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ON GENERALIZED RENEWAL MEASURES AND CERTAIN FIRST PASSAGE TIMES[1]

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Let $X_1, X_2, \dots$ be i.i.d. random variables with common mean $\mu \geq 0$ and associated random walk $S_0 = 0$, $S_n = X_1 + \cdots + X_n$, $n \geq 1$. For a regularly varying function $\phi(t) = t^\alpha L(t)$, $\alpha > -1$ with slowly varying $L(t)$, we consider the generalized renewal function

$$U_\phi(t) = \sum_{n \geq 0} \phi(n) P(S_n \leq t), \qquad t \in \mathbb{R},$$

by relating it to the family $\tau = \tau(t) = \inf\{n \geq 1: S_n > t\}$ $t \geq 0$. One of the major results is that $U_\phi(t) < \infty$ for all $t \in \mathbb{R}$, iff $\phi(t)^{-1} U_\phi(t) \sim 1/(\alpha + 1)\mu^{\alpha+1}$ as $t \to \infty$, iff $E(X_1^-)^2 \phi(X_1^-) < \infty$, provided ϕ is ultimately increasing ($\Rightarrow \alpha \geq 0$). A related result is proved for $U_\phi(t+h) - U_\phi(t)$ and $U_\phi^+(t) = \sum_{n \geq 0} \phi(n) P(M_n \leq t)$, where $M_n = \max_{0 \leq j \leq n} S_j$. Our results form extensions of earlier ones by Heyde, Kalma, Gut and others, who either considered more specific functions ϕ or used stronger moment assumptions. The proofs are based on a regeneration technique from renewal theory and two martingale inequalities by Burkholder, Davis and Gundy.

1. Introduction and results. Let $X_1, X_2, \dots$ be a sequence of i.i.d. real-valued random variables with finite positive mean μ and associated random walk $S_0 = 0$, $S_n = X_1 + \cdots + X_n$ for $n \geq 1$. Given an arbitrary function $\phi: [0, \infty) \to [0, \infty)$,

$$U_\phi = \sum_{n \geq 0} \phi(n) P(S_n \in \cdot) \tag{1.1}$$

defines a so-called generalized renewal measure of $(S_n)_{n \geq 0}$ which for $\phi \equiv 1$ clearly reduces to the ordinary renewal measure. The latter one has been studied in great detail and one may consult the textbook by Asmussen (1987) and the references given therein. For $\phi(t) = t^{-1}$, U_ϕ is called the harmonic renewal measure and its behavior is closely related to that of the ladder variables and ladder heights associated with $(S_n)_{n \geq 0}$. For results in this special case, we refer to Greenwood, Omey and Teugels (1982), Grübel (1986, 1987, 1988) and Alsmeyer (1990). In Heyde (1964, 1966), Kalma (1972), Maejima and Omey (1984), Embrechts, Maejima and Omey (1984) and also in Alsmeyer (1990), the asymptotic behavior of $U_\phi(t)$ and $U_\phi(t+h) - U_\phi(t)$, as $t \to \infty$, is examined for the more general case when $\phi(t)$ is a regularly varying

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function, that is, for $\phi(t) = t^\alpha L(t)$ with some slowly varying $L(t)$. For $\alpha \ge 0$, it has been shown by Heyde (1966) that

$$(1.2)\qquad U_\phi(t) \sim \frac{t\phi(t)}{(\alpha+1)\mu^{\alpha+1}} \quad \text{as } t \to \infty,$$

provided $E(X_1^-)^{\beta+2} < \infty$ for some β such that $\phi(t) = O(t^\beta)$, with equivalence holding if $\phi(t) = t^\alpha$ [and $\beta = \alpha$ then]. Furthermore, for $\alpha > -1$, nonarithmetic X_1 and all $h > 0$,

$$(1.3)\qquad U_\phi(t+h) - U_\phi(t) \sim \frac{h\phi(t)}{\mu^{\alpha+1}} \quad \text{as } t \to \infty,$$

provided $E(X_1^-)^{\beta+1} < \infty$ for $\beta = \alpha$, if $\phi(t) = t^\alpha$ and for $\beta = [\alpha] + 1$ otherwise. This result is due to Kalma (1972) (the first part) and to Maejima and Omey (1984) (the second one); see also Embrechts, Maejima and Omey (1984). The main object of this paper is to sharpen these results by giving the right moment conditions on X_1 equivalent with, respectively, (1.2) and (1.3) for general regularly varying $\phi(t) = t^\alpha L(t)$, $\alpha > 0$. Our proofs are based on a simple device which, for nonnegative $X_1, X_2, \dots$, consists in relating a generalized renewal function $U_\phi(t)$ to a suitable moment function of the family

$$(1.4)\qquad \tau(t) = \inf\{n \ge 1: S_n > t\}, \qquad t \ge 0,$$

of first passage times, that is, to $EH(\tau(t))$ for some appropriate H. This has been done already in Alsmeyer (1990) for harmonic renewal measures and it is also shown there in the final section how to employ the device for general $\phi(t)$. However, for the two-sided case, the situation is more difficult and we have to modify our device and to combine it with appropriate techniques from renewal theory. In particular, we will extend a well-known result on existence of $E\phi(\tau(t))$ and some related quantities from the special class $\phi(t) = t^\alpha$, $\alpha \ge 1$, to the more general class of regularly varying functions being considered here. The result is stated in Theorem 3 in Section 2. Our main results, Theorem 1 and 2, are given below.

Let $\mathscr{RV}_\alpha$ denote the class of all functions ϕ: $[0,\infty) \to [0,\infty)$ which are regularly varying at infinity with exponent α, $\alpha \in \mathbb{R}$, that is,

$$\lim_{t\to\infty} (\phi(tx)/\phi(t)) = x^\alpha \quad \text{for all } x > 0.$$

Then ϕ is of the form $\phi(t) = t^\alpha L(t)$ for some $L \in \mathscr{RV}_0$, the class of nonnegative, slowly varying functions. Typical examples are $\phi(t) = (t+1)^\alpha \log_k^\beta(t+1)$ for arbitrary $\alpha, \beta \in \mathbb{R}$, $k \in \mathbb{N}$ and with $\log_k$ denoting the k-fold iterated logarithm.

THEOREM 1. (a) *Let* $\mu = EX_1 \in (0,\infty)$. *For each ultimately increasing* $\phi \in \mathscr{RV}_\alpha$, $\alpha \ge 0$, *the following statements are then equivalent*:

$$(1.5)\qquad E(X_1^-)^2\phi(X_1^-) < \infty,$$

$$(1.6)\qquad U_\phi(t) < \infty \quad \textit{for all } t \in \mathbb{R},$$

$$(1.7)\qquad U_\phi(t) \sim \frac{t\phi(t)}{(\alpha+1)\mu^{\alpha+1}} \quad \textit{as } t \to \infty.$$

(b) *For ultimately decreasing* $\phi \in \mathscr{RV}_\alpha$, $\alpha \in (-1, 0]$, (1.5) *still implies* (1.6) *and* (1.7).

Clearly, the ultimate monotonicity is an additional requirement only when $\alpha = 0$. For the statement of Theorem 2, let $M_n = \max_{0 \le j \le n} S_j$ for $n \ge 0$ and

$$U_\phi^+(t) = \sum_{n \ge 0} \phi(n) P(M_n \le t). \tag{1.8}$$

Generalized renewal functions for the sequence of maxima were considered by Heyde (1966) who proved (1.11) below under a stronger moment condition on X_1.

THEOREM 2. (a) *Let* $\mu = EX_1 \in (0, \infty)$. *For each ultimately increasing* $\phi \in \mathscr{RV}_\alpha$, $\alpha \ge 0$, *the following statements are equivalent*:

$$EX_1^- \phi(X_1^-) < \infty, \tag{1.9}$$

$$U_\phi^+(t) < \infty \quad \textit{for all } t \in \mathbb{R}, \tag{1.10}$$

$$U_\phi^+(t) \sim \frac{t\phi(t)}{(\alpha + 1)\mu^{\alpha+1}} \quad \textit{as } t \to \infty, \tag{1.11}$$

$$U_\phi(t + h) - U_\phi(t) < \infty \quad \textit{for all } t \in \mathbb{R} \textit{ and } h > 0, \tag{1.12}$$

$$U_\phi(t + h) - U_\phi(t) \sim \frac{h\phi(t)}{\mu^{\alpha+1}} \quad \textit{for all } h > 0 \textit{ as } t \to \infty, \tag{1.13}$$

where for the latter assertion X_1 *is additionally supposed to be nonarithmetic.*

(b) *For ultimately decreasing* $\phi \in \mathscr{RV}_\alpha$, $\alpha \in (-1, 0]$, $\mu \in (0, \infty)$ *is sufficient for validity of* (1.10) *through* (1.13).

REMARKS. (a) An arithmetic version of (1.13) can also be easily formulated.

(b) A look at the proofs of our theorems shows that the equivalences remain true when replacing "for all $t \in \mathbb{R}$" by "for some $t \in \mathbb{R}$" in (1.6) and (1.10).

The proofs of Theorem 1 and 2 are given in Section 3. They are preceded by a number of results in Section 2, the most important of which is Theorem 3 below generalizing some moment results in random walk theory.

2. A basic theorem and further prerequisites. Let $S_* = \min_{n \ge 0} S_n$, $\sigma_- = \inf\{n \ge 1: S_n \le 0\}$ the first weakly descending ladder epoch and S_{σ_-} the associated ladder variable.

THEOREM 3. *Suppose* $\mu = EX_1 \in (0, \infty)$. *For each ultimately increasing* $\phi \in \mathscr{RV}_\alpha$, $\alpha \ge 0$, *the following statements are then equivalent*:

$$EX_1^- \phi(X_1^-) < \infty, \tag{2.1}$$

$$E\phi(|S_*|) < \infty, \tag{2.2}$$

$$E\phi(|S_{\sigma_-}|) 1(\sigma_- < \infty) < \infty, \tag{2.3}$$

$$E\tau(t)\phi(\tau(t)) < \infty \quad \textit{for all (some) } t \ge 0. \tag{2.4}$$

For $\phi(t) = t^\alpha$, $\alpha > 1$, (2.1) $\Leftrightarrow$ (2.2) is a result of Kiefer and Wolfowitz (1956), (2.1) $\Leftrightarrow$ (2.3) of Hogan (1983) and Janson (1986) and finally (2.1) $\Leftrightarrow$ (2.4) of Gut (1974). Our proof is essentially based on Lemma 4 which in turn is furnished by three further ones, two of them being rather elementary and the third one giving two martingale inequalities essentially due to Burkholder, Davis and Gundy (1972). Gut [(1974), Lemma 2.3] used the classical Burkholder inequality for proving the assertions of Lemma 4 for the special ϕ's above. For the functions considered here, however, this inequality is no longer adequate.

Let us begin with some general remarks on the functions ϕ to be considered here. It is known, see Bingham, Goldie and Teugels [(1987), page 44], that for each $\phi \in \mathscr{RV}_\alpha$, there is an infinitely often differentiable $\psi \in \mathscr{RV}_\alpha$ such that $\phi(t) \sim \psi(t)$ as $t \to \infty$ and

$$(2.5) \qquad t^n \psi^{(n)}(t) \sim \alpha(\alpha - 1) \cdots (\alpha - n + 1)\psi(t) \quad \text{for all } n \geq 1.$$

Let $\mathscr{SV}_\alpha$ denote the subclass of all such smooth functions in $\mathscr{RV}_\alpha$. It is not difficult to see that much of our subsequent analysis, in particular for the proof of Theorem 3, can be restricted w.l.o.g. to functions $\phi \in \mathscr{SV}_\alpha$ whenever this helps simplifying the arguments. Note that ϕ is then either ultimately concave or ultimately convex if $\alpha \notin \{0, 1\}$. The next lemma shows that even more regularity assumptions on ϕ can be assumed without loss of generality.

Lemma 1. (a) *Each unbounded, ultimately increasing (and ultimately convex/concave)* $\phi \in \mathscr{SV}_\alpha$, $\alpha \geq 0$, *is asymptotically equal to a function* ψ [*i.e.*, $\phi(t) \sim \psi(t)$, *as* $t \to \infty$] *which is increasing (and convex/concave) and satisfies* $\psi(0) = 0$ *and*

$$(2.6) \qquad \psi(x + y) \leq c(\psi(x) + \psi(y))$$

for all $x, y \in [0, \infty)$ *and a suitable constant* $c \in (0, \infty)$. *In particular*, $\psi(2x) \leq 2c\psi(x)$.

(b) *If* ϕ *is ultimately concave or an element of* $\mathscr{SV}_0$, *then*

$$(2.7) \qquad \phi(b + x_1 + \cdots + x_n) \leq \sum_{j=1}^{n} \phi(b + x_j)$$

holds for all sufficiently large b, *all* $n \geq 1$ *and* $x_1, \ldots, x_n \geq 0$.

Proof. (a) Choose b so large that $\psi(t) = \phi(b + t) - \phi(b)$ is increasing (and convex/concave) with $\psi'(0) = \phi'(b) > 0$. The latter is possible because ϕ is unbounded. Clearly, $\phi(t) \sim \psi(t)$ as $t \to \infty$, and $\psi(0) = 0$. Furthermore, for all $x \geq y \geq 0$,

$$\frac{\psi(x + y)}{\psi(x) + \psi(y)} \leq \frac{\psi(2x)}{\psi(x)}$$

and the right-hand side converges to 2 as $x \to 0$, because $\psi'(0) > 0$. In particular, it remains bounded for $0 \leq y \leq x < \varepsilon$ sufficiently small. However, it does so also for $x \geq \varepsilon > 0$ by the uniform convergence theorem for regularly varying functions; see Bingham, Goldie and Teugels [(1987), page 22]. This easily implies (2.6).

(b) If ϕ is ultimately concave, there is nothing to prove. So let $\phi \in \mathscr{S}\mathscr{V}_0$. Choose some $\beta \in (0,1)$, set $\psi(t) = t^\beta \phi(t)$, $\psi_b(t) = \psi(b+t) - \psi(b)$ and note that, for sufficiently large b, $\psi_b \in \mathscr{S}\mathscr{V}_\beta$ is concave with $\psi_b(0) = 0$. Thus, by applying (2.7) to ψ_b for all $n \geq 1$ and all $x_1, \dots, x_n \geq 0$,

$$\begin{aligned}
\phi(b+s_n) &= \frac{\psi(b+s_n)}{(b+s_n)^\beta} = \frac{\psi_b(s_n) + b^\beta \phi(b)}{(b+s_n)^\beta} \\
&\leq \sum_{i=1}^n \frac{\psi_b(x_i)}{(b+s_n)^\beta} + \frac{b^\beta \phi(b)}{(b+s_n)^\beta} \\
&\leq \sum_{i=1}^n \left(\frac{b+x_i}{b+s_n}\right)^\beta \phi(b+x_i) - \frac{(n-1)b^\beta \phi(b)}{(b+s_n)^\beta} \\
&\leq \sum_{i=1}^n \phi(b+x_i),
\end{aligned} \tag{2.8}$$

where $s_n = x_1 + \cdots + x_n$ and $x_i \leq s_n$ for all $1 \leq i \leq n$ has been used for the last inequality. □

The following two martingale inequalities are due to Burkholder, Davis and Gundy (1972). However, inequality (2.9), though following from their Theorem 3.1, cannot be found there; see instead Chow and Teicher [(1978), Theorem 11.3.2, page 397].

Lemma 2. *Let ψ: $[0,\infty) \to [0,\infty)$ be an increasing, continuous function with $\psi(0) = 0$ and $\psi(2t) \leq c\psi(t)$ for all $t > 0$ and some $c \in (0,\infty)$. Then, for all $\gamma \in [1,2]$, there is a constant $K = K_\gamma \in (0,\infty)$ such that for all martingales $(W_n)_{n \geq 0}$ with increments $Y_1, Y_2, \dots$ and canonical filtration $(\mathscr{F}_n)_{n \geq 0}$, that is, $\mathscr{F}_n = \sigma(W_0, \dots, W_n)$ for $n \geq 0$,*

$$E\psi\Big(\sup_{n\geq 1}|W_n|\Big) \leq K\left(E\psi\left(\left|\sum_{j\geq 1} E\big(|Y_j|^\gamma|\mathscr{F}_{j-1}\big)\right|^{1/\gamma}\right) + E\psi\Big(\sup_{n\geq 1}|Y_n|\Big)\right). \tag{2.9}$$

If ψ is furthermore convex, then even

$$E\psi\Big(\sup_{n\geq 1}|W_n|\Big) \leq KE\psi\left(\left|\sum_{j\geq 1} Y_j^2\right|^{1/2}\right). \tag{2.10}$$

Lemma 3. *Let X_1 be nonnegative and τ be an arbitrary stopping time for $(S_n)_{n\geq 1}$. Then for all measurable ψ: $[0,\infty) \to [0,\infty)$,*

$$E\psi\Big(\max_{n\leq\tau} X_n\Big) \leq E\psi(X_1)E\tau. \tag{2.11}$$

PROOF. The assertion follows immediately because of

$$\psi\Big(\max_{n\le\tau} X_n\Big) \le \sum_{n=1}^{\tau} \psi(X_n)$$

and Wald's identity. □

LEMMA 4. *Let $\psi \in \mathscr{S}\mathscr{V}_\alpha$, $\alpha \ge 0$, be an increasing, continuous function with $\psi(0) = 0$ and $\psi(2t) \le c\psi(t)$ for all $t > 0$ and some $c \in (0,\infty)$. Suppose $\mu = EX_1 = 0$ and $E\phi(|X_1|) < \infty$. Then for all a.s. finite stopping times τ for $(S_n)_{n\ge1}$,*

$$E\psi(|S_\tau|) \le CE(\tau \vee \psi(\tau)). \tag{2.12}$$

Moreover, if $\alpha > 1$, then for each $\beta > 1 \vee \alpha/2$,

$$E\psi(|S_\tau|) \le CE\tau^\beta. \tag{2.13}$$

Finally, if $\alpha = 1$ and ψ is convex, then

$$E\psi(|S_\tau|) \le CE\tau. \tag{2.14}$$

In all three inequalities, C denotes a finite constant which does not depend on τ.

PROOF. Define

$$W_n = S_{\tau\wedge n} = \sum_{j=1}^{n} X_j 1(\tau \ge j), \qquad n \ge 0,$$

which is clearly a martingale with respect to $\mathscr{F}_n = \sigma(S_0,\dots,S_n)$, $n \ge 0$, and it converges a.s. to $W_\infty = S_\tau$. Its increments $Y_n = X_n 1(\tau \ge n)$ satisfy

$$E\big(|Y_n|^\gamma | \mathscr{F}_{n-1}\big) = 1(\tau \ge n)\mu_\gamma \quad \text{a.s.},$$

where $\mu_\gamma = E|X_1|^\gamma$. Thus it follows from Lemma 3 and (2.9) in Lemma 2 with $\gamma = 1$ that

$$\begin{aligned} E\psi(|S_\tau|) \le E\psi\Big(\sup_{n\ge0}|W_n|\Big) &\le K\bigg(E\Big(\mu_1 \sum_{j\ge1} 1(\tau \ge j)\Big) + E\psi\Big(\sup_{j\ge1}|X_j| 1(\tau \ge j)\Big)\bigg) \\ &= K\Big(E\psi(\mu_1\tau) + E\psi\Big(\max_{j\le\tau}|X_j|\Big)\Big) \\ &\le K\big(E\psi(\mu_1\tau) + E\psi(|X_1|)E\tau\big) \le CE(\tau \vee \psi(\tau)) \end{aligned}$$

for a suitable constant C which does not depend on τ. This proves (2.12). If $\alpha > 1$, (2.9) gives for each $\gamma \in [1,2]$, $\gamma < \alpha$ and $\beta > \alpha/\gamma$,

$$\begin{aligned} E\psi(|S_\tau|) \le E\psi\Big(\sup_{n\ge0}|W_n|\Big) &\le K\bigg(E\bigg(\Big|\mu_\gamma \sum_{j\ge1} 1(\tau \ge j)\Big|^{1/\gamma}\bigg) \\ &\qquad + E\psi\Big(\sup_{j\ge1}|X_j| 1(\tau \ge j)\Big)\bigg) \\ &\le K\Big(E\psi\big(\mu_\gamma^{1/\gamma}\tau^{1/\gamma}\big) + E\psi(|X_1|)E\tau\Big) \le CE\tau^\beta, \end{aligned}$$

where $C \in (0, \infty)$ is a constant which does depend on β but not on τ. Since $\gamma \in [1, 2]$, $\gamma < \alpha$ is arbitrary, (2.13) follows.

Finally, if $\psi \in \mathscr{SV}_1$ is convex, then $\psi(b + t^{1/2}) \in \mathscr{SV}_{1/2}$ is clearly concave for sufficiently large b. Using this, the monotonicity of ψ, Wald's identity and (2.10), we obtain

$$\begin{aligned} E\psi(|S_\tau|) &\le E\psi\Big(\sup_{n\ge 0}|W_n|\Big) \le KE\psi\left(\left|\sum_{j\ge 1} X_j^2 1(\tau \ge j)\right|^{1/2}\right) = KE\psi\left(\left|\sum_{j=1}^{\tau} X_j^2\right|^{1/2}\right) \\ &\le KE\psi\left(\left|b + \sum_{j=1}^{\tau} X_j^2\right|^{1/2}\right) \le KE\left(\sum_{j=1}^{\tau} \psi(|b + X_j|)\right) \\ &= KE\psi(b + |X_1|)E\tau, \end{aligned}$$

which shows (2.14).

Proof of Theorem 3. With regard to Lemma 1(a), it is obviously no loss of generality to assume $\phi \in \mathscr{SV}_\alpha$ to be increasing with $\phi(0) = 0$ and $\phi(2t) \le c\phi(t)$ for all $t > 0$ and some $c \in (0, \infty)$. Let G_+ be the distribution of $S_{\tau(0)}$, the first strictly ascending ladder height and $V^+ = \sum_{n\ge 0} G_+^{*(n)}$ the associated renewal measure. Let G_- be the distribution of S_{σ_-} given $\sigma_- < \infty$, $\gamma = P(\sigma_- < \infty)$ and $V^- = \sum_{n \ge 0}\gamma^n G_-^{*(n)}$. Then the following facts are known; see Asmussen [(1987), page 201] and Janson [(1986), Lemma 1]:

$$P(S_{\sigma_-} \le t, \sigma_- < \infty) = \gamma G_-(t) = \int_{(-\infty, t]} V^+(t - x) P(X_1 \in dx), \tag{2.15}$$

$$P(S_* \le t) = (1 - \gamma)V^-(t), \tag{2.16}$$

for all $t < 0$. Using $t/(\mu E\tau(0)) \le V^+(t) \le Ct/(\mu E\tau(0))$ for all $t \ge \varepsilon$ and suitable $\varepsilon, C \in (0, \infty)$, we easily infer from (2.15):

$$P(S_{\sigma_-} \le -t, \sigma_- < \infty) \asymp \int_t^\infty P(X_1^- \ge s)\, ds,$$

where $f(t) \asymp g(t)$ means that $f(t)/g(t)$ stays bounded away from 0 and ∞ as $t \to \infty$. Since $\phi(0) = 0$, $\phi' \ge 0$ and $\int_0^t \phi(s)\, ds \sim (\alpha + 1)^{-1} t\phi(t)$ as $t \to \infty$, we further obtain

$$\begin{aligned} E\phi(|S_{\sigma_-}|)1(\sigma_- < \infty) &= \int_0^\infty \phi'(t) P(S_{\sigma_-} \le -t, \sigma_- < \infty)\, dt \\ &\asymp \int_0^\infty \phi'(t) \int_t^\infty P(X_1^- \ge s)\, ds\, dt = \int_0^\infty \phi(t) P(X_1^- \ge t)\, dt \\ &= E\left(\int_0^{X_1^-} \phi(t)\, dt\right) \asymp EX_1^- \phi(X_1^-), \end{aligned}$$

which proves equivalence of (2.1) and (2.3).

Since $|S_{\sigma_-}| \le |S_*|$ on $\{\sigma_- < \infty\}$ and ϕ is increasing, clearly (2.2) implies (2.1). For the reverse conclusion, let $(W_n)_{n \ge 0}$ be a zero-delayed random walk with i.i.d. increments $Y_1, Y_2, \dots$ having common distribution G_-. Then, by (2.16),

$$E\phi(|S_*|) = (1-\gamma)\sum_{n\geq 0}\gamma^n E\phi(|W_n|),$$

and the desired conclusion obviously follows if we can show that $E\phi(|W_1|) < \infty$ implies $E\phi(|W_n|) = O(n^p)$ as $n \to \infty$ for some $p > 0$. If $\alpha > 1$, then $\xi = EW_1$ is finite, and it follows from (2.6) and Lemma 4 [applied to $(W_j - j\xi)_{j\geq 0}$ and $\tau = n$] that

$$E\phi(|W_n|) \leq C(\phi(n\xi) + E\phi(|W_n - n\xi|)) = O(\phi(n)) \quad \text{as } n \to \infty,$$

where C is a suitable constant. For $\alpha = 1$, $E|W_1|$ may be infinite, but $\xi_\beta = E|W_1|^\beta < \infty$ for each $\beta < 1$. It is not difficult to verify that $\phi(t^{1/\beta})$ still satisfies (2.6) and the conditions in Lemma 4. Thus, since $\sum_j x_j \leq (\sum_j x_j^\beta)^{1/\beta}$,

$$\begin{aligned} E\phi(|W_n|) &\leq E\phi\left(\left|\sum_{j=1}^n |Y_j|^\beta\right|^{1/\beta}\right) \\ &\leq C\left(\phi(n\xi_\beta) + E\phi\left(\left|\sum_{j=1}^n \left(|Y_j|^\beta - \xi_\beta\right)\right|^{1/\beta}\right)\right) = O(\phi(n^{1/\beta})) \end{aligned}$$

follows again by Lemma 4, but now applied to the martingale $(\sum_{j=1}^k(|Y_j|^\beta - \xi_\beta))_{k\geq 1}$ and $\tau = n$. For the remaining α's, ϕ is ultimately subadditive in the sense of Lemma 1(b), whence

$$E\phi(|W_n|) = O(n) \quad \text{as } n \to \infty.$$

So we have proved (2.2) $\Leftrightarrow$ (2.3).

Finally, let us consider equivalence of (2.1) and (2.4). Since $\tau(t)\phi(\tau(t))/t\phi(t) \to \mu^{-\alpha-1}$ a.s. (see Lemma 5 below), we infer from Fatou's lemma that

$$\liminf_{t\to\infty}\frac{E\tau(t)\phi(\tau(t))}{t\phi(t)} \geq \mu^{-\alpha-1}.$$

Thus, assuming (2.4), we obtain (2.1) from

$$\begin{aligned} \infty > E\tau(0)\phi(\tau(0)) &\geq \int_0^\infty E\tau(t)\phi(\tau(t))P(X_1^- \in dt) \\ &\geq C\int_0^\infty t\phi(t)P(X_1^-\in dt) = CEX_1^-\phi(X_1^-), \end{aligned}$$

where C is a suitable constant. Now suppose (2.1), write τ for $\tau(t)$ and let $\phi \in \mathscr{SV}_\alpha$. Put $\Phi(t) = t\phi(t)$, which is then an element of $\mathscr{SV}_{\alpha+1}$. Since truncation of $X_1, X_2, \dots$ by some $c > 0$ enlarges τ without affecting $X_1^-, X_2^-, \dots$ we may assume w.l.o.g. $X_1, X_2, \dots$ to be bounded above, so that (2.1) becomes $E\Phi(|X_1|) < \infty$. The following argument is the same as in Gut [(1974), Lemma 2.3.], where the result is proved for $\phi(t) = t^\alpha$, $\alpha > 1$. Suppose first $\alpha > 0$. Then $E|X_1|^\beta < \infty$ for all $\beta < \alpha + 1$, whence $E\tau^\beta < \infty$ by Theorem 2.1 of Gut (1974). For $\beta \in (1 \vee (\alpha+1)/2, \alpha + 1)$, we thus obtain, by applying (2.13) of

Lemma 4 to $(S_n - n\mu)_{n \geq 0}$,

$$E\Phi(|S_\tau - \mu\tau|) \leq CE\tau^\beta < \infty.$$

But $0 < S_\tau \leq t + c$, whence (2.4) follows.

If $\alpha = 0$, then $\Phi \in \mathscr{SV}_1$ can be assumed convex, because ϕ is increasing. Hence we infer from (2.14) in Lemma 4 that

$$E\Phi(|S_\tau - \mu\tau|) \leq CE\tau < \infty,$$

which again yields the desired conclusion. □

The remainder of this section is devoted to a number of further renewal theoretic prerequisites which are needed for the proofs of Theorem 1 and 2 in Section 3. We begin with a lemma which generalizes a result due to Hatori (1959) and Lai (1975). It is needed for the proof of Lemma 6 below.

LEMMA 5. *Suppose $X_1 > 0$ a.s. and $\phi \in \mathscr{RV}_\alpha$, $\alpha \geq 0$, to be ultimately increasing. Let $Z(t) = \phi(\tau(t))/\phi(t)$ for $t \geq 0$. Then*

$$Z(t) \to \mu^{-\alpha} \quad \textit{a.s. and in mean}. \tag{2.17}$$

In particular, $Z(t)$, $t \geq 1$ are uniformly integrable.

PROOF. Let $\phi(t) = t^\alpha L(t)$ for some $L \in \mathscr{RV}_0$. Writing for short τ instead of $\tau(t)$, the a.s. convergence of $Z(t)$ is obvious from

$$Z(t) = \left(\frac{\tau}{t}\right)^\alpha \frac{L(t \cdot (\tau/t))}{L(t)},$$

the slow variation of Lt and the fact that $\tau/t \to \mu^{-1}$ a.s. as $t \to \infty$.

In order to prove convergence in mean of $Z(t)$, we must prove uniform integrability of $Z(t)$, $t \geq 1$. This, however, is easily concluded from uniform integrability of $(\tau(t)/t)^p$ for all $p > 0$; see Lai (1975).

Let $\sigma_0 = 0$, $\sigma_1 = \tau(0)$, $\sigma_2, \dots$ be the sequence of strictly ascending ladder epochs associated with $(S_n)_{n \geq 0}$ and $\tau_1, \tau_2, \dots$ its increments. Furthermore let

$$\tau^+(t) = \inf\{n \geq 1: S_{\sigma_n} > t\}, \qquad t \geq 0. \tag{2.18}$$

For the proofs of our main theorems, we must know the asymptotic behavior of

$$V_\phi^+(t) = \sum_{n \geq 0} E\phi(\sigma_n) 1(S_{\sigma_n} \leq t), \tag{2.19}$$

which may be viewed as a generalized renewal measure of $(S_{\sigma_n})_{n \geq 0}$ with random coefficients. The required result is given in Lemma 6 below. Observe that

$$V_\phi^+(t) = E\left(\sum_{j=0}^{\tau^+(t)-1} \phi(\sigma_j)\right). \tag{2.20}$$

Thus, the asymptotic behavior of $V_\phi^+(t)$ is linked to that of $\tau^+(t)$. □

LEMMA 6. *Suppose* $\mu \in (0,\infty)$, $\phi \in \mathscr{RV}_\alpha$, $\alpha \geq 0$, *ultimately increasing and* $EX_1^- \phi(X_1^-) < \infty$. *Put* $\nu = E\sigma_1$. *Then*

$$V_\phi^+(t) \sim \frac{t\phi(t)}{(\alpha+1)\nu\mu^{\alpha+1}} \quad \text{as } t \to \infty. \tag{2.21}$$

Furthermore, $\phi(t)^{-1}V_\phi^+((t, t+h])$ *is bounded in* t *for each* $h > 0$ *and, if* X_1 *is nonarithmetic,*

$$V_\phi((t, t+h]) \sim \frac{h\phi(t)}{\nu\mu^{\alpha+1}} \quad \text{as } t \to \infty. \tag{2.22}$$

Assuming only $\mu \in (0,\infty)$,

$$\liminf_{t\to\infty} \frac{V_\phi^+((t,t+h])}{\phi(t)} > 0 \tag{2.23}$$

holds for all $h > d$, *where* d *denotes the span of the distribution of* X_1.

PROOF. Again we may assume w.l.o.g. that ϕ is everywhere increasing and satisfies (2.6). Let $\Phi(t) = t\phi(t)$, write τ^+ for $\tau^+(t)$ and note $t^{-1}\tau^+ \to (\nu\mu)^{-1}$ a.s. We first observe that

$$\begin{aligned} Y(t) &\equiv \frac{1}{\Phi(t)} \sum_{j=0}^{\tau^+-1} \phi(\sigma_j) = \frac{1}{\Phi(t)} \sum_{j=0}^{\tau^+-1} \frac{\phi\big(j\cdot(\sigma_j/j)\big)}{\phi(\nu j)}\phi(\nu j) \\ &\sim \frac{1}{\Phi(t)} \sum_{j=0}^{\tau^+-1} \phi(\nu j) \sim \frac{1}{\Phi(t)} \int_1^{\tau^+} \phi(\nu x)\,dx \\ &\sim \frac{\Phi(\nu\tau^+)}{(\alpha+1)\nu\Phi(t)} \to \frac{1}{(\alpha+1)\nu\mu^{\alpha+1}} \quad \text{a.s. as } t\to\infty, \end{aligned} \tag{2.24}$$

where Lemma 5, the regular variation property of ϕ, the uniform convergence theorem and Karamata's theorem for such functions have been used; see Bingham, Goldie and Teugels [(1987), Chapter 1]. (2.20) and (2.24) show that uniform integrability of $Y(t)$, $t \geq t_0 > 0$, remains to be proved. In the following $C \in (0,\infty)$ denotes a generic constant which may differ from line to line. Using (2.6), we have

$$\begin{aligned} \sum_{j=0}^{\tau^+-1} \phi(\sigma_j) &\leq \tau^+\phi(\sigma_{\tau^+}) \leq C\big(\tau^+\phi(\nu\tau^+) + \tau^+\phi(|\sigma_{\tau^+} - \nu\tau^+|)\big) \\ &\leq C\big(\Phi(\nu\tau^+) + \tau^+|\sigma_{\tau^+} - \nu\tau^+|^{\alpha+\varepsilon}\big) \end{aligned} \tag{2.25}$$

for arbitrary $\varepsilon > 0$. By Lemma 5, $\Phi(\tau^+)/\Phi(t)$, $t \geq 1$ are uniformly integrable. Choosing $\varepsilon < \alpha$,

$$\frac{\tau^+|\sigma_{\tau^+} - \nu\tau^+|^{\alpha+\varepsilon}}{\Phi(t)}, \qquad t \geq t_0 > 0,$$

are also uniformly integrable, because

$$\frac{\tau^+|\sigma_{\tau^+}-\nu\tau^+|^{\alpha+\varepsilon}}{\Phi(t)} \le C\cdot\frac{\tau^+}{t}\cdot\frac{|\sigma_{\tau^+}-\nu\tau^+|^{\alpha+\varepsilon}}{t^{(\alpha+\varepsilon)/2}}.$$

Namely, $(\tau^+/t)^p$, $t \ge 1$, are uniformly integrable for all $p > 0$ by Lemma 5 and

$$\left|\frac{\sigma_{\tau^+}-\nu\tau^+}{t^{1/2}}\right|^{\beta(\alpha+\varepsilon)}, \qquad t \ge t_0,$$

are then uniformly integrable for some $\beta > 1$ and $t_0 > 0$ by Theorem 2 of Chow, Hsiung and Lai (1979), noting that $E\sigma_1^{\beta(\alpha+\varepsilon)} < \infty$ for some $\beta > 1$ follows from $EX_1^-\phi(X_1^-) < \infty$. This together with (2.25) yields the desired conclusion and completes the proof of (2.21).

Now let $\rho(t,h) = \inf\{n \ge 0: S_{\sigma_{\tau^+(t)+n}} - S_{\sigma_{\tau^+(t)}} > h\}$, $h \in \mathbb{R}$ and $R_t = S_{\sigma_{\tau^+(t)}} - t$ the so-called excess over the boundary. Observe that $\rho(t,h) = 0$ for $h \le 0$, that it is independent of $(\tau^+, S_{\sigma_{\tau^+}})$ with the same distribution as $\tau^+(h)$ for $h > 0$ and that $\tau^+(t+h) - \tau^+(t) = \rho(t, h - R_t)$. If X_1 is nonarithmetic, so is S_{σ_1} and R_t converges in distribution to a random variable R_∞, say, which we may assume independent of $(S_n)_{n\ge 0}$. It then follows that $E\tau^+(t - R_\infty) = t/\nu\mu$ for all $t \ge 0$. These are well-known facts from renewal theory and may, for example, be found in Asmussen (1987). We have further:

$$(2.26)\qquad \begin{aligned} E\phi(\sigma_{\tau^+})\rho(t,h-R_t) &\le E\left(\sum_{j=0}^{\rho(t,h-R_t)} \phi(\sigma_{\tau^++j})\right) \\ &= V_\phi^+((t,t+h]) \le E\big(\phi(\sigma_{\tau^+(t+h)})\rho(t,h-R_t)\big). \end{aligned}$$

Since $\rho(t, h-R_t) \le \rho(t,h)$ for all $t \ge 0$ and $\rho(t,h)$ has the same distribution as $\tau^+(h)$ which in turn has moments of arbitrary order, we infer uniform integrability of $\rho(t,h-R_t)^p$, $t \ge 0$, for all $p > 0$. Combining this with uniform integrability of

$$\frac{\Phi(\sigma_{\tau^+})}{\Phi(t)}, \qquad t \ge t_0,$$

(see Lemma 5) we easily conclude that

$$\frac{\phi(\sigma_{\tau^+})\rho(t,h-R_t)}{\phi(t)}, \qquad t \ge t_0,$$

are also uniformly integrable. We do not give more details. It is furthermore obvious that this remains true when replacing τ^+ by $\tau^+(t+h)$. Hence, the above inequality gives boundedness of $\phi(t)^{-1}V_\phi^+((t,t+h])$, for $t \ge 0$. If X_1 is nonarithmetic, then $(\phi(t)^{-1}\phi(\sigma_{\tau^+}), R_t)$ converges in distribution to $(\mu^{-\alpha}, R_\infty)$ as $t \to \infty$, and again this is true also with $\tau^+(t+h)$ instead of τ^+.Consequently, by uniform integrability, the extreme right- and left-hand sides in

(2.26) divided by $\phi(t)$ converge to

$$\mu^{-\alpha} E\tau^{+}(h - R_{\infty}) = \frac{h}{\nu\mu^{\alpha+1}} \quad \text{as } t \to \infty,$$

proving (2.22).

If only $\mu \in (0, \infty)$ is assumed, then (2.26), Fatou's lemma and Lemma 5 yield

$$\liminf_{t\to\infty} \frac{V_{\phi}^{+}((t, t+h])}{\phi(t)} \geq \liminf_{t\to\infty} \frac{E\phi(\tau^{+})\rho(t, h - R_t)}{\phi(t)} = \frac{h}{(\nu\mu)^{\alpha+1}}$$

for all $h > 0$, provided X_1 is nonarithmetic. This proves (2.23).

If X_1 is d-arithmetic, then R_{nd} converges in distribution to a d-arithmetic random variable R_{∞} satisfying $E\tau^{+}(nd - R_{\infty}) = d/\nu\mu$ for all $n \in \mathbb{N}_0$. Here again R_{∞} is assumed to be independent of $(S_n)_{n\geq 0}$. With this modification, the above argument leads to

$$\liminf_{n\to\infty} \frac{V_{\phi}^{+}(\{nd\})}{\phi(nd)} \geq \frac{d}{(\nu\mu)^{\alpha+1}},$$

and this obviously implies (2.23) for d-arithmetic X_1. □

The final lemma of this section is stated without proof, but may be easily derived from Spitzer's formula for maxima [see Chung (1974), Theorem 8.5.1.] or by direct calculation [see Keener (1987), Lemma 2]. Recall that $S_* = \min_{n\geq 0} S_n$.

LEMMA 7. *Suppose $\mu \in (0, \infty)$, let $\nu = E\sigma_1$ and h be a measurable, nonnegative function. Then*

$$Eh(S_*) = \nu^{-1} E\left(\sum_{j=0}^{\sigma_1 - 1} h(S_j)\right). \tag{2.27}$$

In particular,

$$U(t) = \nu E V^{+}(t - S_*) \quad \textit{for all } t \in \mathbb{R}, \tag{2.28}$$

where $U(t)$ and $V^{+}(t)$ denote the ordinary renewal functions of $(S_n)_{n\geq 0}$ and $(S_{\sigma_n})_{n\geq 0}$, respectively.

Note that (2.27) is indeed a consequence of (2.26), because by (3.1) below with $\phi \equiv 1$,

$$U(t) = E\left(\sum_{j=0}^{\sigma_1 - 1} V^{+}(t - S_j)\right).$$

3. Proofs of Theorems 1 and 2. In the following we keep the notation of the previous sections. So $\sigma_0, \sigma_1, \ldots$ denote the strictly ascending ladder epochs of $(S_n)_{n\geq 0}$ whose increments $\tau_1, \tau_2, \ldots$ are i.i.d. with finite mean ν, σ_-

is the first weakly descending ladder epoch with associated ladder height S_{σ_-} and $S_* = \min_{n \ge 0} S_n$. Furthermore, the definitions of $\tau(t)$ in (1.4), $\tau^+(t)$ in (2.18) and of V_ϕ^+ in (2.19) should be recalled.

A well-known technique from renewal theory [see Athreya, McDonald and Ney (1978)] consists in splitting the random walk $(S_n)_{n \ge 0}$ into the cycles

$$\{S_0, \dots, S_{\sigma_1 - 1}\}, \{S_{\sigma_1}, \dots, S_{\sigma_2 - 1}\}, \dots .$$

Doing so with regard to U_ϕ, we obtain

$$\begin{aligned} U_\phi(t) &= \sum_{n \ge 0} E\left(\sum_{j=\sigma_n}^{\sigma_{n+1}-1} \phi(j) 1(S_j \le t) \right) \\ &= \sum_{n \ge 0} E\left(\sum_{j=0}^{\tau_{n+1}-1} \phi(\sigma_n + j) 1(S_{\sigma n + j} \le t) \right) \qquad (3.1) \\ &= \sum_{k \ge 0} \int_{(0,\infty)} E\left(\sum_{j=0}^{\sigma_1 - 1} \phi(k+j) 1(S_j \le t - x) \right) \sum_{n \ge 0} P(\sigma_n = k, S_{\sigma_n} \in dx), \end{aligned}$$

which forms the basic identity for the proofs of Theorems 1 and 2 below. Let us define

$$G(t) = E\left(\sum_{j=0}^{\sigma_1 - 1} 1(S_j \le t) \right), \qquad t \in \mathbb{R}, \tag{3.2}$$

which is the distribution function of a finite measure with total mass $E\sigma_1$ being concentrated on $(-\infty, 0]$.

PROOF OF THEOREM 1. (a) We show (1.6) $\Rightarrow$ (1.5), (1.5) $\Rightarrow$ (1.6) and finally (1.5), (1.6) $\Rightarrow$ (1.7), which obviously proves part (a) of the theorem. Without loss of generality, let $\phi \in \mathscr{S}\mathscr{V}_\alpha$, $\alpha \ge 0$, and everywhere increasing on $[0, \infty)$.

(1.6) $\Rightarrow$ (1.5). By using (3.1), the monotonicity of ϕ, $\sigma_n \ge n$ for all $n \ge 0$ and Lemma 7, we infer

$$\begin{aligned} U_\phi(t) &\ge \sum_{k \ge 0} \phi(k) \int_{(0,\infty)} G(t - x) \sum_{n \ge 0} P(\sigma_n = k, S_{\sigma_n} \in dx) \\ &= \int_{(0,\infty)} G(t-x) V_\phi^+(dx) = \int_{(-\infty, 0]} V_\phi^+(t - x) G(dx) \qquad (3.3) \\ &= E\left(\sum_{j=0}^{\sigma_1 - 1} V_\phi^+(t - S_j) \right) = \nu E V_\phi^+(t - S_*) \ge \nu E U_\phi^*(t - S^*), \end{aligned}$$

where $U_\phi^*(t) = \sum_{n \ge 0} \phi(n) P(S_{\sigma_n} \le t)$. Thus (1.6) implies $EU_\phi^*(t - S_*) \le EV_\phi^+(t - S_*) < \infty$ for all $t \ge 0$. But $(S_{\sigma_n})_{n \ge 0}$ has nonnegative increments, whence by Theorem 1 of Heyde (1966),

$$EU_\phi^*(t - S_*) \asymp E(t - S_*)\phi(t - S_*) \quad \text{as } t \to \infty,$$

and thereby $E|S_*|\phi(|S_*|) < \infty$ follows. As a consequence, (1.5) must hold, according to Theorem 3.

(1.5) ⇒ (1.6). Let $\varepsilon > 0$. Then

$$\begin{aligned}(3.4)\qquad &\sum_{n\geq 0} E\left(\sum_{j=0}^{\tau_{n+1}-1} \phi(\sigma_n + j)1(\tau_{n+1} > \varepsilon\sigma_n)\right)\\ &\leq \sum_{n\geq 0} E\tau_{n+1}\phi(\sigma_{n+1})1(\tau_{n+1} > \varepsilon\sigma_n)\\ &\leq \sum_{n\geq 0} E\sigma_1\phi\left(\left(1+\frac{1}{\varepsilon}\right)\sigma_1\right)1(\sigma_1 > \varepsilon n),\end{aligned}$$

where $\sigma_n \geq n$ for all $n \geq 0$ should be observed for the last inequality. Assuming (1.5), thus $E\sigma_1^2\phi(\sigma_1) < \infty$ by Theorem 3, one can easily check that the final expression in (3.4) equals some finite value $A = A(\varepsilon)$, say. Setting $\tau_n' = \tau_n \wedge (\varepsilon\sigma_{n-1})$ for all $n \geq 1$ and $\phi_a(t) = \phi((1+a)t)$, we therefore conclude from (3.1) for all $t \geq 0$,

$$\begin{aligned}(3.5)\qquad U_\phi(t) &\leq A + \sum_{n\geq 0} E\left(\sum_{j=0}^{\tau_{n+1}'-1} \phi(\sigma_n + j)1(S_{\sigma_n+j} \leq t)\right)\\ &\leq A + \sum_{n\geq 0} E\left(\sum_{j=0}^{\tau_{n+1}-1} \phi_{1/\varepsilon}(\sigma_n)1(S_{\sigma_n+j} \leq t)\right)\\ &= A + \nu EV_{\phi_{1/\varepsilon}}^+(t - S_*) \asymp E(t - S^*)\phi_{1/\varepsilon}(t - S^*),\end{aligned}$$

where the final asymptotic equivalence ($\asymp$) in (3.5) follows by Lemma 6. Since $\phi_{1/\varepsilon}(t) \asymp \phi(t)$, Theorem 3 gives again the desired conclusion.

(1.5), (1.6) ⇒ (1.7). Inequalities (3.3) and (3.5) together give

$$(3.6)\qquad \nu EV_\phi^+(t - S_*) \leq U_\phi(t) \leq A + \nu EV_{\phi_{1/\varepsilon}}^+(t - S_*)$$

for all $t \in \mathbb{R}$ and all $\varepsilon > 0$. Lemma 6, together with $(\phi_{1/\varepsilon}(t)/\phi(t)) \to (1 + (1/\varepsilon))^\alpha$, yields

$$\frac{\nu V_\phi^+(t - S_*)}{t\phi(t)} \to \frac{1}{(\alpha+1)\mu^{\alpha+1}} \quad \text{a.s. as } t \to \infty,$$

$$\frac{\nu V_{\phi_{1/\varepsilon}}^+(t - S_*)}{t\phi(t)} \to \frac{(1 + (1/\varepsilon))^\alpha}{(\alpha+1)\mu^{\alpha+1}} \quad \text{a.s. as } t \to \infty.$$

Supposing w.l.o.g. that $t\phi_{1/\varepsilon}(t)$ satisfies (2.6) in Lemma 1(a), we further obtain

$$(3.7)\qquad \frac{V_{\phi 1/\varepsilon}(t - S_*)}{t\phi(t)} \asymp \frac{(t - S_*)\phi(t - S_*)}{t\phi(t)} \leq C(1 + |S_*|\phi(|S_*|))$$

for all large t and some $C > 0$. Hence (1.7) follows from (3.6) and the dominated convergence theorem.

(b) For decreasing $\phi \in \mathscr{S}\mathscr{V}_\alpha$, $\alpha \in (-1, 0]$, we obviously get (3.3) with reversed inequality sign. Thus, by (3.7) and Fatou's lemma,

$$\limsup_{t\to\infty} \frac{U_\phi(t)}{t\phi(t)} \le \frac{1}{(\alpha+1)\mu^{\alpha+1}}.$$

Conversely, we have

$$U_\phi(t) \ge E\left(\sum_{j=0}^{\tau(t)-1} \phi(j)\right) \sim \frac{1}{(\alpha+1)} E\tau(t)\phi(\tau(t)) \quad \text{as } t \to \infty,$$

whence, again by Fatou's lemma,

$$\liminf_{t\to\infty} \frac{U_\phi(t)}{t\phi(t)} \ge \liminf_{t\to\infty} \frac{E\rho(t)\phi(\tau(t))}{(\alpha+1)t\phi(t)} = \frac{1}{(\alpha+1)\mu^{\alpha+1}}. \qquad \square$$

Proof of Theorem 2. We assume again w.l.o.g. that $\phi \in \mathscr{S}\mathscr{V}_\alpha$, $\alpha \ge 0$, is everywhere increasing and that it satisfies (2.6) of Lemma 1.

(1.9) $\Leftrightarrow$ (1.10). Since $\{M_n \le t\} = \{\tau(t) \ge n\}$ for all n, t, we easily obtain

$$\begin{aligned} U_\phi^+(t) &= \sum_{n\ge 0} \phi(n) P(\tau(t) > n) = \sum_{k\ge 1} P(\tau(t) = k)\left(\sum_{n=0}^{k-1} \phi(n)\right) \\ &\sim \frac{1}{\alpha+1} \sum_{k\ge 1} k\phi(k) P(\tau(t) = k) = \frac{E\tau(t)\phi(\tau(t))}{\alpha+1} \quad \text{as } t\to\infty. \end{aligned} \tag{3.8}$$

Thus, by Theorem 3, $U_\phi^+(t) < \infty$ for all $t \ge 0$ iff (1.9) holds.

(1.9), (1.10) $\Rightarrow$ (1.11). Applying Fatou's lemma to (3.8) yields

$$\liminf_{t\to\infty} \frac{U_\phi^+(t)}{t\phi(t)} \ge \frac{1}{(\alpha+1)\mu^{\alpha+1}}.$$

For the reverse inequality, let $\Phi(t) = \int_0^t \phi(x)\,dx$ which is increasing, convex, an element of $\mathscr{S}\mathscr{V}_{\alpha+1}$ and satisfies $\Phi(t) \sim t\phi t/(\alpha+1)$ as $t \to \infty$. We obtain

$$\begin{aligned} U_\phi^+(t) &= \sum_{n\ge 0} E\left(\sum_{j=\sigma_n}^{\sigma_{n+1}-1} \phi(j) 1(M_n \le t)\right) \\ &= \sum_{n\ge 0} E\left(1(S_{\sigma_n} \le t) \sum_{j=\sigma_n}^{\sigma_{n+1}-1} \phi(j)\right) \\ &\le E\left(\sum_{n\ge 0} 1(\tau^+(t) > n) \int_{\sigma_n}^{\sigma_{n+1}} \phi(x)\,dx\right) \\ &= E\left(\sum_{n\ge 0} 1(\tau^+(t) > n)(\Phi(\sigma_{n+1}) - \Phi(\sigma_n))\right) \\ &= E\Phi(\sigma_{\tau^+(t)}) \le \Phi(\nu E\tau^+(t)), \end{aligned} \tag{3.9}$$

where Jensen's inequality and Wald's identity have been used for the last inequality. But $E\tau^+(t) \sim (\nu\mu)^{-1}t$ by the elementary renewal theorem so that

$$\limsup_{t\to\infty} \frac{U_\phi^+(t)}{t\phi(t)} \le \lim_{t\to\infty} \frac{\Phi(t/\mu)}{t\phi(t)} = \frac{1}{(\alpha+1)\mu^{\alpha+1}},$$

which completes the proof of (1.11).

(1.12) $\Rightarrow$ (1.9). Similar to (3.3), we obviously have

$$\begin{aligned} &U_\phi(t+h) - U_\phi(t) \\ &\quad \ge \nu E V_\phi^+((t-S_*, t+h-S_*]) \quad \text{for all } t \in \mathbb{R} \text{ and } h > 0. \end{aligned} \tag{3.10}$$

Thus, by (2.23) in Lemma 6, for $h > d$, the span of the distribution of X_1, and for all sufficiently large t,

$$U_\phi(t+h) - U_\phi(t) \ge C E_\phi(t - S_*),$$

where $C > 0$ denotes a suitable constant. Consequently, (1.12) implies $E\phi(|S_*|) < \infty$ which in turn implies (1.9) by Theorem 3.

(1.9) $\Rightarrow$ (1.12). Let $\varepsilon > 0$ be arbitrary, $\phi_{1/\varepsilon}$ be as in the proof of Theorem 1, V^+ be the ordinary renewal measure of $(S_{\sigma_n})_{n\ge 0}$ and $I = (t, t+h]$ for some $t \in \mathbb{R}$ and $h > 0$. Then

$$\begin{aligned} &\sum_{n\ge 0} E\left(\sum_{j=0}^{\tau_{n+1}-1} \phi(\sigma_n + j) 1(S_j \in I, \tau_{n+1} \ge \varepsilon\sigma_n)\right) \\ &\quad \le \sum_{n\ge 0} E\left(\phi_{1/\varepsilon}(\tau_{n+1}) \sum_{j=0}^{\tau_{n+1}-1} 1(S_{\sigma_n+j} \in I)\right) \\ &\quad = \int_{(0,\infty)} E\left(\phi_{1/\varepsilon}(\sigma_1) \sum_{j=0}^{\sigma_1-1} 1(S_j \in I - x)\right) V^+(dx) \\ &\quad = E\left(\phi_{1/\varepsilon}(\sigma_1) \sum_{j=0}^{\sigma_1-1} V^+(I - S_j)\right) \le C E\sigma_1 \phi_{1/\varepsilon}(\sigma_1) \end{aligned} \tag{3.11}$$

for some $C > 0$, where uniform boundedness of V^+ on intervals of constant length has been utilized for the last inequality. Now, setting $\tau_n' = \tau_n \wedge (\varepsilon\sigma_{n-1})$ for all $n \ge 1$, we further obtain by a similar calculation as in (3.5),

$$\begin{aligned} U_\phi(t+h) - U_\phi(t) &\le C E\sigma_1\phi_{1/\varepsilon}(\sigma_1) + \sum_{n\ge 0} E\left(\sum_{j=0}^{\tau_{n+1}'-1} \phi(\sigma_n + j) 1(S_{\sigma_n+j} \in I)\right) \\ &\le C E\sigma_1\phi_{1/\varepsilon}(\sigma_1) + \sum_{n\ge 0} E\left(\sum_{j=0}^{\tau_{n+1}-1} \phi_{1/\varepsilon}(\sigma_n) 1(S_{\sigma_n+j} \in I)\right) \\ &= C E\sigma_1\phi(\sigma_1) + \nu E V_{\phi_{1/\varepsilon}}^+(I - S_*) \quad \text{for all } t \ge 0. \end{aligned} \tag{3.12}$$

Using Lemma 6 and Theorem 3, it is obvious now that the right-hand side in (3.12) is finite iff (1.9) holds.

(1.9), (1.12) ⇒ (1.13). Inequalities (3.10) and (3.12) together show that under (1.9), inequality (3.6) persists for bounded intervals, that is,

$$\nu E V_{\phi}^{+}(I - S_{*}) \le U_{\phi}(I) \le A + E V_{\phi}^{+}(I - S_{*}) \tag{3.13}$$

for all $t \in \mathbb{R}$, $h > 0$ and $\varepsilon > 0$ and with a constant A which does not depend on t. Thus, (1.13) follows from Lemma 6 and by a similar discussion as after (3.6). We do not give the details again.

(b) It is not difficult to see and will not be spelled out further that our arguments proving (1.10)–(1.12) for increasing ϕ can easily be extended to decreasing $\phi \in \mathscr{SV}_{\alpha}$, $\alpha \in (-1, 0]$, provided $\mu \in (0, \infty)$. Validity of (1.13) in this case has been shown by Kalma (1972). Surprisingly, we have not been able to find a simple argument for this in the spirit of our proof. Some further comments on this will be given in the following section. □

4. Further conclusions and discussion. A great number of random variables which arise in the study of random walks with positive drift are closely related to X_1^- concerning their moments. This has been proved for the standard case [$\phi(t) = t^{\alpha}$, $\alpha > 0$] by Janson [(1986), Theorem 1]. Combining his calculations with our results, one can easily extend his results to the classes $\mathscr{RV}_{\alpha}$. We will not do this here. Recalling relation (3.8), we directly infer from Theorem 2 the following extension of Theorem 2.3 by Gut (1974).

COROLLARY 1. *Suppose $\mu \in (0, \infty)$ and $\phi \in \mathscr{RV}_{\alpha}$, $\alpha \ge 0$, to be ultimately increasing. Then*

$$\lim_{t \to \infty} \frac{E\tau(t)\phi(\tau(t))}{t\phi(t)} = \mu^{-\alpha - 1} \tag{4.1}$$

holds iff $EX_1^- \phi(X_1^-) < \infty$.

Considering first passage times

$$T(t) = \inf\{n \ge 1 \colon S_n > tf(n)\}, \qquad t \ge 0,$$

with curved, continuous and ultimately increasing boundaries f such that $\lim_{x \to \infty} x^{-1} f(x) = 0$, Gut (1974) has shown that equivalence of (2.1) and (2.4) for $\phi(t) = t^{\alpha}$, $\alpha \ge 1$, remains true with $\tau(t)$ replaced by $T(t)$. This follows simply by squeezing $T(t)$ between two first passage times with horizontal boundaries. As a consequence, one may use Theorem 3 to extend Gut's results to general $t\phi(t)$ with ultimately increasing $\phi \in \mathscr{RV}_{\alpha}$, $\alpha \ge 0$. We omit further details. Let us finally state one further extension in this respect where we do not even need one of our theorems. Janson [(1986), Theorem 3] has proved for the last exit times $\xi(t) = \sup\{n \ge 0 \colon S_n \le t\}$, $t \ge 0$, that

$$E|S_{\xi(t)}|^{\alpha} < \infty \quad \text{for all } t \ge 0 \Leftrightarrow E(X_1^+ \wedge X_2^-)^{\alpha + 1} X_2^- < \infty.$$

A check of his proof shows that again we have the following extension.

COROLLARY 2. *Suppose* $\mu \in (0, \infty)$ *and* $\phi \in \mathscr{RV}_\alpha$, $\alpha \geq 0$, *to be ultimately increasing. Then the following assertions are equivalent.*

$$E(X_1^+ \wedge X_2^-)\phi(X_1^+ \wedge X_2^-)X_2^- < \infty; \tag{4.2}$$

$$E\phi(|S_{\xi(t)}|) < \infty \quad \text{for all } t \geq 0. \tag{4.3}$$

We conjecture that equivalence of (1.5)–(1.7) holds also for ultimately decreasing $\phi \in \mathscr{RV}_\alpha$, $\alpha \in (-1, 0]$. However, we have not been able to prove this. In order to provide a lower bound for $U_\phi(t)$, one is tempted to merely proceed as in the case of increasing ϕ and reverse the inequality signs. Thus, for proving necessity of (1.5), that is, $EX_1^-\phi(X_1^-) < \infty$, one must verify (3.5) with reversed inequality sign. Unfortunately, the constant A arising there has been shown to be finite by just assuming the validity of (1.5), which is now no longer available. So one has to look for an alternative inequality. Even though we have tried a number of such different estimates, they surprisingly all failed to give the desired result. For the same reasons we have not been able to conclude (1.13) from (1.9) by our methods in the decreasing case.

For $\alpha < -1$ the situation is qualitatively different, as one can see from the results by Kalma (1972) and Maejima and Omey (1984). Namely, in this case $U_\phi(t)$ is bounded and there is regular behavior of $(t\phi(t))^{-1}U_\phi(t)$ as $t \to \infty$, connected to the tails of X_1^+ rather than X_1^-. This becomes clear if one observes that the related moment for $\tau(t)$, which is still $E(\tau(t)\Phi(\tau(t))/t\phi(t))$, has $\tau(t)$ in the denominator now, so that convergence of this moment requires a small probability [compared to $\phi(t)^{-1}$] of early stopping. This in turn depends on the tails of X_1^+ as one can easily see.

As a further question of theoretical interest one may ask for necessary and sufficient conditions for the finiteness of $U_\phi(t)$ in the case where μ is infinite or does not exist, but with finite $E\sigma_1$ then, so that $(S_n)_{n \geq 0}$ remains transient. Even though one can still use the techniques in Section 3 from renewal theory, one has to first find a substitute for Theorem 3 which is obviously heavily based on the finiteness of μ.

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