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FIXATION RESULTS FOR THRESHOLD VOTER SYSTEMS

BY RICHARD DURRETT[1] AND JEFFREY E. STEIF[2]

Cornell University

We consider threshold voter systems in which the threshold $\tau > n/2$, where n is the number of neighbors, and we present results in support of the following picture of what happens starting from product measure with density 1/2. The system fixates, that is, each site flips only finitely many times. There is a critical value, θ_c, so that if $\tau = \theta n$ with $\theta > \theta_c$ and n is large then most sites never flip, while for $\theta \in (1/2, \theta_c)$ and n large, the limiting state consists mostly of large regions of points of the same type. In $d = 1$, $\theta_c \approx 0.6469076$ while in $d > 1$, $\theta_c = 3/4$.

1. Introduction. In this paper we will consider two closely related models in which the state at time t is $\xi_t\colon \mathbb{Z}^d \to \{0, 1\}$ and we think of $\xi_t(x)$ as giving the opinion of the voter at x at time t. One system has continuous time and the other discrete time. In continuous time process, called the *threshold voter model*, there is an independent rate one Poisson process $\{T_n^x, n \geq 1\}$ for each lattice point $x \in \mathbb{Z}^d$. At time T_n^x, the voter at x examines the points in her neighborhood $\{y\colon y - x \in \mathscr{N}\}$. If at least τ neighbors have the opposite opinion, then the opinion at x changes, otherwise it stays the same. We say that the system *fixates* if $\lim_{t \to \infty} \xi_t(x)$ exists for all x, that is, each voter changes her opinion only finitely many times.

THEOREM 1. *If* $0 \in \mathscr{N}$ *and* $\tau > (|\mathscr{N}| - 1)/2$, *then starting from any initial configuration, the system fixates almost surely.*

When the threshold is small the system does not fixate but has a *nontrivial stationary distribution*, that is, one that assigns no mass to the absorbing states in which no flips are possible. Here and in what follows $\|y\|_p = (|y_1|^p + \cdots + |y_d|^p)^{1/p}$ for $p \in [1, \infty)$ and $\|y\|_\infty = \sup_i |y_i|$. Durrett [4] has shown:

THEOREM 2. *Suppose* $\mathscr{N} = \{y\colon \|y\|_p \leq r\}$ *and* $\tau = \theta|\mathscr{N}|$ *with* $\theta < 1/4$. *If* r *is large, then there is a nontrivial stationary distribution* μ_r, *which approaches product measure with density* 1/2 *as* $r \to \infty$.

We believe that Theorems 1 and 2 are asymptotically sharp and that in other cases the system *clusters*, that is, as $t \to \infty$, ξ_t converges weakly to $(\delta_0 + \delta_1)/2$, where δ_i is the pointmass on the configuration $\xi(x) \equiv i$.

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CONJECTURE 1. *If* $\mathcal{N} = \{y: \|y\|_p \le r\}$ *and* $\tau = \theta|\mathcal{N}|$ *with* $\theta \in (1/4, 1/2)$, *then for large r the system clusters starting from product measure with density* $1/2$.

Andjel, Mountford and Liggett [1] have recently proved a related result.

THEOREM 3. *If* $d = 1$, $\mathcal{N} = \{y: |y| \le r\}$, *and* $\tau = r$ *then starting from any initial configuration the system clusters.*

To explain the difference between Theorem 2 and Conjecture 1, suppose $\theta < 1/2$ and that we start from product measure with density $1/2$. If $|\mathcal{N}|$ is large, then there will be large regions in which each site is *excited*, that is, has more than $\theta|\mathcal{N}|$ neighbors of each type. Now excited sites flip at rate 1 and the stationary distribution for independent rate 1 flips is product measure with density $1/2$, so this is a self-perpetuating situation. To find out if this situation is stable, we consider what happens if a large ball of 1's appears in the excited region. If the ball is very large, its sides are almost flat and points on the boundary see 1's at $3/4 - \varepsilon$ of their neighbors and 0's at $1/4 + \varepsilon$ of their neighbors. If $\theta < 1/4$, the ball shrinks and product measure with density $1/2$ is stable. If $\theta > 1/4$, then blobs grow and clustering occurs.

The arguments in the last paragraph generalize easily to show that if $\theta < 1/4$, a large excited region will expand, and a comparison with oriented percolation proves Theorem 2. We have not, however, succeeded in turning the ideas above into a proof for $\theta \in (1/4, 1/2)$. The main trouble is that one must understand what happens when two blobs collide. If $d = 1$, $\mathcal{N} = \{y: |y| \le r\}$, and $\tau = r$, this is easy since the boundary between an interval of 1's and an interval of 0's moves like a simple random walk as long as both intervals have length at least $r + 1$.

The heuristic in the last paragraph can be applied to $\theta > 1/2$. Suppose we start from product measure with density $1/2$. Most sites cannot flip but if we look far enough we will find a large ball of 1's. If the ball is very large its sides are almost flat and points on the boundary see 1's at $3/4$ of their neighbors and 0's at $1/4$ of their neighbors. If $\theta < 3/4$, then the ball will grow but if $\theta > 3/4$ it cannot. This calculation leads to

CONJECTURE 2. *Suppose* $d > 1$, $\mathcal{N} = \{y: \|y\|_p \le r\}$, $\tau = \theta|\mathcal{N}|$ *with* $\theta > 1/2$. *Start with product measure with density* $1/2$, *and let* $\xi_\infty = \lim \xi_t$. *As* $r \to \infty$, $\xi_\infty \Rightarrow \nu_{1/2}$ *if* $\theta > 3/4$, $\xi_\infty \Rightarrow (\delta_0 + \delta_1)/2$ *if* $\theta < 3/4$.

We have excluded $d = 1$ because our main result (Theorem 6) suggests that the conclusion is false there. To state that result we turn to discrete time.

The *threshold voter automaton* is a deterministic discrete time process in which at each time n, the voter at x examines the opinions of her neighbors $x + \mathcal{N}$ and changes her opinion if and only if at least τ neighbors have the opposite opinion. Fisch and Gravner [5] have results concerning the asymptotic behavior of this model in one dimension. To state their result we need two definitions. We say that the system is *locally periodic* if

$$\lim_{n\to\infty} \xi_{2n}(x) = \xi_e(x) \qquad \lim_{n\to\infty} \xi_{2n+1}(x) = \xi_o(x)$$

and $\xi_e \neq \xi_o$. We say that the system is *uniformly locally periodic* if in addition $\xi_e = 1 - \xi_o$.

THEOREM 4. *Suppose* $d = 1$ *and* $\mathscr{N} = \{y: |y| \leq r\}$. *The system*:

(i) *is uniformly locally periodic if and only if* $\tau \leq r/2$;
(ii) *is locally periodic if* $\tau \leq r$;
(iii) *fixates if* $\tau = r + 1$ *or* $\tau \geq 5r/4$.

It is natural to conjecture that (ii) and (iii) should be:

(ii′) is locally periodic if and only if $\tau \leq r$;
(iii′) fixates if and only if $\tau \geq r + 1$;

and that (i), (ii′) and (iii′) hold in $d > 1$ if $\mathscr{N} = \{y: \|y\|_p \leq R\}$ and r is replaced by $(|\mathscr{N}| - 1)/2$.

Our main result gives qualitative properties of the "limiting state" in $d = 1$. We put the phrase limiting state in quotes since the limit is not known to exist if $r + 1 < \tau < 5r/4$. We begin with a simple result that holds in all d.

THEOREM 5. *Let* B_k *be the event that the voters at all sites x with* $\|x\|_2 \leq k$ *never change. Suppose* $\mathscr{N} = \{y: \|y\|_p \leq r\}$ *and* $\tau = \theta|\mathscr{N}|$ *with* $\theta > 3/4$. *If we start from product measure with density* $1/2$ *then for all* k, $P(B_k) \to 1$ *as* $r \to \infty$.

To formulate our result in $d = 1$, let

$$c(a) = \log 2 + a \log a + (1 - a)\log(1 - a).$$

The reason for our interest in this quantity is that if S_n is the sum of n independent random variables that are 0 and 1 with equal probability, then for $a > 1/2$,

$$\lim_{n\to\infty} \frac{1}{n} \log P(S_n \geq na) = -c(a), \tag{1.1}$$

$$P(S_n \geq na) \leq e^{-c(a)n} \quad \text{for all } n. \tag{1.2}$$

See, for example, Section 9 of Chapter 1 of [3]. The reader will encounter many constants C and γ whose values are unimportant and will change from line to line, but we reserve lower case c for this special constant.

THEOREM 6. *Let* A_k *be the event that all the voters at all sites x with* $|x| \leq k$ *fixate in the same state. Suppose* $d = 1$, $\mathscr{N} = \{y: |y| \leq r\}$, $\tau = \theta|\mathscr{N}|$, *and start from product measure with density* $1/2$. *Let* θ_c *be the unique solution in*

$(1/2, 3/4)$ *of the equation* $2c(\theta) = c(2\theta - 1)$.

(i) *If* $\theta > \theta_c$, *then for all* k, $P(B_k) \to 1$ *as* $r \to \infty$.
(ii) *If* $\theta \in (1/2, \theta_c)$, *then for all* k, $P(A_k) \to 1$ *as* $r \to \infty$.

Theorem 6 is the main result of the paper. The reason for interest in this result is that it is a prototype for conclusions that one would like to prove for related systems with $\kappa \geq 3$ states. See [6]. Theorem 6 can be explained on the basis of heuristic arguments introduced above. Suppose $\theta < 3/4$. Call an interval of 1's of length $> r$ a *blob*, and call an interval of length $r + 1$ in which there are less than $(2r + 1)\theta - r$ 1's, a *blockade*. It is easy to check that a blockade will stop a blob. To prove (i) it suffices to observe that in order for a blob to form we need a point to be *unsatisfied*, that is, have at least $\theta(2r + 1)$ 1's in its neighborhood. If $2c(\theta) > c(2\theta - 1)$, then as $r \to \infty$ blockades are much more numerous than unsatisfied points and $P(B_k) \to 1$.

To prove the converse in (ii), we have to show that if $2c(\theta) < c(2\theta - 1)$ then: (a) for large r blobs are more numerous than blockades; and (b) that blobs grow until they run into each other. To prove (a) we show that if $\sigma > 0$ and there are at least $(\theta + \sigma)(2r + 1)$ 1's in the neighborhood of a point, then a blob will form with high probability. To prove (b) we show that if a collection of unsatisfied sites does not grow into a blob then the number of sites that flip is smaller than σr with high probability, so a blockade does not form.

In $d > 1$ blob formation is more complicated. However, we think that the answer is simpler.

CONJECTURE 3. *Suppose* $d > 1$, $\mathcal{N} = \{y\colon \|y\|_p \leq r\}$, *and we start from product measure with density* $1/2$. *If* $\theta \in (1/2, 3/4)$, *then for all* k, $P(A_k) \to 1$ *as* $r \to \infty$.

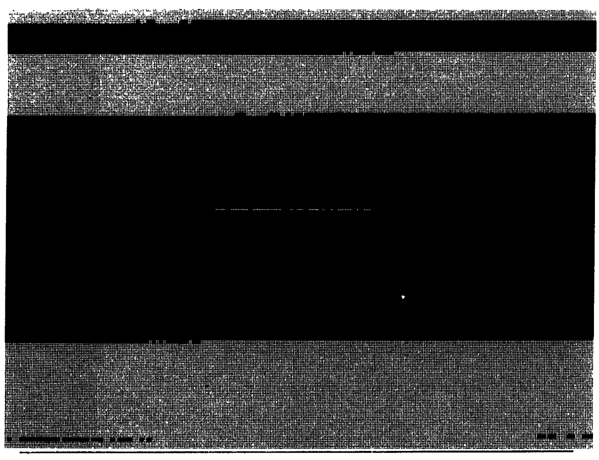

FIG. 1.

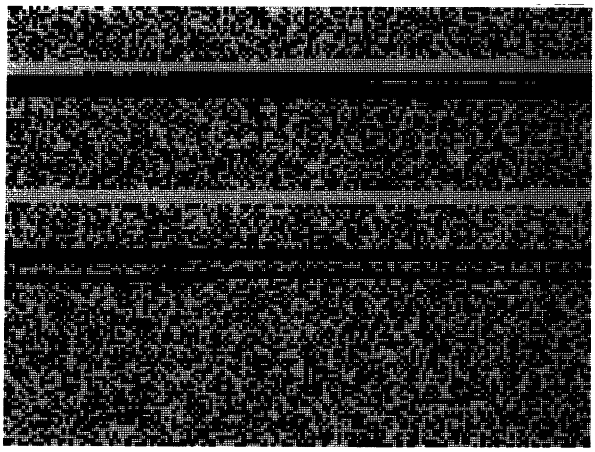

FIG. 2.

(Note that $\theta > 3/4$ is covered by Theorem 5.) The idea behind Conjecture 3 is that in $d > 1$ blobs do not have to go through the very rare "blockades" but can go around them and can grow until they run into another blob.

Conjecture 3 (and Theorem 5) are the discrete time analogue of Conjecture 2. Theorem 5 and part (i) of Theorem 6 generalize in a straightforward way to continuous time. We are convinced that part (ii) of Theorem 6 holds but we do

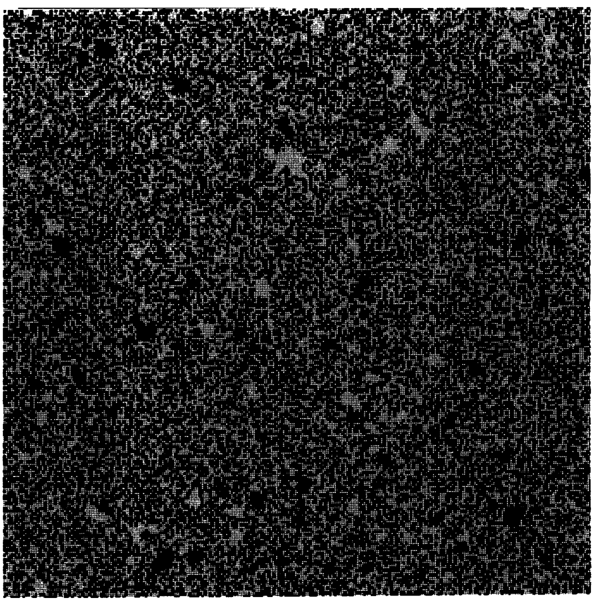

FIG. 3.

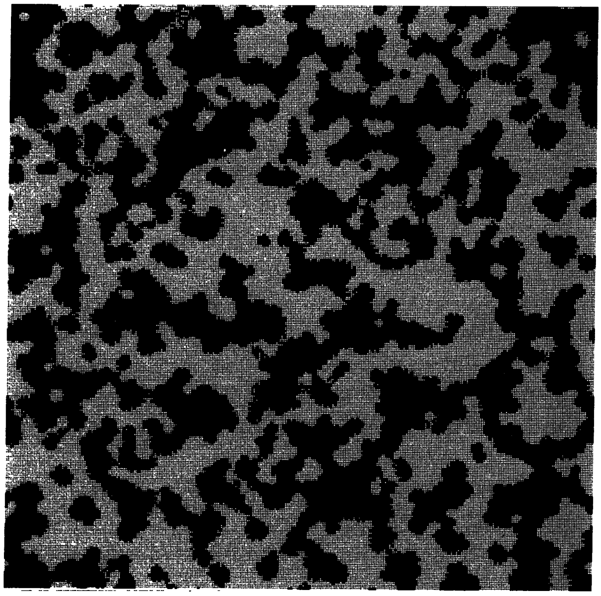

FIG. 4.

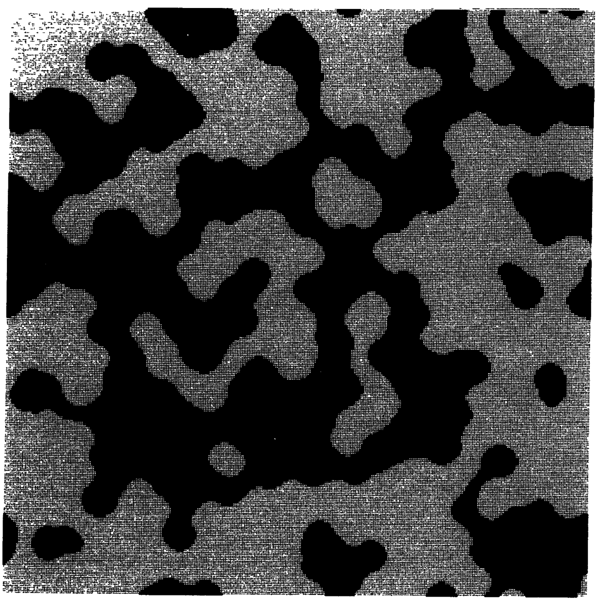

FIG. 5.

not know how to carry out the last part of the argument (i.e., the "Proof of (b)" in Section 6) in continuous time. The rest of the paper is devoted to proofs. Theorem 1 is proved in Section 2, Theorem 5 in Section 3, part (i) of Theorem 6 in Section 4, and part (ii) of Theorem 6 in Section 5.

Before plunging into the proofs we invite the reader to examine the simulation results presented in Figures 1–5. The first two show the threshold voter automaton in one dimension with $\mathscr{N} = \{x: |x| \le 40\}$ run on $\{1, 2, \dots, 19200\}$ with periodic boundary conditions, for example, $1, \dots, 40$ are neighbors of 19200. The sites are arranged in 120 rows of 160 sites and are colored gray or black according to the value of $\lim_{t \to \infty} \xi_t(x)$. Figure 1 shows the system with threshold 52 ($52/81 = 0.6419$). Notice that as suggested by part (ii) of Theorem 6 most of the space consists of solid blobs. A gray blockade can be seen in the middle of the picture. Figure 2 shows threshold 55 ($55/81 = 0.679$). There are six rather large blobs but most of the space consists of sites that have not flipped.

Figures 3–5 are pictures of the t.v.a. in $d = 2$ with $\mathscr{N} = \{x: \|x\|_\infty \le 2\}$. Figure 3 shows threshold 19 ($19/25 = 0.76$) and Figure 4 shows threshold 17 ($17/25 = 0.68$). The contrast in the behavior is consistent with the conjecture that $\theta_c = 3/4$. Finally Figure 5 shows a simulation of the "majority vote" case, threshold 13.

2. Proof of Theorem 1. Our proof is based on an idea of Grannan and Swindle [7]. Let $\delta_{x,y}(t)$ be 1 if $\eta_t(x) \neq \eta_t(y)$, and 0 otherwise, and define the *energy* at time t to be

$$\mathscr{E}_t = \sum_{x, y:\, y - x \in \mathscr{N}} e^{-\varepsilon \|x + y\|_2} \delta_{x,y}(t).$$

Since $\mathscr{E}_0 < \infty$, we can prove Theorem 1 by showing

(2.1) if $\tau > (|\mathscr{N}| - 1)/2$ and ε is small then a flip at x decreases the energy by at least $\gamma(x) > 0$.

To prove (2.1) we note that if $\alpha = |\{y \in x + \mathscr{N}\colon \eta_t(y) \neq \eta_t(x)\}|$ and $N = \sup\{\|x\|_2\colon x \in \mathscr{N}\}$, then the drop in energy due to the flip at x at time t is at least

$$e^{-2\varepsilon \|x\|_2}\left[e^{-\varepsilon N}\alpha - e^{\varepsilon N}(|\mathscr{N}| - 1 - \alpha)\right], \tag{2.2}$$

since even in the worst case all the points in $\{y \in x + \mathscr{N}\colon \eta_t(y) \neq \eta_t(x)\}$ have $\|x + y\|_2 \le 2\|x\|_2 + N$. In order for a flip to occur we must have $\alpha \ge \tau > (|\mathscr{N}| - 1)/2$ and hence $|\mathscr{N}| - 1 - \alpha < \alpha$. Since the last two numbers are integers smaller than $|\mathscr{N}|$, (2.1) follows from (2.2). □

3. Proof of Theorem 5. Let $B_2(y, s) = \{x\colon \|x - y\|_2 \le s\}$. Our first step is a special case of Lemma 2.1 in [4].

LEMMA 3.1. *Suppose $b < 1/2$. There are constants ρ_0 and R_0, so that if $\rho \ge \rho_0$ and $r \ge R_0$, then each site x in $B_2(0, \rho r)$ has $|(x + \mathscr{N}) \cap B_2(0, \rho r)| \ge b|\mathscr{N}|$.*

PROOF. In one dimension we can take $\rho_0 = R_0 = 1$. Turning to dimensions $d > 1$, let q be the volume of $B_p(0,1)$. To prove the result it is convenient to scale space by $1/r$ and translate so that x/r sits at the origin. Any $d-1$ dimensional hyperplane through the origin divides $B_p(0,1)$ into two pieces with volume $q/2$. Let $b < b_2 < b_1 < 1/2$. An application of continuity shows that if $\rho \ge \rho_0$ and $0 \in B_2(y,\rho)$, then the volume of $B_2(y,\rho) \cap B_p(0,1)$ is at least qb_1.

The second and final step is to argue that if r is large, then the lattice behaves like the "continuum limit" considered above. Pick $\varepsilon > 0$ so that if $0 \in B_2(y,\rho)$, then the volume of $B_2(y,\rho-\varepsilon) \cap B_p(0,1-\varepsilon)$ is always larger than qb_2. Then pick R_0 so that $1/R_0 < \varepsilon$ and if $r \ge R_0$, then

$$|B_p(0,r)|/qr^d < b_2/b.$$

Let $\mathscr{X} = (\mathbb{Z}^d/r) \cap B_p(0,1) \cap B_2(y,\rho)$. The first part of the choice of R_0 implies that if $r \ge R_0$, then

$$B_2(y,\rho-\varepsilon) \cap B_p(0,1-\varepsilon) \subset \bigcup_{x\in\mathscr{X}} x + \left[\frac{-1}{2r}, \frac{1}{2r}\right]^d$$

so $r^{-d}|\mathscr{X}| \ge qb_2 \ge r^{-d}b|B_p(0,r)|$, by the second part of the choice of R_0, and the proof of Lemma 3.1 is complete. □

PROOF OF THEOREM 5. Pick b, c so that $2(1-\theta) < 2c < b < 1/2$ and let S_n be the sum of n independent random variables that take values 0 and 1 with equal probability. It follows from the large deviations result quoted in the introduction that as $n \to \infty$,

$$(1/n)\log P(S_{bn} < cn) \to -\gamma < 0.$$

Combining the last observation with Lemma 3.1 we see that if $\rho \ge \rho_0$, then the probability all the sites in $B_2(0,\rho r)$ have at least $c|\mathscr{N}|$ neighbors in $B_2(0,\rho r)$ of each type approaches 1 as $r \to \infty$. When this occurs no site in $B_2(0,\rho r)$ can ever flip [since $c > (1-\theta)$] and the proof is complete. □

4. Proof of Theorem 6, part (i). In view of Theorem 5 we can assume without loss of generality that $\theta_c < \theta \le 3/4$. We begin with some definitions. We say $z \in \mathbb{Z}$ is *unsatisfied* for η if

$$\sum_{y=z-r}^{z+r} 1_{\{\eta(y)\neq\eta(z)\}} \ge (2r+1)\theta$$

So z is unsatisfied if it switches its value at the next step. We say $z \in \mathbb{Z}$ is a *1-blockade* for η if $\sum_{y=z}^{z+r} 1_{\{\eta(y)=1\}} < (2r+1)\theta - r$ (and a *0-blockade* for η if $\sum_{y=z}^{z+r} 1_{\{\eta(y)=0\}} < (2r+1)\theta - r$). The point of a 1-blockade can be seen in the following easily verified fact:

(4.1) Let $z \in \mathbb{Z}$ be a 1-blockade for η. Then every 0 in $[z, z+r]$ remains 0 for all time independent of what happens outside. Conversely, if z is not a blockade and all sites in $[z-r, z-1]$ are 1, then z will flip to 1.

An analogous statement holds for 0-blockades: just interchange the 0's and 1's. It follows from the large deviations result quoted in the introduction that

$$\lim_{r\to\infty} (1/r)\log P(z \text{ is unsatisfied at time } 0) = -2c(\theta), \tag{4.2}$$

$$\lim_{r\to\infty} (1/r)\log P(z \text{ is a 1-blockade at time } 0) = -c(2\theta - 1) \tag{4.3}$$

if $\theta \le 3/4$. Equation (4.3) fails when $\theta > 3/4$, for then $(2\theta - 1) > 1/2$. θ_c is defined by $2c(\theta_c) = c(2\theta_c - 1)$. Since $c(\theta)$ is a strictly increasing function of θ on $[1/2, 1]$, it follows that if $\theta > \theta_c$ there is a $\delta > 0$ so that $c(2\theta - 1) + \delta < 2c(\theta)$. Let $k \ge 1$. To prove the theorem we will define a good event G_r so that:

(a) $\lim_{r\to\infty} P(G_r) = 1$.
(b) When G_r occurs, lattice points $-k, \dots, k$ never change their value.

Let L_r be the smallest even integer $\ge \exp(r\{c(2\theta-1)+\delta\})$ and let $G_r = \bigcap_{i=0}^{4} G_r^i$, where

$$G_r^0 = \{\text{each } y \in [-L_r, L_r] \text{ is satisfied}\},$$

$$G_r^1 = \{\text{there exists } y_1 \in [-L_r, -L_r/2] \text{ with } y_1 \text{ a 1-blockade}\},$$

$$G_r^2 = \{\text{there exists } y_2 \in [-L_r/2, -k] \text{ with } y_2 \text{ a 0-blockade}\},$$

$$G_r^3 = \{\text{there exists } y_3 \in [k, L_r/2] \text{ with } y_3 \text{ a 0-blockade}\},$$

$$G_r^4 = \{\text{there exists } y_4 \in [L_r/2, L_r] \text{ with } y_4 \text{ a 1-blockade}\}.$$

To estimate $P(G_r)$ we begin by observing that since $c(2\theta - 1) + \delta < 2c(\theta)$, (4.2) implies $P(G_r^0) \to 1$ as $r \to \infty$. Turning our attention now to G_r^i, let $z_i = -L_r + (i-1)(r+1)$ for $i = 1, \dots, [L_r/2(r+1)]$. Using an obvious independence we have that

$$\begin{aligned} P(G_r^1) &\ge 1 - P\left(\bigcap_{i=1}^{[L_r/2(r+1)]} \{z_i \text{ is not a 1-blockade}\}\right) \\ &\ge 1 - \left(1 - C\exp(-r\{c(2\theta-1) + \delta/2\})\right)^{[L_r/2(r+1)]} \to 1 \end{aligned}$$

as $r \to \infty$. The last argument applies to G_r^i, $i = 2, 3, 4$ and we have shown (a). To show (b), we note that (4.1) implies that each 0 in $[y_1, y_1 + r] \cup [y_4, y_4 + r]$ will never flip. Since each $y \in [-L_r, L_r]$ is satisfied it follows that no 0 in $[y_1, y_4 + r]$ will ever flip. Similarly, no 1 in $[y_2, y_3 + r]$ will ever flip and the proof is complete. □

5. Proof of Theorem 6, part (ii). Our first goal is to show that the large deviations rate for blobs is the same as for an unsatisfied point at time 0. To prepare for this we begin with:

LEMMA 5.1. *Let $X_1, \dots, X_n$ be independent and take the values 0 and 1 with equal probability and let $S_m = X_1 + \cdots + X_m$. Suppose $\theta > 1/2$, $\sigma \in (0, 1 - \theta)$. There is a constant $\gamma > 0$ so that for all $\alpha \le 1$,*

$$P(S_{\alpha n} \le \theta \alpha n | S_n \ge (\theta + \sigma)n) \le Ce^{-\gamma \alpha n}.$$

REMARK. The intuition underlying this result is that conditional on $S_n \ge (\theta + \sigma)n$, $X_1, \dots, X_n$ behave like independent random variables that are 1 with probability $\theta + \sigma$.

PROOF. The conclusion is trivial for $\alpha < \sigma/2$. If we condition on $S_n = m$, then the distribution of $(X_1, \dots, X_{\alpha n})$ is the same as that of αn draws without replacement from an urn with m 1's and $n - m$ 0's. Such a drawing clearly produces more 1's than drawing with replacement from an urn with $m - \alpha n$ 1's and $n - m$ 0's, but in this case the draws are i.i.d. and the conclusion follows from the obvious generalization of (1.2). To prove the result for $\alpha > \sigma/2$ we begin with three observations. If $\delta > 0$, then (1.1) implies

$$P(S_n \ge n(\theta + \sigma)) \ge Ce^{-(c(\theta+\sigma)+\delta)n}. \tag{5.1}$$

Equation (1.2) implies that if $k \ge m/2$, then

$$P(S_m = k) \le P(S_m \ge k) \le e^{-c(k/m)m}, \tag{5.2}$$

so by symmetry we have for all k that

$$P(S_m = k) \le e^{-c(k/m)m}. \tag{5.3}$$

To bound the conditional probability now we observe that

$$\begin{aligned}
&P(S_{\alpha n} \le \theta \alpha n, S_n \ge n(\theta + \sigma)) \\
&\quad \le \sum_{k \le \theta \alpha n} P(S_{\alpha n} = k, S_n - S_{\alpha n} \ge n(\theta + \sigma) - k) \\
&\quad \le n \sup_{0 \le a \le \theta} \exp\left(-\alpha n c(a) - (1 - \alpha) n c\left(\frac{\theta + \sigma - \alpha a}{1 - \alpha} \right) \right).
\end{aligned}$$

Dividing by $P(S_n \ge n(\theta + \sigma))$ and using (5.1) gives

$$P(S_{\alpha n} \le \theta \alpha n | S_n \ge n(\theta + \sigma)) \le Cne^{-(\varepsilon - \delta)n},$$

where

$$\varepsilon = \inf_{\substack{0 \le a \le \theta \\ \sigma/2 \le \alpha \le 1}} \left\{ \alpha c(a) + (1 - \alpha) c\left(\frac{\theta + \sigma - \alpha a}{1 - \alpha} \right) - c(\theta + \sigma) \right\},$$

which is strictly positive by the strict convexity of $c(\theta)$. Picking $\delta < \varepsilon/2$, the desired result follows. □

Let $S_{a,b}$ be the number of 1's in $[a, b]$ in the initial configuration, let

$$H_\sigma = \{S_{-r,r} \geq (2r+1)(\theta+\sigma)\}$$

and

$$E = \{\text{all sites in } [-r, r] \text{ are 1 at time } 2r\}.$$

LEMMA 5.2. *Let $\theta \in (1/2, 3/4)$ be fixed. For all $\sigma > 0$, there are constants $0 < \gamma, C < \infty$ so that*

$$P(E|H_\sigma) \geq 1 - Ce^{-\gamma r} \quad \textit{for all } r.$$

REMARK. For the rest of this section γ and C will denote positive finite constants whose values are unimportant and will in general change from line to line.

PROOF. We begin by proving that we do not have to worry about sites flipping from 1 to 0. We call a point z *bad* if $(2r+1) - S_{z-r,z+r} \geq (2r+1)\theta$, that is, if the number of 0's is $\geq (2r+1)\theta$. This corresponds to the possibility of a 1 switching to a 0. Let

$$G_1 = \{\text{no } z \in [-3r^2, 3r^2] \text{ is bad at time } 0\}.$$

Since H_σ is an increasing event and G_1^c is a decreasing event, Harris' inequality implies

$$P(G_1^c|H_\sigma) \leq P(G_1^c) \leq (6r^2+1)P(S_{-r,r} \geq \theta(2r+1)) \leq Ce^{-\gamma r}$$

since $\theta > 1/2$. G_1 guarantees that no 1 in $[-3r^2, 3r^2]$ at time 0 will flip to 0 at time 1. Hence no 1 in $[-3r^2+r, 3r^2-r]$ at time 1 will flip to 0 at time 2, and continuing inductively we see that no 1 in $[-2r, 2r]$ will flip to 0 before time r.

If H_σ occurs, then at time 0 we have $S_{x-r,x+r} \geq (2r+1)\theta$ for all $x \in [-\sigma(2r+1), \sigma(2r+1)]$ so at time 1 all $x \in [-\sigma(2r+1), \sigma(2r+1)]$ will be 1. To show that the interval of 1's will continue to grow we divide space into blocks and use induction. Suppose without loss of generality that $\sigma < (3/4 - \theta)$ and $1/\sigma$ is an integer. Pick M so that $1/M < (3/4 - \theta)$, let $\alpha = \sigma/M$, and call $(i\alpha r, (i+1)\alpha r)$ the ith block. Let T_i be the number of 1's in the ith block at time 0. Let $N = 1/\alpha$ and

$$G_2 = \{T_i \geq \theta r\alpha \text{ for } -N \leq i < N\},$$

$$G_3 = \{T_i \geq (\tfrac{1}{2} - \sigma)r\alpha \text{ for } i \in [-2N, -N) \cup [N, 2N)\}.$$

Lemma 5.1 implies

$$P(G_2|H_\sigma) \geq 1 - 2NCe^{-\gamma\alpha r}.$$

Since G_3 is independent of H_σ, (1.1) implies

$$P(G_3|H_\sigma) \geq 1 - 2NCe^{-\gamma\alpha r}.$$

We will now check that when $G = \bigcap_{i=1}^3 G_i$ occurs, all sites in $[-r, r]$ will be 1 at time r. As we observed above, blocks $-2M, \dots, 2M-1$ are filled with 1's at time 1. Let $2M \leq k \leq N$. We will show by induction that block k is filled with 1's at time $k - 2M + 2$. Supposing this is true for $j < k$, we have (on G) the following lower bounds on the number of 1's at time $k - 2M + 1$:

block number	bound
$k - N + 1, \dots, -1$	$\theta\alpha r$
$0, \dots, k - 1$	αr
$k, \dots, N - 1$	$\theta\alpha r$
$N, \dots, N + k - 1$	$(\frac{1}{2} - \sigma)\alpha r.$

So if $x \in [k\alpha r, (k+1)\alpha r]$, and we let $T_{x-r,x+r}$ denote the number of 1's in $[x - r, x + r]$ at time $k - 2M + 1$, we have

$$T_{x-r,x+r} \geq (2N - 1)\theta\alpha r + k\big(1 + \big(\tfrac{1}{2} - \sigma\big) - 2\theta\big)\alpha r.$$

Now $N = 1/\alpha$, $\alpha = \sigma/M$, $k \geq 2M$, and we have assumed $\sigma < (3/4 - \theta)$, $1/M < (3/4 - \theta)$ so

$$T_{x-r,x+r} \geq r(2\theta - \theta\sigma/M + (3/4 - \theta)2\sigma) \geq (2\theta + (3/4 - \theta)\sigma)r.$$

The last inequality shows that x will flip and completes the proof of Lemma 5.2. □

Lemma 5.2 will allow us to conclude that, when $\theta < \theta_c$ and r is large, blobs are more numerous that blockades. To show that blobs will grow until they run into each other, we need to show that unsatisfied sites which do not turn into blobs do not produce blockades. Let

$$D_1 = \{\text{no point in } [-2r, -0.01r) \cup (0.01r, 2r] \text{ changes by time } r\}$$

$$D_2 = \{S_{x,x+r} < (1.97 - 2\theta)(r + 1) \text{ for all } x \in [-r, 0.01r]\}$$

$$D_3 = \{\text{no } x \in [-4r^2, 4r^2] \text{ is bad}\}$$

and $D = \bigcap_{i=1}^{3} D_i$. To see the reason for interest in D observe that (a) from the proof of Lemma 5.2, D_3 guarantees that no 1 in $[-2r, 2r]$ will flip to 0 by time $2r$, and (b) on $D_1 \cap D_2$ each interval $[x, x + r]$ with $x \in [-r, 0.01r]$ has fewer than $(1.99 - 2\theta)(r + 1)$ 1's at time r. (b) says that there is no 0-blockade, and (a) says that there is no 1-blockade (unless there was one at time 0).

Lemma 5.3. *Let $\theta \in (0.50, 0.65)$ be fixed. There are constants $0 < \lambda, C < \infty$ so that*

$$P(E \cup D | H_0) \geq 1 - Ce^{-\lambda r}.$$

Remark. If $E \cup D$ occurs then we say that the unsatisfied site at 0 is *well behaved.*

Proof. From the proof of Lemma 5.2 we see that

$$P(D_3 | H_0) \geq 1 - Ce^{-\gamma r}$$

and on D_3 no 1 in $[-2r, 2r]$ flips to 0 by time $2r$. To bound $P(D_2|H_0)$ we begin by observing that if $\delta > 0$, (1.1) implies

$$P(H_\delta | H_0) \le Ce^{-\gamma r} \tag{5.4}$$

and if $\delta \le 0.005$, using the reasoning from the proof of Lemma 5.1 gives

$$P(S_{x,x+r} \ge (r+1)(\theta + 0.01) | H_\delta^c \cap H_0) \le Ce^{-\gamma r}. \tag{5.5}$$

Since we have assumed $\theta < 0.65$, we have $\theta + 0.01 < 1.97 - 2\theta$ and combining (5.4) and (5.5) gives

$$P(D_2 | H_0) \ge 1 - Ce^{-\gamma r}.$$

Suppose now that D_1^c occurs, let $T \le r$ be the first time such a change occurs, let y be a location $[-2r, 2r] - [-0.01r, 0.01r]$ that flips, and suppose without loss of generality that $y \ge 0$. Let ϕ denote the number of points in $[-0.01r, 0.01r]$ that have flipped by time $T - 1$. Since T is the first time we must have

$$S_{y-r, y+r} + \phi \ge (2r+1)\theta.$$

We will now use the last equation to argue that $\phi \ge \varepsilon r$ for some $\varepsilon > 0$. Since $y \ge 0.01r$ and $S_{r+1, r+y}$ is independent of H_0,

$$P\left(S_{r+1,r+y} \ge \left(\tfrac{1}{2} + \delta\right) y | H_0\right) \le Ce^{-\gamma r}. \tag{5.6}$$

To bound $S_{y-r,r}$ we have to consider two cases. First the trivial case: If $y \in ((2 - 2\delta)r, 2r]$, then

$$S_{y-r,r} \le (2r+1) - y \le 2\delta r,$$

so when $S_{r+1,r+y} \le (\frac{1}{2} + \delta)y \le (\frac{1}{2} + \delta)(2r+1)$ we have

$$S_{y-r,y+r} \le (2r+1)\left(\tfrac{1}{2} + 2\delta\right). \tag{5.7}$$

When $y \in [0.01r, (2 - 2\delta)r]$, $(2r+1) - y \ge 2\delta r$ so arguing as above gives

$$P\left(S_{y-r,r} \ge (\theta + 2\delta)(2r + 1 - y) | H_\delta^c \cap H_0\right) \le Ce^{-\gamma r}. \tag{5.8}$$

Now if we pick $\delta < (\theta - 1/2)/1000$ then

$$\begin{aligned}\left(\tfrac{1}{2} + \delta\right)y + (\theta + 2\delta)(2r + 1 - y) &= (2r+1)\theta + \left(\tfrac{1}{2} - \theta - \delta\right)y + 2(2r+1)\delta \\ &\le (2r+1)\theta - \left(\theta - \tfrac{1}{2}\right)(0.01 - 0.006)r,\end{aligned}$$

since $y \ge 0.01r$ and $(2r+1) \le 3r$. Letting $\sigma = 0.001(\theta - 1/2)$ we see that in either case $\phi \ge \sigma(2r+1)$, that is, at time $T-1$ there are $(\theta + \sigma)(2r+1)$ 1's in $[-r, r]$. The last observation implies that at time T all the sites in $[-\sigma(2r+1), \sigma(2r+1)]$ will be 1 and arguing as in the proof of Lemma 5.2 we see that with probability at least $1 - Ce^{-\gamma r}$ all sites in $[-r, r]$ will be 1 at time $T + r \le 2r$.

Proof of Theorem 6, part (ii). Let F_x be the event that x and $x+1$ fixate in the same state. Translation invariance implies $P(F_x)$ is independent of x so it suffices to show that $P(F_0) \to 1$ as $r \to \infty$. Let λ be the constant in Lemma 5.3 and pick σ, δ small enough so that $2c(\theta + \sigma) + 3\delta$ is smaller than

$c(2\theta - 1)$, $2c(\theta) + \lambda$ and $4c(\theta)$. The first choice is possible since $\theta < \theta_c$, the other two since c is continuous and strictly increasing on $[1/2, 1]$. Let $L_r = \exp(\{2c(\theta + \sigma) + 2\delta\}r)$. These choices are designed so that the following events have high probability:

$$G_1 = \{\text{a blob forms in } [0, L_r]\},$$

$$G_3 = \{\text{there are no blockades in } [0, L_r]\},$$

$$G_5 = \{\text{all points in } [0, L_r] \text{ that are unsatisfied are well behaved}\},$$

$$G_7 = \{\text{there are not two unsatisfied sites in } [0, L_r] \text{ with } 2r + 1 \le |x - y| \le 2r^3 + 4r\}.$$

We define events G_2, G_4, G_6 and G_8 by replacing $[0, L_r]$ by $[-L_r, 0]$ and define

$$G_9 = \{\text{all } x \in [-r^4, r^4] \text{ are satisfied}\}.$$

Let $G = \bigcap_{i=1}^{9} G_i$. We will show that:

(a) $P(G) \to 1$ as $r \to \infty$; and
(b) if G occurs, then $(1/2r^3)\sum_{x=-r^3}^{r^3-1} 1_{F_x} \ge 1 - 3/r$.

Proof of (a). It suffices to show that $P(G_i) \to 1$ for $i = 1, 3, 5, 7, 9$.

(G_1) Note that the events

$$S_{(3i-1)r,(3i+1)r} \ge (\theta + \sigma)(2r + 1), \qquad 1 \le i \le L_r/3r$$

are independent and have probability $\ge \exp(-\{2c(\theta + \sigma) + \delta\}r)$ for large r, so the choice of L_r implies that with high probability the desired inequality will hold for at least one value of i, and the desired conclusion follows from Lemma 5.2.

(G_3) The expected number of blockades in $[0, L_r]$ is smaller than

$$L_r \exp(-c(2\theta - 1)r) \to 0 \quad \text{as } r \to \infty.$$

(G_5) The expected number of unsatisfied sites in $[0, L_r]$ that are not well behaved is smaller than

$$L_r \exp(-2c(\theta)r) C \exp(-\lambda r) \to 0 \quad \text{as } r \to \infty.$$

(G_7) When $|x - y| \ge (2r + 1)$ the events that x and y are unsatisfied are independent so the expected number of unsatisfied pairs satisfying the indicated inequalities is at most

$$(2r^3 + 4r) L_r \exp(-4c(\theta)r) \to 0 \quad \text{as } r \to \infty.$$

(G_9) The expected number of unsatisfied sites in $[-r^4, r^4]$ is at most

$$(2r^4 + 1)\exp(-2c(\theta)r) \to 0 \quad \text{as } r \to \infty.$$

Proof of (b). Let $y_1 \ge 0$ be the first unsatisfied lattice point at time 0 and for $k \ge 2$ let y_k be the first unsatisfied lattice point in $[0, L_r]$ after $y_{k-1} + r^3$. If there is no such point, let $y_k = \infty$. G_7 guarantees us that all unsatisfied points at time 0 in $[0, L_r]$ are contained in $\bigcup_i [y_i - 2r, y_i + 2r]$ and that up until time r^2, the evolutions in different $[y_i - 2r, y_i + 2r]$'s will not affect each other. Let j be the smallest i such that $[y_i - r, y_i + r]$ is a blob at time $2r$. G_1 guarantees us that there is such a j. Since G_3 occurs, there are no 0-blockades or 1-blockades at time 0 in $[0, L_r]$. Consider some $i < j$. Since the interval i is well behaved by G_5 and j is the index of the first blob, it follows that no blockades are produced by the time evolution. We therefore have a blob at y_j and no blockades between itself and the origin which will prevent it from moving to the left. We also, of course, have the analogous situation on the left side of the origin with a blob moving to the right and no blockades in its way.

If these blobs to the left and right are the same type, then it is clear that F_x will occur for all $x \in [-r^3, r^3]$. If the blobs are of opposite type, then without loss of generality, assume that $[y_j - r, y_j + r]$ is a 1-blob at time $2r$ and the blob to the left of the origin is a 0-blob. Define a decreasing sequence of lattice points $\{z_n\}$ as follows. Let $z_1 = \inf\{x \in [y_j - 2r, y_j + 2r]$: x flips at time $0\}$ and for $k \geq 1$ let $z_{k+1} = \inf\{x \in [z_k - r, z_k]$: x flips at time $k\}$. Clearly $z_1 \geq y_j - 2r$ and $z_{k+1} \geq z_k - r$ and so $z_{k+1} \geq y_j - 2r - kr$.

For $l \geq 2r$, let $w_l = \inf\{x$: $[x, y_j + r]$ is a 1-blob at time $l\}$. Now, no 1's will switch to 0 until the effect of the 0-blob to the left of the origin begins. Also, at time $2r$, $[y_j - r, y_j + r]$ are all 1's. From these two facts, it follows that as long as there is no effect of the blob to the left of the origin, each $x \in [z_1, y_j - r - 1]$ sees at least as many 1's at time $2r$ as z_1 saw at time 0 and hence at time $2r + 1$, all $x \in [z_1, y_j + r]$ are in state 1. Similarly, since $z_2 \geq z_1 - r$ and $[z_1, y_j + r]$ is a 1-blob at time $2r + 1$, all $x \in [z_2, z_1]$ see at time $2r + 1$ at least as many 1's as z_2 saw at time 1. Hence $[z_2, y_j + r]$ are all 1's at time $2r + 2$. By induction, $[z_k, y_j + r]$ are all 1's at time $2r + k$. Therefore $z_k \geq w_{2r+k}$ for all $k \geq 1$ and so $z_{2r+k} \geq z_k - 2r^2 \geq w_{2r+k} - 2r^2$. In words, after time $2r$, the leftmost point that flipped as a result of the blob starting at $[y_j - r, y_j + r]$ is at most $2r^2$ points to the left of the growing blob at that point in time.

By applying the same argument to the first blob to the left of the origin, and using the fact that the range of the interaction is $2r + 1$, this last inequality implies that the blobs will eventually be within $4r^2 + 2r + 1$ units, which gives us the desired result,

$$\frac{1}{2r^3} \sum_{x=-r^3}^{r^3-1} 1_{F_x} \geq 1 - \frac{3}{r}. \qquad \square$$

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DEPARTMENT OF MATHEMATICS
CORNELL UNIVERSITY
WHITE HALL
ITHACA, NEW YORK 14853-7901

DEPARTMENT OF MATHEMATICS
AND STATISTICS
GOTHENBERG UNIVERSITY
GOTHENBERG
SWEDEN

[redacted]

[redacted]