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ISOPERIMETRIC INEQUALITIES AND TRANSIENT RANDOM WALKS ON GRAPHS

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The two-dimensional grid Z^2 and any graph of smaller growth rate is recurrent. We show that any graph satisfying an isoperimetric inequality only slightly stronger than that of Z^2 is transient. More precisely, if $f(k)$ denotes the smallest number of vertices in the boundary of a connected subgraph with k vertices, then the graph is transient if the infinite sum $\sum f(k)^{-2}$ converges. This can be applied to parabolicity versus hyperbolicity of surfaces.

1. Introduction. Let G be a connected graph which is locally finite, that is, all vertices have finite degree. We consider a random walk starting at a vertex v, say, such that at any vertex u, the walk proceeds to a neighbour with probability $1/d(u)$, where $d(u)$ is the degree (i.e., the number of neighbours) of u. The graph G is *recurrent* if we revisit v with probability 1. Otherwise G is *transient*. It is well known that the three-dimensional grid Z^3 is transient while Z^2 is recurrent. More generally, Nash-Williams [13] (see also [4, 10]) proved that any graph with smaller growth rate than Z^2 is recurrent. Lyons [10] showed that certain subgraphs of grids are transient provided they grow just a little faster than Z^2. Other results, in terms of isoperimetric inequalities, supporting the statement that Z^2 is, in a sense, an "extreme" recurrent graph, can be derived from work of Fernandez [5], Grigor'yan [7] and Varopoulos [15]. (Varopoulos [16] used results of Gromov [8] to characterize completely the recurrent Cayley graphs.)

We shall carry these results further. If V is a vertex set in G, then ∂V will denote the *boundary* of V, that is, the set of vertices of V having neighbours outside of V. Let f be a nondecreasing positive real function defined on the natural numbers. We say that G satisfies an *f-isoperimetric* inequality if there exists a constant $c > 0$ such that, for each finite vertex set V of G,

$$|\partial V| > cf(|V|).$$

If this inequality holds for all finite vertex sets V which contain a fixed vertex (root) v and induce connected subgraphs in G, we say that G satisfies a *rooted, connected f-isoperimetric inequality*. The main result of this paper says that G is transient if G has bounded degrees and satisfies a rooted, connected f-isoperimetric inequality such that

$$\sum_{k=1}^{\infty} f(k)^{-2} < \infty.$$

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This includes the above-mentioned transience results except that those of Fernandez and Grigor'yan are for the continuous case as well. Using results of Kanai [9], our results can be translated to the continuous case as well but only for surfaces with a bound on the curvature which corresponds to our bound on the degrees. As suggested by the referee, our result also implies a transience result for graphs with no bounds on the degrees when the isoperimetric inequality is formulated in terms of edge sets rather than vertex sets as in [3].

Our general result, which is hopefully of independent interest, is about trees in graphs. It says that every graph of maximum degree 3 satisfying a rooted, connected f-isoperimetric inequality [such that $f(k) \to \infty$ as $k \to \infty$] contains a subdivision of the dyadic tree defined in the next section. If $\sum_{k=1}^{\infty} f(k)^{-2} < \infty$, then that tree is transient.

Note that not every transient graph contains a transient tree; see, for example, [12]. The proof involves Menger's theorem on graphs, but no graph theoretic knowledge will be assumed.

2. Basic concepts: Graphs, Menger's theorem, flows, transience and trees. A graph G is a set $V(G)$ of elements called vertices and a set $E(G)$ of unordered pairs xy of vertices called edges. We say that the edge xy is incident with x and y and that x and y are neighbours. The number of neighbours of x is called the degree of x and is denoted $d(x)$. All graphs in this paper are *locally finite*, which means that all degrees are finite. A path $x_1x_2 \cdots$ is the graph with distinct vertices $x_1, x_2, \dots$ and edges $x_1x_2, x_2x_3, \dots$. A cycle $x_1x_2 \cdots x_nx_1$ is defined analogously. A graph is connected if any two vertices are connected by a path. A component is a maximal connected subgraph. The union of finite (respectively, infinite) components is called the finite (respectively, infinite) part of G. If $S \subseteq V(G) \cup E(G)$, then $G - S$ is obtained from G by deleting S and all edges incident with vertices in S. If $S \subseteq V(G)$, then $G - (V(G) \setminus S)$ is the subgraph induced by S.

We shall use the following version of Menger's theorem, which can be derived from the max-flow-min-cut theorem [6] and which is proved in almost all books on graph theory.

THEOREM 2.1. *Let k be a natural number and A, B vertex sets each of cardinality k in a graph G. Assume that, for each vertex set S with fewer than k vertices, $G - S$ has a path from A to B. Then G has k pairwise disjoint paths from A to B.*

A flow g in a graph G is obtained from G by assigning a direction and a nonnegative real number $g(e)$ to every edge, such that Kirchhoff's current law is satisfied, that is, for each vertex x in G, the sum of flow values in edges entering x equals the sum of flow values in edges leaving x. If Kirchhoff's current law is satisfied for all vertices except one, say v_0, we say that g is a flow of value I from v_0 to (*or from*) infinity if I is the difference between the two sums above at the vertex v_0. The energy $W(g)$ of a flow g is defined as the square sum $\Sigma\, g(e)^2$ taken over all edges.

We shall use the following combinatorial criterion for transience (see, e.g., [4, 10]).

THEOREM 2.2. *A connected, locally finite graph is transient if and only if, for some (and hence each) positive number I and for some (and hence each) vertex v, G has a flow of value I and of finite energy from v to infinity.*

A tree is a connected graph with no cycle. The dyadic tree T_0 is the tree whose vertex set is the disjoint union $S_0 \cup S_1 \cup S_2 \cup \cdots$, where $|S_k| = 2^k$ for $k = 0, 1, \dots$, and each vertex in S_k has two neighbours in S_{k+1} and one neighbour in S_{k-1} for $k = 1, 2, \dots$. A partial dyadic tree is any subtree of T_0 induced by $S_0 \cup S_1 \cup \cdots \cup S_k$ and a subset of S_{k+1}. A subdivision of a graph is obtained by inserting new vertices of degree 2 on the edges. (Any edge may be subdivided any number of times and it may not be subdivided at all.)

3. The main result and its consequences.

THEOREM 3.1. *Let G be a connected graph of maximum degree 3 and with at least one vertex of degree less than 3 such that G satisfies a rooted, connected f-isoperimetric inequality. Assume that $f(k) \to \infty$ as $k \to \infty$. Then G contains a subdivision H of the dyadic tree T_0 and for each $k = 1, 2, \dots$, a set E_k of 2^k edges, such that E_k intersects each of the 2^k paths in H from S_{k-1} to S_k and such that the finite part of $H - E_k$ belongs to the finite part of $G - E_k$.*

We prove Theorem 3.1 in the next section. First we describe its applications to graph transience.

THEOREM 3.2. *Let G be a graph as in Theorem 3.1. If, in addition,*

$$\sum_{k=1}^{\infty} f(k)^{-2} < \infty,$$

then G contains a transient tree.

PROOF. Let H be as in Theorem 3.1. We shall prove that H is transient. We define a flow g of value 1 in H from the vertex v_0 in S_0 to infinity by sending the flow 2^{-k} in the paths from S_{k-1} to S_k, $k \geq 1$. Let a_k denote the number of edges in H with flow 2^{-k}, $k \geq 0$.

For each $k \geq 1$, the finite component of $H - E_k$ belongs to a finite component G_k of $G - E_k$. By the rooted, connected f-isoperimetric inequality (assuming the constant c is 1),

$$2^{k+1} \geq |\partial(V(G_{k+1}))| \geq f(|V(G_{k+1})|) \geq f(a_1 + a_2 + \cdots + a_k).$$

Hence the energy $W(g)$ of the flow g satisfies

$$\begin{aligned} W(g) &= \sum_{k=1}^{\infty} a_k 2^{-2k} \le 4 \sum_{k=1}^{\infty} a_k f(a_1 + a_2 + \cdots + a_k)^{-2} \\ &\le 4 \sum_{k=1}^{\infty} \sum_{i=1}^{a_k} f(a_1 + a_2 + \cdots + a_{k-1} + i)^{-2} \\ &= 4 \sum_{n=1}^{\infty} f(n)^{-2} < \infty. \end{aligned}$$

□

THEOREM 3.3. *Let G be a graph as in Theorem 3.2 except that now G has maximum degree d, $3 < d < \infty$. Then G is transient.*

PROOF. If v is a vertex in G of degree $d(v) = k > 3$, then we let $e_1, e_2, \dots, e_k$ denote the edges incident with v. We blow v up to a path as follows: We delete v but none of $e_1, e_2, \dots, e_k$. Then we add a path $v_1 v_2 \cdots v_k$ and let e_i be incident with v_i in the new graph. We perform this operation for each vertex v of degree greater than 3. The resulting graph G' satisfies a connected rooted f'-isoperimetric inequality [where $f'(n) = f(\lfloor n/d \rfloor)$] and is therefore transient by Theorem 3.2. If g' is a flow in G' of value $I > 0$ and finite energy from a vertex towards infinity, then we let g denote the restriction of g' to $E(G)$. Since G is obtained from G' by contracting edges, g is also a flow of value I. Moreover, $W(g) \le W(g')$. Hence G is transient. □

Let us define an *ε-isoperimetric inequality* as an f-isoperimetric inequality for $f(k) = k^{1/2+\varepsilon}$, where $\varepsilon \ge 0$. A rooted, connected ε-isoperimetric inequality is defined analogously. Results of Grigor'yan [7] and Varopoulos [15] imply that a connected graph of finite maximum degree satisfying an ε-isoperimetric inequality for some $\varepsilon > 0$ is transient. (This is not the case for $\varepsilon = 0$ because of the grid Z^2. For $\varepsilon = 1/2$, it was proved by Dodziuk [2].) Theorem 3.3 implies the following stronger result:

THEOREM 3.4. *Each connected graph of finite maximum degree satisfying a rooted, connected ε-isoperimetric inequality for some $\varepsilon > 0$ is transient.*

We conclude this section with a transience result for graphs with no bounds on the degrees. In this result our graphs are allowed to have multiple edges. If V is a finite vertex set in a locally finite graph G, then we let $\partial_e V$ denote the set of edges from V to $V(G) \setminus V$ and we let $d(V)$ denote the sum of degrees of the vertices in V. Thus

$$d(V) = |\partial_e V| + 2|E(G(V))|.$$

Let f be a nondecreasing positive real function defined on the positive real numbers. We say that G satisfies a rooted, connected f-isoperimetric edge-inequality if there is a vertex v in G and a positive constant c such that, for each finite vertex set in G which induces a connected subgraph containing v, we have

$$|\partial_e V| > cf(d(V)).$$

THEOREM 3.5. *Let G be a connected, locally finite graph satisfying a rooted, connected f-isoperimetric edge-inequality. If*

$$\sum_{k=1}^{\infty} f(k)^{-2} < \infty,$$

then G is transient.

PROOF. There exists a positive constant α such that, for each natural number $n \geq 4$, there exists a graph H_n with n vertices such that all vertices of H_n have degree 3 (except possibly one which has degree 2) and such that, for any partition $V(H_n) = A \cup B$, there are at least $\alpha \min\{|A|, |B|\}$ edges between A and B; see [1]. Now we form a new graph G' as follows. Let u be a vertex of G. If $d(u) \geq 4$, then we replace u by $H^u = H_n$, where $n = d(u)$. The edges in G which are incident with u in G will be incident with distinct vertices in H_n. Then G' has maximum degree less than or equal to 4. We claim that G' satisfies a rooted, connected f'-isoperimetric inequality where

$$f'(n) = \min\{f(n/2), n\}.$$

To see this, we let V' denote any finite vertex set of G'. We shall prove that either $|\partial V'| \geq (\alpha/6)|V'|$ or

$$|\partial V'| \geq (4 + 3/\alpha)^{-1} f((1/2)|V'|).$$

Assume that $|\partial V'| < (\alpha/6)|V'|$. Now let V be the set of those vertices u in G such that

$$|V(H^u) \cap V'| \geq (1/2)|V(H^u)|.$$

If

$$|V(H^u) \cap V'| < (1/2)|V(H^u)|,$$

then

$$|V(H^u) \cap \partial V'| \geq (\alpha/3)|V(H^u) \cap V'|$$

by the partition property of H_n. Hence

$$|V'| \leq d(V) + (3/\alpha)|\partial V'| \leq d(V) + 1/2|V'|. \tag{3.1}$$

If an edge in G belongs to $\partial_e V$ but not to $\partial_e V'$, then the edge joins an H^u with an H^w, where

$$|V(H^u) \cap V'| \geq (1/2)|V(H^u)|$$

and

$$|V(H^w) \cap V'| < (1/2)|V(H^w)|.$$

The number of such edges (for fixed u and w) is at most

$$|V(H^u) \setminus V'| + |V(H^w) \cap V'| \leq (3/\alpha)|\partial V' \cap (V(H^u) \cup V(H^w))|.$$

Hence

$$|\partial_e V| < |\partial_e V'| + (3/\alpha)|\partial V'| \le 4|\partial V'| + (3/\alpha)|\partial V'|. \tag{3.2}$$

By (3.1), (3.2) and the f-isoperimetric edge-inequality,

$$\begin{aligned}|\partial V'| &\ge (4 + 3/\alpha)^{-1}|\partial_e(V)| \ge (4 + 3/\alpha)^{-1} f(d(V)) \\ &\ge (4 + 3/\alpha)^{-1} f((1/2)|V'|).\end{aligned}$$

□

4. Subdivisions of dyadic trees. We now prove Theorem 3.1. Let G be a connected graph of maximum degree 3 satisfying an f-isoperimetric inequality, where $f(k) \to \infty$ as $k \to \infty$. We shall say that a finite subtree T in G is *good* if it is a subdivision of a partial dyadic tree satisfying the following: Let A denote the set of endvertices (i.e., vertices of degree at most 1) in T. If B is a vertex set of cardinality less than $|A|$, then some vertex of $A \setminus B$ belongs to the infinite part of $G - B$. Moreover, $T - A$ is contained in the finite part of $G - A$. In particular, a tree with only one vertex is good. The strategy of the proof is to grow larger and larger good subtrees whose union will have the desired properties. Consider therefore the above good tree T. Then T consists of vertices $S_0 \cup S_1 \cup \cdots \cup S_k \cup A_{k+1}$ (where $|S_i| = 2^i$, $0 \le i \le k$, and $0 \le |A_{k+1}| \le 2^{k+1}$) and paths between consecutive sets in the sequence $S_0, \ldots, A_{k+1}$. Let H denote the infinite part of $G - A$.

Consider first the case where $A \cup V(H)$ has a vertex set B of cardinality $|A|$ such that $B \ne A$ and $A \setminus B$ (and hence also $T - B$) is in the finite part of $G - B$. Let G' be the union of those paths which start in B, terminate in A and have only the ends in common with $A \cup B$. Then G' does not intersect $T - A$ nor the infinite part of $G - B$. In particular, G' is finite. Moreover, G' satisfies the assumption of Menger's theorem (Theorem 2.1). Hence G' contains $|A|$ pairwise disjoint paths from A to B. Now we add these paths to T and obtain a larger good tree in G.

Consider next the case where a set B as in the previous case does not exist. If $|A_{k+1}| = 2^{k+1}$, we let E_{k+1} denote the set of edges in T incident with A_{k+1}, and we let v be any vertex of A_{k+1}. Otherwise we let v be any endvertex of T in S_k. Since T is good, v has at least one neighbour v_1 in H. Actually, v has another neighbour v_2 in H since otherwise, $B = (A \setminus \{v\}) \cup \{v_1\}$ satisfies the assumption in the previous case. As G has maximum degree 3, there is no third neighbour of V in H. Now we obtain a good tree T' by adding v_1, v_2 and the edges vv_1, vv_2 to T.

We start with a good tree consisting of just one vertex v_0 of degree less than or equal to 2. Then, successively we augment our good tree as in the first case above, whenever possible. Otherwise we perform the extension in the second case. (If our good tree has only one vertex and we perform the extension in the first case, we shall not add a path but replace the vertex in the good tree by B.) Since $f(k) \to \infty$ as $k \to \infty$, we must perform the extension in the second case infinitely often. Therefore the union of our good trees satisfies the conclusion of Theorem 3.1. □

5. Transient trees in grid graphs. It may be difficult to decide if a given graph contains a subgraph which satisfies an isoperimetric inequality that guarantees transience. We shall therefore mention a couple of less elegant, but perhaps more applicable, consequences of Theorem 3.1. Instead of finding, for a given graph G, an increasing real function f such that

$$\sum_{k=1}^{\infty} f(k)^{-2} < \infty \tag{5.1}$$

and

$$|\partial V| > f(|V|) \tag{5.2}$$

for each finite vertex set which contains the root r and which induces a connected subgraph in G, it is sufficient to consider first any sequence $V_1, V_2, \dots$ of finite vertex sets such that

$$r \in V_1 \subseteq V_2 \subseteq \cdots$$

and then to find an increasing function f satisfying (5.1) and (5.2) for $V = V_1, V_2, \dots$. Thus we may allow different functions f for different nested sequences $V_1, V_2, \dots$.

A further simplification may arise by just considering one fixed sequence $V_1, V_2, \dots$ such that

$$r \in V_1 \subseteq V_2 \subseteq \cdots$$

and finding a constant $c > 0$ and an increasing function f satisfying (5.1) and (5.2) for $V = V_1, V_2, \dots$ and then describing a subdivision H of the dyadic tree T_0 such that

$$S_0 = \{r\}; \tag{5.3}$$

$$S_i \subseteq V_i \setminus V_{i-1} \quad \text{for } i = 1, 2, \dots; \tag{5.4}$$

$$\text{the subgraph of } G \text{ induced by } V_i \text{ contains all paths from } S_{j-1} \text{ to } S_j \text{ for } 1 \le j \le i; \tag{5.5}$$

$$\text{the number of edges in } G \text{ from } V_i \text{ to } V(G) \setminus V_i \text{ is at most } c2^i,\ i = 0, 1, \dots. \tag{5.6}$$

By the same argument as in the proof of Theorem 3.2, H is transient.

We shall illustrate this by the grid graphs considered by Lyons [10]. Let $x^i(n)$ be a positive nondecreasing integer-valued function defined on the nonnegative integers for $i = 1, 2, \dots, k$. Assume further that $x^i(0) = 1$ and that $x^i(n+1) - x^i(n) \le 1$ for $n \ge 0$ and $i = 1, 2, \dots, k$. Let L be the subgraph of the grid Z^{k+1} induced by $\{(x_1, x_2, \dots, x_k, n)\colon 0 \le x_i < x^i(n)$ for $i = 1, 2, \dots, k\}$. (Two vertices in a grid are neighbours if they differ by 1 in one coordinate and agree in all other coordinates.) Lyons [10] proved that L is transient if and only if

$$\sum_{m=0}^{\infty} \left(\prod_{i=1}^{k} x^i(m) \right)^{-1} < \infty. \tag{5.7}$$

The "only if" part follows from Nash–Williams' recurrence criterion (see [10]). To see the "if" part, define the function f such that

$$f\left(\sum_{n=0}^{m}\prod_{i=1}^{k}x^i(n)\right) = \prod_{i=1}^{k}x^i(m)$$

and f is constant in the interval from $\sum_{n=0}^{m-1}\prod_{i=1}^{k}x^i(n)$ to $-1 + \sum_{n=0}^{m}\prod_{i=1}^{k}x^i(n)$. Then (5.7) is equivalent with $\sum_{n=1}^{\infty}f(n)^{-2} < \infty$.

It is not difficult to find $V_1, V_2, \dots$ and H such that (5.3)–(5.6) hold. Let us indicate this for $k = 2$. We shall describe H such that H uses an edge of L from (x_1, x_2, k) to $(x_1, x_2, k+1)$ iff x_i and x_2 are both divisible by 4. V_k will consist of all vertices of L of third coordinate less than or equal to $g(k)$, where $g(k)$ is the smallest number such that L has greater than or equal to 2^{k+1} edges of the form $(x_1, x_2, g(k))$ $(x_1, x_2, g(k)+1)$, where $x_1 \equiv x_2 \equiv 0 \pmod 4$. Suppose we have already defined $V_1, V_2, \dots, V_k$ and the subset of H in V_k. Assume without loss of generality that $x^1(g(k)+1) \geq x^2(g(k)+1)$. Let the vertices of H of third coordinate $g(k)+1$ be lexicographically ordered with respect to the first and second coordinate. The conditions on H tell how many vertices of L with third coordinate $g(k)+1$ must be in S_{k+1}. To decide which, we take those which are smallest with respect to the above lexicographic ordering. We continue like that for vertices of third coordinate $g(k) + g(k) + 3, \dots$. If $x^2(g(k+1)) > x^1(g(k+1))$, we change the lexicographic order. It is important that the vertices which have to be added to S_{k+1} in a given layer do not form a "dense" set. This construction of H and $V_1, V_2, \dots$ satisfies (5.3)–(5.6). Thus the grid graphs considered by Lyons are transient iff they contain a transient tree.

Some of the transient graphs of Lyons may be considered as only slight "fattenings" of a quadrant of the recurrent Z^2. Another such fattening is the graph M whose vertex set consists of all (x, y, z) in Z_+^3 where two vertices (x, y, z) and (x', y', z') are neighbours if either $x = x' = 0$ and $|y - y'| + |z - z'| = 1$ or $z = z'$ and $|x - x'| + |y - y'| = 1$. In [11] it is shown that M is transient. (This implies that Scherk's surfaces are hyperbolic, a question raised by Osserman [14].) We do not know if M has a transient tree.

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