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REGULARITY OF INFINITELY DIVISIBLE PROCESSES[1]

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We develop new tools that enable us to extend the majorizing measure lower bound to a large class of infinitely divisible processes. We show (in a rigorous sense) that the complexity of these processes is dominated by the complexity of the *positive* infinitely divisible processes.

1. Introduction. A stochastic process (or random field) is a family $(X_t)_{t\in T}$ of random variables indexed by a set T. The objects of study are the regularity properties of the process, in particular of boundedness. Celebrated sufficient conditions (also called upper bounds) include Dudley's metric entropy bound [2] for sub-Gaussian processes, its extension by Pisier [19] and the Preston–Fernique majorizing measure bound for Gaussian processes [6]. In 1985 this author proved that majorizing measures also provide necessary conditions (also called lower bounds) for Gaussian processes [25]. The main purpose of this work is to extend these lower bounds to a large class of infinitely divisible processes.

A stochastic process $(X_t)_{t\in T}$ is called (real, symmetric, without Gaussian component) infinitely divisible if there exists a positive cylindrical measure ν on $\mathbb{R}^T$, called the Lévy measure of the process, such that $\int_{\mathbb{R}^T}|\beta(t)|^2 \wedge 1\, d\nu(\beta) < \infty$ for all $t \in T$, and such that for all families $(\alpha_t)_{t\in T}$ of real numbers all of which, but finitely many, are zero, we have

$$E\exp i\sum_{t\in T}\alpha_t X_t = \exp\left[-\int_{\mathbb{R}^T}\left(1-\cos\left(\sum_{t\in T}\alpha_t\beta(t)\right)\right)d\nu(\beta)\right]. \tag{1.1}$$

By cylindrical measure, we mean that we know the projections of ν on $\mathbb{R}^S$, for S a finite subset of T, and that these are ordinary positive measures. This might already be the place to mention that the main purpose of the paper is to prove inequalities, for which there is no loss of strength in assuming that T is finite. While, for the sake of completeness, we mention some results when T is infinite, it is not our purpose to dwell on the related technicalities.

The reason for which we rule out the Gaussian component is that its role in the questions considered here is elucidated by [25]. The assumption of symmetry might seem more restrictive. This is not the case. This is shown by the well known "symmetrization procedure" that to each process $(X_t)_{t\in T}$ associates the process $(X_t - X'_t)_{t\in T}$, where $(X'_t)_{t\in T}$ is an independent copy of $(X_t)_{t\in T}$.

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If T is a topological space, is the process continuous? Is the process bounded? In this introduction we focus on boundedness; that is, as in the Gaussian case, the key to understanding continuity. Some results concerning continuity are given in Section 6. It must also be pointed out that, in contrast with the Gaussian case, continuity is a somewhat less important issue than boundedness, since the most important infinitely divisible processes (like the Poisson process on $\mathbb{R}$) are not continuous. Since we are dealing with an uncountable family of random variables, each of them defined almost everywhere, it is not completely obvious what we mean by a "bounded" process. The standard way to deal with that difficulty is the notion of "separable process." To avoid technicalities here, let us say that the process is bounded if for each countable subset D of T, we have $\sup_{t \in D} |X_t| \leq \infty$ a.e.

A Gaussian process, that is, a process such that the law of each finite combination $\Sigma \alpha_t X_t$ is Gaussian, is entirely determined by its covariance structure, that is, by the function $E(X_s X_t)$ on $T \times T$. By contrast, an infinitely divisible process is parametrized by a measure on $\mathbb{R}^T$, a much more complicated object. While Gaussian processes are a very rigid structure, this is much less the case for infinitely divisible processes, and even the properties of infinitely divisible real random variables are not completely elucidated [7]. Which function of such parameters should one use to study an infinitely divisible process $(X_t)_{t \in T}$? It turns out that for the purposes of this paper, the fundamental role is played by the family of functions

$$\varphi(s,t,u) = \int_{\mathbb{R}^T} \left(u^2(\beta(s) - \beta(t))^2\right) \wedge 1 \, d\nu(\beta) \tag{1.2}$$

defined for $s,t \in T$, $u \in \mathbb{R}$. Observe that $\varphi(s,t,u) < \infty$ since ν is a Lévy measure. Observe that for given u, $\varphi(s,t,u)^{1/2}$ is a distance on T; it will however be more convenient to work with φ than with $\varphi^{1/2}$.

Our main results involve a regularity assumption that we introduce and discuss now. The assumption will be used in a critical way at several different stages of the proof; it is assumed throughout the paper.

Condition $H(\delta, v_0)$. For some $\delta > 0$, $v_0 > 1$, we have the following. For all $s,t \in T$, all $u > 0$, all $v \geq v_0$, we have

$$\nu(\{\beta; |\beta(s) - \beta(t)| \geq uv\}) \leq v^{-1-\delta} \nu(\{\beta; |\beta(s) - \beta(t)| \geq u\}).$$

To understand this condition, consider the case where ν is concentrated on a ray. It is then the image under the map $x \to x\beta_0$ of a measure μ on $\mathbb{R}$, where $\beta_0 \in \mathbb{R}^T$. In that case, condition $H(\delta, v_0)$ simply means that

$$\forall\, u \in \mathbb{R}^+, \forall\, v \geq v_0, \mu(\{x; |x| \geq uv\}) \leq v^{-1-\delta} \mu(\{x; |x| \geq u\}).$$

A large class of measures that satisfy condition $H(\delta, v_0)$ is obtained by taking mixture of measures on rays that satisfy it.

An important class of infinitely divisible processes is the class of p-stable processes. An infinitely divisible process is a p-stable process if its Lévy measure ν is the image under the map $(x, \beta) \to x\beta$ from $\mathbb{R} \times \mathbb{R}^T$ to $\mathbb{R}^T$ of a product measure $\eta \otimes m$, where m is a probability measure on $\mathbb{R}^T$, and η has density x^{-p-1} with respect to Lebesgue's measure. In the class of p-stable processes, condition $H(\delta, v_0)$ means that $p > 1$.

THEOREM 1.1. *Assume condition $H(\delta, v_0)$. Consider an infinitely divisible process $(X_t)_{t \in T}$ and $M > 0$ such that for each countable subset D of T we have*

$$P\Big(\sup_{s,t \in D} |X_t - X_s| \le M\Big) \ge \tfrac{3}{4}. \tag{1.3}$$

Then there exists a largest integer $i \in \mathbb{Z}$ such that $\varphi(s,t,2^i) \le 1$ for all $s,t \in T$, and there exists a probability measure μ on T with the following property. For $t \in T$, $j \ge i$, we define $n(t,j)$ as the smallest integer $n \ge 0$ for which

$$\mu\big(\{s \in T; \varphi(t,s,2^j) \le 2^n\}\big) \ge e^{-2^n}.$$

Then we have

$$\sup_{t \in T} \sum_{j \ge i} 2^{-j+n(t,j)} \le KM, \tag{1.4}$$

where K depends on δ, v_0 only.

The measure μ is called a majorizing measure by analogy with the Gaussian case, where the first use of these measures was to provide upper bounds, although the name "minorizing measure" could be more appropriate. We do not comment on the μ measurability of the sets $(s \in T; \varphi(t, s, 2^{-j}) \le 2^n\}$. This is because the majorizing measure we will construct is supported by a countable set. Majorizing measures have actually little to do with measure theory; they are a method to put appropriate "weights" on the space that quantify some of its properties.

The statement of the theorem does not relate at first sight to the usual formulations of majorizing measures. The reason is that a "change of variable" has been made. This change of variable helps to formulate the careful interplay between the parameters n, j and μ that is an essential feature here. It is an exercise to see, using a change of variable, that the usual quantity

$$\sup_{t \in T} \int_0^D \left(\log \frac{1}{\mu(B(t,\varepsilon))}\right)^{1/q} d\varepsilon \tag{1.5}$$

(where the balls are for a given metric d on T, and D is the diameter of T) is equivalent to the left-hand side of (1.4), provided $n(j,t)$ is defined as the smallest integer n for which $\mu(B(t, 2^{n/p-j})) \le e^{-2^n}$. (Here $1/p + 1/q = 1$.) It is also an exercise to see that if in the definition of $n(t,j)$, we replace $\varphi(t,s,2^j)$ by $\varphi(t,s,r^j)$ for some $r > 1$, the quantity $\sup_{t \in T} \sum_{j \ge i} r^{-j} 2^{n(t,j)}$ is equivalent to the left-hand side of (1.4). It will be convenient for technical reasons to use an appropriate value of r.

In the case of p-stable processes, one sees easily that (1.2) becomes

$$\varphi(s,t,u) = c(p)u^p d_p^p(s,t),$$

where $c(p)$ depends on p only and

$$d_p(s,t) = \left(\int_{\mathbb{R}^T} |\beta(s) - \beta(t)|^p \, dm(\beta)\right)^{1/p}. \tag{1.6}$$

The condition $\varphi(t,s,2^j) \leq 2^n$ thus becomes $s \in B(t, c^{-1/p}(p)2^{n/p-j})$. Thus in the p-stable case, Theorem 1.1 expresses that one can find μ for which the quantity (1.5) is less than or equal to KM, where K depends on p only. This was first proved in [26]. (The proofs presented here, when specialized to the p-stable case, give a completely new approach.) In [26] a lower bound of a similar nature is also proved for $p = 1$, where the function $\log^{1/q}$ is replaced by an iterated logarithm. The case $p = 1$ is not recovered by Theorem 1.1; but we should mention that the methods used in [26] to obtain the case $p = 1$ are different and quite harder than in the case $p > 1$. Thus, it is not so surprising that, at the level of generality of Theorem 1.1 our proofs require a condition of the type $H(\delta, v_0)$, that means that we "stay away from the case $p = 1$." Whether this is an artifact of our approach, or whether processes failing condition $H(\delta, v_0)$ can have a genuinely more complicated structure, has not been settled at this time.

A considerable difference between the Gaussian case and the infinitely divisible case is that the lower bound of Theorem 1.1 is by no mean an upper bound (except in the considerably easier special situation of harmonic processes recently described in [29]). Thus, how do we know whether Theorem 1.1 catches essential information about the process? That this is indeed the case is expressed by the following theorem, that will be formulated rigorously in Section 7.

We say that a process $(X_t)_{t\in T}$ is a *positive* infinitely divisible process if there exists a (cylindrical) measure ν on $\mathbb{R}^{+T}$, such that $\int_{\mathbb{R}^T} |\beta(t)| \wedge 1 \, d\nu(\beta) < \infty$ for all $t \in T$, and that, for all family of numbers $(\alpha_t)_{t\in T}$, all but finitely many being zero, we have

$$E \exp i \sum_{t\in T} \alpha_t X_t = \exp\left[-\int \left(1 - \exp\left(i \sum_{t\in T} \alpha_t \beta(t)\right)\right) d\nu(\beta)\right]. \tag{1.7}$$

We will call ν the Lévy measure of the process. (Thus, while by "infinitely divisible" process we understand that the process is symmetric, *positive* infinitely divisible processes are positive and certainly not symmetric.)

THEOREM 1.2 (Informal version). *Under condition $H(\delta, v_0)$, if the infinitely divisible process $(X_t)_{t\in D}$ is bounded, it is equal in distribution to the sum of two (not necessarily independent) infinitely divisible processes. The first of these is controlled by a majorizing measure; the boundedness of the second is obviated by that of a certain positive infinitely divisible process.*

By saying that a process is controlled by a majorizing measure, we mean that it belongs to a special class for which the lower bound of Theorem 1.1 is also an upper bound (so its boundedness is obvious).

Roughly speaking, Theorem 1.2 implies that the complexity of infinitely divisible processes arises only from the complexity of positive infinitely divisible processes, since majorizing measures control the rest of the process. The class of positive infinitely divisible processes is obviously much smaller than the class of infinitely divisible processes. It is nonetheless very complicated. Which function of which parameters should one study the boundedness of these processes, or whether such parameters can be found at all, remains the subject of further inquiry; and we feel that progress in that direction, if at all possible, should require the development of radically new ideas.

One can interpret Theorem 1.2 in an intuitive (but somewhat misleading) way. An infinitely divisible process is a compound Poisson process, and the trajectory of the process is the addition of many functions ("jumps") of $\mathbb{R}^T$. There are essentially two different reasons why the process can be bounded. One is that there are many jumps, but these are in different directions, and a lot of cancellation occurs. This is the aspect that is well understood through majorizing measures. Another is that there are just not many jumps, so that even if these jumps are reoriented roughly in the same direction (the positive cone) by replacing them by their absolute values (as functions of $\mathbb{R}^T$) their sum stays bounded. This is the case of processes controlled by bounded positive processes.

We wish now to explain why Theorem 1.1 and, in particular, the methods behind its proof, represent an advance over the case of Gaussian or p-stable processes. We start by recalling the method used for Gaussian processes. The fundamental parameter is the distance $d(s,t) = \|X_s - X_t\|_2$ induced by the process. One introduces a functional $\theta(T)$ that, roughly speaking, measures the size of T with respect to the existence of majorizing measures. The proof articulates in two main steps.

STEP 1. *"Separation" principle.* The space (T, d) contains a "well separated" subset V for which $\theta(V)$ is of order $\theta(T)$.

Here well separated means that, for the induced distance, V is (Lipschitz-isomorphic to) an ultrametric space. The construction of V is done by reiterating a basic step. In this basic step a piece of V is replaced by a finite union of smaller pieces, whose diameters are all of the same order, and of the same order as the distance between themselves.

STEP 2. *Minoration principle in the well separated case.* In that case $E \sup_{s,t \in T} |X_s - X_t|$ is of order at least $\theta(T)$.

This step (proved earlier by Fernique) relies on a very specific property of Gaussian process called Slepian's lemma. (See [9] for a modern proof.) It allows us to find a lower bound of $E \sup_{s,t \in T} |X_s - X_t|$ by comparing it with the corresponding quantity for an auxiliary process, for which it can be easily computed.

One essential common feature of the p-stable case with the Gaussian case is that the fundamental parameter is again a single distance [given by (1.6)]. Thus one can apply Step 1 to that distance, to reduce to the well separated case. The second ingredient is the LePage representation of a p-stable process as a conditionally Gaussian process [12, 15]. The proof is not quite over still, as one lacks control over the (random) majorizing measures for this conditioned process. But this difficulty yields to a sufficient amount of brute force. It should be pointed out that, in the end, the lower bound for p-stable processes of [26], as well as the lower bound for Gaussian processes, relied upon Slepian's lemma.

In the present case of infinitely divisible processes, the fundamental parameter is not a single distance, but a family of distances. Thus we cannot use Step 1, but we have to develop a separation principle involving this family of distances. As with Step 1, it is based on the iteration of a basic step. The main feature now is that there is a precise relationship between the number of pieces to be created at each step, and the distance at which they must be to each other (the distance is measured by an appropriate distance of our family). This new separation principle is an extension of Step 1, as can be seen when using the functions $\varphi(s, t, u) = u^p d(s, t)^p$. It can be compared to Step 1 as a two parameter situation versus a one parameter situation. The basic step in the separation principle is itself obtained by iteration (a large number of times) of a simpler principle. The spirit of this proof is somewhat reminiscent of the proof of an isoperimetric inequality using rearrangements, where a sharp result is obtained by reiterating many times a simple step. The new separation principle is surely the most delicate result of the present paper. The reason is that, in order to get the exact lower bound, no essential loss is permitted at any of the several stages of the proof, and that each of them must, in some sense, capture the exact reality.

After we have succeeded in reducing the situation to a case where there is enough separation, we come upon another obstacle. Infinitely divisible processes do not satisfy a comparison theorem like Slepian's lemma, and cannot be represented as conditionally Gaussian processes. Our proof will instead rely upon the fact that infinitely divisible processes can be represented as a mixture of Bernoulli (or Rademacher) processes. Consider a Bernoulli sequence $(\varepsilon_k)_{k \geq 1}$, that is, the sequence (ε_k) is independent identically distributed, and $P(\varepsilon_k = 1) = P(\varepsilon_k = -1) = 1/2$. For $t = (t_k)_{k \geq 1} \in l^2$, consider the random variable $X_t = \sum_{k \geq 1} t_k \varepsilon_k$. For a subset T of l^2, we call the process $(X_t)_{t \in T}$ a Bernoulli process. Unfortunately Bernoulli processes do not satisfy any comparison principle as strong as Slepian's lemma. A key ingredient of our approach is the discovery that a strong minoration principle for Bernoulli processes can be obtained from a "concentration of measure" property of Bernoulli processes and a much weaker minoration principle. This weaker principle is an approximate version of Sudakov's minoration for Gaussian processes [23, 6]. This observation is potentially important; it allows us to replace the use of Slepian's lemma, that is very specific to Gaussian processes (and on which the previous lower bounds relied), by the use of principles of far greater generality (which could also be used in the Gaussian case instead of Slepian's lemma). One can see the books [16, 11] for a discussion of the "concentration of measure" and the papers [29, 32] for example of Sudakov-type minorations, that can be obtained by a variety of methods. It could be feared that lower bounds would not be obtained beyond the Gaussian situation and its corollaries. This is no longer the case. It is now conceivable that majorizing measure type lower bounds will be obtained in many more situations that could be expected previously, and the shift of perspective that brings this possibility could well be the main contribution of this paper, beyond its specific application to the study of infinitely divisible processes. In order to illustrate the progress brought by the present approach, we would like to describe the results it has enabled us to obtain on a class of processes very different from infinitely divisible processes. Consider $1 < p < \infty$, and an i.i.d. sequence $(h_k)_{k \geq 0}$ of random variables that have density $a_p \exp(-|t|^p)$ with respect to Lebesgue's measure, where a_p is a normalizing constant. For $t \in l^2$, set $X_t = \sum_{k \geq 1} t_k h_k$. Set $q = p/(p-1)$. Denote by $\|\cdot\|_q$ the norm of l^q, and for a subset T of l^q, set

$$\gamma_q(T) = \inf\left(\sup_{t \in T} \int_0^D \left(\log \frac{1}{\mu(B(t,\varepsilon))}\right)^{1/p} d\varepsilon\right),$$

where D is the diameter of T, the ball is for the norm $\|\cdot\|_q$ and the infimum is taken over all probability measures μ on T. We say that two quantities A, B are equivalent if $K^{-1}A \leq B \leq KA$, where K depends on p only. The following is another extension of the results of [25] on Gaussian processes that is seemingly unrelated to the case of infinitely divisible processes.

THEOREM 1.3. (a) *For* $p \leq 2$, $E \sup_{s,t \in T} |X_s - X_t|$ *is equivalent to* $\max(\gamma_q(T), \gamma_2(T))$.

(b) *For* $p \geq 2$, $E \sup_{s,t \in T} |X_s - X_t|$ *is equivalent to*

$$\inf\left\{M \geq 0; \exists\, A \subset l^q,\, B \subset l^2,\, T \subset A + B,\, \gamma_q(A) \leq M,\, \gamma_2(B) \leq M\right\}.$$

One basic fact that could help to understand this statement is that $-\log(P(|\sum_{k \geq 1} t_k h_k| \geq u))$ is of order $u^2 \|t\|_2^{-2}$ for u small but of order $u^p \|t\|_q^{-p}$ for u large. This observation also brings to light the fact that we have a two-parameter situation when $p \neq 2$, in contrast with the one-parameter situation in the Gaussian case $p = 2$.

The most delicate point of that theorem is, given T, to construct A and B in (b). The first part of the proof is to create enough "separation." The separation principle that we prove here turns out to be also suitable for that purpose, by setting now

$$\varphi(s,t,u) = \inf\Big\{a > 0,\ s - t \in u^{-1}\big(a^{1/2}B_2 + a^{1/q}B_q\big)\Big\},$$

where B_2 (respectively, B_q) is the unit ball of l^2 (respectively, l^q). This is not a coincidence, but results from the fact that this separation principle has been designed to handle general situations. Once separation has been obtained, the approach follows the line used here in our treatment of Bernoulli processes. It requires the proof of a new Sudakov-type minoration for the variables h_k, and of a delicate new isoperimetric inequality. In order not to lengthen considerably the present paper, the proofs will be presented elsewhere [31, 32].

We now describe the organization of the paper.

We believe that random choices of signs, and hence Bernoulli processes, are an important structure. The tools concerning them have potential application to a variety of situations. The three main tools, a comparison theorem, a version of Sudakov minoration and the minoration principle are thus presented at the beginning of Section 2. In Section 2 we also describe the representation of infinitely divisible processes as conditional Bernoulli processes that was recently put forward by Rosinski [20], and we prove some inequalities that are basic for the use of that representation. In Section 3, we present the basic functionals that will measure the size of T with respect to the existence of a majorizing measure. The serious work with these functionals starts in Section 4, where we prove the possibility of the basic separation step. In Section 5 we show how to iterate this basic step to construct well separated subsets of T, and we learn how to work with these sets.

In Section 6, we present the central arguments, that make use of all the tools built before to prove Theorem 1.1; we then give a version of Theorem 1.1 for continuous processes. In Section 7, we give a formal meaning to Theorem 1.2, and we deduce it from Theorem 1.1. We conclude by proving a sufficient "bracketing condition" for the boundedness of infinitely divisible processes.

2. Tools. We start by presenting the tools about Bernoulli processes that are central to this paper. While Bernoulli processes do not satisfy comparison theorems as strong as those satisfied by Gaussian processes, they satisfy a weaker principle which turns out to be of crucial importance. We say that a map $f\colon \mathbb{R} \to \mathbb{R}$ is a contraction if $|f(x) - f(y)| \le |x - y|$. The following comparison theorem, that is due to this author, first appeared in [10], with a more complicated proof. The simplifications presented here are the result of a joint effort with Ledoux.

THEOREM 2.1. *Consider contractions $(f_k)_{k\ge1}$ of $\mathbb{R}$ such that $f_k(0) = 0$. Consider a (finite) subset T of l^2. Then*:

(a) *if F is a convex increasing function from $\mathbb{R}$ to $\mathbb{R}$, we have*

$$E \sup_{t\in T} F\Big(\sum_{k\ge1} \varepsilon_k f_k(t_k)\Big) \le E \sup_{t\in T} F\Big(\sum_{k\ge1} \varepsilon_k t_k\Big).$$

(b) *If* $G\colon \mathbb{R}^+ \to \mathbb{R}^+$ *is convex increasing, we have*

$$E \sup_{t\in T} G\left(\left|\sum_{k\geq 1} \varepsilon_k f_k(t_k)\right|\right) \leq 2E \sup_{t\in T} G\left(\left|\sum_{k\geq 1} \varepsilon_k t_k\right|\right).$$

PROOF. (a) A simple approximation argument shows that it suffices to prove that for all $N \geq 1$, we have

$$E \sup_{t\in T} F\left(\sum_{1\leq k\leq N} \varepsilon_k f_k(t_k)\right) \leq E \sup_{t\in T} F\left(\sum_{1\leq k\leq N} \varepsilon_k t_k\right).$$

By iteration, if suffices to show that $E \sup_{t\in T} F(\sum_{1\leq k\leq N} \varepsilon_k t_k)$ decreases when t_1 is replaced by $f_1(t_1)$. If we condition with respect to $\varepsilon_2, \dots, \varepsilon_n$, we are reduced to showing that if T is a subset of $\mathbb{R}^2$, and f is a contraction on $\mathbb{R}$ such that $f(0) = 0$, then

$$E \sup_{t\in T} F(\varepsilon_1 f(t_1) + t_2) \leq E \sup_{t\in T} F(\varepsilon_1 t_1 + t_2),$$

where $t = (t_1, t_2)$. We show that for all t and s in T the right-hand side is always larger than

$$I = \tfrac{1}{2}F(f(t_1) + t_2) + \tfrac{1}{2}F(-f(s_1) + s_2).$$

We observe that we may assume

$$f(t_1) + t_2 \geq f(s_1) + s_2, \tag{2.1}$$

$$-f(s_1) + s_2 \geq -f(t_1) + t_2. \tag{2.2}$$

We distinguish the following cases.

CASE 1. $s_1 \geq 0$, $t_1 \geq 0$. We assume to begin with that $t_1 \geq s_1$, and we show that

$$2I \leq F(t_1 + t_2) + F(-s_1 + s_2).$$

Set $a = -f(s_1) + s_2$, $b = -s_1 + s_2$, $a' = t_1 + t_2$, $b' = f(t_1) + t_2$, so we would like to prove that

$$F(b') + F(a) \leq F(a') + F(b),$$

or

$$F(a) - F(b) \leq F(a') - F(b'). \tag{2.3}$$

Since f is a contraction with $f(0) = 0$, and $s_1 \geq 0$, we have $|f(s_1)| \leq s_1$. Thus $a \geq b$, and, by (2.1), $b' \geq b$. Since $s_1 \leq t_1$, by contraction we have $f(t_1) - f(s_1) \leq t_1 - s_1$, so that

$$a - b = s_1 - f(s_1) \leq t_1 - f(t_1) = a' - b'.$$

Since F is convex and increasing, for all positive x, the map $F(\cdot + x) - F(\cdot)$ is increasing. Thus, for $x = a - b \geq 0$, since $b \leq b'$, we get

$$F(a) - F(b) \leq F(b' + (a - b)) - F(b').$$

Using that $b' + (a - b) \leq a'$ yield (2.3). When $s_1 \geq t_1$, the argument is similar, changing t into s and f into $-f$.

CASE 2. $t_1 \leq 0$, $s_1 \leq 0$. It is completely similar to the preceding one.

CASE 3. $t_1 \geq 0$, $s_1 \leq 0$. Since $f(t_1) \leq t_1$, $-f(s_1) \leq -s_1$, we have that

$$2I \leq F(t_1 + t_2) + F(-s_1 + s_2)$$

and the result follows.

CASE 4. $t_1 \leq 0$, $s_1 \geq 0$. This is similar to Case 3.

This completes the proof of (a).

(b) Set $x^+ = \max(x, 0)$, $x^- = \max(-x, 0)$. Thus $|x| = x^+ + x^-$. We can obviously assume $G(0) = 0$, so that

$$G\left(\left|\sum_{k \geq 1} \varepsilon_k f_k(t_k)\right|\right) = G\left(\left(\sum_{k \geq 1} \varepsilon_k f_k(t_k)\right)^+\right) + G\left(\left(\sum_{k \geq 1} \varepsilon_k f_k(t_k)\right)^-\right)$$

and

$$\sup_{t \in T} G\left(\left|\sum_{k \geq 1} \varepsilon_k f_k(t_k)\right|\right) \leq \sup_{t \in T} G\left(\left(\sum_{k \geq 1} \varepsilon_k f_k(t_k)\right)^+\right) + \sup_{t \in T} G\left(\left(\sum_{k \geq 1} \varepsilon_k f_k(t_k)\right)^-\right).$$

The two pieces on the right have the same distribution. Thus

$$E \sup_{t \in T} G\left(\left|\sum_{k \geq 1} \varepsilon_k f_k(t_k)\right|\right) \leq 2E \sup_{t \in T} G\left(\left(\sum_{k \geq 1} \varepsilon_k f_k(t_k)\right)^+\right).$$

We now use (a) for the function $F(x) = G(x^+)$. We get

$$E \sup_{t \in T} G\left(\left(\sum_{k \geq 1} \varepsilon_k f_k(t_k)\right)^+\right) \leq E \sup_{t \in T} G\left(\left(\sum_{k \geq 1} \varepsilon_k t_k\right)^+\right) \leq E \sup_{t \in T} G\left(\left|\sum_{k \geq 1} \varepsilon_k t_k\right|\right). \qquad \square$$

We now turn toward a version of Sudakov's minoration for Bernoulli processes. For $t \in \mathbb{R}^N$, we set

$$\|t\|_2 = \left(\sum_{k \geq 1} t_k^2\right)^{1/2}, \qquad \|t\|_\infty = \sup_{k \geq 1} |t_k|, \qquad \|t\|_1 = \sum_{k \geq 1} |t_k|$$

and we denote by B_2, B_∞, B_1, the unit balls for these norms, respectively. Given two sets $T, D \subset \mathbb{R}^{\mathbb{N}}$, we denote by $N(T, D)$ the minimum number of translates of D by elements of T needed to cover T. For convenience we denote by K a numerical constant that may vary at each occurrence. On the other hand, $K_1, K_2, \dots$ denote specified constants.

Consider a sequence $(g_k)_{k \geq 1}$ of independent standard normal random variables. Sudakov's minoration states that, for a (finite) subset T of l^2, and all $\varepsilon > 0$, we have

$$E \sup_{t \in T} \left| \sum_{k \geq 1} g_k t_k \right| \geq \frac{\varepsilon}{K} \left(\log N(T, \varepsilon B_2)\right)^{1/2}. \tag{2.4}$$

The proof of our version of Sudakov's minoration for Bernoulli processes has two main steps. In the first stage, we extend (2.4) when we control $\|t\|_\infty$ for $t \in T$. For convenience we set, for a (finite) subset T of l^2

$$b(T) = E \sup_{t \in T} \left| \sum_{k \geq 1} \varepsilon_k t_k \right|.$$

The following was motivated by a (dimension dependent) result of [4].

PROPOSITION 2.2. *There exists a universal constant K such that for any $\varepsilon > 0$, and any subset T of l^2 such that $\|t\|_\infty \leq \varepsilon^2 / Kb(T)$ for $t \in T$, we have*

$$\varepsilon\left(\log N(T, \varepsilon B_2)\right)^{1/2} \leq Kb(T). \tag{2.5}$$

PROOF.

STEP 1. The essential step is to show that if $T \subset B_2$ and $\|t\|_\infty \leq 1/Kb(T)$ for all $t \in T$, then

$$\left(\log N(T, \tfrac{1}{2} B_2)\right)^{1/2} \leq Kb(T). \tag{2.6}$$

There is nothing to prove if $T \subset (1/4)B_2$, so we can assume that there is an element t of G for which $\|t\|_2 \geq 1/4$. Then, by Khintchin's inequality [24] we have

$$b(T) \geq E \left| \sum_{k \geq 1} \varepsilon_k t_k \right| \geq 1/4\sqrt{2}.$$

(The use of the best constant is essentially irrelevant.) Consider a parameter $s > 0$; set

$$h_k = g_k 1_{\{|g_k| > s\}}, \qquad f_k = g_k 1_{\{|g_k| \leq s\}}.$$

For any subset U of T we have

$$E \sup_{t \in U} \left| \sum_{k \geq 1} g_k t_k \right| \leq E \sup_{t \in U} \left| \sum_{k \geq 1} f_k t_k \right| + E \sup_{t \in U} \left| \sum_{k \geq 1} h_k t_k \right|.$$

Since f_k is symmetric and $\|f_k\|_\infty \leq s$, the "comparison principle" as in [9] implies that $E \sup_{t \in U} |\sum_{k \geq 1} f_k t_k| \leq sE \sup_{t \in U} |\sum_{k \geq 1} \varepsilon_k t_k|$, so we have

$$E \sup_{t \in U} \left| \sum_{k \geq 1} g_k t_k \right| \leq sb(T) + E \sup_{t \in U} \left| \sum_{k \geq 1} h_k t_k \right|. \tag{2.7}$$

Consider now $\lambda \neq 0$. Since the random variable h_i is symmetric, we can write

$$E \exp \lambda h_i = 1 + \lambda^2 \varphi_s(\lambda) \leq \exp\big(\lambda^2 \varphi_s(\lambda)\big),$$

where

$$\varphi_s(\lambda) = E \frac{\exp \lambda h_i + \exp - \lambda h_i - 2}{2\lambda^2}.$$

Since the function $x^{-2}(e^x + e^{-x} - 2)$ increases, φ_s increases on $\mathbb{R}^+$. By Lebesgue's dominated convergence theorem, we have $\lim_{s \to \infty} \varphi_s(1) = 0$. By (2.4) there exists a number K_1 such that

$$E \sup_{t \in U} \Big| \sum_{k \geq 1} g_k t_k \Big| \geq \frac{1}{K_1} \big(\log N(U, \tfrac{1}{2} B_2)\big)^{1/2} \tag{2.8}$$

for all $U \subset l^2$. We fix s so that $\varphi_s(1) \leq 1/36K_1^2$, and we prove (2.6) with $K = 24sK_1^2$.

For $t \in l^2$ we have, for each k,

$$E \exp \lambda t_k h_k \leq \exp \lambda^2 t_k^2 \varphi_s(t_k \lambda) \leq \exp \lambda^2 t_k^2 \varphi_s(\|t\|_\infty |\lambda|),$$

since φ_s increases on $\mathbb{R}^+$. Thus we have, if $t \in B_2$,

$$E \exp \lambda \sum_{k \geq 1} t_k h_k = \prod_{k \geq 1} E \exp \lambda t_k h_k \leq \exp \lambda^2 \varphi_s(\|t\|_\infty \lambda).$$

By Chebyshev's exponential inequality $P(Z \geq x) \leq \exp - \lambda x E \exp \lambda Z$, this yields

$$P\Big(\Big| \sum_{k \geq 1} t_k h_k \Big| \geq x \Big) \leq 2 \exp\big(-\lambda x + \lambda^2 \varphi_s(\|t\|_\infty \lambda)\big). \tag{2.9}$$

Consider now a subset $U \subset T$. Set $N = \operatorname{card} U$ and suppose that $(\log N)^{1/2} \leq 4K_1 sb(T)$ and $N \geq 3$. We use (2.9) with

$$\lambda = 6K_1 \sqrt{\log N} \leq 24K_1^2 sb(T).$$

Since $\|t\|_\infty \leq 1/24K_1^2 sb(T)$ for $t \in T$, we have

$$P\Big(\Big| \sum_{k \geq 1} t_k h_k \Big| \geq x \Big) \leq 2 \exp\big(-\lambda x + \lambda^2 \varphi_s(1)\big)$$

and thus, since $\lambda^2 \varphi_s(1) \leq \log N$, we have

$$P\Big(\sup_{t \in U} \Big| \sum_{k \geq 1} t_k h_k \Big| \geq x \Big) \leq 2N^2 \exp(-\lambda x).$$

Thus, for any $y \geq 0$,

$$E\Big(\sup_{t \in U} \Big| \sum_{k \geq 1} t_k h_k \Big| \Big) = \int_0^\infty P\Big(\sup_{t \in U} \Big| \sum_{k \geq 1} t_k h_k \Big| > x \Big) \, dx$$

$$\leq y + \int_y^\infty 2N^2 \exp(-\lambda x) \leq y + \frac{2N^2}{\lambda} \exp - (\lambda y).$$

We use this for $y = \sqrt{\log N}/3K_1$. Since $\lambda y = 2\log N$, we get

$$(2.10)\quad E\left(\sup_{t\in U}\left|\sum_{k\geq 1} t_k h_k\right|\right) \leq y + \frac{2}{\lambda} = \frac{\sqrt{\log N}}{3K_1} + \frac{2}{6K_1\sqrt{\log N}} \leq \frac{2\sqrt{\log N}}{3K_1},$$

since $\log N \geq 1$. Suppose now that $N(U,(1/2)B_2) = N$. Combining (2.7), (2.8) and (2.10) we have

$$\frac{1}{K_1}\sqrt{\log N} \leq sb(T) + \frac{2}{3K_1}\sqrt{\log N},$$

so that $\sqrt{\log N} \leq 3K_1 sb(T)$.

In conclusion, we have shown that if $U \subset T$ satisfies $N(U,(1/2)B_2) = \operatorname{card} U = N \geq 3$, then we cannot have $3K_1 sb(T) < \sqrt{\log N} \leq 4K_1 sb(T)$. Consider the smallest integer N such that $\sqrt{\log N} > 3K_1 sb(T)$. Since $b(T) \geq 1/4\sqrt{2}$, we can assume $K_1 sb(T) \geq 1$, so that $\sqrt{\log N} \leq 4K_1 sb(T)$ and $N \geq 3$. We claim that $N(T,(1/2)B_2) < N$ [so that $(\log N(T,(1/2)B_2))^{1/2} \leq 3K_1 sb(T)$]. Indeed, otherwise we could construct by immediate induction points $t_1,\dots,t_N \in T$ such that $\|t_i - t_j\|_2 > 1/2$ for $i \neq j$, so that, if $U = \{t_1,\dots,t_N\}$, we have $\operatorname{card} U = N = N(U,(1/2)B_2)$. But we have shown that this is impossible.

Step 2. We now get the conclusion by a (standard) iteration procedure. We denote by K_2 the constant of (2.6).

For $t \in T$, $l \in \mathbb{Z}$, we have

$$N\big(T \cap (t + 2^{-l}B_2), 2^{-l-1}B_2\big) = N\big((T - t) \cap 2^{-l}B_2, 2^{-l-1}B_2\big).$$

Since $b(T-t) \leq 2b(T)$, we see from (2.6) by homogeneity that

$$N\big(T \cap (t + 2^{-l}B_2), 2^{-l-1}B_2\big) \leq \exp K_2^2 2^{2l+2} b(T)^2$$

provided

$$\forall\, t \in T,\qquad \|t\|_\infty \leq \frac{1}{K_2 2^{2l+2} b(T)}.$$

We now note that

$$\begin{aligned} N(T, 2^{-k}B_2) &\leq \prod_{l<k}\sup_{t\in T} N\big(T \cap (t + 2^{-l}B_2), 2^{-l-1}B_2\big) \\ &\leq \exp K_2^2 \sum_{l<k} 2^{2l+2} b(T)^2 \\ &\leq \exp K_2^2 2^{2k+3} b(T)^2 \end{aligned}$$

provided $\|t\|_\infty \leq (K_2 2^{2k} b(T))^{-1}$ for all $t \in T$. Given $\varepsilon > 0$, consider the smallest integer k such that $2^{-k} \leq \varepsilon$, so that $2^k \leq 2/\varepsilon$. Thus

$$N(T, \varepsilon B_2) \leq \exp\left[K_2^2 2^5 \varepsilon^{-2} b(T)^2\right]$$

provided

$$\|t\|_\infty \leq \left(4K_2 b(T)\right)^{-1}\varepsilon^2 \leq \left(K_2 2^{2k} b(T)\right)^{-1}$$

for all $t \in T$. □

The main difficulty in finding lower bounds for Bernoulli processes is that $b(B_1) = 1$, so that $b(b(T)B_1) = b(T)$. Thus, knowing $b(T)$, we cannot expect any information on T unless we stay away from $b(T)B_1$. This is the idea of our statement.

THEOREM 2.3. There exists a numerical constant K with the following property. Consider a finite set $T \subset l^2$, and $\varepsilon > 0$. Set $D = Kb(T)B_1 + \varepsilon B_2$. Then

$$\varepsilon(\log N(T, D))^{1/2} \leq Kb(T). \tag{2.11}$$

PROOF. It is instructive to compare that statement with (2.5). It is simple to see (see the proof of Corollary 2.7) that

$$D \cap \frac{\varepsilon^2}{b(T)} B_\infty \subset K\varepsilon B_2$$

so that in the case where $T \subset (\varepsilon^2/b(T))B_\infty$, (2.11) is actually equivalent to (2.5). The idea in the general case is to reduce (2.11) to (2.5) applied to a new set T' that satisfies the hypothesis of Proposition 2.2. We denote by K_3 the constant of that proposition, and we prove the theorem for $K = 8K_3$. We set $a = \varepsilon^2/Kb(T)$. For $j \geq 1$, we set

$$f_j(u) = \max(0, \min(a, u - (j-1)a)).$$

Thus $f_j(0) = 0$, and f_j is a contraction. We set $f_0 = 0$, and for $j \leq 0$ we set $f_{-j}(a) = -f_j(-a)$.

We observe that for any $u, u' \in \mathbb{R}$ we have

$$\sum_j \left| f_j(u) - f_j(u') \right| = |u - u'|. \tag{2.12}$$

Another elementary property is the following. Given $u, u' \in \mathbb{R}$, we can find v, v' such that

$$\sum_j \left| f_j(u) - f_j(u') \right|^2 = |u - v|^2 + |u' - v'|^2 + a|v - v'|. \tag{2.13}$$

(For example, if $0 \leq u' < u$, take $v = v' = u$ if $[u/a] = [u'/a]$; while, if $[u'/a] < [u/a]$, take $v' = a([u'/a] + 1)$, $v = a[u/a]$.)

Consider the map $\psi\colon \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}\times\mathbb{Z}}$ given by $\psi(t) = (f_j(t_k))_{j\in\mathbb{Z},\, k\geq 0}$. It follows from (2.13) that if $\|\psi(t) - \psi(t')\|_2 \leq \varepsilon/2$, we can write $t - t' = x^1 + x^2 + y$, where $\|x^1\|_2 \leq \varepsilon/2$, $\|x^2\|_2 \leq \varepsilon/2$, $\|y\|_1 \leq \varepsilon^2/4a \leq Kb(T)$, so that $t - t' \in \varepsilon B_2 + Kb(T)B_1$. Thus we have

$$N(T, D) \leq N\left(\psi(T), \frac{\varepsilon}{2}B_2\right).$$

Consider now a doubly indexed Bernoulli sequence $(\varepsilon_{kj})_{k\geq 1,\, j\in\mathbb{Z}}$ and another Bernoulli sequence $(\varepsilon'_k)_k$; we assume the sequences (ε_{kj}) and (ε'_k) to be independent. Then, by symmetry,

$$b(\psi(T)) = E\sup_{t\in T}\Bigg|\sum_{\substack{k\geq 1\\ j\in\mathbb{Z}}} \varepsilon_{kj} f_j(t_k)\Bigg| = E\sup_{t\in T}\Bigg|\sum_{k\geq 1}\varepsilon'_k\Bigg(\sum_{j\in\mathbb{Z}}\varepsilon_{kj} f_j(t_k)\Bigg)\Bigg|.$$

Now, for every choice of signs ε_{kj} and of t, t' in T, it follows from (2.12) that

$$\Bigg|\sum_j \varepsilon_{kj} f_j(t_k) - \sum_j \varepsilon_{kj} f_j(t'_k)\Bigg| \leq |t_k - t'_k|.$$

Thus Theorem 2.1(b), used conditionally on the sequence ε_{kj}, gives

$$E\sup_{t\in T}\Bigg|\sum_{k\geq 1}\varepsilon'_k\Bigg(\sum_j \varepsilon_{kj} f_j(t_k)\Bigg)\Bigg| \leq 2b(T),$$

so that $b(\psi(T)) \leq 2b(T)$. Now, by construction, for all t, k, j,

$$|f_j(t_k)| \leq a = \frac{\varepsilon^2}{8K_3 b(T)} \leq \frac{(\varepsilon/2)^2}{K_3 b(\psi(T))}$$

so that by Proposition 2.2 applied to $\psi(T)$ in $\mathbb{R}^{\mathbb{N}\times\mathbb{Z}}$ we have

$$\varepsilon\left(\log N\left(\psi(T), \frac{\varepsilon}{2}B_2\right)\right)^{1/2} \leq 2K_3 b(\psi(T)) \leq 4K_3 b(T). \qquad \square$$

Corollary 2.4. *There exists a universal constant K with the following property. Consider N points $t^1, \dots, t^N \in l^2$, and suppose that $t^i - t^j \notin AB_1 + \varepsilon B_2$ for $i, j \leq N$, $i \neq j$. Then*

$$E\sup_{i\leq N}\sum_{k\geq 1}\varepsilon_k t^i_k \geq \frac{1}{K}\min\left(A, \varepsilon\sqrt{\log N}\right). \tag{2.14}$$

Proof. We denote by K_4 the constant of Theorem 2.3, and we prove (2.14) for $K = 2K_4$.

Set $T = \{t_1, \dots, t_N\}$. If $b(T) \leq A/K_4$, we have

$$D = K_4 b(T)B_1 + \varepsilon B_2 \subset AB_1 + \varepsilon B_2$$

so that $N(T, D) = N$, and thus by (2.11) we have $b(T) \geq (1/K_4)\varepsilon\sqrt{\log N}$.

This proves that

$$b(T) \geq \frac{1}{K_4} \min\left(A, \varepsilon\sqrt{\log N}\right). \tag{2.15}$$

Since $E\Sigma_{k \geq 1}\varepsilon_k t_k^1 = 0$, we do not change the left-hand side of (2.14) by replacing t^i by $t^i - t^1$. Thus we can assume that $t^1 = 0$. Then

$$\sup_{i \leq N} \sum_{k \geq 1} \varepsilon_k t_k^i = \sup_{i \leq N} \left(\sum_{k \geq 1} \varepsilon_k t_k^i \right)^+ .$$

Now

$$\sup_{i \leq N} \left| \sum_{k \geq 1} \varepsilon_k t_k^i \right| \leq \sup_{i \leq N} \left(\sum_{k \geq 1} \varepsilon_k t_k^i \right)^+ + \sup_{i \leq N} \left(\sum_{k \geq 1} \varepsilon_k t_k^i \right)^- .$$

By symmetry the two terms on the right have the same distribution. Taking expectations we get

$$b(T) \leq 2E \sup_{i \leq N} \sum_{k \geq 1} \varepsilon_k t_k^i .$$

Combined with (2.15) this completes the proof. □

A third important fact about Bernoulli processes is their strong "concentration of measure" properties. For a subset T of l^2, we set $\sigma(T) = \sup_{t \in T} \|t\|_2$. The following is proved in [28].

Theorem 2.5. *Consider a (finite) subset T of l^2, and the random variable $Y = \sup_{t \in T} \Sigma_{k \geq 1}\varepsilon_k t_k$. Denote by M a median of Y, that is, $P(Y \geq M) \geq 1/2$, $P(Y \leq M) \geq 1/2$. Then*

$$P(|Y - M| \geq u) \leq 4 \exp\left(- \frac{u^2}{8\sigma^2(T)} \right). \tag{2.16}$$

For a subset T of l^2, we set

$$b_0(T) = E \sup_{t \in T} \sum_{k \geq 1} \varepsilon_k t_k .$$

A simple consequence of (2.16) is that $b_0(T) = E(Y) \leq M + K\sigma(T)$; so that

$$P\left(\sup_{t \in T} \sum_{k \geq 1} \varepsilon_k t_k \leq b_0(T) - K\sigma(T)(1 + u) \right) \leq 4 \exp(-u^2). \tag{2.17}$$

This fact will be used through the simple, but crucial following observation.

Proposition 2.6. *Consider $T = \{t^1, \dots, t^N\} \subset l^2$. For $i \leq N$, consider $T_i \subset l^2$. Set $\sigma = \sup_{i \leq N} \sigma(T_i)$. Consider $T' = \bigcup_{i \leq N}(t^i + T_i)$. Then we have*

$$b_0(T') \geq b_0(T) + \min_{i \leq N} b_0(T_i) - K\sigma\sqrt{\log N} .$$

COMMENT. This will be used in the case where $K\sigma\sqrt{\log N} \le (1/2)b_0(T)$ and will allow by induction the proof of strong lower bounds for Bernoulli processes.

PROOF. Set $S = \min_{i \le N} b_0(T_i)$. For $u \ge 0$, set

$$\Omega_u = \left\{\forall\, i \le N,\ \sup_{t \in T_i} \sum_{k \ge 1} \varepsilon_k t_k \ge S - K\sigma(1+u)\right\}.$$

It follows from (2.17) that $P(\Omega_u) \ge 1 - 4N\exp(-u^2)$. We observe that for $\omega \in \Omega_u$ we have

$$\sup_{t \in T'} \sum_{k \ge 1} \varepsilon_k(\omega) t_k \ge \sup_{t \in T} \sum_{k \ge 1} \varepsilon_k(\omega) t_k + S - K\sigma(1+u). \tag{2.18}$$

Define the random variable h by

$$h(\omega) = \inf\{u > 0;\ \omega \in \Omega_u\}$$

so that

$$P(h \ge u) \le 1 - P(\Omega_u) \le 4N \exp - u^2$$

and $Eh \le K\sqrt{\log N}$.

Taking expectations in (2.18), we get

$$\begin{aligned} b_0(T') &\ge b_0(T) + S - K\sigma(1 + Eh) \\ &\ge b_0(T) + S - K\sigma\sqrt{\log N}. \end{aligned}$$ □

We now combine Corollary 2.4 and Proposition 2.6.

COROLLARY 2.7. *Consider $T = \{t^1, \dots, t^N\} \subset l^2$. Suppose that for some $a, b > 0$, and all $i, j \le N$, $i \ne j$ we have*

$$\sum_{k \ge 1} \left|t_k^i - t_k^j\right|^2 \wedge a^2 \ge b^2.$$

For $i \le N$, consider $T_i \subset l^2$. Set $\sigma = \sup_{i \le N} \sigma(T_i)$. Consider $T' = \bigcup_{i \le N}(t^i + T_i)$. Then we have

$$b_0(T') \ge \frac{1}{K}\min\left(b\sqrt{\log N}, \frac{b^2}{a}\right) + \min_{i \le N} b_0(T_i) - K\sigma\sqrt{\log N}.$$

COMMENT. This will be used for b of order $a\sqrt{\log N}$ and $\sigma \ll K^2 b$.

PROOF. In view of Corollary 2.4 and Proposition 2.6, it suffices to show that if t satisfies $\sum_{k \ge 1} t_k^2 \wedge a^2 \ge b^2$, then $t \notin (b^2/4a)B_1 + (b/4)B_2$. Suppose indeed that $t_k = s_k + u_k$, where $\sum s_k^2 < b^2/4$, $\sum u_k \le b^2/4a$. Then

$$|t_k| \wedge a \le |u_k| \wedge a + |s_k|$$

so that

$$t_k^2 \wedge a^2 \leq 2\left(u_k^2 \wedge a^2 + s_k^2\right)$$

and since $\sum u_k^2 \wedge a^2 \leq a \sum u_k$ we get $\sum t_k^2 \wedge a^2 < b^2$. □

We now turn to the representation of infinitely divisible processes as a mixture of Bernoulli processes that is essential to our approach. We denote by $(\tau_k)_{k \geq 1}$ the sequence of arrival times of a Poisson process of parameter 1, that is, $\tau_k = \sum_{1 \leq i \leq k} \Gamma_i$, where the sequence Γ_i is i.i.d. and $P(\Gamma_i \geq u) = e^{-u}$. Consider now a finite index set T and a measurable function G: $\mathbb{R}^+ \times \mathbb{R}^T \to \mathbb{R}^T$. Consider a probability measure m on $\mathbb{R}^T$, and denote Lebesgue's measure on $\mathbb{R}^+$ by λ. We denote by $(Y_k)_{k \geq 1}$ an i.i.d. sequence distributed like m, by $(\varepsilon_k)_{k \geq 1}$ a Bernoulli sequence, and we assume that each of the sequences (τ_k), (Y_k), (ε_k) is independent of the others.

The following theorem has a long history, and was brought to light in the present formulation in [20].

THEOREM 2.8. *Denote by ν the image measure of $\lambda \otimes m$ by G.*

(a) *Suppose that $\int_{\mathbb{R}^T} \beta(t)^2 \wedge 1 \, d\nu(\beta) < \infty$ for all $t \in T$. Then the series $\sum_{k \geq 1} \varepsilon_k G(\tau_k, Y_k)$ converges in $\mathbb{R}^T$* a.s. *and its law is the law of the (symmetric) infinitely divisible process of Lévy measure ν.*

(b) *Suppose that ν (or, equivalently, m) is supported by $(\mathbb{R}^+)^T$, and that $\int_{\mathbb{R}^T} |\beta(t)| \wedge 1 \, d\nu(\beta) < \infty$ for all $t \in T$. Then the series $\sum_{k \geq 1} G(\tau_k, Y_k)$ converges in $\mathbb{R}^T$* a.s. *and its law is the law of the positive infinitely divisible process of Lévy measure ν.*

Suppose now that we are given ν. There are many ways to represent ν as the image of $\lambda \otimes$ m under a measurable transformation G. One way, of special convenience, was pointed out in [20] (see [21] for other applications). Consider a probability measure m such that $\nu \ll m$ (interestingly enough, the choice of m is irrelevant). Consider $g = d\nu/dm$, a Radon–Nikodym derivative of ν with respect to m, and set $G(u, \beta) = \beta 1_{[0, g(\beta)]}(u)$. Throughout the paper we set $R(u, \beta) = 1_{[0, g(\beta)]}(u)$. We observe the crucial fact that $R(\cdot, \beta)$ is nonincreasing, and that $R(\cdot, \beta) \in \{0, 1\}$. Thus, if f is a real-valued function such that $f(0) = 0$, we have $f(R(u, \beta)v) = R(u, \beta) f(v)$.

It follows from Theorem 2.8 that

$$\sum_{k \geq 1} \varepsilon_k R(\tau_k, Y_k) Y_k \tag{2.19}$$

is distributed like the infinitely divisible process $(X_t)_{t \in T}$ of Lévy measure ν. [Respectively, $\sum_{k \geq 1} R(\tau_k, Y_k) Y_k$ is distributed like the positive infinitely divisible process $(X_t)_{t \in T}$ of Lévy measure ν.] This representation will be called Rosinski's representation of the process. It should be stressed that this representation uses, in a rather subtle way, much information about ν.

For simplicity we set $R_k = R(\tau_k, Y_k)$. An important fact is that conditionally on τ_k, the sequence $(R_k Y_k)_{k\geq 1}$ is independent. In our proofs using this representation of infinitely divisible processes, we will first condition with respect to (τ_k). The influence of the sequence (τ_k) will be felt only through the following two numbers:

$$\alpha^+ = \sup_{k\geq 1} \tau_k/k, \qquad \alpha^- = \inf_{k\geq 1} \tau_k/k, \tag{2.20}$$

(that are well defined a.e. by the law of large numbers). We will then show, conditionally on (τ_k), that with large probability, the sequence $(R_k Y_k)$ has some desirable configuration. We will then condition on a sequence (Y_k) for which this configuration occurs, and we will work with the Bernoulli process $\sum \varepsilon_k R_k Y_k$. When working along that scheme, it is convenient to assume that the basic probability space is a product $\Omega_0 \times \Omega \times \Omega_1$ provided with a product probability $\Pr = P_0 \otimes P \otimes Q$, and that for $\overline{\omega} = (\omega_0, \omega, \omega_1) \in \Omega_0 \times \Omega \times \Omega_1$, we have $\tau_k(\overline{\omega}) = \tau_k(\omega_0)$, $Y_k(\overline{\omega}) = Y_k(\omega)$, $\varepsilon_k(\overline{\omega}) = \varepsilon_k(\omega_1)$. The reason for these names is that the simplest names go to the most frequently used object. We will not distinguish in which space the expectation E operator is, since this should be clear from the context. With a slight abuse of notations, E, P will also denote conditional expectation and probability in $\Omega_0 \times \Omega$ with respect to ω_0.

We now prove some simple inequalities that are basic to studying the sequence $(R_k Y_k)$ conditionally on ω_0.

The following is obvious.

LEMMA 2.9. *Consider $\alpha > 0$ and a nonincreasing function f on $\mathbb{R}^+$. Then*

$$\alpha \sum_{k\geq 1} f(\alpha k) \leq \int_0^\infty f(x)\, d\lambda(x) \leq \alpha\Big(f(0) + \sum_{k\geq 1} f(\alpha k)\Big).$$

LEMMA 2.10. *Consider a measurable function $f \geq 0$ on $\mathbb{R}^T$, such that $f(0) = 0$. Then*

$$\frac{1}{\alpha^+}\int_{\mathbb{R}^T} f(\beta)\, d\nu(\beta) - \sup f \leq \sum_{k\geq 1} E(f(R_k Y_k)) \leq \frac{1}{\alpha^-}\int_{\mathbb{R}^T} f(\beta)\, d\nu(\beta).$$

PROOF. We fix $\beta \in \mathbb{R}^T$. Since $R(\cdot, \beta)$ decreases,

$$\sum_{k\geq 1} R(\tau_k, \beta) f(\beta) \leq \sum_{k\geq 1} R(\alpha^- k, \beta) f(\beta)$$

$$\leq \frac{1}{\alpha^-}\int_0^\infty R(x, \beta) f(\beta)\, d\lambda(x),$$

by Lemma 2.9. In a similar fashion,

$$\sum_{k\geq 1} R(\tau_k,\beta)f(\beta) \geq \sum_{k\geq 1} R(\alpha^+ k,\beta)f(\beta)$$

$$\geq \frac{1}{\alpha^+}\int_0^\infty R(x,\beta)f(\beta)\,d\lambda(x) - \sup f.$$

We now apply these inequalities for $\beta = Y_1$, and we take expectations. Since the sequence (Y_k) is equidistributed, we have $E(f(R(\tau_k, Y_1)Y_1)) = E(f(R_k Y_k))$. Since Y_1 has law m, we get

$$\sum_{k\geq 1} E(f(R_k Y_k)) \leq \frac{1}{\alpha^-}\int R(x,\beta)f(\beta)\,d\lambda(x)\,dm(\beta)$$

$$\sum_{k\geq 1} E(f(R_k Y_k)) \geq \frac{1}{\alpha^+}\int R(x,\beta)f(\beta)\,d\lambda(x)\,dm(\beta) - \sup f.$$

To conclude, we note that since ν is the image of $\lambda \otimes m$ under the map $(x,\beta) \to R(x,\beta)\beta$, we have, since $f(0) = 0$,

$$\int R(x,\beta)f(\beta)\,d\lambda(x)\,dm(\beta) = \int f(R(x,\beta)\beta)\,d\lambda(x)\,dm(\beta)$$

$$= \int f(\beta)\,d\nu(\beta).$$ □

The following elementary inequality will be crucial.

PROPOSITION 2.11. *Consider independent random variables* $0 \leq W_k \leq 1$. *Then* (a) *If* $A \leq (1/4)\sum_{k\geq 1}E(W_k)$ *we have*

$$P\left(\sum_{k\geq 1} W_k \leq A\right) \leq \exp(-A).$$

(b) *If* $A \geq 4\sum_{k\geq 1}E(W_k)$ *we have*

$$P\left(\sum_{k\geq 1} W_k \geq A\right) \leq \exp\left(-\frac{A}{2}\right).$$

PROOF. (a) Observe that $\exp - x \leq 1 - x/2$ for $x \leq 1$. Thus

$$E\exp(-W_k) \leq 1 - \tfrac{1}{2}EW_k \leq \exp - \left(\tfrac{1}{2}EW_k\right)$$

and

$$E\exp\left(-\sum_{k\geq 1} W_k\right) \leq \exp - \left(\tfrac{1}{2}\sum_{k\geq 1} EW_k\right).$$

The result thus follows from the inequality

$$P(Z \leq A) \leq \exp(AE)\exp(-Z)$$

used for $Z = \sum_{k\geq 1} W_k$.

(b) Observe that $\exp x \le 1 + 2x$ for $x \le 1$; so, as before,

$$E \exp\left(\sum_{k \ge 1} W_k\right) \le \exp 2\left(\sum_{k \ge 1} EW_k\right)$$

and now use the inequality

$$P(Z \ge A) \le (\exp[-A]) E \exp Z. \qquad \square$$

PROPOSITION 2.12. (a) *Suppose that* $2\alpha^+ \le \varphi(s,t,u)$. *Then*

$$P\left(\sum_{k \ge 1} R_k |Y_k(s) - Y_k(t)|^2 \wedge \frac{1}{u^2} \le \frac{\varphi(s,t,u)}{8\alpha^+ u^2}\right) \le \exp\left(-\frac{\varphi(s,t,u)}{8\alpha^+}\right).$$

(b) *For* $A \ge 4\varphi(s,t,u)/\alpha^-$, *we have*

$$P\left(\sum_{k \ge 1} R_k |Y_k(s) - Y_k(t)|^2 \wedge \frac{1}{u^2} \ge \frac{A}{u^2}\right) \le \exp\left(-\frac{A}{2}\right).$$

PROOF. Set $W_k = R_k u^2 |Y_k(s) - Y_k(t)|^2 \wedge 1$. We use Lemma 2.10 with $f(\beta) = u^2|\beta(s) - \beta(t)|^2 \wedge 1$ and the definition of φ to get

$$\frac{1}{2\alpha^+}\varphi(s,t,u) \le \frac{1}{\alpha^+}\varphi(s,t,u) - 1 \le \sum_{k \ge 1} EW_k \le \frac{1}{\alpha^-}\varphi(s,t,u).$$

We then use Proposition 2.11. $\square$

PROPOSITION 2.13. *Assume* $H(\delta, v_0)$. *Consider* $s, t \in T$. *Set*

$$W_k = R_k |Y_k(s) - Y_k(t)|.$$

Then

$$\sum_{k \ge 1} EW_k^{1+\delta/2} 1_{\{W_k \ge u\}} \le \frac{1}{\alpha^-} K u^{1+\delta/2} \varphi(s,t,u^{-1}),$$

where K *depends on* δ, v_0 *only.*

PROOF. It follows from Lemma 2.10, used for the function $f(\beta) = |\beta(s) - \beta(t)|^{1+\delta/2} 1_{\{|\beta(s)-\beta(t)| \ge u\}}$, that

$$\sum_{k \ge 1} EW_k^{1+\delta/2} 1_{\{W_k \ge u\}} \le \frac{1}{\alpha^-} \int_{\{|\beta(s)-\beta(t)| \ge u\}} |\beta(s) - \beta(t)|^{1+\delta/2} \, d\nu(\beta)$$
$$= \frac{1}{\alpha^-} \int_u^\infty x^{1+\delta/2} \, d\mu(x),$$

where μ is the image of ν on $\mathbb{R}$ under the map $\beta \to |\beta(s) - \beta(t)|$. Integrating by parts, we get

$$\int_u^\infty x^{1+\delta/2} \, d\mu(x) = (1 + \delta/2) \int_u^\infty x^{\delta/2} \mu([x, \infty)) \, d\lambda(x) + u^{1+\delta/2} \mu([u, \infty)).$$

Condition $H(\delta, v_0)$ implies that $\mu([xv,\infty)) \le v^{-1-\delta}\mu([x,\infty))$ for $v \ge v_0$. Thus $I = \int_u^\infty x^{\delta/2}\mu([x,\infty))\,dx$ is finite. Now

$$I \le \int_u^{v_0 u} x^{\delta/2}\mu([x,\infty))\,dx + \int_{v_0 u}^{\infty} x^{\delta/2}\mu([x,\infty))\,dx$$

$$\le u(v_0 - 1)(v_0 u)^{\delta/2}\mu([u,\infty)) + v_0^{-\delta/2} I$$

by making the change of variable $x = v_0 v$. Thus

$$I \le \left(1 - v_0^{-\delta/2}\right)^{-1} u^{1+\delta/2} v_0^{1+\delta/2}\mu([u,\infty))$$

and the conclusion follows from the fact that

$$\mu([u,\infty)) = \nu(\{|\beta(s) - \beta(t)| \ge u\}) \le \int u^{-2}|\beta(s) - \beta(t)|^2 \wedge 1\,d\nu(\beta)$$

$$= \varphi(s,t,u^{-1}). \qquad \square$$

LEMMA 2.14. *Consider independent positive random variables* $(W_k)_{k\ge 1}$ *and* $0 < \delta \le 2$. *Suppose that*

$$u^{1+\delta/2}S \ge \sum_{k\ge 1} EW_k^{1+\delta/2}1_{\{W_k \ge u\}}.$$

Then

$$P\left(\sum_{k\ge 1} W_k 1_{\{W_k\ge u\}} \ge 4uS\right) \le KS^{-\delta/2}.$$

PROOF. Observe first that, since $W_k^{1+\delta/2}1_{\{W_k\ge u\}} \ge u^{\delta/2}W_k 1_{\{W_k\ge u\}}$, we have $\sum_{k\ge 1} EW_k 1_{\{W_k\ge u\}} \le uS$. There is nothing to prove if $S \le 1$, so we can suppose $S \ge 1$. We set

$$G_k = W_k 1_{\{u\le W_k\le uS\}} - E\left(W_k 1_{\{u\le W_k \le uS\}}\right).$$

Thus

$$P\left(\sum_{k\ge 1} W_k 1_{\{W_k\ge u\}} \ge 4uS\right) \le P\left(\sum_{k\ge 1} G_k \ge 3uS\right) + \sum_{k\ge 1} P(W_k \ge uS).$$

Now

$$\sum_{k\ge 1} P(W_k \ge uS) \le (uS)^{-1-\delta/2} \sum_{k\ge 1} EW_k^{1+\delta/2}1_{\{W_k\ge u\}} \le S^{-\delta/2}.$$

Also, we note that $|G_k| \le uS$, so that

$$EG_k^2 \le \|G_k\|_\infty^{1-\delta/2} E|G_k|^{1+\delta/2}$$

$$\le K(uS)^{1-\delta/2} EW_k^{1+\delta/2}1_{\{W_k\ge u\}}.$$

Thus we have

$$\begin{aligned}P\Big(\sum_{k\ge 1} G_k \ge 3uS\Big) &\le \frac{1}{(3uS)^2} E\Big(\sum_{k\ge 1} G_k\Big)^2\\ &\le \frac{1}{(3uS)^2}\sum_{k\ge 1} EG_k^2\\ &\le K(uS)^{-1-\delta/2}\sum_{k\ge 1} EW_k^{1+\delta/2}1_{\{W_k\ge u\}}\\ &\le KS^{-\delta/2}.\end{aligned}$$ □

3. The functionals θ_i and majorizing measures. A main tool in the proof of Theorem 1.1 is the introduction of a (family of) functional(s) that measures the size of a subset of T with respect to the existence of a majorizing measure. The first choice that comes to mind would be the infimum of the right-hand side of (1.4) over all possible choices of μ. The functional thus obtained, however, does not seem to have the desirable regularity properties. A difficulty of the same nature arose in the case of Gaussian processes. It was solved by defining the functional through ultrametric distances that refine the canonical metric of the process. The same idea works here, but the idea of ultrametricity will appear only implicitly through the use of partitions on X.

We consider $r \ge 8$, that is, the smallest power of 2 such that $r \ge v_0$ and

$$r^{-\delta} \le 2^{-8}. \tag{3.1}$$

The use of this condition will become apparent in Section 4. Thus, r that will remain fixed throughout the paper depends only on δ and v_0. From now on, we denote by K a constant that depends only on δ and v_0, and that may change at each occurrence. Note that r is such a constant.

For $j \in \mathbb{Z}$, $s, t \in T$, we set

$$\varphi_j(s,t) = \varphi(s,t,r^j).$$

Our first task is to observe an essential consequence of the condition $H(\delta, v_0)$.

LEMMA 3.1. *For $s, t \in T$, $j \in \mathbb{Z}$, we have*

$$\varphi_j(s,t) \le r^{-1-\delta}\varphi_{j+1}(s,t). \tag{3.2}$$

PROOF. Consider $s, t \in T$, $v \ge v_0$, $u > 0$. Denoting by μ the image measure on $\mathbb{R}^+$ of β under the map $\beta \to |\beta(s) - \beta(t)|$, we get

$$\begin{aligned}\varphi(s,t,vu) &= \int (|\beta(s) - \beta(t)|uv)^2 \wedge 1\, d\nu(\beta)\\ &= \int_{\mathbb{R}^+} (uvx)^2 \wedge 1\, d\mu(x).\end{aligned}$$

We denote by $f(x)$ the (left) derivative of $x^2 \wedge 1$, and we integrate by parts to obtain

$$\varphi(s,t,vu) = \int_{\mathbb{R}^+} uvf(uvx)\mu([x,\infty))\,d\lambda(x)$$

$$\geq v^{1+\delta}\int_{\mathbb{R}^+} uvf(uvx)\mu([vx,\infty))\,d\lambda(x),$$

where the inequality follows from condition $H(\delta, v_0)$. We now make the change of variable $y = vx$, and integrate by parts again to get

$$\varphi(s,t,vu) \geq v^{1+\delta}\varphi(s,t,u), \tag{3.3}$$

from which (3.2) follows since $r \geq v_0$. □

For a set $A \subset T$, $j \in \mathbb{Z}$ we set

$$D_j(A) = \sup_{s,t\in A} \varphi_j(s,t).$$

It thus follows from (3.2) that

$$D_j(A) \leq r^{-1-\delta}D_{j+1}(A). \tag{3.4}$$

Consider a subset U of T and $i \in \mathbb{Z}$. By a "sequence of partitions $(\mathscr{A}_j)_{j\geq i}$ of U," we mean throughout the paper an increasing sequence of finite partitions of U, such that $\mathscr{A}_i$ is trivial, that is, $\mathscr{A}_i = \{U\}$. Given $t \in U$, we denote by $A_j(t)$ the unique element of $\mathscr{A}_j$ that contains t. This notation, as well as its obvious variations, will be used throughout the paper. For example, if the sequence of partitions of U is denoted $(\mathscr{B}_j)_{j\geq i}$, $B_j(t)$ denotes the element of $\mathscr{B}_j$ that contains t.

Throughout the rest of the paper we denote by h the function given by $h(t) = \log(1/t)$ for $t \leq 1$ and $h(t) = 0$ for $t \geq 1$. Thus $h(1) = 0$.

Consider a probability measure μ on U, such that all the sets of each partition $\mathscr{A}_j$, $j \geq i$, are μ-measurable. Consider the quantity

$$\sup_{t\in U}\sum_{j\geq i} r^{-j}\Big(D_j\big(A_j(t)\big) + h\big(\mu\big(A_j(t)\big)\big)\Big). \tag{3.5}$$

We define the functional $\theta_i(U)$ as the infimum, over all possible choices of the sequence of partitions $(\mathscr{A}_j)_{j\geq i}$ and of the probability μ of the quantity (3.5). One of the basic ideas in the definition of this functional is that the two terms $D_j(A_j(t))$ and $h(\mu(A_j(t)))$ have opposite influence; indeed the first term (respectively, the second term) decreases (respectively, increases) when $\mathscr{A}_j$ increases.

Let us observe that since $\mu(U) = 1$, and $h(1) = 0$, the term in (3.5) corresponding to $j = i$ is equal to $r^{-j}D_j(U)$ for all t in U. Let us also observe that when μ is not a probability measure, but is a positive measure of mass less than or equal to 1, then the quantity (3.5) dominates $\theta_i(U)$.

We introduce this family of functionals instead of a single function for technical reasons; $\theta_i(U)$ measures what happens when "we start at level i."

The functional of true interest to us will be $\theta_i(U)$, where i is the largest for which $D_i(U) \le 1$.

The next two results relate the size of a set when measured by the functionals θ_i, and when measured by a majorizing measure. We present them to motivate our formulation of majorizing measures, and to help the reader to get acquainted with the functionals θ_i; but none of them is central to the paper.

Throughout the paper we set

$$B_j(t,a) = \{s \in T; \varphi_j(s,t) \le a\}.$$

We have observed that $\varphi_j^{1/2}$ is a distance; thus, in particular, for $s, t, u \in T$, we have

$$\varphi_j(s,u) \le 4\max(\varphi_j(s,t), \varphi_j(t,u)). \tag{3.6}$$

This implies in particular that

$$D_j(B_j(t,a)) \le 4a. \tag{3.7}$$

A noteworthy consequence of the following result is that majorizing measures can always be replaced by the considerably more convenient discrete structure associated to the functionals θ_i.

THEOREM 3.2. *Consider a probability measure μ on T and $i \in \mathbb{Z}$. For $j \ge i$, we define*

$$n(t,j) = \inf\{n \ge 0; \mu(B_j(t,2^n)) \ge e^{-2^n}\}.$$

Set

$$M = \sup_{t\in T} \sum_{j \ge i} r^{-j} 2^{n(t,j)}.$$

Then we have

$$\theta_{i-1}(T) \le K(M + r^{-i+1} D_{i-1}(T)).$$

COMMENT. This will be used in the case $D_{i-1}(T) \le 1$. Since $M \ge r^{-i}$, the result then becomes $\theta_{i-1}(t) \le KM$.

PROOF. Denote by $\mathscr{A}_{i-1}$ the trivial partition of T. We construct an increasing sequence of finite partitions $(\mathscr{A}_j)_{j\ge i}$ of T, and positive measures ν_j on T, that satisfies the following conditions:

(3.8) For $j \ge i$, $A \in \mathscr{A}_j$, there exists $n(A,j) \ge 0$ such that $n(t,j) = n(A,j)$ whenever $t \in A$.

(3.9) For $j \ge i$, $A \in \mathscr{A}_j$, we have $D_j(A) \le 2^{n(A,j)+4}$.

(3.10) For $j \ge i$, $A \in \mathscr{A}_j$, consider the set $B \in \mathscr{A}_{j-1}$ such that $A \subset B$. Then $\nu_j(A) \ge \exp(-2^{n(A,j)+1})\nu_{j-1}(B)$.

$$\nu_j(T) \le 1. \tag{3.11}$$

The construction proceeds by induction on $j \geq i$. If we set $\nu_{j-1}(T) = 1$, the case $j = i$ is identical to the general case; thus we show how to construct $\mathscr{A}_j$ and ν_j once $\mathscr{A}_{j-1}$ and ν_{j-1} have been constructed, $j \geq i$.

For $n \geq 0$, we set

$$T_n = \{t \in T; n(t, j) = n\}.$$

We can assume $D_{i-1}(T) < \infty$ (otherwise there is nothing to prove). Then $D_j(T) < \infty$, and only finitely many sets T_n are not empty.

Thus, by definition, we have

$$\forall\, t \in T_n, \qquad \mu\big(B_j(t, 2^n)\big) \geq e^{-2^n}. \tag{3.12}$$

Consider a set $X_n \subset T_n$, that is maximal subject to the condition that the sets $B_j(t, 2^n)$ are disjoint for $t \in X_n$. It follows from (3.12) that card $X_n \leq e^{2^n}$. Consider now $y \in T_n$. By the maximality of X_n, we have $B_j(t, 2^n) \cap B_j(y, 2^n) \neq \varnothing$ for some $t \in X_n$. It follows from (3.6) that $y \in B_j(t, 2^{n+2})$. Thus the sets $B_j(t, 2^{n+2})$ for $t \in X_n$ cover T_n and from (3.7) we have $D_j(B_j(t, 2^{n+2})) \leq 2^{n+4}$.

To construct $\mathscr{A}_j$ we show how to partition a given set B of $\mathscr{A}_{j-1}$. First, we partition B in the sets $B \cap T_n$, $n \geq 0$. This is a finite partition since only finitely many sets T_n are not empty. We have shown that T_n can be covered by at most $\exp 2^n$ sets A for which $D_j(A) \leq 2^{n+4}$. Thus $B \cap T_n$ can be partitioned in at most $\exp 2^n$ sets with the same property. On each of these sets $n(\cdot, j)$ is constant equal to n. For each of these sets, we decide that ν_j gives mass $\exp(-2^{n+1})\nu_{j-1}(B)$ to an arbitrary point of the set. Thus we have

$$\nu_j(A) \leq \Bigg(\sum_{n \geq 0} e^{2^n} e^{-2^{n+1}}\Bigg)\nu_{j-1}(B) = \sum_{n \geq 0} e^{-2^n}\nu_{j-1}(B) \leq \nu_{j-1}(B).$$

Since $\nu_{j-1}(T) \leq 1$, this shows that $\nu_j(T) \leq 1$, and completes the construction.

To finish the proof, set $\nu' = \sum_{j \geq i} 2^{i-j-1}\nu_j$ so that $\|\nu'\| \leq 1$, and we consider a probability measure $\nu \geq \nu'$. Consider now $t \in T$. By (3.8) we have $n(A_j(t), j) = n(t, j)$ for all $j \geq i$. Using (3.10) inductively we thus get

$$\nu_j\big(A_j(t)\big) \geq \exp\Bigg(-\sum_{i \leq k \leq j} 2^{n(t,k)+1}\Bigg)$$

and thus

$$\nu\big(A_j(t)\big) \geq \exp\Bigg(-j + i - 1 - \sum_{i \leq k \leq j} 2^{n(t,k)+1}\Bigg).$$

Thus

$$h\big(\nu\big(A_j(t)\big)\big) \leq j - i + 1 + \sum_{i \leq k \leq j} 2^{n(t,k)+1}. \tag{3.13}$$

We observe that since $M \geq r^{-i}$ we have

$$\sum_{j \geq i} r^{-j}(j - i + 1) \leq Kr^{-i} \leq KM.$$

It follows from (3.9) and (3.13) that for all $t \in T$ we have

$$\sum_{j \geq i} r^{-j}\Big(D_j\big(A_j(t)\big) + h\big(\nu\big(A_j(t)\big)\big)\Big)$$

$$\leq \sum_{j \geq i} r^{-j}\Big(2^{n(t,j)+4} + j - i + 1 + \sum_{i \leq k \leq j} 2^{n(t,k)+1}\Big)$$

$$\leq KM + K\sum_{k \geq i}\Big(\sum_{j \geq k} r^{-j}\Big)2^{n(t,k)+1} \leq KM.$$

Thus

$$\sum_{j \geq i-1} r^{-j}\Big(D_j\big(A_j(t)\big) + h\big(\nu\big(A_j(t)\big)\big)\Big) \leq r^{-i+1}D_{i-1}(T) + KM.$$

This completes the proof. □

In the converse direction, it is rather clear that the size of T, measured by the existence of a majorizing measure, is at most of the order of the size of T measured by some θ_i, at least when $D_i(T) \leq 1$. Indeed, consider a sequence of partitions $(\mathscr{A}_j)_{j \geq i}$ and a probability μ on T. Given $t \in T$, $j \geq i$, consider n such that

$$2^n \geq D_j\big(A_j(t)\big) + h\big(\mu\big(A_j(t)\big)\big).$$

Then $A_j(t) \subset B_j(t, 2^n)$, and thus $\mu(A_j(t)) \leq \mu(B_j(t, 2^n))$. Since $h(\mu(A_j(t))) \leq 2^n$, we have

$$\mu\big(B_j(t, 2^n)\big) \geq \mu\big(A_j(t)\big) \geq e^{-2^n}.$$

Thus, if we define

$$n(t,j) = \inf\big\{n \geq 0; \mu\big(B_j(t, 2^n)\big) \geq e^{-2^n}\big\},$$

we see that

$$2^{n(t,j)} \leq 1 + 2\Big(D_j\big(A_j(t)\big) + h\big(\mu\big(A_j(t)\big)\big)\Big)$$

and thus

$$\sum_{j \geq i} r^{-j}2^{n(t,j)} \leq 2r^{-i} + 2\sum_{j \geq i} r^{-j}\Big(D_j\big(A_j(t)\big) + h\big(\mu\big(A_j(t)\big)\big)\Big).$$

Our next result is a stronger form of the same principle. It asserts that majorizing measures exist on T as soon as we control $\theta_i(F)$ for all finite subsets F of T.

THEOREM 3.3. *Set*

$$M = \sup\{\theta_i(F); F \subset T, F \text{ finite}\}.$$

Then we can find a probability measure μ on T such that if for $j \geq i$, $t \in T$ we *set*

$$(3.14)\qquad n(t,j) = \inf\{n \ge 0; \mu(B_j(t,2^n)) \ge e^{-2^n}\},$$

then we have

$$\sup_{t\in T}\sum_{j\ge i} r^{-j}2^{n(t,j)} \le K(M + r^{-i}).$$

PROOF.

STEP 1. We show that given $k \ge i$, there exists a finite set $G_k \subset T$ such that

$$(3.15)\qquad \forall\, t \in T, \exists\, s \in G_k, \qquad \varphi_k(s,t) \le 2Mr^k.$$

Indeed, consider a finite subset G of T such that

$$(3.16)\qquad \forall\, s,t \in G,\, s \ne t, \qquad \varphi_k(s,t) > 2Mr^k.$$

Observe that if $A \subset G$ is not reduced to one point, then $D_k(A) > 2Mr^k$. Since $\theta_i(G) \le M$, we can find a sequence of partitions $(\mathscr{A}_j)_{j\ge i}$ of G and a probability measure ν on G such that

$$\forall\, t \in G, \qquad \sum_{j\ge i} r^{-j}\Big(D_j(A_j(t)) + h\big(\nu(A_j(t))\big)\Big) \le 2M.$$

For $j = k$, we must have $A_j(t) = \{t\}$, so that $h(\nu(\{t\})) \le 2r^kM$ and thus $\nu(\{t\}) \ge e^{-2r^kM}$, and thus card $G \le e^{2r^kM}$.

Consider now a subset G_k of T, that is maximal with respect to condition (3.16). It satisfies (3.15). Since each finite subset of G_k is of cardinality less than or equal to e^{2r^kM}, G_k is finite.

We conclude that for all $k \ge 1$, there exists a finite set $F_k \subset T$ such that

$$(3.17)\qquad \forall\, t \in T, \exists\, s \in F_k, \qquad \varphi_k(s,t) \le 1.$$

Indeed, it follows from (3.2) that one can take $F_k = G_l$ whenever $r^{(l-k)(1+\delta)} \ge 2Mr^l$.

STEP 2. Since $\theta_j(F_k) \le M$, we can find a sequence of partitions $(\mathscr{A}_j^k)_{j\ge i}$ of F_k and a probability measure ν_k on F_k such that for all $t \in F_k$ we have

$$(3.18)\qquad \sum_{j\ge i} r^{-j}\Big(D_j(A_j^k(t)) + h\big(\nu_k(A_j^k(t))\big)\Big) \le 2M.$$

Given $t \in F_k$, we denote by $n_k(t,j)$ the smallest integer n for which

$$(3.19)\qquad D_j(A_j^k(t)) + h\big(\nu_k(A_j^k(t))\big) \le 2^n.$$

Thus we have

$$(3.20)\qquad \forall\, t \in F_k, \qquad \sum_{j\ge i} r^{-j}2^{n_k(t,j)} \le 4M.$$

As observed before the proof of the theorem, we have $A_j^k(t) \subset B_j(t, 2^{n_k(t,j)})$ so that

$$\nu_k\big(B_j(t, 2^{n_k(t,j)})\big) \geq e^{-2^{n_k(t,j)}}. \tag{3.21}$$

We consider now a map u_j from T into F_j such that $\varphi_j(s, u_j(s)) \leq 1$ for all $s \in T$. Consider $s \in T$, $k \geq i$. Set $t = u_k(s)$, so that $\varphi_k(s,t) \leq 1$, and $\varphi_j(s,t) \leq 1$ for $i \leq j \leq k$. Consider now $n \geq 0$, $x \in T$. Since $\varphi_j(x, u_j(x)) \leq 1$, $\varphi_j(s,t) \leq 1$, it follows from (3.6) that

$$\begin{aligned} x \in B_j(t, 2^n) &\Rightarrow \varphi_j\big(t, u_j(x)\big) \leq 2^{n+1} + 2 \\ &\Rightarrow \varphi_j\big(s, u_j(x)\big) \leq 2^{n+2} + 6 \leq 2^{n+5}. \end{aligned}$$

Thus

$$u_j\big(B_j(t, 2^n)\big) \subset B_j(s, 2^{n+5}). \tag{3.22}$$

Denote by μ_j^k the image of ν_k by the map u_j. Combining (3.21) and (3.22) we have, for all $j \leq k$,

$$\mu_j^k\big(B_j(s, 2^{n_k(t,j)+5})\big) \geq e^{-2^{n_k(t,j)}}. \tag{3.23}$$

Step 3. The point of replacing ν_k by μ_j^k is that μ_j^k is supported by the finite set G_j that does not depend on k, and that we can now use a limit argument. Consider an ultrafilter $\mathscr{U}$ on $\mathbb{N}$. We set for $s \in T$:

$$\mu_j = \lim_{k \to \mathscr{U}} \mu_j^k; \qquad m(s,j) = \lim_{k \to \mathscr{U}} n_k(u_k(s), j).$$

It should be clear from (3.20) and (3.23) that we have

$$\forall s \in T, \qquad \sum_{j \geq i} r^{-j} 2^{m(s,j)} \leq 4M, \tag{3.24}$$

$$\mu_j\big(B_j(s, 2^{m(s,j)+5})\big) \geq e^{-2^{m(s,j)}}. \tag{3.25}$$

Consider the probability measure $\mu = \sum_{j \geq i} 2^{i-j-1} \mu_j$. It follows from (2.25) that

$$\mu\big(B_j(s, 2^{m(s,j)+5})\big) \geq \exp\big(-j + i - 2^{m(s,j)}\big).$$

If we recall the definition of $n(s,j)$ given in (3.14), we see that $n(s,j) \leq n$ whenever

$$2^{m(s,j)+5} \leq 2^n; \qquad 2^{m(s,j)} + j - i + 1 \leq 2^n.$$

One can find n that satisfies these conditions, and moreover

$$2^n \leq 2^{m(s,j)+5} + 2(j - i + 1).$$

It then follows from (3.24) that

$$\forall s \in T, \qquad \sum_{j \geq i} r^{-j} 2^{n(s,j)} \leq K(M + r^{-i}). \qquad \square$$

4. The basic separation step. In this section we will study $\theta_i(U)$ for finite subsets U of T. It will be convenient (but unessential) to use the fact that when U is finite, there exists a sequence $(\mathscr{A}_j)_{j\geq i}$ of partitions of U and a probability measure μ on U such that the quantity (3.5) equals $\theta_i(U)$. We will say that the sequence $(\mathscr{A}_j)_{j\geq i}$ and μ *achieve* $\theta_i(U)$.

LEMMA 4.1. *If* $V\subset U$, *then* $\theta_i(V)\leq\theta_i(U)-r^{-i}D_i(U)+r^{-i}D_i(V)$.

PROOF. Consider a sequence $(\mathscr{A}_j)_{j\geq i}$ of partitions of U and a probability measure μ on U that achieve $\theta_i(U)$. We define a map f from U to V as follows. Given $t\in U$, if $A_j(t)\cap V\neq\varnothing$ for all $j\geq i$, since U is finite, we have $V\cap\bigcap_{j\geq i}A_j(t)\neq\varnothing$, and we take $f(t)\in V\cap\bigcap_{j\geq i}A_j(t)$. Otherwise there exists a largest $j\geq i$ such that $V\cap A_j(t)\neq\varnothing$, and we take $f(t)\in V\cap A_j(t)$.

The basic property of f is that $f(U)\subset V$ and $f(A_j(t))\subset A_j(t)$ whenever $t\in V$. Indeed, if $s\in A_j(t)$, then $A_j(s)=A_j(t)$, so that $V\cap A_j(s)\neq\varnothing$, and by definition of f we have $f(s)\in A_j(s)=A_j(t)$.

Consider now the probability ν on V which is the image of μ by f, that is, $\nu(B)=\mu(f^{-1}(B))$ for $B\subset V$. For $t\in V$, we have $f^{-1}(A_j(t)\cap V)\supset A_j(t)$, and thus

$$\nu\big(A_j(t)\cap V\big)\geq\mu\big(A_j(t)\big). \tag{4.1}$$

The sequence of partitions $(\mathscr{A}_j)_{j\geq i}$ of U induces by restriction a sequence of partitions $(\mathscr{B}_j)_{j\geq i}$ of V. For $t\in V$, we have $B_j(t)=A_j(t)\cap V$. Since $D_j(B_j(t))\leq D_j(A_j(t))$, it follows from (4.1) that for all $t\in V$ we have

$$\sum_{j\geq i+1}r^{-j}\Big(D_j\big(B_j(t)\big)+h\Big(\nu\big(B_j(t)\big)\Big)\Big)$$

$$\leq\sum_{j\geq i+1}r^{-j}\Big(D_j\big(A_j(t)\big)+h\Big(\mu\big(A_j(t)\big)\Big)\Big)\leq\theta_i(U)-r^{-i}D_i(U).$$

The result follows. □

REMARK. The equation above shows that

$$\theta_i(Y)\leq\sup_{t\in V}\sum_{j\geq i}r^{-j}\Big(D_j\big(B_j(t)\big)+h\Big(\nu\big(B_j(t)\big)\Big)\Big)$$

$$\leq r^{-i}D_i(V)+\sup_{t\in V}\sum_{j\geq i+1}r^{-j}\Big(D_j\big(A_j(t)\big)+h\Big(\mu\big(A_j(t)\big)\Big)\Big).$$

LEMMA 4.2. $\theta_{i+1}(U)\leq\theta_i(U)+r^{-i-1}D_{i+1}(U)$.

PROOF. Consider an increasing sequence $(\mathscr{A}_j)_{j\geq i}$ of partitions of U and a probability measure μ on U that achieve $\theta_i(U)$. We set $\mathscr{B}_j=\mathscr{A}_j$ for $j\geq i+2$, and take for $\mathscr{B}_{i+1}$ the trivial partition of U. Then, for $t \in U$ we have

$$\sum_{j \ge i+1} r^{-j}\Big(D_j\big(B_j(t)\big) + h\big(\mu\big(B_j(t)\big)\big)\Big)$$
$$= r^{-i-1}D_{i+1}(U) + \sum_{j \ge i+2} r^{-j}\Big(D_j\big(A_j(t)\big) + h\big(\mu\big(A_j(t)\big)\big)\Big)$$
$$\le r^{-i-1}D_{i+1}(U) + \theta_i(U). \qquad \square$$

The next lemma should be compared with [25], Lemma 7.

LEMMA 4.3. *Suppose that we are given a finite covering $U = \bigcup_{l \in L} U_l$ of U. Consider numbers $w(U_l) \ge 0$, [we shall call $w(U_l)$ the weight of U_l], such that $\sum_{l \in L} w(U_l) \le 1$. Then*

$$\theta_i(U) \le r^{-i}D_i(U) + \max_{l \in L}\big(\theta_{i+1}(U_l) + 2r^{-i-1}h\big(w(U_l)\big)\big).$$

PROOF. It follows from Lemma 4.1 that we can actually assume that the sets $(U_l)_{l \in L}$ form a partition of U. For $l \in L$, consider a sequence of partitions $(\mathscr{A}_j^l)_{j \ge i+1}$ of U_l and a probability measure μ_l on U_l that achieve $\theta_{i+1}(U_l)$. We define the sequence of partitions $(\mathscr{A}_j)_{j \ge i}$ of U as follows: $\mathscr{A}_i$ is the trivial partition of U, and for $j \ge i+1$, $\mathscr{A}_j$ is generated in the obvious manner by the partitions $(\mathscr{A}_j^l)$ of U_l. Thus for $t \in U_l$ and $j \ge i+1$, we have $A_j(t) = A_j^l(t)$. We set $\mu' = \sum_{l \in L} w(U_l)\mu_l$. This is a positive measure, and $\|\mu'\| \le 1$, so that we can consider a probability measure $\mu \ge \mu'$.

For $t \in U_l$, we have

$$\sum_{j \ge i} r^{-j}\Big(D_j\big(A_j(t)\big) + h\big(\mu\big(A_j(t)\big)\big)\Big)$$
$$\le r^{-i}D_i(U) + \sum_{j \ge i+1} r^{-j}\Big(D_j\big(A_j^l(t)\big) + h\big(w_l\mu_l\big(A_j^l(x)\big)\big)\Big)$$
$$\le r^{-i}D_i(U) + \theta_{i+1}(U^l) + \Big(\sum_{j \ge i+1} r^{-j}\Big)h(w_l)$$

since $h(xy) \le h(x) + h(y)$. Since we assume $r \ge 2$, we have $\sum_{j \ge i+1} r^{-j} \le 2r^{-i-1}$. The result follows. $\square$

Throughout the rest of the paper, we set

$$\xi = 2^{-3}; \qquad \eta = 2^{-7}; \qquad \gamma = \tfrac{1}{8}. \tag{4.2}$$

The reason why we like to give names to these numerical values is that using these values might obscure the role of these quantities in the proofs; and we feel that it is clearer to think of these quantities as parameters, and to point out in the course of the proofs why the choices of (4.2) are relevant.

The following is an essential ingredient.

PROPOSITION 4.4. *There exists a constant* K_5 *that depends only on* δ, v_0, *with the following property. Consider* $U \subset T$, *and let* $p \geq 6$ *be such that* $D_i(U) \leq 2^{p+1}$. *Then one of the following occurs.*

CASE A. *We can find* $V \subset U$, $n \geq p$ *such that*

$$D_{i+1}(V) \leq 2^{n+1}, \tag{4.3}$$

$$\theta_i(U) - \theta_{i+1}(V) \leq K_5 r^{-i-1} 2^n, \tag{4.4}$$

$$\forall\, t \in V, \quad \theta_{i+1}(V \cap B_{i+1}(t, \eta 2^n)) < \theta_{i+1}(V) - 2K_5 r^{-i-1} 2^n. \tag{4.5}$$

CASE B. *We can find* $V \subset U$ *such that* $D_{i+1}(V) \leq 2^p$ *and*

$$\theta_{i+1}(V) \geq \theta_i(U) - r^{-i} 3 \cdot 2^p. \tag{4.6}$$

CASE C. *We can find* $n \geq p$, N *such that* $\log N \geq \gamma 2^n$ *and points* $(t_l)_{l \leq N}$ *of* U *in such a way that*

$$k \neq l \Rightarrow \varphi_{i+1}(t_k, t_l) \geq \xi 2^n, \tag{4.7}$$

$$\forall\, l \leq N, \quad \theta_{i+1}(U \cap B_{i+1}(t_l, \eta 2^n)) \geq \theta_i(U) - 2K_5 r^{-i-1} 2^n. \tag{4.8}$$

COMMENT 1. One should note in Case C the relationship between the logarithm of the number of pieces constructed, and how well these points are separated by φ_{i+1} (both of the same order). This is an essential feature of the present theory, and possibly the major difference with the work of [25].

COMMENT 2. What we want to achieve is Case C; in that case we find $\exp(\gamma 2^n)$ sets $U \cap B_{i+1}(t_l, \eta 2^n)$ which are separated for φ_{i+1} [as follows from (4.7) and (3.6)] and for which we do not lose more on θ_i than the correct order $r^{-i} 2^n$. Unfortunately this cannot always be achieved in one step (this phenomenon is not an artifact of the method of proof, but an essential feature of the structure we investigate). If we fail to achieve Case C, either we are in Case B which means that there exists a rather small subset V of U for which we do not lose too much on θ_i, or in Case A, which means that we have a subset V of U, for which we do not lose too much on θ_i, which has the very strong additional property (4.5) [note that there is of factor 2 in front of K_5 in (4.5) but not in (4.4)].

PROOF. We will show that $K_5 = 4r$ works. Consider a sequence of partitions $(\mathscr{A}_j)_{j \geq i}$ and a probability μ on U that achieve $\theta_i(U)$. We set

$$W_0 = \{A \in \mathscr{A}_{i+1};\, D_{i+1}(A) + h(\mu(A)) \leq 2^p\}$$

and for $n \geq p$ we define

$$W_n = \{A \in \mathscr{A}_{i+1}; 2^n < D_{i+1}(A) + h(\mu(A)) \leq 2^{n+1}\}.$$

For $A \in W_0$, we have $\mu(A) \geq \exp - 2^p$, so that card $W_0 \leq \exp 2^p$. In a similar fashion we have card $W_n \leq \exp 2^{n+1}$.

For $n \geq p$, we denote by U_n the union of W_n. For each $n \geq p$, we perform in U_n the following construction by induction. We set $T_{n,0} = \varnothing$. Having constructed $T_{n,0}, \dots, T_{n,k}$, we consider a point $t_{n,k+1}$ in $U_n \setminus \bigcup_{l \leq k} T_{n,l}$ such that

$$\theta_{i+1}(U_n \cap B_{i+1}(t_{n,k+1}, \eta 2^n))$$

is as large as possible. We then set

$$T_{n,k+1} = U_n \cap B_{i+1}(t_{n,k+1}, \xi 2^n) \setminus \bigcup_{l \leq k} T_{n,l}.$$

The construction continues as long as $U_n \setminus \bigcup_{l < k} T_{n,l}$ is not empty. We denote by k_n the largest integer k for which $t_{n,k}$ is defined and satisfies

$$\theta_{i+1}(U_n \cap B_{i+1}(t_{n,k}, \eta 2^n)) \geq \theta_k(U) - 2K_5 r^{-i-1} 2^n. \tag{4.9}$$

If either $U_n = \varnothing$ or (4.9) fails for $k = 1$, we set $k_n = 0$. For $n \geq p$, $A \in W_n$, we set

$$A' = A \setminus \bigcup_{l \leq k_n} T_{n,l}.$$

When A' is not empty, we see by (4.9) that

$$\theta_{i+1}(U_n \cap B_{i+1}(t_{n,k+1}, \eta 2^n)) < \theta_i(U) - 2K_5 r^{-i-1} 2^n.$$

By definition of $t_{n,k+1}$, we have

$$\forall\, t \in A', \qquad \theta_{i+1}(U_n \cap B_{i+1}(t, \eta 2^n)) < \theta_i(U) - 2K_5 r^{-i-1} 2^n. \tag{4.10}$$

The proof proceeds by contradiction. We assume that none of the cases occur, and we will conclude using Lemma 4.2 that $\theta_i(U) < \theta_i(U)$, a contradiction.

Consider $n \geq p$, and $A \in W_n$. We have $D_{i+1}(A) \leq 2^{n+1}$. It follows from (4.10) that

$$\forall\, t \in A', \qquad \theta_{i+1}(A' \cap B_{i+1}(t, \eta 2^n)) < \theta_i(U) - 2K_5 r^{-i-1} 2^n. \tag{4.11}$$

We assume that Case A does not occur. Thus $V = A'$ must fail one of (4.3) to (4.5). Since it satisfies (4.3) and (4.5), it must fail (4.4) and thus

$$\theta_{i+1}(A') < \theta_i(U) - K_5 r^{-i-1} 2^n. \tag{4.12}$$

Consider $A \in W_0$; we have $D_{i+1}(A) \leq 2^p$. Since we assume that Case B fails we must have

$$\theta_{i+1}(A) < \theta_i(U) - r^{-i} 3 \cdot 2^p. \tag{4.13}$$

We observe that by construction, for $l < k$ we have $t_{n,k} \notin B_{i+1}(t_{n,l}, \xi 2^n)$, so that $\varphi_{i+1}(t_{n,k}, t_{n,l}) > \xi 2^n$. By construction the sequence

$$\theta_{i+1}(U_n \cap B_{i+1}(t_{n,k}, \eta 2^n)), \qquad k = 1, 2, \dots$$

decreases. Thus, by the definition of k_n, for $k \le k_n$ we have

$$\theta_{i+1}(U_n \cap B_{i+1}(t_{n,k}, \eta 2^n)) \ge \theta_i(U) - 2K_5 r^{-i-1} 2^k.$$

Since we assume that Case C does not occur, this means that $\log k_n \le \gamma 2^n$. □

LEMMA 4.5. *For all $t \in U_n$ we have*

$$\theta_{i+1}(U_n \cap B_{i+1}(t, \xi 2^n)) \le \theta_i(U) - r^{-i} D_i(U) - 2^{n-1} r^{-i-1}.$$

PROOF. For $t \in U$, we have

$$\sum_{j \ge i} r^{-j}\Big(D_j\big(A_j(t)\big) + h\Big(\mu\big(A_j(t)\big)\Big)\Big) \le \theta_i(U).$$

The term for $j = i$ is $r^{-i} D_i(U)$. If $t \in U_n$, the term for $j = i + 1$ is greater than or equal to $r^{-i-1} 2^n$, so that we have

$$\sum_{j \ge i+2} r^{-j}\Big(D_j\big(A_j(t)\big) + h\Big(\mu\big(A_j(t)\big)\Big)\Big) \le \theta_i(U) - r^{-i} D_i(U) - r^{-i-1} 2^n.$$

It follows from the remark after Lemma 4.1 that

$$\theta_{i+1}(U_n \cap B_{i+1}(t, \xi 2^n)) \le \theta_i(U) - r^{-i} D_i(U) - r^{-i-1} 2^n + r^{-i-1} D_{i+1}(B_{i+1}(t, \xi 2^n)).$$

Now (3.7) shows that

$$D_{i+1}(B_{i+1}(t, \xi 2^n)) \le 4\xi 2^n = 2^{n-1}.$$ □

Since $T_{n,k} \subset U_n \cap B_{i+1}(t_{n,k}, \xi 2^n)$, it follows from Lemma 4.1 that

(4.14) $$\forall\, n \ge p, \forall\, k \le k_n, \qquad \theta_{i+1}(T_{n,k}) \le \theta_i(U) - r^{-i} D_i(U) - r^{-i-1} 2^{n-1}.$$

We consider the covering of U given by the sets $A \in W_0$, the sets $T_{n,k}$ for $n \ge p$, $k \le k_n$, and the sets A' for $A \in W_n$, $n \ge p$. We define now the appropriate weights of these sets, in order to apply Lemma 4.3.

For $A \in W_0$, we set $w(A) = \exp(-2^p - 1)$. For $n \ge p$, $k \le k_n$, we set $w(T_{n,k}) = \exp(-n - \gamma 2^n)$; and for $A \in W_n$, we set $w(A') = \exp(-n - 2^{n+1})$. Since card $W_0 \le \exp 2^p$, we have

$$\sum_{A \in W_0} w(A) \le 1/e.$$

Since $k_n \le \exp(\gamma 2^n)$, we have

$$\sum_{n \ge p,\, k \le k_n} w(T_{n,k}) \le \sum_{n \ge p} e^{-n} \le e^{-p+1}.$$

Since card $W_n \le \exp 2^{n+1}$, we have

$$\sum_{A \in W_n,\, n \ge p} w(A') \le \sum_{n \ge p} e^{-n} \le e^{-p+1}.$$

Since $p \ge 6$, the total sum of the weights is then less than or equal to 1, and we can use Lemma 4.3 to obtain that

$$\theta_i(U) \le r^{-i} D_i(U) + \max(G_1, G_2, G_3), \tag{4.15}$$

where

$$G_1 = \max_{A \in W_0} \left(\theta_{i+1}(A) + r^{-i-1}(2 + 2^{p+1})\right),$$

$$G_2 = \max_{n \ge p,\, k \le k_n} \left(\theta_{i+1}(T_{n,k}) + 2r^{-i-1}(n + \gamma 2^n)\right),$$

$$G_3 = \max_{n \ge p,\, A \in W_n} \left(\theta_{i+1}(A') + 2r^{-i-1}(n + 2^{n+1})\right).$$

It follows from (4.13) that, since $r \ge 4$, $p \ge 1$,

$$\begin{aligned} G_1 &< \theta_i(U) - r^{-i} 3 \cdot 2^p + r^{-i-1}(2 + 2^{p+1}) \\ &\le \theta_i(U) - r^{-i} 2^{p+1} \le \theta_i(U) - r^{-i} D_i(U). \end{aligned}$$

It follows from (4.14) that

$$G_2 \le \max_{n \ge p} \left(\theta_i(U) - r^{-i} D_i(U) - r^{-i-1}(2^{n-1} - 2n - \gamma 2^{n+1})\right). \tag{4.16}$$

Since $\min_{n \ge 6} 2^{n-2} - 2n > 0$ and since $\gamma = 1/8$, we get

$$G_2 < \theta_i(U) - r^{-i} D_i(U). \tag{4.17}$$

From (4.12) we have

$$\begin{aligned} G_3 &\le \max_{n \ge p} \left(\theta_i(U) - K_5 r^{-i-1} 2^n + r^{-i-1}(2n + 2^{n+2})\right) \\ &= \theta_i(U) - \min_{n \ge p} r^{-i-1}\left(K_5 2^n - 2n - 2^{n+2}\right). \end{aligned}$$

We recall that $K_5 = 4r$. Since $r \ge 8$, we have $4r2^n - 2n - 2^{n+2} \ge 3r2^n$ for $n \ge 1$. We thus get

$$G_3 \le \theta_i(U) - 3r^{-i} 2^p < \theta_i(U) - r^{-i} D_i(U).$$

If we recall (4.15)–(4.17), we see that we have reached the desired contradiction that $\theta_i(U) < \theta_i(U)$. □

We now show why when Case A occurs, condition (4.5) is precious information.

Corollary 4.6. *Consider $U \subset T$, and $p \ge 6$ such that $D_i(U) \le 2^{p+1}$. Suppose that*

$$\forall\, t \in U, \qquad \theta_i(U \cap B_i(t, \eta 2^p)) < \theta_i(U) - 2K_5 r^{-i} 2^p. \tag{4.18}$$

Then one of the following occurs:

CASE A. There exists n with $2^n \geq r2^{p+1}$ and a subset V of U such that $D_{i+1}(V) \leq 2^{n+1}$ and

$$\theta_{i+1}(V) \geq \theta_i(U) - K_5 r^{-i-1} 2^n, \tag{4.19}$$

$$\forall\, t \in V, \quad \theta_{i+1}(V \cap B_{i+1}(t, \eta 2^n)) < \theta_{i+1}(V) - 2K_5 r^{-i-1} 2^n. \tag{4.20}$$

CASE B. There exists n such that $2^n \geq r2^{p+1}$, an integer N such that $\log N \geq \gamma 2^n$, and points $(t_l)_{l \leq N}$ of U that satisfy

$$k \neq l \Rightarrow \varphi_{i+1}(t_k, t_l) \geq \xi 2^n, \tag{4.21}$$

$$\forall\, l \leq N, \quad \theta_{i+1}(U \cap B_{i+1}(t_l, \eta 2^n)) \geq \theta_i(U) - 2K_5 r^{-i-1} 2^n. \tag{4.22}$$

COMMENT. Under condition (4.18), either we achieve separation (Case B), or we find ourselves in the same situation again (Case A), but in a more dramatic fashion, since $r^{-i-1}2^n \geq 2(r^{-i}2^p)$.

PROOF. For $s \in B_{i+1}(t, r2^{p+1})$, we have from (3.2) that

$$\varphi_i(s,t) \leq r^{-1-\delta}\varphi_{i+1}(s,t) \leq r^{-\delta}2^{p+1} \leq \eta 2^p$$

by (3.1) and since $\eta = 2^{-7}$. [This point is a crucial use of condition $H(\delta, \nu_0)$.] Thus

$$B_{i+1}(t, r2^{p+1}) \subset B_i(t, \eta 2^p).$$

It thus follows from (4.18) that if a subset V of U satisfies $D_{i+1}(V) \leq r2^{p+1}$, we must have $\theta_i(V) < \theta_i(U) - K_5 r^{-i} 2^{p+1}$. By Lemma 4.2 we have, since $K_5 = 4r$,

$$\begin{aligned}\theta_{i+1}(V) &\leq \theta_i(V) + r^{-i-1}D_{i+1}(V)\\ &< \theta_i(U) - K_5 r^{-i}2^{p+1} + r^{-i}2^{p+1}\\ &\leq \theta_i(U) - r^{-i}(3r2^{p+1}).\end{aligned}$$

Since we have assumed that r is a power of 2, there exists q such that $2^q = r2^{p+1}$, and $q \geq 6$. We have shown that

$$D_{i+1}(V) \leq 2^q \Rightarrow \theta_{i+1}(V) < \theta_i(U) - 3r^{-i}2^q. \tag{4.23}$$

Since $D_i(U) \leq 2^{p+1} \leq 2^{q+1}$, we can use Proposition 4.4 for q instead of p; and (4.23) means that Case B of that proposition cannot occur. Thus either Case A or C of that proposition occurs, which is the content of Corollary 4.6. □

We are now ready to obtain the basic separation step in the case where $\theta_i(U) \gg r^{-i}D_i(U)$.

THEOREM 4.7. *There exists a constant K_6, depending only on δ, v_0, with the following property. Consider $U \subset T$, $i \in \mathbb{Z}$. Assume that $\theta_i(U) \geq$*

$K_6 r^{-i}(D_i(U)+1)$. Then we can find $i' > i$, $n \in \mathbb{N}$, a subset V of U, points $(t_l)_{l \le N}$ of V, where $\log N \ge \gamma 2^n$, *such that*

$$D_{i'-1}(V) \le 2^{n+1}, \tag{4.24}$$

$$l \ne k \Rightarrow \varphi_{i'}(t_l, t_k) \ge \xi 2^n, \tag{4.25}$$

$$\forall\, l \le N, \quad \theta_{i'}(V \cap B_{i'}(t_l, \eta 2^n)) \ge \theta_i(U) - K_6\big(r^{-i}(D_i(U)+1) + r^{-i'}2^n\big). \tag{4.26}$$

PROOF. The idea of the proof is to iterate the application of Proposition 4.4 and Corollary 4.6. Separation will be obtained at the last step; the problem is of course to control enough of the loss on the functionals θ_i at the different stages to recover (4.26).

Consider the smallest $p \ge 6$ such that $D_i(U) \le 2^{p+1}$.

STEP 1. We construct by induction a decreasing sequence $U_i = U \supset U_{i+1} \supset \cdots \supset U_{i_1}$ of subsets of U that satisfy the following properties:

$$\text{for } i \le j \le i_1, \quad D_j(U_j) \le 2^{p+1}, \tag{4.27}$$

$$\text{for } i \le j < i_1, \quad \theta_{j+1}(U_{j+1}) \ge \theta_j(U_j) - 3r^{-j}2^p. \tag{4.28}$$

This construction is done by induction; it continues as long as possible. It might happen that the first step of the construction is not possible. In that case we set $i_1 = i$.

We now show that the construction has to stop eventually, provided K_6 is large enough. Suppose, for contradiction, that the construction never stops. Since U is finite, the sets U_j are eventually equal to a given fixed V. We then have $D_j(V) \le 2^{p+1}$ for j large enough. From (3.4) we see that $D_j(V) = 0$ for all j. This implies that $\theta_j(V) = 0$ (as is seen using a sequence consisting only of the trivial partitions of V). Summation of the inequalities (4.28) for $j > i$ then yields $\theta_i(U) < 6r^{-i}2^p$. The definition of p shows that $2^p \le 2^6(D_i(U) + 1)$. We can and do assume $K_6 \ge 6 \cdot 2^6$. We thus get $\theta_i(U) < K_6(D_i(U)+1)$, contrary to our assumption.

STEP 2. We now construct by induction a decreasing sequence $U_{i_1} \supset U_{i_1+1} \supset \cdots \supset U_{i_2}$ of subsets of U_{i_1}, and integers $n_{i_1}, n_{i_1+1}, \dots, n_{i_2}$, such that the following conditions hold for $i_1 < j \le i_2$:

$$D_j(U_j) \le 2^{n_j+1}, \tag{4.29}$$

$$\theta_{j-1}(U_{j-1}) - \theta_j(U_j) \le K_5 r^{-j} 2^{n_j}, \tag{4.30}$$

$$\forall\, t \in U_j, \quad \theta_j\big(U_j \cap B_j(t, \eta 2^{n_j})\big) < \theta_j(U_j) - 2K_5 r^{-j}2^{n_j}, \tag{4.31}$$

$$n_{i_1+1} \ge p; \quad \forall\, j, i_1 + 1 \le j < i_2, \quad 2^{n_{j+1}} \ge r 2^{n_j+1}. \tag{4.32}$$

The construction is done by induction as long as possible. It might happen that U_{i_1+1} cannot be found, in which case we set $i_2 = i_1$. It is clear from (4.31) and (4.32) that the construction has to stop eventually, since $r^{-j-1}2^{n_{j+1}} \geq 2(r^{-j}2^{n_j})$.

For clarity we distinguish cases, and do not avoid redundancy.

CASE 1. We have $i_1 = i_2 = i$. We appeal to Proposition 4.4. We see that Case B cannot occur, since otherwise we could perform the first step of the construction in Step 1. We see that Case A cannot occur, for otherwise we could perform the first step of the construction in Step 2. Thus Case C must occur. This is exactly the conclusion we want to establish with $V = U$, $i' = i + 1$.

CASE 2. We have $i_2 = i_1 > i$. We apply Proposition 4.4 to U_{i_1}. We argue as before that Case B cannot occur, for otherwise we could perform the construction of Step 1 past i_1. Also, Case A cannot occur, for otherwise we would be able to perform the first step of the construction in Step 2. Thus Case C must occur, that is we can find $n \geq p$ and N with $\log N \geq \gamma 2^n$ and points $(t_l)_{l \leq N}$ of U_{i_1}, such that

$$(4.33)\qquad l \neq k \Rightarrow \varphi_{i_1+1}(t_k, t_l) \geq \xi 2^n,$$

$$(4.34)\qquad \forall\, l \leq N,\qquad \theta_{i_1+1}\big(U_{i_1} \cap B_{i_1+1}(t_l, \eta 2^n)\big) \geq \theta_{i_1}(U_{i_1}) - 2K_5 r^{-i_1-1}2^n.$$

We set $V = U_{i_1}$, $i' = i_1 + 1$. Thus

$$D_{i'-1}(V) = D_{i_1}(U_{i_1}) \leq 2^{p+1} \leq 2^{n+1}.$$

It remains to prove (4.26). We sum the inequalities (4.28) for $i \leq j < i_1$ to obtain

$$(4.35)\qquad \theta_{i_1}(U_{i_1}) \geq \theta_i(U) - 6r^{-i}2^p.$$

Combining with (4.34) we get

$$(4.36)\qquad \forall\, l \leq N,\qquad \theta_{i'}(V \cap B_{i'}(t_l, \eta 2^n)) \geq \theta_i(U) - 6r^{-i}2^p - 2K_5 r^{-i}2^n.$$

We have already observed that $2^p \leq 2^6(D_i(U) + 1)$. Since we can and do assume $K_6 \geq 2K_5$, $K_6 \geq 6 \cdot 2^6$, (4.26) follows from (4.36).

CASE 3. We have $i_2 > i_1 = i$. This case is similar to but simpler than the next case so we do not detail it.

CASE 4. We have $i_2 > i_1 > i$. Since $i_2 > i_1$, we can use (4.31) for $j = i_2$. This gives

$$\forall\, t \in U_{i_2},\qquad \theta_{i_2}\big(U_{i_2} \cap B_{i_2}(t, \eta 2^{n_{i_2}})\big) < \theta_{i_2}(U_{i_2}) - 2K_5 r^{-i_2}2^{n_{i_2}}.$$

This is (4.18) used for U_{i_2} instead of U, i_2 instead of i, n_{i_2} instead of p. Since $D_{i_2}(U_{i_2}) \leq 2^{n_{i_2}+1}$ by (4.29), we can use Corollary 4.6. The Case A of that corollary cannot occur, for otherwise the construction of Step 2 would not stop at i_2 since we assume that the construction continues as long as possible. Thus Case B must occur: that is we can find n such that $2^n \geq r2^{n_{i_2}+1}$, an integer N such that $\log N \geq \gamma 2^n$, and points $(t_l)_{l \leq N}$ that satisfy

$$k \neq l \Rightarrow \varphi_{i_2+1}(t_k, t_l) \geq \xi 2^n, \tag{4.37}$$

$$\forall\, l \leq N, \quad \theta_{i_2+1}\big(U_{i_2} \cap B_{i_2+1}(t_l, \eta 2^n)\big) \geq \theta_{i_2}\big(U_{i_2}\big) - 2K_5 r^{-i_2-1} 2^n. \tag{4.38}$$

We set $V = U_{i_2}$, $i' = i_2 + 1$. Thus

$$D_{i'-1}(V) = D_{i_2}\big(U_{i_2}\big) \leq 2^{n_{i_2}+1} \leq 2^{n+1}.$$

It remains to prove (4.26). We have already observed that (4.32) implies that

$$\forall\, j \geq i_1 + 1, \quad r^{-j-1} 2^{n_{j+1}} \geq 2\big(r^{-j} 2^{n_j}\big). \tag{4.39}$$

We sum the relations (4.30) for $i_1 < j \leq i_2$ to get

$$\theta_{i_1}\big(U_{i_1}\big) - \theta_{i_2}\big(U_{i_2}\big) \leq K_5 \sum_{i_1 < j \leq i_2} 2^{n_j} r^{-j}.$$

Combining with (4.39) gives

$$\theta_{i_1}\big(U_{i_2}\big) \geq \theta_{i_1}\big(U_{i_1}\big) - 2K_5 r^{-i_2} 2^{n_{i_2}}.$$

Combining with (4.35) and (4.38), and using the fact that $2^n \geq r 2^{n_{i_2}}$, we get

$$\begin{aligned} \forall\, l \leq N, \quad & \theta_{i'}\big(V \cap B_{i'}(t_l, \eta 2^n)\big) \\ & \geq \theta_i(U) - 2K_5 r^{-i_2-1} 2^n - 2K_5 r^{-i_2} 2^{n_{i_2}} - 6 r^{-i} 2^p \\ & \geq \theta_i(U) - 4K_5 r^{-i'} 2^n - 6 r^{-i} 2^p \end{aligned}$$

and we conclude as in Case 2. □

5. Indexed trees. The results of the previous section will have to be applied recursively to construct appropriate families of sets in U. We now introduce the vocabulary necessary to describe these constructions. A *tree of subsets* of U is a (finite) family $\mathscr{F}$ of subsets such that any two subsets are either disjoint or comparable for the inclusion. The tree has a largest element, namely U. The main reason for which we include U in $\mathscr{F}$ is to make legitimate the name of tree; but actually we will be interested mainly in $\mathscr{F}_- = \mathscr{F} \setminus \{U\}$. An *indexed* tree is a tree such that to each element A of $\mathscr{F}_-$ we associate two indexes $i(A) \in \mathbb{Z}$, $n(A) \geq 0$ in such a way that

$$A \subset B, \quad A \neq B \quad \Rightarrow \quad i(A) > i(B). \tag{5.1}$$

We say that A is the *father* of B if $C \supset B$, $C \neq B \Rightarrow C \supset A$. Only U has no father. We say that B is the *son* of A if A is the father of B. If A has no sons, it is called an endpoint. The set of endpoints of $\mathscr{F}$ is denoted by $\mathscr{F}_e$.

For simplicity, we will call a *tree* an indexed tree that has the following additional properties:

(5.2) If B, B' are brothers, $i(B) = i(B')$; $n(B) = n(B')$.

(5.3) If B, B' are brothers, $t \in B$, $t' \in B'$, then $\varphi_i(t, t') \geq \eta 2^{n(B)}$, where $i = i(B) = i(B')$.

(5.4) To each set $B \in \mathscr{F}_-$ can be associated a set $\overline{B}$, called the antecedent of B, in such a way that: (a) if B, B' are brothers, $\overline{B} = \overline{B}'$; (b) if A is the father of B, $A \supset \overline{B} \supset B$. (c) $D_{i(B)-1}(\overline{B}) \leq 2^{n(B)+1}$.

Observe that, in general, $\overline{B}$ does not belong to $\mathscr{F}$. The intuition for the idea of antecedent is that the sons of A will be obtained by application of Theorem 4.7. The antecedent of the sons of A is the set V of that theorem, the role of which is to ensure that the sons of A are close to each other.

The quantity of fundamental importance, that will help measure the size of $\mathscr{F}$ for our purposes, is the *depth* of $\mathscr{F}$, that is defined as follows:

$$d(\mathscr{F}) = \inf\left\{B \in \mathscr{F}_e; \sum_{B \subset A \in \mathscr{F}_-} r^{-i(A)} 2^{n(A)}\right\}.$$

If $\mathscr{F} = \{U\}$, we set $d(\mathscr{F}) = 0$. We say that $\mathscr{F}$ is *α-full* if it satisfies the following condition. For $B \in \mathscr{F}_-$, the number of brothers of B is greater than or equal to $\exp \alpha 2^{n(B)}$. [The quantity $\alpha d(\mathscr{F})$ is our actual measure of size of $\mathscr{F}$.]

We can now state and prove our basic separation result. In order to be able to use that theorem for other purposes (in particular for the proof of Theorem 1.3) let us recall that the only properties of φ required are (3.3) and (3.6).

THEOREM 5.1. *For each finite set $U \subset T$, $i \in \mathbb{Z}$, there exists a γ-full tree $\mathscr{F}$ on U such that $d(\mathscr{F}) \geq (2K_6)^{-1}(\theta_i(U) - K_6 r^{-i}(D_i(U) + 1))$, where K_6 is the constant of Theorem* 4.7.

PROOF. There is nothing to prove unless $\theta_i(U) \geq K_6 r^{-i}(D_i(U) + 1)$. In that case we can use Theorem 4.7 to find $i' > i$, $n \in \mathbb{N}$, a subset V of U such that $D_{i'-1}(V) \leq 2^{n+1}$, and points $(t_l)_{l \leq N}$ of V, where $\log N \geq \gamma 2^n$, which satisfy the following properties:

$$k \neq l \Rightarrow \varphi_{i'}(t_k, t_l) \geq \xi 2^n, \tag{5.5}$$

$$\begin{aligned} \forall\, l \leq N, \quad & \theta_{i'}(V \cap B_{i'}(t_l, \eta 2^n)) \\ & \geq \theta_i(U) - K_6\big(r^{-i}(D_i(U) + 1) + r^{-i'} 2^n\big). \end{aligned} \tag{5.6}$$

Consider the sets $A_l = V \cap B_{i'}(t_l, \eta 2^n)$. In the tree $\mathscr{F}$, the sets A_l are the sons of U. Their antecedent is V, and $i(A_l) = i'$, $n(A_l) = n$.

Since $\xi = 2^n \eta$, it is simple to see from (3.6) and (5.5) that $s \in A_k$, $t \in A_l$, $k \neq l \Rightarrow \varphi_i(s, t) \geq \eta 2^n$. Thus (5.3) and (5.4) hold.

The proof amounts to reiterating this operation in each set A_l, then in each of its sons and so on, but since U is finite, we can also argue by induction on card U. Since card $A_l <$ card U, we can find on each set A_l a γ-full tree $\mathscr{F}_l$ such that

$$d(\mathscr{F}_l) \geq \frac{1}{2K_6}\big(\theta_{i'}(A_l) - K_6 r^{-i'}(D_{i'}(A_l) + 1)\big). \tag{5.7}$$

We note that $D_{i'}(A_l) \leq 4\eta 2^n \leq 2^{n-1}$, so that $D_{i'}(A_l) + 1 \leq 2^n$. Combining (5.7) with (5.6) we get

$$d(\mathscr{F}_l) \geq \frac{1}{2K_6}\big(\theta_i(U) - K_6 r^{-i}(D_i(U) + 1)\big) - r^{-i'}2^n. \tag{5.8}$$

We now set $\mathscr{F} = \{U\} \cap \bigcup_{l \leq N} \mathscr{F}_l$. The indexes being defined in the obvious manner, it is clear that this is a γ-full indexed tree. It is also clear that

$$\begin{aligned} d(\mathscr{F}) &\geq r^{-i'}2^n + \inf_{l \leq N} d(\mathscr{F}_l) \\ &\geq \frac{1}{2K_6}\big(\theta_i(U) - K_6 r^{-i}(D_i(U) + 1)\big). \end{aligned}$$ □

The trees we have constructed in Theorem 5.1 do not have enough separation for our purposes; the required separation will be obtained by constructing appropriate subtrees. We say that a tree $\mathscr{F}'$ contained in $\mathscr{F}$ is a subtree of $\mathscr{F}$ if whenever A and B are brothers in $\mathscr{F}'$ they are brothers in $\mathscr{F}$ (although the common father in $\mathscr{F}'$ is in general different from the common father in $\mathscr{F}$) and if, moreover, when $A \in \mathscr{F}'$, both $i(A)$ and $n(A)$ have the same value whether A is seen as an element of $\mathscr{F}'$ or as an element of $\mathscr{F}$.

Definition 5.2. A tree $\mathscr{F}$ is called L-normalized if it has the following property:

$$B \subset A \in \mathscr{F}_-, \qquad B \neq A \quad \Rightarrow \quad r^{-2i(A)}2^{n(A)} \geq L r^{-2i(B)}2^{n(B)}.$$

Proposition 5.3. *A γ-full tree $\mathscr{F}$ on U contains an L-normalized γ-full subtree $\mathscr{F}'$ on U such that $d(\mathscr{F}') \geq d(\mathscr{F})/2L\sqrt{r}$.*

Proof. Consider the smallest integer q such that $L \leq r^{q/2}$. Thus $r^{q/2} \leq L\sqrt{r}$.

We first select $A \in \mathscr{F}$ such that

$$r^{-3i(A)/2}2^{n(A)}$$

is as large as possible. We put U, A and its brothers (that we denote by $A_1, \dots, A_N$) in $\mathscr{F}'$. Consider $B \in \mathscr{F}$ such that $B \subset A$ and $i(B) \leq i(A) + q$. We show that

$$\sum_{B \subset C \in \mathscr{F}_-} r^{-i(C)}2^{n(C)} \leq 2\sqrt{r}\, L r^{-i(A)}2^{n(A)}. \tag{5.9}$$

Indeed, by the choice of A we have

$$r^{-3i(C)/2} \le r^{-3i(A)/2} 2^{n(A)}$$

and thus

$$r^{-i(C)} 2^{n(C)} \le r^{(i(C)-i(A))/2} r^{-i(A)} 2^{n(A)}.$$

Thus

$$\sum_{B \subset C \in \mathscr{F}_-} r^{-i(C)} 2^{n(C)} \le \sum_{l \le q} r^{l/2} r^{-i(A)} 2^{n(A)}.$$

Now, since $r \ge 4$,

$$\sum_{l \le q} r^{l/2} \le 2r^{q/2} \le 2\sqrt{r}\, L,$$

so that (5.9) follows. For each $l \le N$, consider now a subset $B_l \subset A_l$, $B_l \in \mathscr{F}$ such that $i(B_l) \le i(A_l) + q$, and that $i(B_l)$ is as large as possible. Consider the γ-full tree on B_l given by $\mathscr{F}^l = \{C \in \mathscr{F}\colon C \subset B_l\}$. Given $C \in \mathscr{F}_e^l$, we have, by (5.9),

$$\begin{aligned} d(\mathscr{F}) &\le \sum_{C \subset D \in \mathscr{F}_-} r^{-i(D)} 2^{n(D)} = \sum_{C \subset D \in \mathscr{F}_-^l} r^{-i(D)} 2^{n(D)} + \sum_{B_l \subset D \in \mathscr{F}_-} r^{-i(D)} 2^{n(D)} \\ &\le \sum_{C \subset D \in \mathscr{F}_-^l} r^{-i(D)} 2^{n(D)} + 2L\sqrt{r}\, r^{-i(A_l)} 2^{n(A_l)}. \end{aligned}$$

Thus

$$\sum_{C \subset D \in \mathscr{F}_-^l} r^{-i(D)} 2^{n(D)} \ge d(\mathscr{F}) - 2L\sqrt{r}\, r^{-i(A_l)} 2^{n(A_l)}$$

and, since $C \in \mathscr{F}_e^l$ is arbitrary, we have

$$d(\mathscr{F}^l) \ge d(\mathscr{F}) - 2L\sqrt{r}\, r^{-i(A_l)} 2^{n(A_l)}. \tag{5.10}$$

Proceeding again by induction over the cardinality of U, we can assume that, by the induction hypothesis, $\mathscr{F}^l$ contains a γ-full L-normalized subtree $\mathscr{G}^l$ on B_l such that

$$d(\mathscr{G}^l) \ge d(\mathscr{F}^l)/(2L\sqrt{r}). \tag{5.11}$$

We now set

$$\mathscr{F}' = \{U\} \cup \bigcup_{l \le N} \{A_l\} \cup \{\mathscr{G}^l \setminus \{B_l\}\}.$$

It is clear that this is a γ-full tree; and

$$d(\mathscr{F}') \ge r^{-i(A)} 2^{n(A)} + \inf_l d(\mathscr{G}^l).$$

Thus, by (5.10) and (5.11), we indeed have $d(\mathscr{F}') \ge d(\mathscr{F})/(2L\sqrt{r})$. It remains to show that $\mathscr{F}'$ is L-normalized. Since $\mathscr{G}^l$ is L-normalized, it suffices to compare $2^{-2i(A)} 2^{n(A)}$ and $r^{-2i(B)} 2^{n(B)}$ for $B \subset B_l$, $B \in \mathscr{G}_-^l$. By the definition of B_l we have $i(B) > i(A_l) + q$. By the definition of A we have

$$r^{-3i(A_l)/2} 2^{n(A_l)} \ge r^{-3i(B)/2} 2^{n(B)}$$

so that, since $i(B) - i(A_l) > q$, we have

$$\begin{aligned} r^{-2i(A_l)}2^{n(A_l)} &\geq r^{(i(B)-i(A_l))/2}r^{-2i(B)}2^{n(B)} \\ &\geq r^{q/2}r^{-2i(B)}2^{n(B)} \\ &\geq Lr^{-2i(B)}2^{n(B)}. \end{aligned}$$ □

It is worthwhile to note the obvious fact that a subtree of an L-normalized tree is L-normalized.

DEFINITION 5.4. Consider an integer $s \geq 0$. A tree $\mathscr{F}$ is called *s-increasing* if it satisfies the following condition:

$$(5.12) \quad \forall A, B \in \mathscr{F}_-, B \subset A, \quad B \neq A \quad \Rightarrow \quad n(B) \geq n(A) + s.$$

When $s = 1$, we simply say increasing instead of 1-increasing. Here is another extraction principle.

PROPOSITION 5.5. *A γ-full tree $\mathscr{F}$ contains an s-increasing γ-full subtree $\mathscr{F}'$ such that $d(\mathscr{F}') \geq 2^{-s}d(\mathscr{F})$.*

PROOF. Denote by $(A_l)_{l \leq N}$ the sons of U, and put them in $\mathscr{F}'$. For $l \leq N$, consider $B_l \subset A_l$, $B_l \in \mathscr{F}$ such that $n(B_l) \geq n(A_l) + s$, and that $i(B_l)$ is as small as possible. [If, for some $l \leq N$, no such B_l can be found, the computation below will show that $d(\mathscr{F}) \leq 2^s r^{-i(A_l)}2^{n(A_l)}$, so that $\mathscr{F}' = \{A_l, l \leq N\} \cup \{U\}$ works.] Denote by $\overline{B}_l$ the antecedent of B_l. We have

$$(5.13) \quad \sum_{\overline{B}_l \subset B \in \mathscr{F}_-} r^{-i(B)}2^{n(B)} \leq 2^{n(A_l)+s-1} \sum_{j \geq i(A_l)} r^{-j} \leq 2^s r^{-i(A_l)}2^{n(A_l)}.$$

Consider now the tree $\mathscr{F}^l = \{B \subset \overline{B}_l, B \in \mathscr{F}\} \cup \{U\}$. From (5.13) we see as in the proof of the preceding proposition that

$$d(\mathscr{F}^l) \geq d(\mathscr{F}) - 2^s r^{-i(A_l)}2^{n(A_l)},$$

and we argue by induction over the cardinality of U as in the proof of that proposition. □

It is worthwhile to note the obvious fact that a subtree of an s-increasing tree is s-increasing. For a tree $\mathscr{F}$, we set

$$n(\mathscr{F}) = \inf\{n(A), A \in \mathscr{F}_-\}.$$

Observe that if $\mathscr{F}'$ is a subtree of $\mathscr{F}$, then $n(\mathscr{F}') \geq n(\mathscr{F})$.

PROPOSITION 5.6. *Consider $q \geq 0$. A γ-full tree $\mathscr{F}$ on U contains a γ-full subtree $\mathscr{F}'$ such that $n(\mathscr{F}') \geq q$ and $d(\mathscr{F}') \geq d(\mathscr{F}) - r^{-i}2^q$, where $i = \inf\{i(A)\colon A \in \mathscr{F}_-\}$.*

PROOF. We need only consider the case where $n(\mathscr{F}) < q$. If $n(A) < q$ for $A \in \mathscr{F}_-$, then (by the computation below) $d(\mathscr{F}) \leq r^{-i}2^q$, and there is nothing to prove. Otherwise, consider $A \in \mathscr{F}_-$ such that $n(A) \geq q$ and $i(A)$ is as small as possible. Denote by $\bar{A}$ the antecedent of A. We have

$$\sum_{\bar{A}\subset B\in\mathscr{F}_-} r^{-i(B)}2^{n(B)} \le \sum_{j\ge i} r^{-j}2^{q-1} \le r^{-i}2^q.$$

Set $\mathscr{F}' = \{B \in \mathscr{F};\ B \subset \bar{A}\} \cup \{U\}$. For $C \in \mathscr{F}_e'$, we have

$$\sum_{C\subset B\in\mathscr{F}_-'} r^{-i(B)}2^{n(B)} + \sum_{\bar{A}\subset B\in\mathscr{F}_-} r^{-i(B)}2^{n(B)} = \sum_{C\subset B\in\mathscr{F}_-} r^{-i(B)}2^{n(B)} \ge d(\mathscr{F})$$

since $C \in \mathscr{F}_e$. Thus

$$\sum_{C\subset B\in\mathscr{F}_-'} r^{-i(B)}2^{n(B)} \ge d(\mathscr{F}) - r^{-i}2^q,$$

that is, $d(\mathscr{F}') \ge d(\mathscr{F}) - r^{-i}2^q$. □

We now study random subsets of a tree $\mathscr{F}$. The next result shows that under mild conditions these subsets contain a subtree essentially as large as $\mathscr{F}$.

THEOREM 5.7. *Consider $\kappa > 0$. Consider an s-increasing κ-full tree $\mathscr{F}$ on U. Assume that $\kappa 2^{n(\mathscr{F})} \ge 2$. Consider $b, c > 0$, and assume that $cs \ge 2$. Consider a random subset $\mathscr{G}$ of $\mathscr{F}$. Assume that*

$$\forall A \in \mathscr{F}_-, \qquad P(A \in \mathscr{G}) > 1 - b2^{-cn(A)}. \tag{5.14}$$

Consider the following event:

$$\Omega^{\mathscr{F}} = \Big\{\exists\, \mathscr{G}' \subset \mathscr{G};\ \mathscr{F}' = \mathscr{G}' \cup \{U\} \text{ is a } \frac{\kappa}{2}\text{-full subtree of } \mathscr{F} \text{ such that } d(\mathscr{F}') = d(\mathscr{F})\Big\}.$$

Then

$$P(\Omega^{\mathscr{F}}) \ge 1 - 4b2^{-cn(\mathscr{F})}.$$

PROOF. We argue by induction on the cardinality of U. Set $n = n(\mathscr{F})$. Consider the sons $A_1, \dots, A_N$ of U. Thus $n(A_l) = n$ for $l \le N$. Set $i = i(A_1)$. For $l \le N$, set $\mathscr{F}^l = \{B \in \mathscr{F},\ B \subset A_l\}$, and set $\mathscr{G}^l = \mathscr{F}^l \cap \mathscr{G}$. Consider the events $\Omega^{1,l} = \{A_l \in \mathscr{G}\}$,

$$\Omega^{2,l} = \{\exists\ \mathscr{G}'^l \subset \mathscr{G}^l \text{ such that } \mathscr{H}^l = \mathscr{G}'^l \cup \{A_l\} \text{ is a } \frac{\kappa}{2}\text{-full subtree of } \mathscr{F}^l \text{ with } d(\mathscr{H}^l) = d(\mathscr{F}^l)\}.$$

From (5.14) we have $P(\Omega^{1,l}) \ge 1 - b2^{-cn}$. Since $\mathscr{F}$ is s-increasing, we have $n(\mathscr{F}^l) \ge n + s$. So, by the induction hypothesis, and since $2^{-cs} \le 1/4$, we have

$$P(\Omega^{2,l}) \ge 1 - 4b2^{-c(n+s)} \ge 1 - b2^{-cn}.$$

Consider the random set

$$S = \{l \le N;\ \Omega^{2,l} \text{ and } \Omega^{1,l} \text{ occur}\}.$$

We set

$$\Omega' = \left\{\operatorname{card} S \ge \frac{N}{2}\right\}.$$

Since $P(l \notin S) \le 2b2^{-cn}$, we see that

$$P(\Omega') \ge 1 - 4b2^{-cn}.$$

Thus, it suffices to show that $\Omega' \subset \Omega^{\mathscr{F}}$. For $l \in S$, consider the subtree $\mathscr{H}^l$ of $\mathscr{F}^l$ given by the definition of $\Omega^{2,l}$, so that $\mathscr{H}^l \subset \mathscr{G}$. Set

$$\mathscr{F}' = \bigcup_{l \in S} \mathscr{H}^l \cup \{U\}.$$

We have

$$\begin{aligned} d(\mathscr{F}') &= r^{-i}2^n + \inf_l d(\mathscr{H}^l) \\ &= r^{-i}2^n + \inf_l d(\mathscr{F}^l) = d(\mathscr{F}). \end{aligned}$$

To see that $\mathscr{F}'$ is $\kappa/2$ full, it suffices to show that $\log \operatorname{card} S \ge (\kappa/2)2^n$. Since $\log N \ge \kappa 2^n \ge 2$, we have $\log N/2 \ge (1/2)\log N$, so that

$$\log \operatorname{card} S \ge \log \frac{N}{2} \ge \frac{1}{2}\log N \ge \frac{\kappa}{2}2^n. \qquad \square$$

DEFINITION 5.8. We say that a tree $\mathscr{F}$ is *balanced* if

$$\max_{B \in \mathscr{F}_e} \sum_{B \subset A \in \mathscr{F}_-} r^{-i(A)}2^{n(A)} \le 2d(\mathscr{F}).$$

PROPOSITION 5.9. *A κ-full tree $\mathscr{F}$ on U contains a κ-full balanced subtree $\mathscr{F}'$ such that $d(\mathscr{F}') \ge d(\mathscr{F})$.*

PROOF.

CASE 1. There exists $A \in \mathscr{F}_-$ such that $r^{-i(A)}2^{n(A)} \ge d(\mathscr{F})$. Consider the subtree $\mathscr{F}'$ of $\mathscr{F}$ that consists of U, A and its brothers. It is obviously γ-full, balanced and satisfies $d(\mathscr{F}') \ge d(\mathscr{F})$.

CASE 2. For all $A \in \mathscr{F}_-$ we have $r^{-i(A)}2^{n(A)} < d(\mathscr{F})$. Set

$$\mathscr{F}' = \left\{B \in \mathscr{F};\ \sum_{B \subset A \in \mathscr{F}_-} r^{-i(A)}2^{n(A)} \le 2d(\mathscr{F})\right\}.$$

This is obviously a γ-full subtree of $\mathscr{F}$. For $B \in \mathscr{F}'$, so in particular for $B \in \mathscr{F}'_e$, we have

$$\sum_{B \subset A \in \mathscr{F}_-} r^{-i(A)}2^{n(A)} \le 2d(\mathscr{F}),$$

so it suffices to show that $d(\mathscr{F}') \geq d(\mathscr{F})$. Consider $B \in \mathscr{F}_e'$. If $B \in \mathscr{F}_e$, we have

$$\sum_{B \subset A \in \mathscr{F}_-} r^{-i(A)} 2^{n(A)} \geq d(\mathscr{F})$$

by definition of $d(\mathscr{F})$. If $B \notin \mathscr{F}_e$, consider a son C of B. Since $C \notin \mathscr{F}'$, we have

$$\sum_{C \subset A \in \mathscr{F}_-} r^{-i(A)} 2^{n(A)} > 2d(\mathscr{F}),$$

so that

$$r^{-i(C)} 2^{n(C)} + \sum_{B \subset A \in \mathscr{F}_-} r^{-i(A)} 2^{n(A)} > 2d(\mathscr{F})$$

and hence

$$\sum_{B \subset A \in \mathscr{F}_-} r^{-i(A)} 2^{n(A)} > d(\mathscr{F})$$

since $r^{-i(C)} 2^{n(C)} < d(\mathscr{F})$. □

We end this section with a technical point.

LEMMA 5.10. *Consider $\kappa > 0$ and a κ-full increasing tree $\mathscr{F}$ on U. Set $n = n(\mathscr{F})$ and assume that $\kappa 2^n \geq 1$. For $B \in \mathscr{F}_-$, denote by $N(B)$ the number of brothers of B and assume that* $\log N(B) \leq 2\kappa 2^{n(B)}$ [*since $\mathscr{F}$ is κ-full, we have* $\log N(B) \geq \kappa 2^{n(B)}$]. *Then*

$$\sum_{B \in \mathscr{F}_-} \frac{1}{N(B)^4} \leq 2\exp(-3\kappa 2^n).$$

PROOF. This will be again shown by induction on the cardinality of U. Denote by $(A_l)_{l \in N}$ the sons of U. For $l \leq N$, set $\mathscr{F}^l = \{B \in \mathscr{F},\ B \subset A_l\}$ and $n_l = n(\mathscr{F}^l)$. Since $\mathscr{F}$ is increasing, we have $n_l \geq n + 1$. Using this and the induction hypothesis, we have

$$\begin{aligned}
\sum_{B \in \mathscr{F}_-} \frac{1}{N(B)^4} &= \frac{N}{N^4} + \sum_{l \leq N} \sum_{B \in \mathscr{F}_-^l} \frac{1}{N(B)^4} \\
&\leq \frac{1}{N^3} + \sum_{l \leq N} 2\exp(-3\kappa 2^{n_l}) \\
&\leq \frac{1}{N^3} + 2N\exp(-3\kappa 2^{n+1}) \\
&\leq \frac{1}{N^3} + 2\exp(-4\kappa 2^n) \\
&\leq 2\exp(-3\kappa 2^n),
\end{aligned}$$

since $2\exp - (\kappa 2^n) \leq 2/e \leq 1$. □

6. The minoration. Before we start the serious work, let us prove a simple (known) fact that will show how the quantity M of Theorem 1.1 controls $D_i(T)$.

LEMMA 6.1. *Consider a Lévy measure ν on $\mathbb{R}$, and a random variable X that satisfies*

$$\forall\, t \in \mathbb{R}, \qquad E \exp itX = \exp - \int_{\mathbb{R}} (1 - \cos tx)\, d\nu(x), \tag{6.1}$$

$$P(|X| \le M) \ge \tfrac{3}{4}. \tag{6.2}$$

Then $\int_{\mathbb{R}} (x/M)^2 \wedge 1\, d\nu(x) \le K$, where K is universal.

PROOF. Observe that $\cos ux \ge 1/2$ for $|x| \le M$, $|u| \le 1/M$. Thus we have

$$E \cos uX \ge -\tfrac{1}{4} + \tfrac{3}{4} \cdot \tfrac{1}{2} = \tfrac{1}{8}.$$

For $u \le 1/M$ we now have from (6.1) that

$$\int_{\mathbb{R}} (1 - \cos ux)\, d\nu(x) \le \log 8.$$

Integrating for $0 \le u \le 1/M$, we get

$$\int_{\mathbb{R}} \left(1 - \frac{\sin x/M}{x/M}\right) d\nu(x) \le \log 8.$$

We observe that for some K we have $u^2 \wedge 1 \le K\,(1 - \sin u/u)$. The result follows. □

It follows from Lemma 6.1 that (1.3) implies that $D_j(T) \le K$ whenever $r^j \le 1/M$. We denote by i the largest integer $i \in \mathbb{Z}$ for which $D_i(T) \le 1$. Since $D_{i+1}(T) > 1$ by definition of i, we have from (3.2) that $D_{i+p_0}(T) > K$ where p_0 depends on δ, v_0 only. Thus from Lemma 6.1 we have $r^{i+p_0} > 1/M$, which implies $r^{-i} \le KM$.

To prove Theorem 1.1, it suffices by Theorem 3.2 to prove that $\theta_i(U) < KM$ for each finite subset U of T. It might be already useful to mention that since $D_i(U) \le 1$, and $r^{-i} \le KM$, there is nothing to prove unless $\theta_i(U) \gg r^{-i}(D_i(U) + 1)$. This is why this term in Theorem 5.1 is not an obstacle. We now consider two parameters $q \ge 0$, $L > 0$, to be determined later. We first use Theorem 5.1 to find a γ-full tree $\mathscr{F}^1$ on U such that

$$d(\mathscr{F}^1) \ge \frac{1}{K}\big(\theta_i(U) - Kr^{-i}(D_i(U) + 1)\big) \ge \frac{1}{K}\big(\theta_i(U) - 2Kr^{-i}\big).$$

Consider the smallest integer s such that $s \ge 1$, $s\delta \ge 4$. We use Proposition 5.6 to find a γ-full subtree $\mathscr{F}^2$ of $\mathscr{F}^1$ such that $n(\mathscr{F}^2) \ge q$ and

$$d(\mathscr{F}^2) \ge \frac{1}{K}\big(\theta_i(U) - Kr^{-i}(2 + 2^q)\big) \ge \frac{1}{K}\big(\theta_i(U) - Kr^{-i}2^{q+2}\big).$$

We then use Propositions 5.3 and 5.5 to find an s-increasing L-normalized subtree $\mathcal{F}^3$ of $\mathcal{F}^2$ on U such that

$$d(\mathcal{F}^3) \geq \frac{1}{LK}\big(\theta_i(U) - Kr^{-i}2^q\big). \tag{6.3}$$

Consider now another parameter $\kappa \leq \gamma$. It should be obvious that if we have $\kappa 2^q \geq 1$, then [since $n(\mathcal{F}^3) \geq q$] there exists a κ-full subtree $\mathcal{F}$ of $\mathcal{F}^3$, with $d(\mathcal{F}) = d(\mathcal{F}^3)$, where for each $B \in \mathcal{F}$, if $N(B)$ denotes the number of brothers of B, we have

$$\kappa 2^{n(B)} \leq \log N(B) \leq 2\kappa 2^{n(B)}. \tag{6.4}$$

To each point $A \in \mathcal{F}$, we associate a point $t_A \in A$ as follows. If $A \in \mathcal{F}_e$, we pick any $t_A \in A$. Otherwise, we consider a son B of A, and the antecedent $\overline{B}$ of B (that does not depend on B), and we pick any $t_A \in \overline{B}$.

Since $t_A \in A$ it follows from (5.3) that

$$\text{if } B_1, B_2 \text{ are brothers, we have } \varphi_{i(B_1)}\big(t_{B_1}, t_{B_2}\big) \geq \eta 2^{n(B_1)}. \tag{6.5}$$

It follows from (5.4)(c), since $t_A \in \overline{B}$, that we have:

$$\text{if } A \text{ is the father of } B,\ \varphi_{i(B)-1}(t_A, t_B) \leq 2^{n(B)+1}. \tag{6.6}$$

We now use the notations of Section 2. We first take $\alpha > 0$ large enough that $P_0(\Omega'_0) \geq 3/4$, where

$$\Omega'_0 = \left\{\omega_0 \in \Omega_0;\ \alpha^+(\omega_0) = \sup_{k\geq 1}\frac{\tau_k(\omega_0)}{k} \leq \alpha,\ \alpha^-(\omega_0) = \inf_{k\geq 1}\frac{\tau_k(\omega_0)}{k} \geq \frac{1}{\alpha}\right\}.$$

By hypothesis we have

$$P_0 \otimes P \otimes Q\Big(\sup_{s,t\in U}|X_s - X_t| \geq M\Big) \leq 1/4$$

so, by Fubini's theorem, we can find $\omega_0 \in \Omega'_0$ such that, conditionally on ω_0, we have

$$P \otimes Q\bigg(\sup_{s,t\in U}\Big|\sum_{k\geq 1}\varepsilon_k R_k Y_k(s) - \sum_{k\geq 1}\varepsilon_k R_k Y_k(t)\Big| \geq M\bigg) \leq \tfrac{1}{3}. \tag{6.7}$$

This ω_0 will stay fixed from that point on.

The value of κ is now determined as the largest number less than or equal to γ for which $12\kappa \leq \eta/8\alpha$. This is a universal constant; the motivation for this choice will occur in the proof of the next result.

Lemma 6.2. *Assume that* $\kappa 2^q \geq 1$ *and that*

$$\eta 2^q \geq 2\alpha. \tag{6.8}$$

Consider the event $\Omega^1 \subset \Omega$ *determined by the following conditions. Whenever* $B_1, B_2 \in \mathcal{F}$ *are brothers, we have*

$$\sum_{k\geq 1} R_k\big|Y_k(t_{B_1}) - Y_k(t_{B_2})\big|^2 \wedge r^{-2i(B_1)} \geq \frac{\eta}{8\alpha} r^{-2i(B_1)}2^{n(B_1)}. \tag{6.9}$$

Whenever A is the father of B in $\mathscr{F}$, *we have*

$$\sum_{k\geq 1} R_k |Y_k(t_B) - Y_k(t_A)|^2 \wedge r^{-2i(B)+2} \leq \alpha r^{-2i(B)+2} 2^{n(B)+3}. \tag{6.10}$$

Then $P(\Omega^1) \geq 1 - 4\exp(-3\kappa 2^q)$.

PROOF. It follows from Proposition 2.12(a) and (6.5) and (6.8) that given B_1, B_2, (6.9) holds with probability at least $1 - \exp(-(\eta/8\alpha)2^{n(B_1)})$. Proposition 2.12(b) and (6.6) imply that given A, B, (6.9) occurs with probability at least $1 - \exp(-\alpha 2^{n(B)+2})$.

Thus the probability that (6.9) occurs for all the couples of brothers of a given B, and that (6.10) occurs for all the brothers of B, is at least

$$\begin{aligned} 1 - N(B)^2 \exp\left(-\frac{\eta}{8\alpha}2^{n(B)}\right) - N(B)\exp(-\alpha 2^{n(B)+2}) \\ \geq 1 - 2N(B)^2 \exp\left(-\frac{\eta}{8\alpha}2^{n(B)}\right). \end{aligned}$$

From (6.4) this is at least, by the choice of κ,

$$\begin{aligned} 1 - 2\exp\left(-2^{n(B)}\left[\frac{\eta}{8\alpha} - 4\kappa\right]\right) &= 1 - 2\exp(-8\kappa 2^{n(B)}) \\ &\geq 1 - \frac{2}{N(B)^4}. \end{aligned}$$

Thus $P(\Omega^1) \geq 1 - \sum_{B\in\mathscr{F}_-} 2N(B)^{-4}$, and the conclusion follows from Lemma 5.10, since $\kappa 2^q \geq 1$. □

We consider now a new parameter $v \geq v_0$, to be determined later. For $B \in \mathscr{F}_-$, we set $c(B) = vr^{-i(B)+1}$. Consider $B \in \mathscr{F}_-$ and its father A. We set

$$W_k = W_k^B = R_k|Y_k(t_B) - Y_k(t_A)|. \tag{6.11}$$

It follows from Proposition 2.13 that

$$\sum_{k\geq 1} EW_k^{1+\delta/2} 1_{\{W_k \geq c(B)\}} \leq K\alpha c(B)^{1+\delta/2} \varphi\left(t_A, t_B, c(B)^{-1}\right). \tag{6.12}$$

It follows from (6.6) and (3.3) that

$$\varphi\left(t_A, t_B, v^{-1}r^{i(B)+1}\right) \leq v^{-1-\delta}\varphi_{i(B)-1}(t_A, t_B) \leq v^{-1-\delta}2^{n(B)+1}.$$

Combining with (6.12) this gives

$$\sum_{k\geq 1} EW_k^{1+\delta/2} 1_{\{W_k \geq c(B)\}} \leq c(B)^{1+\delta/2}(Kv^{-1-\delta}2^{n(B)})$$

(we have absorbed α into K since α is universal). We now appeal to Lemma 2.15 to get

$$P\left(\sum_{k\geq 1} W_k 1_{\{W_k \geq c(B)\}} \geq Kc(B)v^{-1-\delta}2^{n(B)}\right) \leq K(v^{-1-\delta}2^{n(B)})^{-\delta/2}.$$

Thus we get, by definition of $c(B)$,

$$(6.13)\quad P\Big(\sum_{k\geq 1} W_k 1_{\{W_k \geq c(B)\}} \geq K_7 v^{-\delta} r^{-i(B)} 2^{n(B)}\Big) \leq K_7 v^{\delta(1+\delta)/2} 2^{-\delta n(B)/2}.$$

We consider now the random subset $\mathscr{G}$ of $\mathscr{F}$ given by, for $B \in \mathscr{F}_-$,

$$B \in \mathscr{G} \quad\Leftrightarrow\quad \sum_{k\geq 1} W_k^B 1_{\{W_k^B \geq c(B)\}} \leq K_7 v^{-\delta} r^{-i(B)} 2^{n(B)},$$

where W_k^B is given by (6.11). If we assume

$$(6.14)\qquad \kappa \cdot 2^q \geq 2,$$

we see from (6.13) that we can use Theorem 5.7 with $c = \delta/2$, $b = K_7 v^{\delta(1+\delta)/2}$ (since $cs \geq 2$). Thus the event

$$\Omega^2 = \Big\{\exists\, \mathscr{G}' \subset \mathscr{G},\ \mathscr{F}^4 = \mathscr{G}' \cup \{U\} \text{ is a } \frac{\kappa}{2}\text{-full subtree of } \mathscr{F},\ d(\mathscr{F}^4) = d(\mathscr{F})\Big\}$$

satisfies $P(\Omega^2) \geq 1 - 4K_7 v^{\delta(1+\delta)/2} 2^{-\delta q/2}$.

We have shown that, under (6.8) and (6.14), the event $\Omega^3 = \Omega^1 \cap \Omega^2$ satisfies

$$(6.15)\qquad P(\Omega^3) \geq 1 - 4\exp(-3\kappa 2^q) - 4K_7 v^{\delta(1+\delta)/2} 2^{-\delta q/2}.$$

We now come to the central part of the argument.

THEOREM 6.3. *Given numbers w_1, w_2, w_3, κ one can find numbers v, L, n, S depending only on w_1, w_2, w_3, κ, with the following property. Consider any increasing, balanced, L-normalized $\kappa/2$-full subtree of sets $\mathscr{F}'$ of $\mathscr{F}$, such that $n(\mathscr{F}') \geq n$, such that to each $A \in \mathscr{F}'$ is associated a point $y_A = (y_{A,k})_{k\geq 1}$ of $\mathbb{R}^{\mathbb{N}}$, in such a way that the following properties hold, where we set $g_i(x) = x 1_{\{|x| \geq v r^{-i+1}\}}$.*

Whenever B_1, B_2 are brothers in $\mathscr{F}'$, we have

$$(6.16)\qquad \sum_{k\geq 1} |y_{B_1,k} - y_{B_2,k}|^2 \wedge r^{-2i(B_1)} \geq w_1 r^{-2i(B_1)} 2^{n(B_1)}.$$

If A is the father of B,

$$(6.17)\qquad \sum_{k\geq 1} |y_{B,k} - y_{A,k}|^2 \wedge r^{-2i(B)+2} \leq w_2 r^{-2i(B)} 2^{n(B)},$$

$$(6.18)\qquad \sum_{k\geq 1} \big|g_{i(B)}(y_{B,k} - y_{A,k})\big| \leq w_3 v^{-\delta} r^{-i(B)} 2^{n(B)}.$$

Then we have

$$(6.19)\qquad Q\Big(\sup_{B\in \mathscr{F}_e'} \sum_{k\geq 1} \varepsilon_k (y_{B,k} - y_{U,k}) \geq S d(\mathscr{F}')\Big) \geq \tfrac{1}{2}.$$

Before we prove that statement, we show how to conclude the proof of Theorem 1.1. We set

$$w_1 = \frac{\eta}{8\alpha}, \qquad w_2 = 8\alpha r^2, \qquad w_3 = K_7.$$

These numbers depend on δ, v_0 only. We denote by v, L, n the numbers produced by Theorem 6.3. This is what determines the parameters v, L. To determine the parameter q, we set it as the smallest $q \geq n$ that satisfies (6.8), (6.14) and

$$4\exp(-3\kappa 2^q) + 4K_7 v^{\delta(1+\delta)/2} 2^{-\delta q/2} \leq \tfrac{1}{4}. \tag{6.20}$$

All the parameters are now fixed. By (6.15) and (6.20), we see that $P(\Omega^3) \geq 3/4$. By Fubini's theorem and (6.7) we can fix $w \in \Omega^3$ such that

$$Q\left(\sup_{s,t\in U}\left|\sum_{k\geq 1}\varepsilon_k\big(R_k Y_k(s) - R_k Y_k(t)\big)\right| \leq M\right) > 1/2, \tag{6.21}$$

where $R_k = R_k(\tau_k(\omega_0), Y_k(\omega))$, $Y_k(s) = Y_k(\omega)(s)$.

We consider the tree $\mathscr{F}^4$ given in the definition of Ω^2. For $A \in \mathscr{F}^4$, we set $y_{A,k} = Y_k(\omega)(t_A)$. Since $\omega \in \Omega^1$, (6.9) holds: This implies (6.16), and (6.10) implies (6.17). The definition of $\mathscr{G}$ and Ω^2 show that (6.18) hold. Since $\mathscr{F}^4$ is increasing, L-normalized, $\kappa/2$-full and satisfies $n(\mathscr{F}^4) \geq q \geq n$, by Proposition 5.9 we can find a balanced $\kappa/2$-full subtree $\mathscr{F}'$ of $\mathscr{F}^4$ with $d(\mathscr{F}') = d(\mathscr{F}^4)$. We can apply Theorem 6.3, and (6.19) holds. But (6.21) implies

$$Q\left(\sup_{B\in\mathscr{F}_-}\left|\sum_{k\geq 1}\varepsilon_k(y_{B,k} - y_{U,k})\right| \leq M\right) > 1/2,$$

and comparing with (6.19) gives $Sd(\mathscr{F}') \leq M$. We recall that $d(\mathscr{F}') = d(\mathscr{F}^4) = d(\mathscr{F}^3)$, so that, by (6.3),

$$\frac{S}{LK}\left(\theta_i(U) - Kr^{-i}2^q\right) \leq M,$$

which can be rewritten as $\theta_i(U) \leq K(M + r^{-i}) \leq KM$ since $r^{-i} \leq KM$. This completes the proof of Theorem 1.1.

Proof of Theorem 6.3. For $B \in \mathscr{F}'_-$, we denote its father by fB. We write the "chaining"

$$y_B - y_U = \sum_{B\subset C\in F'_-} y_C - y_{fC}.$$

An essential idea is to split the differences $y_C - y_{fC}$ into what, roughly speaking, is their l^1 part and their l^2 part. We set, for $x \in \mathbb{R}$,

$$f_j(x) = x - g_j(x) = x 1_{\{|x| \leq vr^{-j+1}\}}.$$

We write

$$y_C - y_{fC} = x_C^1 + x_C^2,$$

where

$$x_C^1 = \left(g_{i(C)}(y_{C,k} - y_{fC,k})\right)_{k \ge 1}, \tag{6.22}$$

$$x_C^2 = \left(f_{i(C)}(y_{C,k} - y_{fC,k})\right)_{k \ge 1} \tag{6.23}$$

and we write

$$z_B^1 = \sum_{B \subset C \in \mathscr{F}'_-} x_C^1, \tag{6.24}$$

$$z_B^2 = \sum_{B \subset C \in \mathscr{F}'_-} x_C^2, \tag{6.25}$$

so that $y_B - y_U = z_B^1 + z_B^2$. By (6.18) we have

$$\|x_C^1\|_1 \le w_3 v^{-\delta} r^{-i(C)} 2^{n(C)}.$$

By (6.24) we have

$$\|z_B^1\|_1 \le w_3 v^{-\delta} \sum_{B \subset C \in \mathscr{F}'_-} r^{-i(C)} 2^{n(C)} \le 2 w_3 v^{-\delta} d(\mathscr{F}'),$$

since $\mathscr{F}'$ is balanced. Thus, for any signs $(\varepsilon_k)_{k \ge 1}$, we have

$$\sup_{B \in \mathscr{F}'_-} \left| \sum_{k \ge 1} \varepsilon_k z_{B,k}^1 \right| \le 2 w_3 v^{-\delta} d(\mathscr{F}')$$

and thus

$$\sup_{B \in \mathscr{F}'_-} \sum_{k \ge 1} \varepsilon_k (y_{B,k} - y_{U,k}) \ge \sup_{B \in \mathscr{F}'_-} \sum_{k \ge 1} \varepsilon_k z_{B,k}^2 - 2 w_3 v^{-\delta} d(\mathscr{F}'). \tag{6.26}$$

The core of the proof will be to obtain lower bounds on $\sum_{k \ge 1} \varepsilon_k z_{B,k}^2$ using Proposition 2.6. We collect inequalities for that purpose. It follows from (6.17) that

$$\begin{aligned} \|x_C^2\|_2^2 &\le \sum_{k \ge 1} |y_{C,k} - y_{fC,k}|^2 \wedge v^{2r - 2i(C) + 2} \\ &\le v^2 \left(\sum_{k \ge 1} |y_{C,k} - y_{fC,k}|^2 \wedge r^{-2i(C)+2} \right) \\ &\le w_2 v^2 r^{-2i(C)} 2^{n(C)}. \end{aligned} \tag{6.27}$$

Lemma 6.4. *Suppose that*

$$v^{1+\delta} \ge \frac{4 w_3}{r w_1}. \tag{6.28}$$

Then, whenever B_1, B_2 are brothers, if we set $j = i(B)$, $m = n(B)$ we have

$$\sum_{k \ge 1} |x_{B_1,k}^2 - x_{B_2,k}^2|^2 \wedge r^{-2j} \ge w_1 r^{-2j} 2^{m-1}. \tag{6.29}$$

Proof. We denote by A the common father of B_1, B_2. For $l = 1, 2$, we have $x_{B_l,k}^2 = y_{B_l,k} - y_{A,k}$ unless $|y_{B_l,k} - y_{A,k}| > v r^{-j+1}$. The number N_l of indices k for which $|y_{B_l,k} - y_{A,k}| > vr^{-j+1}$ satisfies

$$N_l \le \frac{1}{vr^{-j+1}} \sum_{k\ge 1} \left|g_j(y_{B_l,k} - y_{A,k})\right| \le w_3 v^{-1-\delta} r^{-1} 2^m$$

by (6.18). For at most $N_1 + N_2$ exceptional indices we can have $x^2_{B_1,k} - x^2_{B_2,k} \ne y_{B_1,k} - y_{B_2,k}$. It follows that

$$\sum_{k\ge 1} \left|x^2_{B_1,k} - x^2_{B_2,k}\right|^2 \wedge r^{-2j} \ge \sum_{k\ge 1} |y_{B_1,k} - y_{B_2,k}|^2 \wedge r^{-2j} - r^{-2j}(N_1 + N_2)$$

$$\ge r^{-2j} 2^m \left[w_1 - \frac{2}{r} w_3 v^{-1-\delta} \right]$$

by (6.16). □

We denote by $A_1, \dots, A_N$ the sons of U in $\mathscr{F}'$. Set $j = i(A_1)$, $m = n(A_1)$. Since $\mathscr{F}'$ is $\kappa/2$-full and by (6.4) we have

$$\frac{\kappa}{2} 2^m \le \log N \le 2\kappa 2^m. \tag{6.30}$$

For $l \le N$, set $t_l = x^2_{A_l}$. Set

$$T_l = \left\{ z^2_B - x^2_{A_l};\ B \subset A_l,\ B \in \mathscr{F}'_e \right\}.$$

It follows from (6.25) and (6.27) that, for $B \in \mathscr{F}'_e$, $B \subset A_l$ we have

$$\|z^2_B - x^2_{A_l}\|_2 = \left\| \sum_{B\subset C\subset A_l,\, C\ne A_l} x^2_C \right\|_2$$

$$\le \sum_{B\subset C\subset A_l,\, C\ne A_l} w_2^{1/2} v r^{-i(C)} 2^{n(C)/2}.$$

Since $\mathscr{F}'$ is L-normalized, for $C \subset D$, $C \ne D$, we have (since we can assume $L \ge 4$)

$$r^{-i(C)} 2^{n(C)/2} \le L^{-1/2} r^{-i(D)} 2^{n(D)/2}$$

and thus

$$\sum_{B\subset C\subset A_l,\, C\ne A_l} r^{-i(C)} 2^{n(C)/2} \le \sum_{p\ge 1} L^{-p/2} r^{-j} 2^{m/2}$$

$$\le \frac{2}{\sqrt{L}} r^{-j} 2^{m/2}.$$

We set $\sigma = (1/\sqrt{L}) w_2^{1/2} v r^{-j} 2^{m/2+1}$. Thus we have shown that $\|z\|_2 \le \sigma$ for $z \in T_l$. Note that the argument also shows that

$$\forall\, B \in F'_e, \qquad \|z^2_B\|_2 \le w_2^{1/2} v r^{-j} 2^{m/2+1}. \tag{6.31}$$

We denote by K_8 the constant of Corollary 2.7. By (6.29), we can use this corollary for $a = r^{-j}$, $b^2 = w_1 r^{-2j} 2^{m-1}$.

We obtain, using (6.30),

$$E \sup_{B \in \mathscr{F}_e'} \sum_{k \geq 1} \varepsilon_k z_{B,k}^2 \geq \frac{1}{K_8} \min\left[\left(\frac{\kappa w_1}{4}\right)^{1/2}, \frac{w_1}{2}\right] r^{-j} 2^m - K_8 \sigma (2\kappa 2^m)^{1/2} + H, \tag{6.32}$$

where

$$H = \min_{l \leq N} E \sup_{x \in T_l} \sum_{k \geq 1} \varepsilon_k x_k.$$

If we recall the value of σ, the left side of (6.32) is at least

$$\left[\frac{1}{K_8} \min\left[\left(\frac{\kappa w_1}{4}\right)^{1/2}, \frac{w_1}{2}\right] - \frac{K_8}{\sqrt{L}} (8 w_2 \kappa)^{1/2} v\right] r^{-j} 2^m + H.$$

It follows that there is a function $L(w_1, w_2, v)$, such that if

$$L = L(w_1, w_2, v), \tag{6.33}$$

we have

$$E \sup_{B \in \mathscr{F}_e'} \sum_{k \geq 1} \varepsilon_k z_{B,k}^2 \geq w_4 r^{-j} 2^m + H, \tag{6.34}$$

where w_4 depends on w_1, κ only. If we use (6.34), and argue by induction on card U in a way by now familiar, we get

$$E \sup_{B \in \mathscr{F}_e'} \sum_{k \geq 1} \varepsilon_k z_{B,k}^2 \geq w_4 d(\mathscr{F}'). \tag{6.35}$$

The constant w_4 depends on w_1, but not on v. This is essential for our success. It should be pointed out that the use of Proposition 2.2 would not be sufficient for that purpose [the quantity in front of $d(\mathscr{F}')$ would decay as v^{-1}], and that the improvement of Theorem 2.3 is used in an essential way here.

We now appeal to (2.17) to see, by (6.31) and (6.35), that

$$Q\left(\sup_{B \in \mathscr{F}_e'} \sum_{k \geq 1} \varepsilon_k z_{B,k}^2 \geq w_4 d(\mathscr{F}') - K w_2^{-1/2} v r^{-j} 2^{m/2+1}\right) \geq 1/2.$$

Thus by (6.26),

$$Q\left(\sup_{B \in \mathscr{F}_e'} \sum_{k \geq 1} \varepsilon_k (y_{B,k} - y_{U,k}) \geq (w_4 - 2 w_3 v^{-\delta}) d(\mathscr{F}') - K w_2^{1/2} v r^{-j} 2^{m/2+1}\right) \geq 1/2. \tag{6.36}$$

We take for v the smallest number that satisfies (6.28) and

$$2 w_3 v^{-\delta} \leq w_4/2$$

and we determine L by (6.33). Since $d(\mathscr{F}') \geq r^{-j} 2^m$ we have

$$r^{-j} 2^{m/2} \leq 2^{-m/2} d(\mathscr{F}') \leq 2^{-n/2} d(\mathscr{F}'),$$

and (6.36) becomes

$$Q\left(\sup_{B\in\mathscr{F}_e'}\sum_{k\geq 1}\varepsilon_k(y_{B,k}-y_{U,k})\geq\left[\frac{w_4}{2}-Kw_2^{1/2}v2^{-n/2+1}\right]d(\mathscr{F}')\right)\geq\frac{1}{2}.$$

We now take for n the smallest value such that

$$K2^{-n/2+1}w_2^{1/2}v\leq\frac{w_4}{4}$$

and we have proved Theorem 6.3 with $S = w_4/4$. □

We will end this section by some results related to continuity. In [25], Section 3, it is shown how to deduce, in the Gaussian case, results for continuity from results for boundedness. The proof of Theorem 6.5 below from Theorem 1.1 is a rather simple adaptation of these ideas; as we do not wish to lengthen this paper with routine material, we omit the proof.

Consider a compact metric space (T,τ), and an infinitely divisible process $(X_t)_{t\in T}$. Consider the distance d on T given by $d(s,t) = (\int_{\mathbb{R}^T}|\beta(s)-\beta(t)|^2\wedge 1\,d\nu(t))^{1/2}$. For simplicity we say that $(X_t)_{t\in T}$ is uniformly continuous for a certain metric if the random variables X_t have been defined everywhere in such a way that for almost every ω, the trajectories $t\to X_t(\omega)$ are uniformly continuous. The following result is proved as in the Gaussian case ([25], page 127), and shows that d is what really matters.

PROPOSITION 6.5. *$(X_t)_{t\in T}$ is uniformly continuous for τ if and only if d is τ continuous and $(X_t)_{t\in T}$ is uniformly continuous for d.*

THEOREM 6.6. *Suppose that $(X_t)_{t\in T}$ is uniformly continuous for d. Then there exists a probability measure μ on T with the following property. Denote by i the largest integer for which $D_i(T)\leq 1$. For $j\geq i$, $t\in T$, define $n(t,j)$ as the smallest integer for which*

$$\mu\big(B_j(t,2^n)\big)\geq e^{-2^n}.$$

Then

$$\lim_{j\to\infty}\sup_{t\in T}\sum_{l\geq j}r^{-l}2^{n(t,l)}=0.$$

7. The decomposition theorem. Our first task is to describe rigorously the two classes of processes involved in Theorem 2.1.

DEFINITION 7.1. We say that the infinitely divisible process $(X_t)_{t\in T}$ with Lévy measure ν is L-controlled by a majorizing measure if we can find an increasing sequence $(\mathscr{A}_j)_{j \geq i}$ of partitions of T, (with $\mathscr{A}_i = \{T\}$) and a probability measure μ on T that satisfy

$$(7.1) \qquad \forall\, t \in T, \qquad \sum_{j \geq i} r^{-j}\Big(D_j\big(A_j(t)\big) + h\big(\mu\big(A_j(t)\big)\big)\Big) \leq L,$$

$$(7.2) \qquad \forall\, A \in \mathscr{A}_j, \forall\, s, t \in A, \qquad |\beta(s) - \beta(t)| \leq r^{-j} \qquad \nu \text{ a.s.}$$

COMMENT 1. The essential new point of that definition is (7.2). (See [17], where a similar idea occurs.)

COMMENT 2. The condition $D_i(T) < \infty$ implies that ν is a Lévy measure if we know that $\int_{\mathbb{R}^T} \beta(t)^2 \wedge 1\, d\nu(\beta) < \infty$ for at least one $t \in T$.

The motivation of that definition is the following.

THEOREM 7.2. If $(X_t)_{t \in T}$ is L-controlled by a majorizing measure, then $E \sup_{s,t \in T} |X_s - X_t| \leq KL$.

PROOF. In the case where T is infinite, by the conclusion we simply mean that $E \sup_{s,t \in U} |X_s - X_t| \leq KL$ when $U \subset T$ is finite. The proof of Lemma 4.1 shows how to transport μ to U so that (7.1) holds for $t \in U$. Thus to prove Theorem 7.1 we can assume that T is finite.

We observe that there is no loss of generality to assume that $\mathscr{A}_{i+1}$ is not trivial [by replacing the sequence $(\mathscr{A}_j)_{j \geq i}$ by $(\mathscr{A}_j)_{j \geq i'}$ where i' is the largest integer such that $\mathscr{A}_{i'}$ is trivial]. This implies that $r^{-i} \leq KL$, since at least one element A of $\mathscr{A}_{i+1}$ must satisfy $h(\mu(A)) \geq \log 2$.

For $j \geq i$, $A \in \mathscr{A}_j$, let us pick any point $t_A \in A$. If $j > i$, we denote by A' the unique element of $\mathscr{A}_{j-1}$ that contains A. By (7.2) we have $|\beta(t_A) - \beta(t_{A'})| \leq r^{-j+1}\nu$ a.s. We now consider Rosinski's representation of the process $(X_t)_{t \in U}$.

We observe that $\sup_{s,t \in T} |\varepsilon_1 R_1 Y_1(s) - \varepsilon_1 R_1 Y_1(t)| \leq r^{-i} \leq KL$ by (7.2). Thus it is enough to show that $E \sup_{s,t \in T} |Z_s - Z_t| \leq KL$, where $Z_t = \sum_{k \geq 2} \varepsilon_k R_k Y_k(t)$. The point of dropping the first term in the series representation is that we will be able to work with $\alpha' = \inf_{k \geq 2} \tau_k / k$ instead of α^-, and that $1/\alpha'$ is integrable, while $1/\alpha^-$ is not. We set $G_{A,k} = R_k(Y_k(t_A) - Y_k(t_{A'}))$. The proof of Proposition 2.12(b) shows that for $V \geq (4/\alpha')D_{j-1}(A')$ we have, conditionally on w_0,

$$P\bigg(\sum_{k \geq 2} G_{A,k}^2 \geq r^{-2j+2} V\bigg) \leq \exp\bigg(-\frac{V}{2}\bigg).$$

We use this with

$$(7.3) \qquad V = V_A = \frac{4}{\alpha'} D_{j-1}(A') + 2h(\mu(A)) + 2(j-i) + 2v,$$

where $v \geq 0$ is a parameter. We thus obtain

$$P\left(\sum_{k\geq 2} G_{A,k}^2 \geq r^{-2j+2} V_A\right) \leq \mu(A) e^{-(j-i)-v}.$$

We observe that $\sum_{A\in\mathscr{A}_{j+1}} \mu(A) \leq 1$, $\sum_{j>i} e^{-(j-i)} \leq 1$. Thus the event

$$\Omega(v) = \left\{\omega; \forall j > i, \forall A \in \mathscr{A}_j, \sum_{k\geq 2} G_{A,k}^2 \leq r^{-2j+2} V_A\right\}$$

is such that $P(\Omega(v)) \geq 1 - e^{-v}$.

We recall the "sub-Gaussian inequality"

$$P\left(\left|\sum_{k\geq 1} \varepsilon_k x_k\right| \geq u\right) \leq 2\exp\left(-\frac{u^2}{2\sum_{k\geq 1} x_k^2}\right).$$

We use this inequality conditionally on $w_0, w \in \Omega(v)$, to get

$$Q\left(\left|\sum_{k\geq 2} \varepsilon_k G_{A,k}\right| \geq r^{-j+1} V_A\right) \leq 2\exp\left(-\frac{\left(r^{-j+1}V_A\right)^2}{2r^{-2j+2}V_A}\right) = 2\exp\left(-\frac{V_A}{2}\right).$$

Thus, if we consider the event

$$\Omega'(v) = \left\{\overline{\omega}; \forall j > i, \forall A \in \mathscr{A}_j, \left|\sum_{k\geq 2} \varepsilon_k G_{A,k}\right| \leq r^{-j+1} V_A\right\}$$

we have, arguing as before, that $\Pr(\Omega'(v)) \leq 1 - 3\exp(-v)$ where we recall that $\Pr = P_1 \otimes P \otimes Q$.

Consider $t \in T$. We set $t_i = t_T$, $t_j = t_{A_j(t)}$ for $j \geq i$. Thus

$$Z_t - Z_{t_i} = \sum_{j\geq i+1} Z_{t_j} - Z_{t_{j-1}}.$$

Now, on $\Omega'(v)$,

$$|Z_{t_j} - Z_{t_{j-1}}| = \left|\sum_{k\geq 2} \varepsilon_k G_{A_j(t)}\right| \leq r^{-j+1} V_{A_j(t)}$$

and thus

$$|Z_t - Z_{t_i}| \leq \sum_{j\geq i+1} r^{-j+1} V_{A_j(t)}.$$

Calculation using (7.1) and (7.3) shows that, since $r^{-i} \leq KL$,

$$\begin{aligned}\sum_{j\geq i+1} r^{-j+1} V_{A_j(t)} &\leq \frac{4}{\alpha'} \sum_{j\geq i+1} r^{-j+1}\left(D_{j-1}\left(A_{j-1}(t)\right)\right)\\ &\quad + 2\sum_{j\geq i+1} r^{-j+1} h\left(\mu\left(A_j(t)\right)\right) + r^{-i}(K+v)\\ &\leq K\left(v + \frac{1}{\alpha'}\right)L.\end{aligned}$$

Thus $\sup_{s,t\in T}|Z_t - Z_s| \le K(v + 1/\alpha')L$ on $\Omega'(v)$. If we set

$$v(\overline{\omega}) = \inf\{v; \overline{\omega} \in \Omega'(v)\},$$

we thus have

$$\sup_{s,t\in T} |Z_t - Z_s|(\overline{\omega}) \le K\left(v(\overline{\omega}) + \frac{1}{\alpha'(\overline{\omega})}\right)L,$$

and the result follows since $Ev \le K$, $E(1/\alpha') \le K$. □

Consider a symmetric infinitely divisible process $(X_t)_{t\in T}$ of Lévy measure ν. Assume that $\int_{\mathbb{R}^T}|\beta(t)| \wedge 1\, d\nu(\beta) < \infty$ for all $t \in T$. The measure ν', image of ν under the map $\beta \to |\beta|$ satisfies $\int_{\mathbb{R}^T}|\beta(t)| \wedge 1\, d\nu'(\beta) < \infty$ for each $t \in T$, and is supported by $(\mathbb{R}^+)^T$. We can thus define a positive infinitely divisible process $(|X|_t)_{t\in T}$ of Lévy measure ν' that satisfies (1.6).

Suppose that $U \subset T$ is finite, and that we have Rosinski's representation (in distribution) of $(X_t)_{t\in U}$ as $\sum_{k\ge1}\varepsilon_k R_k Y_k$. It follows from Theorem 2.8 that $(|X|_t)_{t\in U}$, in distribution, is equal to $\sum_{k\ge1}R_k|Y_k|$, since, with the notation of that theorem, its Lévy measure ν' is the image of $\lambda \otimes m$ under the map $(x,\beta) \to R(x,\beta)|\beta| = |R(x,\beta)\beta|$, while the Lévy measure ν of $(X_t)_{t\in U}$ is the image of $\lambda \otimes m$ under the map $(x,\beta) \to R(x,\beta)\beta$. Thus ν' is the image of ν under the map $\beta \to |\beta|$.

We observe that

$$\sup_{t\in U}\left|\sum_{k\ge1}\varepsilon_k R_k Y_k(t)\right| \le \sup_{t\in U}\sum_{k\ge1}R_k|Y_k|(t) \tag{7.4}$$

for all finite sets $U \subset T$. Thus the boundedness of $(|X|_t)_{t\le T}$ makes it obvious that $(X_t)_{t\in T}$ is bounded. By the expression "$(X_t)_{t\in T}$ is controlled by a positive process" used in the introduction, we had in mind the case where the process $(|X|_t)_{t\in T}$ itself is bounded.

A somewhat surprising fact is that the boundedness of $\sup_{t\in T}|X_t|$, together with the finiteness of $\sup_{t\in T}\int_{\mathbb{R}^T}|\beta(t)|\,d\nu(\beta)$, controls the boundedness of $|X|_t$.

THEOREM 7.3. *Given $\varepsilon > 0$, we can find a number $K(\varepsilon)$ depending only ε, such that*

$$P\left(\sup_{t\in T}|X_t| \ge M\right) \le \frac{\varepsilon}{25}$$

$$\Rightarrow P\left(\sup_{t\in T}|X|_t \ge K(\varepsilon)\left(M + \sup_{t\in T}\int|\beta(t)|\,d\nu(\beta)\right)\right) \le \varepsilon.$$

PROOF. We will not attempt in the proof to give a sharp dependence of $K(\varepsilon)$ on ε. We can assume that T is finite; thus we can use Rosinski's representation $\sum_{k\ge1}\varepsilon_k R_k Y_k(t)$ of $(X_t)_{t\in T}$.

Consider $\omega_0 \in \Omega_0$ fixed, and define a by

$$P \otimes Q\left\{\sup_{t\in T}\left|\sum_{k\geq 1} \varepsilon_k R_k Y_k(t)\right| \leq M\right\} = 1 - a. \tag{7.5}$$

Using Fubini's theorem, we see that $P(\Omega') \geq 1 - 4a$ where

$$\Omega' = \left\{\omega; Q\left(\sup_{t\in T}\left|\sum_{k\geq 1} \varepsilon_k R_k(\omega) Y_k(\omega)(t)\right| \leq M\right) \geq 3/4\right\}.$$

We now use the fact that for vectors x_k in a Banach space, we have

$$Q\left(\left\|\sum \varepsilon_k x_k\right\| \leq M\right) \geq 3/4 \Rightarrow E\left\|\sum \varepsilon_k x_k\right\| \leq KM.$$

This follows, for example, from the version of Theorem 2.5 given in [28] and the inequality of [9], page 31, equation (3). Thus we have that conditionally on ω_0 and on $\omega \in \Omega'$,

$$E\left(\sup_{t\in T}\left|\sum_{k\geq 1} \varepsilon_k R_k Y_k(t)\right|\right) \leq KM.$$

We now use Theorem 2.1(b) with $f_i(x) = |x|$; we get, conditionally on ω_0 and $\omega \in \Omega'$,

$$E\left(\sup_{t\in T}\left|\sum_{k\geq 1} \varepsilon_k R_k |Y_k(t)|\right|\right) \leq KM.$$

Thus, for any $\eta > 0$, using Fubini's theorem again, we have

$$P \otimes Q\left(\sup_{t\in T}\left|\sum_{k\geq 1} \varepsilon_k R_k |Y_k(t)|\right| \geq \frac{KM}{\eta}\right) \leq 4a + \eta.$$

Consider now a sequence (Z_k), distributed like $(R_k|Y_k|)_{k\geq 1}$ but independent of all the other sequences used. It is defined on a new probability space (Ω', P'). Then we have

$$P' \otimes Q\left(\sup_{t\in T}\left|\sum_{k\geq 1} \varepsilon_k Z_k(t)\right| \geq \frac{KM}{\eta}\right) \leq 4a + \eta$$

so that

$$P' \otimes P \otimes Q\left(\sup_{t\in T}\left|\sum_{k\geq 1} \varepsilon_k (R_k|Y_k(t)| - Z_k(t))\right| \geq \frac{2KM}{\eta}\right) \leq 8a + 2\eta.$$

The sequences $(\varepsilon_k(R_k|Y_k| - Z_k))_{k\geq 1}$ and $R_k|Y_k| - Z_k$ are equidistributed. So we have

$$P' \otimes P\left(\sup_{t\in T}\left|\sum_{k\geq 1} (R_k|Y_k|(t) - Z_k(t))\right| \geq \frac{2KM}{\eta}\right) \leq 8a + 2\eta.$$

If we set

$$\Omega^2 = \left\{(\omega', \omega) \in \Omega' \times \Omega;\ \sup_{t\in T}\left|\sum_{k\ge 1}(R_k|Y_k|(t) - Z_k(t))\right| \le \frac{2KM}{\eta}\right\},$$

we thus have $P' \otimes P(\Omega^2) \ge 1 - 8a - 2\eta$.

Set

$$\Omega^3 = \{\omega \in \Omega;\ P'(\{\omega';\ (\omega', \omega) \in \Omega^2\}) \ge 1/2\}.$$

Thus $P(\Omega^3) \ge 1 - 16a - 4\eta$ by Fubini's theorem. By Lemma 2.10, we have

$$\forall\, t \in T, \qquad E\sum_{k\ge 1} Z_k(t) = E\sum_{k\ge 1} R_k|Y_k(t)| \le \frac{1}{\alpha^-}\int|\beta(t)|\,d\nu(\beta).$$

Thus if $\omega \in \Omega^3$ and $t \in T$ we can find ω' with $(\omega', \omega) \in \Omega^2$ and

$$\sum_{k\ge 1} Z_k(\omega')(t) \le \frac{2}{\alpha^-}\int|\beta(t)|\,d\nu(\beta).$$

This implies

$$\sum_{k\ge 1} R_k(\omega)|Y_k(\omega)(t)| \le \frac{2KM}{\eta} + \frac{2}{\alpha^-}\sup_{t\in T}\int|\beta(t)|\,d\nu(\beta).$$

This holds for $t \in T$, $\omega \in \Omega^3$. Thus

$$(7.6)\qquad P\left(\sup_{t\in T}\sum_{k\ge 1} R_k|Y_k(t)| \ge \frac{2KM}{\eta} + \frac{2}{\alpha^-}\sup_{t\in T}\int|\beta(t)|\,d\nu(\beta)\right) \le 16a + 4\eta.$$

If now we know that

$$P\left(\sup_{t\in T}|X_t| \ge M\right) \le \eta,$$

using the fact that (7.5) implies (7.6) conditionally on ω_0, we get by Fubini's theorem that

$$P\left(\sup_{t\in T}|X|_t \ge \frac{2KM}{\eta} + \frac{2}{\alpha^-}\sup_{t\in T}\int|\beta(t)|\,d\nu(\beta)\right) \le 20\eta.$$

Since $\alpha^- > 0$ a.e., the result follows, by taking $\eta = \varepsilon/25$, $K(\varepsilon) \ge 50M/\varepsilon$ large enough that $P(2/\alpha^- \ge K(\varepsilon)) \le \varepsilon/100$. □

We now can state and prove the decomposition theorem.

THEOREM 7.4. *Consider an infinitely divisible process $(X_t)_{t\in T}$, and assume that condition $H(\delta, v_0)$ holds. Consider M such that for each finite subset U of T we have*

$$P\left(\sup_{t\in U}|X_t| \ge M\right) \le \tfrac{1}{4}.$$

Then we can write (in distribution) $X_t = X_t^1 + X_t^2$, *where* $(X_t^1)_{t\in T}$, $(X_t^2)_{t\in T}$ *are infinitely divisible processes with the following properties*:

(a) $(X_t^1)_{t\in T}$ *is KM-controlled by a majorizing measure and* $X_{t_0}^1 = X_{t_0}$ *for a certain* $t_0 \in T$.

(b) $(X_t^2)_{t\in T}$ *has the following property. Given* $\varepsilon > 0$ *and* $M_\varepsilon \geq M$, *for all finite subsets* U *of* T *we have*

$$(7.7)\quad P\Big(\sup_{s,t\in U}|X_t - X_s| \geq M_\varepsilon\Big) \leq \frac{\varepsilon}{50} \Rightarrow P\Big(\sup_{t\in U}|X^2|_t \geq K(\varepsilon)M_\varepsilon\Big) \leq \varepsilon.$$

The constant K *depends on* δ, v_0 *only*; *the constant* $K(\varepsilon)$ *depends on* ε, δ, v_0 *only.*

COMMENT 1. The decomposition is not unique, and the two pieces are not independent. If $(X_t)_{t\in T}$ is p-stable, neither $(X_t^1)_{t\in T}$ nor $(X_t^2)_{t\in T}$ will be p-stable.

COMMENT 2. If T is reduced to one point, the theorem is void, as $X^2 = 0$.

COMMENT 3. Since $X_{t_0}^1 = X_{t_0}$, Theorem 7.2 shows that

$$E\sup_{t\in U}|X_t^1 - X_{t_0}^1| \leq KM$$

for all finite subsets U of T. In other words, the boundedness of $(X_t^1)_{t\in T}$ is obvious from the fact that it is controlled by a majorizing measure. The boundedness of (X_t^2) is made obvious by the boundedness of $(|X^2|_t)$ [see (7.4)]. Not only is the boundedness of $(X_t)_{t\in T}$ obvious from the decomposition, it is also determined in a quantitative way. This statement is somewhat obscured by the fact that we have no very simple measure of boundedness, so that we have to use the quantiles of $\sup_{t\in U}|X_t|$ in (7.7) to express that $\sup_{t\in U}|X^2|_t$ is controlled by $\sup_{t\in U}|X_t|$. We have felt reluctant to add unnecessary conditions just in order to ensure the integrability of $\sup_{t\in T}|X_t|$ and get a simpler statement. But if we would do that, then (b) would become

$$E\sup_{t\in U}|X^2|_t \leq KE\sup_{t\in U}|X_t|,$$

so that the magnitude of $E\sup_{t\in U}|X_t|$ would be evident from the decomposition. More precisely, we would obtain that $E\sup_{t\in T}|X_t|$ is of order

$$\inf\Big\{M;\ X_t = X_t^1 + X_t^2,\ X_t^1 \text{ is } M\text{-controlled by a majorizing measure, } E\sup_{t\in T}|X^2|_t \leq M\Big\}.$$

PROOF. We combine Theorem 1.1 and Theorem 3.2 to obtain an increasing sequence of partitions $(\mathscr{A}_j)_{j\geq i}$ of T and a probability measure μ on T such that

$$\sup_{t\in T}\sum_{j\geq i} r^{-j}\Big(D_j\big(A_j(t)\big)+h\big(\mu\big(A_j(t)\big)\big)\Big)\leq KM. \tag{7.8}$$

We observe that (as in the proof of Theorem 7.2) there is no loss of generality to assume $r^{-i}\leq KM$. By induction over j, for each $A\in\mathscr{A}_j$, we selected a point $s_A\in A$. We do this in such a way that when $A\subset B$, $B\in\mathscr{A}_{j-1}$, if $s_B\in A$, we set $s_A=s_B$. For $t\in T$, we set $\pi_j(t)=s_{A_j(t)}$. We observe the relations

$$l\geq j\Rightarrow \pi_l\big(\pi_j(t)\big)=\pi_j(t);\qquad l\leq j\Rightarrow \pi_l\big(\pi_j(t)\big)=\pi_l(t).$$

We set $S=\{s_A,\ A\in\mathscr{A}_j,\ j\geq i\}$. We will avoid many technical difficulties by first restricting the process to S. We observe that for $t\in S$, we have $\pi_k(t)=t$ for k large enough.

For $\beta\in\mathbb{R}^S$, $t\in S$, we define

$$l(\beta,t)=\inf\big\{l\geq i;\ \big|\beta(\pi_l(t))-\beta(\pi_{l+1}(t))\big|>\tfrac14 r^{-l}\big\}$$

whenever the set on the right is not empty. When that set is empty, we set $l(\beta,t)=\infty$.

We now study the map $\beta\to\bar\beta$ from $\mathbb{R}^S$ to $\mathbb{R}^S$ given by

$$\bar\beta(t)=\beta\big(\pi_{l(\beta,t)}(t)\big),$$

where we set $\pi_\infty(t)=t$. For $l\geq i$, we define the set $V(l,t)$ by

$$\begin{aligned}V(l,t)&=\{\beta\in\mathbb{R}^S;\ l(\beta,t)=l\}\\&=\Big\{\beta\in\mathbb{R}^S;\ \forall\, i\leq j<l,\ \big|\beta\big(\pi_j(t)\big)-\beta\big(\pi_{j+1}(t)\big)\big|\leq\tfrac14 r^{-l},\\&\qquad\qquad \big|\beta\big(\pi_l(t)\big)-\beta\big(\pi_{l+1}(t)\big)\big|>\tfrac14 r^{-l}\Big\},\\V(\infty,t)&=\{\beta\in\mathbb{R}^S,\ l(\beta,t)=\infty\}\\&=\Big\{\beta\in\mathbb{R}^S,\ \forall\, j\geq i,\ \big|\beta\big(\pi_j(t)\big)-\beta\big(\pi_{j+1}(t)\big)\big|\leq\tfrac14 r^{-j}\Big\}.\end{aligned}$$

LEMMA 7.5. *Let $t\in S$, $\beta\in V(l,t)$. If $j\leq l$, we have $\bar\beta(\pi_j(t))=\beta(\pi_j(t))$. If $j>l$, we have $\bar\beta(\pi_j(t))=\beta(\pi_l(t))$.*

PROOF. If $j\leq l$, since $\pi_k(\pi_j(t))=\pi_{\min(k,j)}(t)$, we have

$$\Big|\beta\Big(\pi_k\big(\pi_j(t)\big)\Big)-\beta\Big(\pi_{k+1}\big(\pi_j(t)\big)\Big)\Big|\leq\tfrac14 r^{-k}$$

for all $k\geq i$ so that by definition, $\bar\beta(\pi_j(t))=\beta(\pi_j(t))$.

If $j>l$, since $\pi_k(\pi_j(x))=\pi_k(x)$ for $k\leq j$, the result follows from the definition of $\bar\beta$. □

LEMMA 7.6. *For $t\in S$, $\beta\in\mathbb{R}^S$, $j\geq i$, we have*

$$\big|\bar\beta\big(\pi_{j+1}(t)\big)-\bar\beta\big(\pi_j(t)\big)\big|\leq\big|\beta\big(\pi_{j+1}(t)\big)-\beta\big(\pi_j(t)\big)\big|1_{\{|\beta(\pi_{j+1}(t))-\beta(\pi_j(t))|\leq r^{-j}/4\}}.$$

PROOF. From Lemma 7.5 we can have $\bar{\beta}(\pi_{j+1}(t)) \neq \bar{\beta}(\pi_j(t))$ only when $\beta \in V(l,t)$ for some $l \geq j+1$. But in that case

$$\bar{\beta}(\pi_{j+1}(t)) = \beta(\pi_{j+1}(t)), \qquad \bar{\beta}(\pi_j(t)) = \beta(\pi_j(t))$$

and $|\beta(\pi_{j+1}(t)) - \beta(\pi_j(t))| \leq (1/4)r^{-j}$ since $\beta \in V(l,t)$. □

LEMMA 7.7. *For $A \in \mathscr{A}_j$, $s,t \in A$, we have*

$$|\bar{\beta}(s) - \bar{\beta}(t)| \leq r^{-j}.$$

PROOF. Since $\pi_j(s) = \pi_j(t)$, it suffices to show that $|\bar{\beta}(t) - \bar{\beta}(\pi_j(t))| \leq (1/2)r^{-j}$. Since $t \in S$, $t = \pi_k(t)$ for k large enough, so that by Lemma 7.6,

$$|\bar{\beta}(t) - \bar{\beta}(\pi_j(t))| \leq \sum_{k \geq j} |\bar{\beta}(\pi_{k+1}(t)) - \bar{\beta}(\pi_k(t))| \leq \sum_{k \geq j} \tfrac{1}{4} r^{-k} \leq \tfrac{1}{2} r^{-j}. \quad \square$$

Given $t \in S$, the value of $\bar{\beta}(t)$, $\beta \in \mathbb{R}^S$ depends only on the values $\beta(\pi_k(t))$ and there are only finitely many of the points $\pi_k(t)$. Thus it is routine to see that we can define a (cylindrical) measure $\bar{\nu}$ on $\mathbb{R}^S$, as the image of ν under the map $\beta \to \bar{\beta}$. (Since no problems arise when T is finite, we leave the details of that point to the reader.) We now proceed to prove that $\bar{\nu}$ is a Lévy measure, and that the process $(X_t^1)_{t \in S}$ with Lévy measure $\bar{\nu}$ is *KM*-controlled by a majorizing measure. From Lemma 7.7 we have $|\beta(s) - \beta(t)| \leq r^{-j}\bar{\nu}$ a.e. when $s,t \in A \in \mathscr{A}_j$.

Consider the distance d on S given by

$$d(s,t)^2 = \int |\beta(s) - \beta(t)|^2 \, d\bar{\nu}(\beta)$$
$$= \int |\bar{\beta}(s) - \bar{\beta}(t)|^2 \, d\nu(\beta).$$

From Lemma 7.6, we have

$$(7.9)\qquad \begin{aligned} r^{2j} d(\pi_{j+1}(t), \pi_j(t))^2 &\leq \int r^{2j} |\bar{\beta}(\pi_{j+1}(t)) - \bar{\beta}(\pi_j(t))|^2 \, d\nu(\beta) \\ &\leq \int r^{2j} |\beta(\pi_{j+1}(t)) - \beta(\pi_j(t))|^2 \wedge 1 \, d\nu(\beta) \\ &\leq D_j(A_j(t)). \end{aligned}$$

Here we should point out that if in condition (7.1), one replaces $D_j(A_j(t))$ by the smaller quantity $\varphi_j(\pi_{j+1}(t), \pi_j(t))$, the proof of Theorem 7.2 still works. If we had settled for this weaker definition of "controlled by a majorizing measure," the proof would be essentially finished. But we must check condition (7.1) for

$$\bar{D}_j(A) = \sup_{s,s' \in A} \int r^{2j} |\beta(s) - \beta(s')|^2 \wedge 1 \, d\bar{\nu}(\beta).$$

There seems to be no reason why the sequence of partitions $(\mathscr{A}_l)$ would work. This sequence of partitions must be refined, and this will require significant extra work. Since this refinement must take place on T, we must work on T. From (7.9) we have

$$(7.10)\qquad \begin{aligned} d\big(\pi_j(t),\pi_{j-1}(t)\big) &\le r^{-j}D_j\big(A_j(t)\big)^{1/2} \\ &\le r^{-j/2}\Big(r^{-j}D_j\big(A_j(t)\big)\Big)^{1/2}. \end{aligned}$$

Thus, using (7.8) and Cauchy–Schwarz,

$$(7.11)\qquad \sum_{j\ge i} d\big(\pi_j(t),\pi_{j+1}(t)\big) \le Kr^{-i/2}M^{1/2} \le KM.$$

We now extend the cylindrical measure $\bar{\nu}$ to a cylindrical measure ν' on $\mathbb{R}^T$ in the following way. Given a finite set $U=\{t_1,\dots,t_n\}$ of T, the projection of ν' on $\mathbb{R}^U$ is defined as follows: We consider $j\ge i$, the projection of $\bar{\nu}$ on $\mathbb{R}^{U_j}$, where $U_j=\{\pi_j(t_1),\dots,\pi_j(t_n)\}$. We transport this projection on $\mathbb{R}^U$ in the obvious manner, and we take the limit in measure as $j\to\infty$. [That this limit exists follows from (7.11).]

Consider a probability measure μ' on T that gives mass $2^{-j+i+1}\mu(A)$ to the point s_A, for all $j\ge i$ and all $A\in\mathscr{A}_j$. For $t\in T$, we define

$$n(t,j)=\inf\{n\ge 0;\ \mu'\big(B(t,r^{-j}2^{n/2})\big)\ge e^{-2^n}\},$$

where $B(t,\varepsilon)=\{s\in T;\ d(t,s)\le\varepsilon\}$. We proceed to prove that

$$(7.12)\qquad \sup_{t\in T}\sum_{j\ge i} r^{-j}2^{n(t,j)} \le KM.$$

For that purpose, we fix $t\in T$. For $j\ge i$, we set $a_j=d(\pi_j(t),\pi_{j+1}(t))$ and $b_j=r^{-j}h(2^{-j+i+1}\mu(A_j(t)))$. Thus, by (7.8), and since $r^{-i}\le KM$, we have

$$(7.13)\qquad \sum_{j\ge i} b_j \le KM.$$

The ball $B(t,\sum_{k\ge j}a_k)$ contains $s_{A_j(t)}$. Thus, we have

$$\mu'\left(B\left(t,\sum_{k\ge j}a_k\right)\right)\ge \mu'(s_{A_j(t)})\ge e^{-r^jb_j}.$$

The definition of $n(t,j)$ then shows that

$$2^{n(t,j)}\le\max\left(1,r^{2j}\left(\sum_{k\ge j}a_k\right)^2,r^jb_j\right).$$

Thus

$$(7.14)\qquad r^{-j}2^{n(t,j)}\le r^{-j}+r^j\left(\sum_{k\ge j}a_k\right)^2+b_j.$$

By (7.10), we have $a_k\le r^{-k/2}c_k^{1/2}$, where $c_k=r^{-k}D_k(A_k(t))$. By convexity of the function $x \to x^2$, we have

$$r^j\Big(\sum_{k\geq j} a_k\Big)^2 \leq \Big(\sum_{k\geq j} r^{(j-k)/2}c_k^{1/2}\Big) \leq K\sum_{k\geq j} r^{(j-k)/2}c_k.$$

Since $\sum_{k\geq i} c_k \leq KM$ by (7.8), (7.12) follows from (7.13) and (7.14).

We consider now $\overline{\varphi}_j(s,t) = r^{2j}d^2(s,t)$, and we denote by $\overline{\theta}_j$, $\overline{D}_j$ the quantities defined like θ_j, D_j, but using $\overline{\varphi}_j$ instead of φ_j.

By Theorem 3.2, we have

$$\overline{\theta}_i(T) \leq K\big(M + r^{-i}\overline{D}_i(T)\big).$$

By (7.10), we have $d(t,\pi_i(t)) \leq Kr^{-i/2}M^{1/2}$. Since for $s,t\in T$ we have $\pi_i(s) = \pi_i(t)$, we have $d(s,t) \leq Kr^{-i/2}M^{1/2}$. Thus

$$\overline{D}_i(T) = \sup_{s,t\in T} r^{2i}d^2(s,t) \leq Kr^iM,$$

and hence $\overline{\theta}_i(T) \leq KM$. This means that one can find an increasing sequence $(B_j)_{j\geq i}$ of partitions of T and a probability measure μ'' on T such that

$$(7.15)\qquad \forall\, t\in T,\qquad \sum_{j\geq i} r^{-j}\big(\overline{D}_j(B_j(t)) + h\big(\mu''(B_j(t))\big)\big) \leq KM.$$

Consider the sequence $(\mathscr{C}_j)_{j\geq i}$ of partitions of T, where $\mathscr{C}_j$ is generated by $\mathscr{A}_j$ and $\mathscr{B}_j$. We have

$$s,t\in C\in\mathscr{C}_j \Rightarrow |\beta(s)-\beta(t)| \leq r^{-j}\qquad \overline{\nu}\text{ a.s.}$$

Since $C_j(t)\subset B_j(t)$, we also have

$$\forall\, t\in T,\qquad \sum_{j\geq i} r^{-j}\overline{D}_j\big(C_j(t)\big) \leq KM.$$

Consider a probability measure $\overline{\mu}$ on T that give mass $\geq 2^{-j+i-1}\mu(A)\mu''(B)$ to an arbitrary point of $A\cap B$, whenever $A\in\mathscr{A}_j$, $B\in\mathscr{B}_j$, $A\cap B\neq\varnothing$. Thus

$$\overline{\mu}\big(C_j(t)\big) \geq 2^{-j+i-1}\mu\big(A_j(t)\big)\mu''\big(B_j(t)\big)$$

and hence, by (7.8) and (7.15),

$$\forall\, t\in T,\qquad \sum_{j\geq i} r^{-j}h\big(\overline{\mu}(C_j(t))\big) \leq KM.$$

This completes the proof that $(X_t^1)_{t\in S}$ is KM-controlled by a majorizing measure. □

LEMMA 7.8. *For all* $t\in S$, $\int|\beta(t)-\overline{\beta}(t)|\,d\nu(\beta) \leq KM$.

PROOF. We have

$$\int|\beta(t)-\overline{\beta}(t)|\,d\nu(\beta) = \sum_{j\geq i}\int_{V(j,t)}\big|\beta(t)-\beta\big(\pi_j(t)\big)\big|\,d\nu(\beta),$$

since $\overline{\beta}(t) = \beta(\pi_j(t))$ on $V(j,t)$, and $\overline{\beta}(t) = \beta(t)$ on $V(\infty,t)$. For $\beta\in V(j,t)$, we have

$$\left|\beta\left(\pi_{j+1}(t)\right) - \beta\left(\pi_j(t)\right)\right| \geq \tfrac{1}{4}r^{-j}$$

so that either we have

$$\left|\beta(t) - \beta\left(\pi_j(t)\right)\right| \geq \tfrac{1}{8}r^{-j}$$

or we have

$$\left|\beta(t) - \beta\left(\pi_{j+1}(t)\right)\right| \geq \tfrac{1}{8}r^{-j} \geq \left|\beta(t) - \beta\left(\pi_j(t)\right)\right|.$$

Thus

$$\int_{V(j,t)} \left|\beta(t) - \beta\left(\pi_j(t)\right)\right| d\nu(\beta) \leq H_j + H_{j+1},$$

where

$$H_j = \int \left|\beta(t) - \beta\left(\pi_j(t)\right)\right| 1_{\{|\beta(t) - \beta(\pi_j(t))| \geq r^{-j}/8\}}\, d\nu(\beta).$$

It follows from condition $H(v_0, \delta)$, (by an argument given in the proof of Proposition 2.14) that

$$\begin{aligned} H_j &\leq Kr^{-j}\nu\left(\left\{\left|\beta(t) - \beta\left(\pi_j(t)\right)\right| \geq \tfrac{1}{8}r^{-j}\right\}\right) \\ &\leq 64Kr^{-j} \int \left|\beta(t) - \beta\left(\pi_j(t)\right)\right|^2 r^{2j} \wedge 1\, d\nu(\beta) \\ &\leq Kr^{-j}\varphi_j\left(t, \pi_j(t)\right) \leq Kr^{-j} D_j\left(A_j(t)\right), \end{aligned}$$

so that the conclusion of the lemma follows from (7.8).

We denote by ν^2 the image of ν under the map $\beta \to \beta - \bar{\beta}$. It follows from Lemma 7.8 that

$$\sup_{t \in S} \int |\beta(t)|\, d\nu^2(\beta) \leq KM.$$

We denote by $(X_t^2)_{t \in S}$ the process with Lévy measure ν^2.

Consider now a finite set $U \subset S$. Set $U' = \{\pi_j(t);\ j \geq i,\ t \in U\}$. This is a finite set. Consider a representation $\sum_{k \geq 1} \varepsilon_k R_k Y_k$ of $(X_t)_{t \in U'}$. For $t \in U'$, $\beta \in \mathbb{R}^S$, $\bar{\beta}(t)$ depends only the value of β on U', so we can define the map $\beta \to \bar{\beta}$ on $\mathbb{R}^{U'}$.

Thus, as was discussed after Theorem 2.8, $\sum_{k \geq 1} \varepsilon_k R_k \bar{Y}_k$ is distributed like $(X_t^1)_{t \in U'}$ while $\sum_{k \geq 1} \varepsilon_k R_k (Y_k - \bar{Y}_k)$ is distributed like $(X_t^2)_{t \in U'}$. Also

$$\sum_{k \geq 1} \varepsilon_k R_k Y_k = \sum_{k \geq 1} \varepsilon_k R_k \bar{Y}_k + \sum_{k \geq 1} \varepsilon_k R_k \left(Y_k - \bar{Y}_k\right), \tag{7.16}$$

so that we can find three processes $(Z_t^i)_{t \in U}$, $0 \leq i \leq 2$ such that $(Z_t^0)_{t \in U} =_{\mathscr{D}} (X_t)_{t \in U}$, $(Z_t^i)_{t \in U} =_{\mathscr{D}} (X_t^i)_{t \in U}$ for $i = 1, 2$ and $Z_t^0 = Z_t^1 + Z_t^2$. This is what we mean by $X_t =_{\mathscr{D}} X_t^1 + X_t^2$. We observe that, by definition, $\pi_l(t_T) = t_T$ for all

$l \geq i$, so that $\bar{\beta}(t_T) = \beta(t_T)$. Thus, by (7.16),

$$\begin{aligned}\sup_{t \in U} |X_t^2| &\leq \sup_{t, s \in U'} |X_t^2 - X_s^2| \\ &=_{\mathscr{D}} \sup_{t, s \in U'} |Z_t^2 - Z_s^2| \\ &\leq \sup_{t, s \in U'} |Z_t^0 - Z_s^0| + \sup_{s, t \in U'} |Z_t^1 - Z_s^1|.\end{aligned}$$

Thus

$$\begin{aligned}P\Big(\sup_{t \in U}|X_t^2| \geq 2u\Big) &\leq P\Big(\sup_{t, s \in U}|X_s - X_t| \geq u\Big) \\ &\quad + P\Big(\sup_{t, s \in U}|X_s^1 - X_t^1| \geq u\Big).\end{aligned} \tag{7.17}$$

By Theorem 7.2, the second term on the right is less than or equal to KM/u. To prove (7.7), if $P(\sup_{t, s \in U}|X_s - X_t| \geq M_\varepsilon) \leq \varepsilon/50$, it follows from (7.12) that $P(\sup_{t \in U}|X_t^2| \geq 50KM_\varepsilon/\varepsilon) \leq \varepsilon/25$ so the result follows from Theorem 7.3 and Lemma 7.8.

It remains to define $(X_t^2)_{t \in T}$. For that purpose, we simply show that for each t, the sequence $(X^2_{\pi_j(t)})_{j \geq i}$ converges in measure. We observe that, by (7.8), we have $\varphi_j(t, \pi_j(t)) \leq r^j M$, so that, by (3.2) we have $\lim_{j \to \infty} \varphi_i(t, \pi_j(t)) = 0$. Also, by (7.11) we have $\lim_{t \to \infty} d(t, \pi_j(t)) = 0$. An easily proved converse of Lemma 6.1 then shows that the sequences $(X_{\pi_j(t)})_{j \geq i}, (X^1_{\pi_j(t)})_{j \geq i}$ converge in measure respectively to X_t and X_t^1, so that the sequence $(X^2_{\pi_j(t)})_{j \geq i}$ converges by taking the difference. This completes the proof of Theorem 7.4. □

Our last result is a "bracketing theorem" in the spirit of [1, 10, 18].

THEOREM 7.9. *Consider an infinitely divisible process* $(X_t)_{t \in T}$. *Consider an increasing sequence of partitions* $(\mathscr{A}_j)_{j \geq i}$ *of* T. *For* $A \subset T$, *we set*

$$\Delta_j(A) = \max\Big(D_j(A), \sup\Big\{\nu\Big(\Big\{\beta;\ \max_{s, t \in U}|\beta(s) - \beta(t)| \geq r^{-j-1}/4\Big\}\Big);\ U \subset A, U \text{ finite}\Big\}\Big).$$

Suppose that

$$\forall\, t \in T, \qquad \sum_{j \geq i} r^{-j}\Big(\Delta_j\big(A_j(t)\big) + h\Big(\mu\big(A_j(t)\big)\Big)\Big) \leq M. \tag{7.18}$$

Then, given $\varepsilon > 0$, *we can find a number* $K(\varepsilon)$, *depending on* ε *only, such that for all finite* $U \subset T$, *we have*

$$P\Big(\sup_{s, t \in U}|X_t - X_s| \geq K(\varepsilon)(M + r^{-i} + G)\Big) \leq \varepsilon, \tag{7.19}$$

where $G = \sup_{U \text{ finite}} \int \sup_{s, t \in U}(|\beta(s) - \beta(t)| 1_{\{|\beta(s) - \beta(t) \geq r^{-i}/4\}}) \, d\nu(\beta)$.

COMMENT 1. We now have a sufficient condition for boundedness at the expense of replacing the D_j by the much larger quantity Δ_j. Observe also that there is no condition $H(\delta, v_0)$ here.

COMMENT 2. Observe that

$$\Delta_j(A) \le K \sup\left\{\int_{\mathbb{R}^T}\left(\max_{s,t\in U}|\beta(s)-\beta(t)|^2\right)r^{2j} \wedge 1\, d\nu(\beta); U \subset A, U \text{ finite}\right\}.$$

PROOF. We keep the notations of Theorem 7.4. As in this theorem, we can replace T by S. We define $l(\beta, t), \bar{\nu}, (X_t^1)_{t\in S}$ as before. Since $D_j \le \Delta_j$, the proof of Theorem 7.4 shows that X_t^1 is KM-controlled by a majorizing measure. Thus it suffices to prove (7.14) with (X_t^2) instead of (X_t).

Consider a finite set $U \subset S$. Set $U' = \{\pi_j(t);\ t \in U,\ j \ge i\}$. This is a finite set. For $A \subset T$, $A \cap U' = \emptyset$, we set

$$f_A(\beta) = \sup\{|\beta(s) - \beta(t)|; s, t \in A \cap U'\}.$$

$$W_{A,j} = \{\beta; \exists\, s, t \in A \cap U', |\beta(s) - \beta(t)| \ge r^{-j-1}/4\}.$$

For $t \in U$, by definition of $\bar{\beta}$ we have, setting $l = l(\beta, t)$,

$$|\beta(t) - \bar{\beta}(t)| = |\beta(t) - \beta(\pi_l(t))| \le f_{A_l(t)}(\beta)$$

and also, by definition of $l(\beta, t)$,

$$\tfrac{1}{4}r^{-l} \le |\beta(\pi_l(t)) - \beta(\pi_{l+1}(t))| \le f_{A_l(t)}(\beta).$$

Thus

$$|\beta(t) - \bar{\beta}(t)| \le f_{A_l(t)}(\beta)1_{\{f_{A_l}(t) \ge r^{-l}/4\}}(\beta).$$

If $f_S(\beta) \ge (1/4)r^{-i}$, we have

$$|\beta(t) - \bar{\beta}(t)| \le f_S(\beta)1_{\{f_S \ge r^{-i}/4\}}(\beta).$$

If $f_S(\beta) < (1/4)r^{-i}$, consider the largest $i \le j \le l-1$ such that $f_{A_j(t)}(\beta) < (1/4)r^{-j}$. If $j = l - 1$, we have, since

$$\tfrac{1}{4}r^{-j} \ge f_{A_j(t)}(\beta) \ge f_{A_l(t)}(\beta) \ge \tfrac{1}{4}r^{-l},$$

that

$$f_{A_l(t)}(\beta) \le \tfrac{1}{4}r^{-l+1}1_{\{f_{A_l}(t) \ge r^{-l}/4\}}(\beta).$$

If $j < l - 1$, then

$$f_{A_j(t)}(\beta) \ge f_{A_{j+1}(t)}(\beta) \ge \tfrac{1}{4}r^{-j-1},$$

so that $\beta \in W(t, j)$, where for simplicity we write $W(t, j) = W_{A_j(t), j}$. Thus we have shown that

$$(7.20)\qquad |\beta(t)-\bar\beta(t)| \le f_S(\beta)1_{\{f_S\ge r^{-i}/4\}}(\beta) + \sum_{j\ge i}\frac{r^{-j+1}}{4}1_{W(t,j)}(\beta).$$

We now appeal to the representation $\sum_{k\ge1}\varepsilon_k R_k Y_k$ of the process $(X_t)_{t\in U}$. As already observed, (X_t^2) has the representation $\sum_{k\ge1}\varepsilon_k R_k(Y_k-\bar Y_k)$. It follows from (7.20) that

$$(7.21)\qquad \begin{aligned}\Big|\sum_{k\ge1}\varepsilon_k R_k\big(Y_k(t)-\bar Y_k(t)\big)\Big| &\le \sum_{k\ge1}R_k|Y_k(t)-\bar Y_k(t)|\\ &\le \sum_{k\ge1}R_k f_S(Y_k)1_{\{f_S\ge r^{-i}/4\}}(Y_k)+\sum_{j\ge i}\sum_{k\ge1}\frac{r^{-j+1}}{4}R_k 1_{W(t,j)}(Y_k).\end{aligned}$$

The conclusion is now simple. We work conditionally on $\omega_0\in\Omega_0$. We have, from Lemma 2.10 and the definition of Δ_j, that for $A\in\mathscr{A}_j$,

$$\sum_{k\ge1}E\big(R_k1_{W_{A,j}}(Y_k)\big)\le\frac{1}{\alpha^-}\nu(W_{A,j})\le\frac{1}{\alpha^-}\Delta_j(A_j(t)),$$

$$E\Big(\sum_{k\ge1}R_k f_S(Y_k)1_{\{f_S\ge r^{-i}/4\}}(Y_k)\Big)\le\frac{1}{\alpha^-}\int f_S(\beta)1_{\{f_S\ge r^{-i}/4\}}(\beta)\,d\nu(\beta).$$

By Proposition 2.12(b), if we set

$$V_A=\frac{4}{\alpha^-}\Delta_j(A)+2(h(\mu(A))+j-i+v+1),$$

where v is a parameter, we have

$$(7.22)\qquad P\Big(\sum_{k\ge1}R_k1_{W_{A,j}}(Y_k)\ge V_A\Big)\le\mu(A)e^{-(j-i+1+v)}.$$

As in the proof of Theorem 7.2, it follows from (7.22) that the event

$$\Omega(v)=\Big\{\omega;\forall j\ge i,\forall A\in\mathscr{A}_j,\ \sum_{k\ge1}R_k1_{W_{A,j}}(Y_k)\le V_A\Big\}$$

has a probability $\ge 1-e^{-v}$. For $\omega\in\Omega(v)$, we have from (7.21),

$$\sum_{k\ge1}R_k|Y_k(t)-\bar Y_k(t)|\le\sum_{k\ge1}R_k f_S(Y_k)1_{\{f_S\ge r^{-i}/4\}}(Y_k)+\sum_{j\ge i}\frac{r^{-j+1}}{4}V_{A_j(t)}.$$

By definition of V_A and (7.18), the second sum is less than or equal to $K(M/\alpha^-+r^{-i}+v)$. The conclusion follows easily. □

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