^2}{\varepsilon}, \tag{3.13}$$

$$f_\varepsilon(x,y) \leq C\left(\varepsilon + \frac{|x-y|^2}{\varepsilon}\right), \tag{3.14}$$

$$\langle \gamma(x), p\rangle \leq C\frac{|x-y|^2}{\varepsilon} \quad \textit{if } \langle y-x, \gamma(x)\rangle \geq -\theta|y-x|, \tag{3.15}$$

$$\langle \gamma(y), q\rangle \leq C\frac{|x-y|^2}{\varepsilon} \quad \textit{if } \langle x-y, \gamma(y)\rangle \geq -\theta|x-y|, \tag{3.16}$$

$$|p+q| \leq C\frac{|x-y|^2}{\varepsilon}, \tag{3.17}$$

$$|p| \vee |q| \leq C\frac{|x-y|}{\varepsilon}, \tag{3.18}$$

$$D^2 f_\varepsilon(x,y) \leq \frac{C}{\varepsilon}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + \frac{C|x-y|^2}{\varepsilon}\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \tag{3.19}$$

where $C < \infty$ *is a constant that is independent of* $\varepsilon > 0$. *There is a* C^2 *function* $h(x)$ *on* $\overline{G}$ *such that* $h(x) \geq 0$ *and*

$$\langle \gamma, Dh(x)\rangle \geq 1 \quad \textit{for all } x \in \partial G \textit{ and } \gamma \in r(x). \tag{3.20}$$

(*Case* 2.) *There exists an open set* W *containing* $\overline{G}$ *and a* $C^{2,+}$ *function* $g(x,r)$ *on* $W \times \mathbb{R}^N$ *which for each fixed* x *is* C^1 *as a mapping* $r \to g(x,r)$. *Furthermore, there are constants* $C < \infty$ *and* $\theta > 0$ *such that*

$$g(x,0) = 0, \tag{3.21}$$

$$g(x,r) \geq |r|^2, \tag{3.22}$$

$$\langle D_r g(x,r), \gamma_i(x)\rangle \geq 0 \quad \textit{if } \langle r, n_i(x)\rangle \geq -\theta|r|, \tag{3.23}$$

$$|p| \leq C|r|^2, \qquad |q| \leq C|r| \quad \textit{if } (p,q) \in D^+ g(x,r), \tag{3.24}$$

and for any $x \in \overline{G}$, $r \in \mathbb{R}^N$, *there is* $(p,q) \in D^+ g(x,r)$ *such that*

$$\left((p,q), C\begin{pmatrix} |r|^2 I & 0 \\ 0 & I \end{pmatrix}\right) \in D^{2,+} g(x,r). \tag{3.25}$$

There exists an open set W *containing* $\overline{G}$ *and a family of* $C^{2,+}$ *functions*

$\{f_\varepsilon(x,y)\colon \varepsilon > 0\}$ *defined on* $W \times W$ *such that if* $(p,q) \in D^+ f_\varepsilon(x,y)$, *then*

$$f_\varepsilon(x,y) \geq \frac{|x-y|^2}{\varepsilon}, \tag{3.26}$$

$$f_\varepsilon(x,y) \leq C\frac{|x-y|^2}{\varepsilon}, \tag{3.27}$$

$$\langle \gamma_i(x), p\rangle \leq C\frac{|x-y|^2}{\varepsilon} \quad \textit{if } \langle y-x, n_i(x)\rangle \geq -\theta|y-x|, \tag{3.28}$$

$$\langle \gamma_i(x), q\rangle \leq C\frac{|x-y|^2}{\varepsilon} \quad \textit{if } \langle x-y, n_i(y)\rangle \geq -\theta|x-y|, \tag{3.29}$$

$$|p+q| \leq C\frac{|x-y|^2}{\varepsilon}, \tag{3.30}$$

$$|p| \vee |q| \leq C\frac{|x-y|}{\varepsilon}, \tag{3.31}$$

$\forall\, x, y \in W$, $\exists\, (p,q) \in D^+ f_\varepsilon(x,y)$ *such that*

$$\left((p,q), \frac{C}{\varepsilon}\begin{pmatrix} I & -I \\ -I & I\end{pmatrix} + \frac{C|x-y|^2}{\varepsilon}\begin{pmatrix} I & 0 \\ 0 & I\end{pmatrix}\right) \in D^{2,+} f_\varepsilon(x,y), \tag{3.32}$$

where $C < \infty$ *is a positive constant that is independent of* $\varepsilon > 0$. *There is a* C^2 *function* $h(x)$ *on* $\overline{G}$ *such that* $h(x) \geq 0$ *and*

$$\langle \gamma, Dh(x)\rangle \geq 1 \quad \textit{for all } x \in \partial G \textit{ and } \gamma \in r(x). \tag{3.33}$$

REMARK 3.3. The existence of such functions is proved in [3] and [5]. Although the functions for Case 2 have less regularity than those of Case 1, they are still sufficient for our purposes. The functions h appearing in (3.20) and (3.33) can be obtained in the form $h = c_1\bar{h} + c_2$, where c_1 and c_2 are constants and the functions $\bar{h}$ for the two cases are constructed in [3], Proof of Corollary 2.3, and [5], Lemma 3.2. In [3], proof of Theorem 4.1, and [5], Lemma 4.4, functions g satisfying (3.9)–(3.12) and (3.21)–(3.25) are shown to exist. Families $\{f_\varepsilon, \varepsilon > 0\}$ satisfying (3.13), (3.15)–(3.19) and (3.26), (3.28)–(3.32) are constructed in [3], Theorem 4.1, and [5], Theorem 4.1. It should be remarked that the directions $\gamma(x)$ and $\gamma_i(x)$ are assumed to point out of G in [3] and [5], rather than into G as in the present paper and that outward rather than inward normals are used. We note the following consequence of the assumptions made on ∂G in the two cases. For a suitable choice of $\theta \in (0,1)$ ($\theta^2 > 1 - b^2$ in Case 1, arbitrary in Case 2), there exists $\delta > 0$ such that $x \in \partial G$, $y \in \overline{G}$ and $|x-y| \leq \delta$ imply $\langle y-x, \gamma(x)\rangle \geq -\theta|y-x|$ in Case 1 and $\langle y - x, n_i(x)\rangle \geq -\theta|y-x|$ for $i \in I(x)$ in Case 2. The functions f_ε are actually obtained from the functions g by defining

$$f_\varepsilon(x,y) = \varepsilon g\left(x, \frac{x-y}{\varepsilon}\right).$$

Therefore (3.14) and (3.27) follow from (3.12), (3.24) and

$$\begin{aligned} f_\varepsilon(x,y) &= \varepsilon \int_0^1 \left[\frac{d}{ds} g\left(x, s\frac{x-y}{\varepsilon}\right)\right] ds + \varepsilon g(x,0) \\ &\le \varepsilon \int_0^1 \left| D_r g\left(x, s\frac{x-y}{\varepsilon}\right)\right| \left|\frac{x-y}{\varepsilon}\right| ds + \varepsilon g(x,0) \\ &\le C\frac{|x-y|^2}{\varepsilon} + \varepsilon g(x,0). \end{aligned}$$

For information on related functions as well as the original motivation, existence proofs and uses of these functions we refer the reader to [5] and [3].

4. Existence of solutions to the SP. In this section we will prove the existence of solutions to the SP under Cases 1 and 2 described in Section 2. We will first prove a few preliminary lemmas: Lemmas 4.1 and 4.3 for Case 1 and Lemma 4.4 for Case 2. We then prove existence of a solution when ψ is smooth (Lemma 4.5), obtain equicontinuity estimates (Lemma 4.7) and extend to all continuous ψ (Theorem 4.8).

For $x \in \mathbb{R}^N$ define

$$d(x) = d(x,G), \qquad v(x) = d(x)^2.$$

LEMMA 4.1. *Under the assumptions of Case* 1, *there is a constant* $\nu > 0$ *and a neighborhood* U *of* ∂G *such that*

$$\langle \gamma(x), Dd(x)\rangle \le -\nu \quad a.e.\ in\ U \setminus \overline{G}. \tag{4.1}$$

REMARK 4.2. Note that the Lipschitz continuity of $d(x)$ implies the a.e. differentiability of $d(x)$.

PROOF. We first note that the assumptions given for Case 1 imply that the boundary of G is relatively well behaved. Using a simple contradiction argument, it follows from (3.1) and (3.2) that there is $b > 0$ such that for all $x \in \partial G$,

$$\bigcup_{0 \le t \le b} B(x + t\gamma(x), bt) \subset \overline{G}.$$

Therefore by (3.1) if we fix $z \in \partial G$ we can choose $b > 0$ so that

$$\forall\, x \in B(z,b), \forall\, y \in B(z,4b) \cap \partial G, \qquad \bigcup_{0 \le t \le b} B(y + t\gamma(x), bt) \subset \overline{G}. \tag{4.2}$$

Fix $x \in B(z,b) \setminus \overline{G}$. Let $0 < t < b$ and set $y = x - t\gamma(x)$. Choose $w \in \partial G$ so that $d(y) = |y - w|$. Note that $d(y) \le |z - y| \le |z - x| + t|\gamma(x)| \le 2b$ and so $|w - z| \le |w - y| + |y - z| \le 4b$. Therefore, by (4.2),

$$B(w + t\gamma(x), bt) \subset \overline{G}.$$

This last inclusion implies

$$d(x) \le |x-u| \quad \text{for } u \in B(w+t\gamma(x), bt),$$

and since $x - t\gamma(x) = y$,

$$\begin{aligned} d(x)^2 &\le \left| x - w - t\gamma(x) - bt\frac{y-w}{|y-w|} \right|^2 \\ &= \left| y - w - bt\frac{y-w}{|y-w|} \right|^2 \\ &= |y-w|^2 - 2bt|y-w| + (bt)^2 \\ &= (d(y) - bt)^2. \end{aligned}$$

It follows that for t sufficiently small,

$$d(x) \le d(y) - bt.$$

Assume now that d is differentiable at x and set $p = Dd(x)$. Then for $y = x - t\gamma(x)$,

$$\begin{aligned} d(y) &= d(x) + \langle p, y-x \rangle + o(|y-x|) \\ &\le d(y) - bt - t\langle p, \gamma(x) \rangle + o(t). \end{aligned}$$

Sending $t \downarrow 0$ we obtain $\langle p, \gamma(x) \rangle \le -b$. □

Let ρ_δ satisfy

$$\rho_\delta \ge 0, \qquad \operatorname{supp} \rho_\delta \subset B(0,\delta), \qquad \int_{\mathbb{R}^N} \rho_\delta \, dx = 1, \qquad \rho_\delta \in C^\infty.$$

Define

$$v_\delta = v * \rho_\delta, \qquad d_\delta = d * \rho_\delta,$$

where $*$ denotes convolution.

LEMMA 4.3. *Under the assumptions of Case* 1, *for sufficiently small* $\delta > 0$ *there is a constant* $\nu > 0$ *and a neighborhood* U *of* ∂G *such that*

$$\langle \gamma(x), Dv_\delta(x) \rangle \le -\nu d_\delta(x) \quad \text{in } U \setminus \overline{G}. \tag{4.3}$$

PROOF. By Lemma 4.1, there exists a neighborhood U of ∂G and a constant $\nu > 0$ such that

$$\langle \gamma(x), Dv(x) \rangle = 2\langle \gamma(x), Dd(x) \rangle d(x) \le -2\nu d(x) \quad \text{a.e. in } U \setminus \overline{G}.$$

Therefore, whenever $B(x, \delta) \subset U$,

$$\begin{aligned}\langle \gamma(x), D v_\delta(x)\rangle &= \left\langle \gamma(x), \int_{\mathbb{R}^N} D\rho_\delta(x-y) v(y)\,dy\right\rangle \\ &= \left\langle \gamma(x), \int_{\mathbb{R}^N} \rho_\delta(x-y) Dv(y)\,dy\right\rangle \\ &= \int_{\mathbb{R}^N} [\langle \gamma(y), Dv(y)\rangle + \langle \gamma(x) - \gamma(y), Dv(y)\rangle] \rho_\delta(x-y)\,dy \\ &\le \int_{\mathbb{R}^N} [-2\nu\, d(y) + 2L\delta\, d(y)] \rho_\delta(x-y)\,dy \\ &= 2[-\nu + L\delta]\, d_\delta(x),\end{aligned}$$

where L is the Lipschitz constant of γ over the compact set $\overline{U}$. □

LEMMA 4.4. *Consider the situation of Case 2. Let $b > 0$ be chosen so that $d(x, G_i)^2$ is a C^1 function for $d(x, G_i) < b$ and for $i \in I$. Then there exist $c \in (0, b)$, $\nu > 0$ and smooth (C^1) functions $a_i(x)$, $i \in I$, satisfying $a_i(x) \ge 0$ for $i \in I$, $a_i(x) = 0$ for $i \notin I(x)$ and*

$$\left\langle \sum_{i \in I} a_i(x)\gamma_i(x), D_x d(x, G_j)\right\rangle \le -\nu/2 \quad \text{for } j \in I(x)$$

and for all x satisfying $\sum_{i \in I} d(x, G_i) < b$.

PROOF. This follows under the pointwise assumption given by (3.6), the compactness of ∂G and the upper semicontinuity of $I(x)$ by a partition of unity argument. □

LEMMA 4.5. *Let $\psi \in C^1[0, T]$ satisfy $\psi(0) \in \overline{G}$. Then under the assumptions of either Case 1 or Case 2 of Section 2, there exists $(\phi, \eta) \in (H^1(0, T))^2$ such that (ϕ, η) solves the SP.*

REMARK 4.6. Given that $\eta \in H^1(0, T)$, we may write

$$d|\eta|(t) = |\dot{\eta}(t)|\,dt, \quad \text{where } \dot{\eta} = \frac{d\eta}{dt}.$$

PROOF. (*Case 1.*) Suppose that $\varepsilon > 0$. Consider the ODE,

$$\dot{\phi}_\varepsilon(t) = \frac{1}{\varepsilon} d(\phi_\varepsilon(t))\gamma(\phi_\varepsilon(t)) + \dot{\psi}(t), \qquad 0 \le t \le T, \tag{4.4}$$
$$\phi_\varepsilon(0) = \psi(0).$$

Choose a neighborhood U and a constant $\nu > 0$ as in Lemma 4.3. Choose a function $\xi\colon \mathbb{R} \to \mathbb{R}$ so that

$$\xi \in C^\infty, \qquad \xi(r) = r, \qquad 0 \le r \le c, \qquad \xi(r) = 2c, \qquad r \ge 3c,$$

(4.5)
$$0 \le \frac{d\xi(r)}{dr} = \xi'(r) \le 1, \qquad r \in \mathbb{R},$$

where $c > 0$ is a small constant such that $v(x) = d^2(x) \le 4c \Rightarrow x \in U \cup \overline{G}$. Define (recall that v_δ was defined immediately before Lemma 4.3)

$$V(t) = \xi\big(v_\delta(\phi_\varepsilon(t))\big) \quad \text{for } 0 \le t \le T.$$

Then, noting that $\phi_\varepsilon(t) \notin U \cup \overline{G} \Rightarrow \xi'(v_\delta(\phi_\varepsilon(t))) = 0$, we have

$$\begin{aligned} \dot V &= \xi'(v_\delta(\phi_\varepsilon))\langle Dv_\delta(\phi_\varepsilon), \dot\phi_\varepsilon\rangle \\ &= \xi'(v_\delta(\phi_\varepsilon))\Big\langle Dv_\delta(\phi_\varepsilon), \frac{1}{\varepsilon}d(\phi_\varepsilon)\gamma(\phi_\varepsilon) + \dot\psi\Big\rangle \\ &\le -\frac{\nu}{\varepsilon}\xi'(v_\delta(\phi_\varepsilon))d(\phi_\varepsilon)d_\delta(\phi_\varepsilon) + \xi'(v_\delta(\phi_\varepsilon))|\dot\psi||Dv_\delta(\phi_\varepsilon)|. \end{aligned}$$

Integrating this, we obtain

$$\xi(v_\delta(\phi_\varepsilon(t))) + \frac{\nu}{\varepsilon}\int_0^t \xi'(v_\delta(\phi_\varepsilon))d(\phi_\varepsilon)d_\delta(\phi_\varepsilon)\,ds \le \int_0^t \xi'(v_\delta(\phi_\varepsilon))|\dot\psi||Dv_\delta(\phi_\varepsilon)|\,ds$$

for all $0 \le t \le T$. Sending $\delta \downarrow 0$, we get

$$\begin{aligned} &\xi(v(\phi_\varepsilon(t))) + \frac{\nu}{\varepsilon}\int_0^t \xi'(v(\phi_\varepsilon))v(\phi_\varepsilon)\,ds \\ &\quad\le 2\int_0^t \xi'(v(\phi_\varepsilon))|\dot\psi|d(\phi_\varepsilon)\,ds \\ &\quad\le 2\left(\int_0^t \xi'(v(\phi_\varepsilon))|\dot\psi|^2\,ds\right)^{1/2}\left(\int_0^t \xi'(v(\phi_\varepsilon))v(\phi_\varepsilon)\,ds\right)^{1/2}. \end{aligned}$$

The boundedness of $(\int_0^t \xi'(v(\phi_\varepsilon))|\dot\psi|^2\,ds)^{1/2}$, (4.5) and the inequality above imply

$$\frac{1}{\varepsilon}\left[\int_0^T \xi'(v(\phi_\varepsilon(s)))v(\phi_\varepsilon(s))\,ds\right]^{1/2} \le C(T),$$

where $C(T) > 0$ is independent of $\varepsilon > 0$. Thus we have

(4.6)
$$\xi(v(\phi_\varepsilon(t))) + \frac{\nu}{\varepsilon}\int_0^t \xi'(v(\phi_\varepsilon(s)))v(\phi_\varepsilon(s))\,ds \le C_1(T)\varepsilon,$$

where $C_1(T) > 0$ is independent of $\varepsilon > 0$. From (4.6) and (4.5), we may assume that $\varepsilon > 0$ has been chosen small enough to ensure

(4.7)
$$\sup_{0\le t\le T} v(\phi_\varepsilon(t)) \le c.$$

Equation (4.6) now tells us that

$$(4.8)\qquad \frac{1}{\varepsilon}d(\phi_\varepsilon(t))^2 + \frac{\nu}{\varepsilon^2}\int_0^t d(\phi_\varepsilon(s))^2\,ds \le C_1(T)$$

whenever $0 \le t \le T$. Now define (for $0 \le t \le T$)

$$l_\varepsilon(t) = \frac{1}{\varepsilon}d(\phi_\varepsilon(t)), \qquad \eta_\varepsilon(t) = \int_0^t l_\varepsilon(s)\gamma(\phi_\varepsilon(s))\,ds.$$

From (4.8), $\{\eta_\varepsilon\}$ and $\{l_\varepsilon\}$ are bounded in $H^1(0,T)$ and $L^2(0,T)$, respectively. Thus, we may assume that as $\varepsilon \downarrow 0$,

$$l_\varepsilon \to l \quad \text{weakly in } L^2(0,T),$$
$$\eta_\varepsilon \to \eta \quad \text{in } C[0,T],$$
$$\dot\eta_\varepsilon \to \dot\eta \quad \text{weakly in } L^2(0,T),$$

where $\eta \in H^1(0,T)$ and $l \in L^2(0,T)$. From (4.4) it follows that

$$\phi_\varepsilon \to \phi \quad \text{in } C[0,T],$$
$$\dot\phi_\varepsilon \to \dot\phi \quad \text{weakly in } L^2(0,T),$$

where $\phi \in H^1(0,T)$.

It follows from (4.4) that $\phi(t) = \psi(t) + \eta(t)$ and from (4.8) that $\phi(t) \in \overline{G}$ for $0 \le t \le T$. Since $\dot\eta_\varepsilon = l_\varepsilon\gamma(\phi_\varepsilon)$, we also have $\dot\eta = l\gamma(\phi)$ and therefore

$$|\eta|(t) = \int_0^t |\dot\eta(s)|\,ds = \int_0^t l(s)\,ds$$

and

$$\eta(t) = \int_0^t \gamma(\phi(s))l(s)\,ds = \int_0^t \gamma(\phi(s))|\dot\eta(s)|\,ds$$

whenever $0 \le t \le T$. Let $\mathcal{T}$ be the open subset of $[0,T]$ given by $\{0 \le t \le T: \phi(t) \in G\}$. Since for any fixed $t \in \mathcal{T}$, we have $l_\varepsilon(t) = (1/\varepsilon)d(\phi_\varepsilon(t)) = 0$ when ε is sufficiently small, we find that $l(t) = 0$ on $\mathcal{T}$. Thus

$$|\eta|(t) = \int_0^t I_{\{\phi(s)\in\partial G\}}l(s)\,ds = \int_0^t I_{\{\phi(s)\in\partial G\}}|\dot\eta(s)|\,ds.$$

Therefore (ϕ, η) solve the SP.

(*Case 2.*) Only minor changes are needed in the proof above. Define $\tilde d(x) = \sum_{i\in I} d(x, G_i)$. Then (4.4) is replaced by

$$\dot\phi_\varepsilon(t) = \frac{1}{\varepsilon}\tilde d(\phi_\varepsilon(t))\left[\sum_{i\in I} a_i(\phi_\varepsilon(t))\gamma_i(\phi_\varepsilon(t))\right] + \dot\psi(t), \qquad 0 \le t \le T,$$
$$\phi_\varepsilon(0) = \psi(0),$$

where the functions $a_i(\cdot)$ are from Lemma 4.4. Define ξ as for Case 1, where $3c > 0$ satisfies the conditions of c in Lemma 4.4, $\tilde v(x) = \tilde d^2(x)$ and $V(t) =$

$\xi(\tilde{v}(\phi_\varepsilon(t)))$. We now obtain (using Lemma 4.4)

$$\dot{V} \le -\frac{\nu}{\varepsilon}\xi'(\tilde{v}(\phi_\varepsilon))\tilde{v}(\phi_\varepsilon) + \xi'(\tilde{v}(\phi_\varepsilon))|\dot{\psi}||D\tilde{v}(\phi_\varepsilon)|,$$

from which we obtain (4.8) as in the case above with $\tilde{d}$ replacing d. If we now define

$$l_{i,\varepsilon}(t) = \frac{1}{\varepsilon}a_i(\phi_\varepsilon(t))\tilde{d}(\phi_\varepsilon(t)), \qquad \eta_{i,\varepsilon}(t) = \int_0^t l_{i,\varepsilon}(s)\gamma_i(\phi_\varepsilon(s))\,ds,$$

$$\eta_\varepsilon(t) = \sum_{i\in I}\eta_{i,\varepsilon}(t),$$

then the proof follows the proof of Case 1 with only notational changes. Note that the property $a_i(x) = 0$ for $i \notin I(x)$ guarantees that the limit of $\eta_\varepsilon(\cdot)$ satisfies part 5 of the definition of a solution to the SP. □

We next obtain estimates of the modulus of continuity of solutions to the SP. For $y \in C[0,T]$ and $0 \le s < t \le T$, let $\|y\|_{s,t} = \max_{s\le\tau\le t}|y(\tau) - y(s)|$.

Lemma 4.7. *Let A be a compact subset of $C[0,T]$. Then* (i) $s = \sup\{|\eta|(T)$: $(\psi + \eta, \eta)$ *solves the SP for* $\psi \in A\} < \infty$; (ii) *the set* $\{\phi$: (ϕ, η) *solves the SP for* $\psi \in A\}$ *is precompact.*

Proof. For a set $S \subset \mathbb{R}^N$, let $N_c(S)$ denote $\{x: \inf_{y\in S}|x-y| < c\}$. Using the compactness of ∂G and an open covering argument, one may show that there exist $c > 0$, a finite collection $\{S_i, 1 \le i \le L\}$ of open subsets of G and vectors $\{v_i, 1 \le i \le L\}$ such that $\bigcup_i S_i = G$ and such that if $x \in N_c(S_i) \cap \partial G$, then $\langle\gamma, v_i\rangle > c$ for all $\gamma \in r(x)$. We may assume that the diameter of each of the sets $N_c(S_i)$ is bounded above by $\delta > 0$, where δ satisfies the conditions of Remark 3.3.

Let $\psi \in A$ be given and let i_0 be such that $\psi(0) = \phi(0) \in S_{i_0}$. Define T_1 to be the smaller of T and $\inf\{t \in [0,T]$: $\phi(t) \notin N_c(S_{i_0})\}$ and define i_1 so that $\phi(T_1) \in S_{i_1}$. Continue in this way to define $\{T_m, i_m\}$ whenever $T_{m-1} < T$. Since $\langle\gamma, u_i\rangle > c$ for all $\gamma \in r(x)$, $x \in N_c(S_i) \cap \partial G$,

$$\begin{aligned}\left\langle \phi(T_m) - \phi(T_{m-1}), v_{i_{m-1}}\right\rangle &- \left\langle \psi(T_m) - \psi(T_{m-1}), v_{i_{m-1}}\right\rangle \\ &= \int_{T_{m-1}}^{T_m}\left\langle \gamma(s), v_{i_{m-1}}\right\rangle d|\eta|(s) \\ &\ge c\big(|\eta|(T_m) - |\eta|(T_{m-1})\big).\end{aligned}$$

Since A is compact, the set $\{\psi(t)$: $t \in [0,T], \psi \in A\}$ is bounded. Of course, it is also true that the set $\{\phi(t)$: $t \in [0,T]$, $(\phi, \phi - \psi)$ solve the SP, $\psi \in A\}$ is bounded. Thus there is $M < \infty$ such that

$$|\eta|(T_m) - |\eta|(T_{m-1}) \le M.$$

We will prove below that for any $\psi \in A$,

$$\|\eta\|_{T_{m-1},\tau} \le C\left(\tfrac{1}{2}\|\psi\|^{3/2}_{T_{m-1},\tau} + \tfrac{3}{2}\|\psi\|^{1/2}_{T_{m-1},\tau}\right), \qquad T_{m-1} \le \tau \le T_m, \tag{4.9}$$

whenever $T_{m-1} < T$. Assuming this estimate, we now complete the proof of the lemma. Using (4.9) and the compactness of A there is $\varepsilon > 0$ such that

$$\|\psi\|_{T_{m-1},T_{m-1}+\varepsilon} \vee \|\eta\|_{T_{m-1},T_{m-1}+\varepsilon} \le c/3,$$

which implies

$$\|\phi\|_{T_{m-1},T_{m-1}+\varepsilon} \le 2c/3.$$

The definition of the sequence T_m then implies that $T_m - T_{m-1} \ge \varepsilon$. Thus part (i) follows with $s \le MT/\varepsilon$. Part (ii) also follows from the bound $T_m - T_{m-1} \ge \varepsilon$ and (4.9).

We now turn to the proof of (4.9). In order to simplify the notation we will let $\phi(T_{m-1}) = x$ and $T_{m-1} = 0$. Since it is only the increments of ψ over $[T_{m-1}, T_m]$ that are important, we can simplify the notation further by letting $\psi(T_{m-1}) = x$, $\eta(T_{m-1}) = 0$ and $|\eta|(T_{m-1}) = 0$. Recall that $c \le \delta$ and that δ satisfies the conditions of Remark 3.3.

(*Case 1.*) Let g be the function described in Theorem 3.2 for Case 1 and let C be the corresponding positive constant. Define $B_\varepsilon(t) = \varepsilon g(x, -\eta(t)/\varepsilon)$, $a(t) = |\eta|(t)$ and $E(t) = e^{-2Ca(t)}$. Then since $g(x,0) = 1$,

$$\begin{aligned}
B_\varepsilon(\tau)E(\tau) &= \int_0^\tau d(B_\varepsilon(u)E(u)) + \varepsilon \\
&= \int_0^\tau [E(u)\, dB_\varepsilon(u) + B_\varepsilon(u)\, dE(u)] + \varepsilon \\
&= \int_0^\tau E(u)\, dB_\varepsilon(u) - 2C\int_0^\tau B_\varepsilon(u)\, E(u)\, da(u) + \varepsilon \\
&= I_1 + I_2 + \varepsilon.
\end{aligned}$$

We then write

$$\begin{aligned}
I_1 &= -\int_0^\tau E(u)\langle D_r g(x, -\eta(u)/\varepsilon), d\eta(u)\rangle \\
&= -\int_0^\tau E(u) \sum_{j=1}^{3} d\theta_j(u),
\end{aligned}$$

where

$$\begin{aligned}
d\theta_1(u) &= \langle D_r g(\phi(u), -[\phi(u) - x]/\varepsilon), \gamma(u)\rangle\, da(u), \\
d\theta_2(u) &= \langle D_r g(x, -\eta(u)/\varepsilon) - D_r g(\phi(u), -\eta(u)/\varepsilon), d\eta(u)\rangle, \\
d\theta_3(u) &= \langle D_r g(\phi(u), -\eta(u)/\varepsilon) - D_r g(\phi(u), -[\phi(u) - x]/\varepsilon), d\eta(u)\rangle.
\end{aligned}$$

Using (3.11) for the $d\theta_1$ term and (3.12) for the rest, we obtain

$$
\begin{aligned}
I_1 &\le \int_0^\tau E(u)\left\{\frac{C}{\varepsilon}|\phi(u)-x||\eta(u)| + \frac{C}{\varepsilon}|\psi(u)-x|\right\} da(u) \\
&\le C\int_0^\tau E(u)\left\{\frac{1}{\varepsilon}|\eta(u)|^2 + \frac{1}{\varepsilon}|\psi(u)-x||\eta(u)| + \frac{1}{\varepsilon}|\psi(u)-x|\right\} da(u) \\
&\le 2C\int_0^\tau E(u)\left\{\frac{1}{\varepsilon}|\eta(u)|^2 + \frac{1}{\varepsilon}|\psi(u)-x|^2 + \frac{1}{\varepsilon}|\psi(u)-x|\right\} da(u) \\
&\le 2C\int_0^\tau E(u)B_\varepsilon(u)\,da(u) + \frac{2C}{\varepsilon}\{\|\psi\|_{0,\tau}^2 + \|\psi\|_{0,\tau}\}\int_0^\tau E(u)\,da(u),
\end{aligned}
$$

where the last inequality is due to (3.10). Combining with I_2 we have

$$
\begin{aligned}
B_\varepsilon(\tau) &\le \left[\frac{2C}{\varepsilon}\{\|\psi\|_{0,\tau}^2 + \|\psi\|_{0,\tau}\}\int_0^\tau e^{-2Ca(u)}\,da(u) + \varepsilon\right]e^{2Ca(\tau)} \\
&\le \left[\frac{1}{\varepsilon}\{\|\psi\|_{0,\tau}^2 + \|\psi\|_{0,\tau}\} + \varepsilon\right]e^{2Ca(\tau)}.
\end{aligned}
$$

Using (3.10), for $\varepsilon > 0$,

$$
\begin{aligned}
|\eta(\tau)| &\le \frac{1}{2}\left(\varepsilon + \frac{1}{\varepsilon}|\eta(\tau)|^2\right) \le \frac{\varepsilon}{2} + \frac{1}{2}B_\varepsilon(\tau) \\
&\le \frac{\varepsilon}{2} + \left[\frac{1}{2\varepsilon}\{\|\psi\|_{0,\tau}^2 + \|\psi\|_{0,\tau}\} + \frac{\varepsilon}{2}\right]e^{2Ca(\tau)}.
\end{aligned}
$$

Recall that $a(\tau) \le M$ is a bound that is independent of the path $\psi \in A$. If $\|\psi\|_{0,\tau} = 0$, there is nothing to prove. If this is not the case, (4.9) is proved by substituting $\varepsilon = \|\psi\|_{0,\tau}^{1/2}$.

(*Case 2.*) Since the function g for Case 2 does not have the same regularity properties as the corresponding function for Case 1, an approximation argument is used. A similar approximation argument will be used in the next section in the proof of Theorem 5.1 for Case 2.

Let $\varepsilon > 0$, the path ψ and the solution (ϕ, η) be fixed. Let $R = (\|\phi\|_{0,\tau} \vee \|\eta\|_{0,\tau})/\varepsilon$. We will approximate $g(x, r)$ on the open domain $W \times B(0, R+1)^0$ by use of the sup-convolution: For $\beta > 0$,

$$
g^\beta(x,r) = \sup\left\{g(y,s) - \frac{1}{2\beta}(|x-y|^2 + |r-s|^2) : y \in W,\ s \in B(0, R+1)^0\right\}.
$$

The functions g^β are $C^{1,1}(\overline{G} \times B(0,R))$ (for all $\beta > 0$ sufficiently small) and $g^\beta(x,r) \to g(x,r)$ in $C(\overline{G} \times B(0,R))$. Furthermore, if $\beta > 0$ is sufficiently small, then for all $(x,r) \in \overline{G} \times B(0,R)$,

$$
(4.10) \qquad (p,q) = Dg^\beta(x,r) \Rightarrow (p,q) \in D^+g(x + \beta p, r + \beta q).
$$

Let C be the constant associated to g in Theorem 3.2. Then (4.10) and (3.24) imply

$$|p| \le 4C|r|^2, \qquad |q| \le 2C|r| \tag{4.11}$$

whenever $0 < \beta < 1/2C$.

We now return to the proof of Case 1, defining $B_\varepsilon^\beta(t)$, $E^\beta(t)$ and so on, as we did $B_\varepsilon(t)$, $E(t)$ and so on, there, save that we substitute g^β for g. In the calculation of the upper bound for I_1, the only significant difference occurs in the estimation of the $d\theta_1^\beta$ term. Let $p = D_x g^\beta(\phi(u), -[\phi(u) - x]/\varepsilon)$, $q = D_r g^\beta(\phi(u), -[\phi(u) - x]/\varepsilon)$. We use

$$\begin{aligned} &\langle q, \gamma(u)\rangle\, da(u) \\ &\quad = \langle D_r g(\phi(u) + \beta p, -[\phi(u) - x]/\varepsilon + \beta q), \gamma(u)\rangle\, da(u). \end{aligned} \tag{4.12}$$

Let $\gamma(u) = \sum_{i \in I} a_i(u)\gamma_i(\phi(u))$ define the functions $a_i(u)$, $i \in I$, and then define $\tilde{\gamma}(u) = \sum_{i \in I} a_i(u)\gamma_i(\phi(u) + \beta p)$. Then using (3.23) (see also Remark 3.3),

$$\begin{aligned} &-\int_0^\tau E(u)\langle D_r g^\beta(\phi(u), -[\phi(u) - x]/\varepsilon), \gamma(u)\rangle\, da(u) \\ &\quad \le -\int_0^\tau E(u)\langle D_r g(\phi(u) + \beta p, -[\phi(u) - x]/\varepsilon + \beta q), \\ &\qquad\qquad \gamma(u) - \tilde{\gamma}(u)\rangle\, da(u). \end{aligned}$$

The continuity properties of the γ_i and this last display imply

$$\limsup_{\beta \to 0} \left[-\int_0^\tau E(u)\, d\theta_1^\beta(u) \right] \le 0.$$

It follows that by using the same estimates as in Case 1 and then sending $\beta \to 0$, we obtain

$$B_\varepsilon(\tau) \le \frac{1}{\varepsilon}\{\|\psi\|_{0,\tau}^2 + \|\psi\|_{0,\tau}\} e^{2Ca(\tau)}.$$

We may therefore proceed exactly as in Case 1. □

We now come to the main result of this section.

THEOREM 4.8. *Let $\psi \in C[0,T]$ satisfy $\psi(0) \in \overline{G}$. Then under the assumptions of either Case* 1 *or* 2 *of Section* 2, *there exists a solution (ϕ, η) of the SP.*

PROOF. The proof is very similar to that of Costantini ([1], Theorem 2.8). An extension that can handle the case of paths with discontinuities appears in [4]. Let $\psi_n \in C^1[0,T]$ be such that $\sup_{t \in [0,T]}|\psi_n(t) - \psi(t)| \to 0$. According to Lemma 4.5, a solution (ϕ_n, η_n) corresponding to ψ_n exists for each n. By Lemma 4.7, we may assume

$$\sup_n |\eta_n|(T) \le s < \infty. \tag{4.13}$$

By the Arzela–Ascoli theorem, we may assume there is $\eta \in C[0,T]$ such that $\sup_{t\in[0,T]}|\eta_n(t) - \eta(t)| \to 0$. According to (4.13) we must have $|\eta|(T) \le s$. Let $\phi = \psi + \eta$. Clearly, parts 1, 2 and 3 of Definition 2.1 hold. For each n, define the measure μ_n on $[0,T] \times \overline{G} \times B(0,1)$ by

$$\mu_n([0,t] \times A) = \int_{[0,t]} I_{\{(\phi_n(s),\gamma_n(s)) \in A\}}\, d|\eta_n|(s),$$

where A is Borel. Then

$$|\eta_n|(t) = \mu_n([0,t] \times \overline{G} \times B(0,1))$$

for all $0 \le t \le T$ and

$$\eta_n(t) = \int_{[0,t]\times\overline{G}\times B(0,1)} \gamma\, d\mu_n.$$

Since $|\eta_n|(T) \le s < \infty$ for all n, we may assume a subsequence of μ_n (again denoted by μ_n) tends weakly to some measure μ. Then weak convergence and continuity of η imply

$$\eta(t) = \int_{[0,t]\times\overline{G}\times B(0,1)} \gamma\, d\mu. \tag{4.14}$$

Define $\lambda(t) = \mu([0,t] \times \overline{G} \times B(0,1))$. Define the sets

$$\Sigma_1 = [0,T] \times G \times B(0,1),$$

$$\Sigma_2 = [0,T] \times \{(x,\gamma) : x \in \partial G,\ \gamma \notin r(x)\},$$

$$\Sigma_3^\delta = \{(t,x) : t \in [0,T],\ |x - \phi(t)| > \delta\} \times B(0,1).$$

For every n,

$$\mu_n(\Sigma_1) = \mu_n(\Sigma_2) = 0$$

and for sufficiently large n,

$$\mu_n(\Sigma_3^\delta) = 0.$$

Since the sets $\Sigma_1 \cup \Sigma_2$ and Σ_3^δ are relatively open [due to our definition of $r(x)$],

$$\mu(\Sigma_1 \cup \Sigma_2) = \mu(\Sigma_3^\delta) = 0$$

for all $\delta > 0$. Thus

$$\mu(\Sigma_1 \cup \Sigma_2) = \mu(\Sigma_3) = 0, \tag{4.15}$$

where $\Sigma_3 = \{(t,x) : x \ne \phi(t)\} \times B(0,1)$. Finally, since $\phi(0) = \psi(0)$, we may assume that $\mu(\{0\} \times \overline{G} \times B(0,1)) = 0$. Then (4.14) and (4.15) and the discussion above imply

$$\eta(t) = \int_{\{(s,\phi,\gamma):\, \phi \in \partial G,\, \phi = \phi(s),\, \gamma \in r(\phi(s))\}} \gamma \, d\mu$$

$$= \int_{(0,t]} I_{\{\phi(s) \in \partial G\}} \int_{r(\phi(s))} \gamma p(s; d\gamma)\, d\lambda(s),$$

where $p(\cdot\,; A)$ is a nonnegative λ-measurable function for each Borel set A. Thus $d|\eta|$ is absolutely continuous with respect to $d\lambda$. This fact and (4.15) imply part (iv) of Definition 2.1. Since the set $\{\alpha\gamma,\ 0 \le \alpha \le 1,\ \gamma \in r(x)\}$ is convex for each $x \in \partial G$, there is a measurable function $c(s)$ such that

$$\gamma(s) = c(s) \int_{r(\phi(s))} \gamma p(s; d\gamma) \in r(\phi(s)) \quad \text{if } \phi(s) \in \partial G$$

almost surely with respect to $d\lambda$. Clearly,

$$\eta(t) = \int_{(0,t]} \gamma(s)\, d|\eta|(s),$$

which gives part (v) of Definition 2.1. □

5. Solutions of SDER. The reader is referred to the book of Karatzas and Shreve [7] for questions regarding the terminology and notation used below. Let $(\Omega, \mathscr{F}, P)$ be a complete probability space and let $\{\mathscr{F}_t,\ t \ge 0\}$ be a filtration satisfying the usual conditions. Let $M = (M_i)$ be an r-dimensional continuous $\mathscr{F}_t$-martingale satisfying

(5.1) $$d\langle M_i, M_j\rangle(t) \le C\, dt \quad \text{for some constant } C < \infty.$$

Let $\sigma_{ij}(x)$ and $b_i(x)$, $(i,j) \in \{1,\dots,N\} \times \{1,\dots,r\}$, be real-valued functions on $\mathbb{R}^N$ satisfying a Lipschitz continuity assumption:

(5.2) $$|\sigma_{ij}(x) - \sigma_{ij}(y)| \vee |b_i(x) - b_i(y)| \le K|x-y|$$

for $(i,j) \in \{1,\dots,N\} \times \{1,\dots,r\}$ and $x, y \in \overline{G}$. Let X and Y be continuous $\mathscr{F}_t$-semimartingales and let k be a continuous $\mathscr{F}_t$-adapted bounded variation process. Assume that the triple (X, Y, k) satisfy (for $t \ge 0$ and a.s.)

(5.3) $$Y(t) = x + \int_0^t b(X)\, ds + \int_0^t \sigma(X)\, dM + k(t),$$

$$X(t) \in \overline{G}, \qquad Y(t) \in \overline{G},$$

$$|k|(t) = \int_{(0,t]} I_{\{Y(s) \in \partial G\}}\, d|k|(s), \qquad k(t) = \int_{(0,t]} \gamma(s)\, d|k|(s),$$

where x is some fixed value in $\overline{G}$ and $\gamma(s) \in R(Y(s))$, $d|k|$ a.s. Let (X', Y', k') be another triple satisfying the same conditions save that x is replaced by $x' \in \overline{G}$.

THEOREM 5.1. *There is a constant $C < \infty$ such that for all $0 \le t \le T$,*

$$E\left(\sup_{0\le s\le t} |Y(s) - Y'(s)|^2\right) \le C\left\{|x - x'|^2 + \int_0^t E\left(\sup_{0\le u\le s} |X(u) - X'(u)|^2\right) ds\right\}.$$

Before proving this theorem we present several lemmas, some of which are simply the statement of well-known results. We also note at this point that the existence and uniqueness of solutions to SDER will follow easily from the result above by a standard fixed point argument. We state this result as a corollary.

COROLLARY 5.2. *Assume the conditions of Case* 1 *or Case* 2 *and also that* (5.2) *holds. Then a strong solution to the associated SDER exists and is unique in the strong sense.*

OUTLINE OF THE PROOF. Given the estimate of Theorem 5.1, the proof follows the same outline as that given for [8], Theorem 4.3. For the sake of completeness, we will indicate the main ideas. Assume that given a continuous $\mathscr{F}_t$-adapted semimartingale X, there are processes Y and k satisfying (5.3) with w replacing M, where Y is a continuous $\mathscr{F}_t$-adapted semimartingale and k is a continuous $\mathscr{F}_t$-adapted process of bounded variation. Then the classical Picard iteration technique (e.g., [7], Section 5.2), together with the estimate given in Theorem 5.1, completes the argument. Thus the existence of such Y and k is all that remains. Obviously the processes may be defined on a pathwise basis. The main problem is verifying the adaptedness property. Next assume that it is possible to show the existence in distribution of a solution to (5.3). Then this weak existence, together with the strong uniqueness implied by Theorem 5.1, implies existence of Y and k with the required properties ([7], Proof of Corollary 3.23). Thus existence in distribution is all that remains.

Let ψ be a bounded variation path that starts inside $\overline{G}$. Then a simpler version of the argument used to derive Theorem 5.1 shows there is at most one solution (ϕ, η) to the SP for ψ. Let

$$S(t) = x + \int_0^t b(X(s))\, ds + \int_0^t \sigma(X(s))\, dw(s),$$

and let $\{S_n(\cdot),\ n \in \mathbb{N}\}$ be a sequence of continuous bounded variation $\mathscr{F}_t$-adapted semimartingales which converges uniformly to $S(\cdot)$. Let (Y_n, k_n), $n \in \mathbb{N}$, be defined pathwise as the solutions of the SP for S_n, $n \in \mathbb{N}$. Then the constructive nature of the existence proof of solutions to the SP, together with the uniqueness of solutions to the SP for paths of bounded variation, implies the processes (Y_n, k_n), $n \in \mathbb{N}$, are $\mathscr{F}_t$-adapted. Thus, in order to prove existence in distribution of solutions to (5.3), all that is required is tightness of the joint distribution of $(Y_n, S_n, k_n, |k_n|)$, $n \in \mathbb{N}$, on the appropriate product space.

This tightness follows from the assumed convergence of S_n to S and Lemma 4.7. □

We now begin the proof of Theorem 5.1.

LEMMA 5.3. *For $f \in C^2(\overline{G} \times \overline{G})$, we have (for $t \geq 0$ and a.s.)*

$$
\begin{aligned}
&f(Y(t), Y'(t)) \\
&\quad = f(x, x') + \int_0^t \langle D_x f(Y, Y'), \sigma(X)\, dM \rangle \\
&\qquad + \int_0^t \langle D_y f(Y, Y'), \sigma(X')\, dM \rangle + \int_0^t \langle D_x f(Y, Y'), b(X) \rangle\, ds \\
&\qquad + \int_0^t \langle D_y f(Y, Y'), b(X') \rangle\, ds + \int_0^t \langle D_x f(Y, Y'), \gamma(Y) \rangle\, d|k|(s) \\
&\qquad + \int_0^t \langle D_y f(Y, Y'), \gamma(Y') \rangle\, d|k'|(s) + \int_0^t \operatorname{tr} d\Sigma,
\end{aligned}
$$

where

$$
\begin{aligned}
d\Sigma = \Big\{ &\sigma(X)^T D_x^2 f(Y, Y') \sigma(X) + \sigma(X)^T D_y D_x f(Y, Y') \sigma(X') \\
&+ \sigma(X')^T D_x D_y f(Y, Y') \sigma(X) + \sigma(X')^T D_y^2 f(Y, Y') \sigma(X') \Big\}\, d\langle M \rangle(s),
\end{aligned}
$$

and where T denotes transpose, tr *denotes the trace and $d\langle M \rangle(s) = (d\langle M_i, M_j \rangle(s))_{1 \leq i, j \leq r}$.*

PROOF. This follows from Itô's formula. □

Let S^r denote the set of all real $r \times r$ symmetric matrices.

LEMMA 5.4. *Let $A = (a_{ij}) \in S^r$, $B = (b_{ij}) \in S^r$ be nonnegative definite. Then*

$$\operatorname{tr} AB \geq 0.$$

LEMMA 5.5. *Let $A = (a_{ij}) \in C([0, T], S^r)$ satisfy $A(t) \geq 0$ for all $0 \leq t \leq T$. Let $B = (b_{ij}) \in C([0, T], S^r)$ and assume each b_{ij} is of bounded variation on $[0, T]$ for $i, j = 1, \ldots, r$ and that*

$$B(t) \geq B(s) \quad \text{for all } 0 \leq s \leq t \leq T$$

(i.e., B is nondecreasing). Then

$$\int_0^T \operatorname{tr} A(t)\, dB(t) \geq 0.$$

PROOF. This follows from Lemma 5.4 and an easy approximation argument. □

LEMMA 5.6. *$\langle M\rangle(t)$ is nondecreasing (in the sense of nonnegative definite matrices and a.s.).*

PROOF. See the proof of [7], Theorem 3.4.2. □

Let $\lambda > 0$ be some constant to be chosen later and let $f_\varepsilon(x,y)$ and $h(x)$ be the functions described in Theorem 3.2 for Case 1. We define $v \in C^2(\overline{G} \times \overline{G})$ by

$$v(x,y) = e^{-\lambda[h(x)+h(y)]} f_\varepsilon(x,y).$$

Although v depends on ε and λ, we will omit this dependence from the notation. We will also separately define

$$u(x,y) = e^{-\lambda[h(x)+h(y)]}.$$

Note that we always have $u(x,y) \le 1$.

LEMMA 5.7. *There is $K_1(\lambda) < \infty$ such that for all $x, y \in \overline{G}$,*

$$D^2 v(x,y) \le K_1(\lambda)\left[\frac{1}{\varepsilon}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + \left(\varepsilon + \frac{|x-y|^2}{\varepsilon}\right)\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}\right].$$

PROOF. Let $\xi_1, \xi_2 \in \mathbb{R}^N$ and $\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$. Then by (3.14), (3.17), (3.18) and (3.19),

$$\begin{aligned}
\langle D^2 v \xi, \xi\rangle &= \langle u D^2 f_\varepsilon \xi, \xi\rangle + 2\langle Df_\varepsilon, \xi\rangle\langle Du, \xi\rangle + \langle f_\varepsilon D^2 u \xi, \xi\rangle \\
&\le \frac{C}{\varepsilon}\left\langle \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}\xi, \xi\right\rangle + \frac{C|x-y|^2}{\varepsilon}\left\langle \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}\xi, \xi\right\rangle \\
&\quad + 2\big(\langle D_x f_\varepsilon, \xi_1\rangle + \langle D_y f_\varepsilon, \xi_2\rangle\big)|Du|\,|\xi| \\
&\quad + C\left(\varepsilon + \frac{|x-y|^2}{\varepsilon}\right)\|D^2 u\|\,|\xi|^2 \\
&\le \frac{C}{\varepsilon}|\xi_1 - \xi_2|^2 + \frac{(C + C\|D^2 u\|)|x-y|^2}{\varepsilon}|\xi|^2 + \varepsilon C\|D^2 u\|\,|\xi|^2 \\
&\quad + 2\big(\langle D_x f_\varepsilon + D_y f_\varepsilon, \xi_1\rangle + \langle D_y f_\varepsilon, \xi_2 - \xi_1\rangle\big)|Du|\,|\xi| \\
&\le \frac{C}{\varepsilon}|\xi_1 - \xi_2|^2 + \frac{(C + C\|D^2 u\|)|x-y|^2}{\varepsilon}|\xi|^2 + \varepsilon C\|D^2 u\|\,|\xi|^2 \\
&\quad + 2C|Du|\left(\frac{|x-y|^2|\xi|^2}{\varepsilon} + \frac{|x-y||\xi_1 - \xi_2|\,|\xi|}{\varepsilon}\right) \\
&\le \frac{K_1(\lambda)}{\varepsilon}|\xi_1 - \xi_2|^2 + \frac{K_1(\lambda)|x-y|^2}{\varepsilon}|\xi|^2 + K_1(\lambda)\varepsilon|\xi|^2.
\end{aligned}$$

□

LEMMA 5.8. *There is $K_2(\lambda) < \infty$ such that for all x, y, ξ and $\nu \in \overline{G}$, the quantity*

$$\begin{pmatrix} \sigma(\xi) \\ \sigma(\nu) \end{pmatrix}^T D^2 v(x,y) \begin{pmatrix} \sigma(\xi) \\ \sigma(\nu) \end{pmatrix}$$
$$= \sigma(\xi)^T D_x^2 v(x,y)\sigma(\xi) + \sigma(\xi)^T D_y D_x v(x,y)\sigma(\nu)$$
$$+ \sigma(\nu)^T D_x D_y v(x,y)\sigma(\xi) + \sigma(\nu)^T D_y^2 v(x,y)\sigma(\nu)$$

is bounded above by

$$K_2(\lambda)\left[\varepsilon + \frac{1}{\varepsilon}\left(|\xi - \nu|^2 + |x - y|^2\right)\right] I.$$

PROOF. Using Lemma 5.7, we compute

$$\begin{pmatrix} \sigma(\xi) \\ \sigma(\nu) \end{pmatrix}^T D^2 v(x,y) \begin{pmatrix} \sigma(\xi) \\ \sigma(\nu) \end{pmatrix}$$
$$\leq \frac{K_1(\lambda)}{\varepsilon} \begin{pmatrix} \sigma(\xi) \\ \sigma(\nu) \end{pmatrix}^T \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} \begin{pmatrix} \sigma(\xi) \\ \sigma(\nu) \end{pmatrix}$$
$$+ K_1(\lambda)\left[\varepsilon + \frac{|x-y|^2}{\varepsilon}\right] \begin{pmatrix} \sigma(\xi) \\ \sigma(\nu) \end{pmatrix}^T \begin{pmatrix} \sigma(\xi) \\ \sigma(\nu) \end{pmatrix}$$
$$= \frac{K_1(\lambda)}{\varepsilon}(\sigma(\xi) - \sigma(\nu))^T(\sigma(\xi) - \sigma(\nu))$$
$$+ K_1(\lambda)\left[\varepsilon + \frac{|x-y|^2}{\varepsilon}\right]\left(\sigma(\xi)^T\sigma(\xi) + \sigma(\nu)^T\sigma(\nu)\right)$$
$$\leq K_2(\lambda)\left[\varepsilon + \frac{1}{\varepsilon}\left(|\xi - \nu|^2 + |x - y|^2\right)\right] I. \qquad \square$$

PROOF OF THEOREM 5.1. Define the stopping time

$$\tau = \inf\{s \in [0,T]: |Y(s) - Y'(s)| \geq \delta\},$$

where $\delta > 0$ is from Remark 3.3. It is sufficient to prove the theorem with the t appearing in the conclusion of the theorem replaced by $t \wedge \tau$. To see this, let B be the diameter of the smallest ball that contains G. Then (assuming without loss that $B/\delta \geq 1$)

$$E\left(\sup_{0\leq s\leq t} |Y(s) - Y'(s)|^2\right) \leq \left(\frac{B}{\delta}\right)^2 E\left(\sup_{0\leq s\leq t\wedge\tau} |Y(s) - Y'(s)|^2\right).$$

To simplify the notation, we write t in place of $t \wedge \tau$, with the understanding that we may assume $|Y(s) - Y'(s)| \leq \delta$ for $0 \leq s \leq t$.

(*Case 1.*) Let v be the C^2 function on $\overline{G} \times \overline{G}$ defined immediately before Lemma 5.7. In the proof, quantities that are finite but depend on λ will be denoted by $C(\lambda)$. C will denote a finite constant that does *not* depend on λ. Then we obtain [using Lemmas 5.3, 5.5, 5.6 and 5.8 for the first inequality, (3.14), (3.15), (3.16), (3.20) and (5.1) for the second inequality and (3.17) and (3.18) for the last inequality]

$$
\begin{aligned}
&Ev(Y(t), Y'(t)) \\
&\quad \le v(x,x') + E\int_0^t \langle D_x v(Y,Y'), b(X)\rangle\, ds + E\int_0^t \langle D_y v(Y,Y'), b(X')\rangle\, ds \\
&\qquad + E\int_0^t \langle D_x v(Y,Y'), \gamma(Y)\rangle\, d|k|(s) + E\int_0^t \langle D_y v(Y,Y'), \gamma(Y')\rangle\, d|k'|(s) \\
&\qquad + E\int_0^t C(\lambda)\left(\varepsilon + \frac{|X-X'|^2}{\varepsilon} + \frac{|Y-Y'|^2}{\varepsilon}\right) \operatorname{tr} I\, d\langle M\rangle \\
&\quad = v(x,x') + E\int_0^t \langle u(Y,Y')D_x f_\varepsilon(Y,Y') + f_\varepsilon(Y,Y')D_x u(Y,Y'), b(X)\rangle\, ds \\
&\qquad + E\int_0^t \langle u(Y,Y')D_y f_\varepsilon(Y,Y') + f_\varepsilon(Y,Y')D_y u(Y,Y'), b(X')\rangle\, ds \\
&\qquad + E\int_0^t \langle u(Y,Y')D_x f_\varepsilon(Y,Y') + f_\varepsilon(Y,Y')D_x u(Y,Y'), \gamma(Y)\rangle\, d|k|(s) \\
&\qquad + E\int_0^t \langle u(Y,Y')D_y f_\varepsilon(Y,Y') + f_\varepsilon(Y,Y')D_y u(Y,Y'), \gamma(Y')\rangle\, d|k'|(s) \\
&\qquad + C(\lambda)E\int_0^t \left(\varepsilon + \frac{|X-X'|^2}{\varepsilon} + \frac{|Y-Y'|^2}{\varepsilon}\right) \sum_{i=1}^r d\langle M_i\rangle \\
&\quad \le v(x,x') + E\int_0^t \langle u(Y,Y')\big(D_x f_\varepsilon(Y,Y') + D_y f_\varepsilon(Y,Y')\big), b(X)\rangle\, ds \\
&\qquad + E\int_0^t \langle u(Y,Y')D_y f_\varepsilon(Y,Y'), b(X') - b(X)\rangle\, ds + C(\lambda)\varepsilon \\
&\qquad + C(\lambda)E\int_0^t \frac{|Y-Y'|^2}{\varepsilon}\, ds + CE\int_0^t u(Y,Y')\frac{|Y-Y'|^2}{\varepsilon}\, d|k|(s) \\
&\qquad + CE\int_0^t u(Y,Y')\frac{|Y-Y'|^2}{\varepsilon}\, d|k'|(s) \\
&\qquad - \lambda E\int_0^t f_\varepsilon(Y,Y')u(Y,Y')\langle Dh(Y), \gamma(Y)\rangle\, d|k|(s) \\
&\qquad - \lambda E\int_0^t f_\varepsilon(Y,Y')u(Y,Y')\langle Dh(Y'), \gamma(Y')\rangle\, d|k'|(s)
\end{aligned}
$$

$$+ C(\lambda)E\int_0^t\left(\varepsilon + \frac{|X - X'|^2}{\varepsilon} + \frac{|Y - Y'|^2}{\varepsilon}\right) ds$$

$$\leq v(x, x') + C(\lambda)\varepsilon + C(\lambda)E\int_0^t\left(\frac{|Y - Y'|^2}{\varepsilon} + \frac{|Y - Y'|\,|X - X'|}{\varepsilon}\right) ds$$

$$+(C - \lambda)E\int_0^t u(Y, Y')\frac{|Y - Y'|^2}{\varepsilon}\, d|k|(s)$$

$$+(C - \lambda)E\int_0^t u(Y, Y')\frac{|Y - Y'|^2}{\varepsilon}\, d|k'|(s)$$

$$+C(\lambda)E\int_0^t\left(\varepsilon + \frac{|X - X'|^2}{\varepsilon}\right) ds.$$

Letting $\lambda = C$ (and dropping the λ notation from the constants) we get

$$Ev(Y(t), Y'(t)) \leq v(x, x') + C\varepsilon + CE\int_0^t\left(\frac{|X - X'|^2}{\varepsilon} + \frac{|Y - Y'|^2}{\varepsilon}\right) ds.$$

Hence,

$$E\left(\frac{|Y(t) - Y'(t)|^2}{\varepsilon}\right) \leq C\left\{\varepsilon + \frac{1}{\varepsilon}\left(|x - x'|^2 + E\int_0^t\left[|X - X'|^2 + |Y - Y'|^2\right] ds\right)\right\}.$$

If we multiply this by ε and then send ε to zero we get

$$E|Y(t) - Y'(t)|^2 \leq C\left\{|x - x'|^2 + \int_0^t\left[E|X - X'|^2 + E|Y - Y'|^2\right] ds\right\}.$$

From Doob's inequality we obtain

$$E\left(\sup_{0\leq s\leq t}|Y(t) - Y'(t)|^2\right) \leq C\left\{|x - x'|^2 + \int_0^t\left[E|X - X'|^2 + E|Y - Y'|^2\right] ds\right\},$$

where the new constant C depends on the Lipschitz constant of the coefficients b and σ. Finally, Gronwall's inequality implies that given $T < \infty$, there is $C < \infty$ so that for $0 \leq t \leq T$,

$$E\left(\sup_{0\leq s\leq t}|Y(s) - Y'(s)|^2\right) \leq C\left\{|x - x'|^2 + \int_0^t E\left(\sup_{0\leq r\leq s}|X(r) - X'(r)|^2\right) ds\right\}.$$

(*Case 2.*) As in the proof of Lemma 4.1, the weaker regularity properties of the functions $\{f_\varepsilon,\ \varepsilon > 0\}$ in Case 2 force the use of an approximation argument. If the functions for Case 2 were of class C^2, the proof of Lemmas 5.7 and 5.8 and Theorem 5.1 would be the same as those for Case 1 save that the corresponding inequalities for Case 2 from Theorem 3.2 would be used [the right-hand side of the inequalities appearing in these lemmas could actually take a slightly simpler form since the extra ε term appearing in (3.14) is not present in (3.27)]. We break the approximation argument down into two stages.

If the f_ε were $C^{1,1}$. Let ρ_α satisfy

$$\rho_\alpha \geq 0, \qquad \operatorname{supp} \rho_\alpha \subset B(0, \alpha), \qquad \int_{\mathbb{R}^N} \rho_\alpha \, dx = 1, \qquad \rho_\alpha \in C^\infty,$$

and define $f_\varepsilon^\alpha = f_\varepsilon * \rho_\alpha$. Then each f_ε^α is C^∞ and $f_\varepsilon^\alpha \to f_\varepsilon$ in C^1. The key property of these approximations is

$$D^2 f_\varepsilon^\alpha(x, y) \leq \frac{C}{\varepsilon}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + \frac{C|x-y|^2}{\varepsilon}\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$$

for $(x, y) \in \overline{G} \times \overline{G}$ and all $\alpha > 0$ sufficiently small, which is implied by (3.32). If we now define v^α as v was with f_ε^α replacing f_ε, Lemmas 5.7 and 5.8 follow as before for the function v^α. We therefore obtain the first inequality in the proof of Theorem 5.1 for Case 1, save that v^α replaces v. Upon letting $\alpha \to 0$, we obtain the inequality as written, and may now proceed as before.

For the f_ε as given. As in the proof of Lemma 4.7, we approximate the f_ε by sup-convolution. For $\beta > 0$, let

$$f_\varepsilon^\beta(x, y) = \sup_{(r, s) \in W \times W} \left\{ f_\varepsilon(r, s) - \frac{1}{2\beta}\left(|x - r|^2 + |y - s|^2\right) \right\}.$$

Then the f_ε^β are of class $C^{1,1}$, $f_\varepsilon^\beta \to f_\varepsilon$ in $C(\overline{G} \times \overline{G})$ as $\beta \to 0$ and for all β sufficiently small,

$$\begin{aligned} &((p, q), A) \in D^{2,+} f_\varepsilon^\beta(x, y) \\ &\qquad \Rightarrow ((p, q), A) \in D^{2,+} f_\varepsilon(x + \beta p, y + \beta q). \end{aligned} \tag{5.4}$$

For fixed $\varepsilon > 0$, the Lipschitz property of f_ε implies that (p, q) is bounded independently of x and y. We let $C(\varepsilon)$ denote a quantity that satisfies $0 < C(\varepsilon) < \infty$, but whose value may change from line to line. Let $f_\varepsilon^{\beta,\alpha} = f_\varepsilon^\beta * \rho_\alpha$. The key property of the $f_\varepsilon^{\beta,\alpha}$ [which follows from (5.4) and (3.32)] is

$$D^2 f_\varepsilon^{\beta,\alpha} \leq \frac{C}{\varepsilon}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + \frac{C|x-y|^2}{\varepsilon}\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} + \frac{C(\varepsilon)|\beta|^2}{\varepsilon}\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$

If we repeat the estimates of Lemmas 5.7 and 5.8, use Itô's formula to compute the analogue of the first inequality in the proof of Theorem 5.1 and then send $\alpha \to 0$, we obtain

$$Ev^{\beta}(Y(t), Y'(t))$$

$$\leq v^{\beta}(x,y) + E\int_0^t \langle D_x v^{\beta}(Y,Y'), b(X)\rangle\, ds + E\int_0^t \langle D_y v^{\beta}(Y,Y'), b(X')\rangle\, ds$$

$$+E\int_0^t \langle D_x v^{\beta}(Y,Y'), \gamma(Y)\rangle\, d|k|(s) + E\int_0^t \langle D_y v^{\beta}(Y,Y'), \gamma(Y')\rangle\, d|k'|(s)$$

$$+E\int_0^t C(\lambda)\left(\frac{|X-X'|^2}{\varepsilon} + \frac{|Y-Y'|^2}{\varepsilon} + \frac{C(\varepsilon)|\beta|^2}{\varepsilon}\right)\operatorname{tr} I\, d\langle M\rangle,$$

where v^{β} is defined as v was but with f_{ε} replaced by f_{ε}^{β}.

We now examine the terms involving $D_x v^{\beta} = uD_x f_{\varepsilon}^{\beta} + f_{\varepsilon}^{\beta} D_x u$ or $D_y v^{\beta}$. All terms involving only f_{ε}^{β} (and not Df_{ε}^{β}) can be handled exactly as before since we will ultimately send $\beta \to 0$ before sending $\varepsilon \to 0$ and since $f_{\varepsilon}^{\beta} \to f_{\varepsilon}$ in $C(\overline{G} \times \overline{G})$ as $\beta \to 0$. We next observe that (5.4) and (3.30) imply

$$E\int_0^t \langle u(Y,Y') D_x f_{\varepsilon}^{\beta}(Y,Y'), b(X)\rangle\, ds$$

$$+E\int_0^t \langle u(Y,Y') D_y f_{\varepsilon}^{\beta}(Y,Y'), b(X')\rangle\, ds$$

$$\leq E\int_0^t \Big\langle u(Y,Y')\big(D_x f_{\varepsilon}^{\beta}(Y,Y') + D_y f_{\varepsilon}^{\beta}(Y,Y')\big), b(X)\Big\rangle\, ds$$

$$+ E\int_0^t \langle u(Y,Y') D_y f_{\varepsilon}^{\beta}(Y,Y'), b(X') - b(X)\rangle\, ds$$

$$\leq CE\int_0^t \left(\frac{|Y-Y'|^2 + C(\varepsilon)|\beta|^2}{\varepsilon} + \frac{(|Y-Y'| + C(\varepsilon)|\beta|)|X-X'|}{\varepsilon}\right) ds.$$

Therefore these terms will take the same form as in Case 1 when $\beta \to 0$. Lastly, there are the terms of the type

$$E\int_0^t \langle u(Y,Y') D_x f_{\varepsilon}^{\beta}(Y,Y'), \gamma(Y)\rangle\, d|k|(s),$$

$$E\int_0^t \langle u(Y,Y') D_y f_{\varepsilon}^{\beta}(Y,Y'), \gamma(Y')\rangle\, d|k'|(s).$$

We consider only the first term, since the second is treated in the same way. Let $\gamma(Y(s)) = \sum_{i \in I} a_i(s)\gamma_i(Y(s))$. Define $\tilde{\gamma}(Y(s)) = \sum_{i \in I} a_i(s)\gamma_i(Y(s) + \beta p)$, where $p = D_x f_{\varepsilon}^{\beta}(Y(s), Y'(s))$ and $q = D_y f_{\varepsilon}^{\beta}(Y(s), Y'(s))$ have a bound that depends on ε but not on β. Then by using (3.28) [and Remark 3.3], we obtain

$$\begin{aligned}
&E\int_0^t \langle u(Y,Y')D_x f_\varepsilon^\beta(Y,Y'),\gamma(Y)\rangle\, d|k|(s)\\
&\quad\le CE\int_0^t u(Y,Y')\frac{|Y-Y'|^2}{\varepsilon}\,d|k|(s)\\
&\qquad + E\int_0^t \langle u(Y,Y')D_x f_\varepsilon^\beta(Y,Y'),\gamma(Y)-\tilde{\gamma}(Y)\rangle\, d|k|(s).
\end{aligned}$$

The continuity properties of the γ_i imply the last term tends to 0 as $\beta \to 0$. Therefore, upon sending $\beta \to 0$, we obtain

$$\begin{aligned}
&Ev(Y(t),Y'(t))\\
&\quad\le v(x,x') + C(\lambda)E\int_0^t\left(\frac{|Y-Y'|^2}{\varepsilon}+\frac{|Y-Y'|\,|X-X'|}{\varepsilon}\right)ds\\
&\qquad+(C-\lambda)E\int_0^t u(Y,Y')\frac{|Y-Y'|^2}{\varepsilon}\,d|k|(s)\\
&\qquad+(C-\lambda)E\int_0^t u(Y,Y')\frac{|Y-Y'|^2}{\varepsilon}\,d|k'|(s)\\
&\qquad+C(\lambda)E\int_0^t \frac{|X-X'|^2}{\varepsilon}\,ds,
\end{aligned}$$

and we may now proceed as in Case 1. □

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