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RESIDUAL LIFE TIME AT GREAT AGE[1]

BY A. A. BALKEMA AND L. DE HAAN

University of Amsterdam and
Erasmus University

The asymptotic behaviour of the residual life time at time t is investigated (for $t \to \infty$). We derive weak limit laws and their domains of attraction and treat rates of convergence and moment convergence. The presentation exploits the close similarity with extreme value theory.

0. Introduction. Consider a light bulb. It has a certain life time X, which is a random variable with probability distribution F. Suppose the distribution tail $R(x) = 1 - F(x) = P\{X > x\}$ is positive for all x. After having burned t hours there remains a residual life time with distribution tail R_t defined by

$$R_t(x) = 1 - F_t(x) = P\{X - t > x \mid X > t\}\,. \tag{1}$$

It is of considerable interest to know the asymptotic behavior of these residual life time distributions for $t \to \infty$. Natural questions: What are the possible limit distribution types? For each such limit distribution G, what is the domain of attraction, $D_r(G)$ (= the set of all distribution functions F such that F_t, suitably normed, converges to G)? What is the speed of convergence?

This paper gives fairly complete answers to all these questions. Most of the proofs are of a rather technical analytical nature. We have therefore collected basic notation and some general results in the introduction. The five sections which follow the introduction may then be read independently of each other.

Consider again the distribution tail $R_t(x)$ of the residual life time. If X has an exponential tail, $R(x) = e^{-\lambda x}$, then $R_t(x) = e^{-\lambda x}$ for all $t > 0$. It is well known that this characterizes the exponential distribution. If R_t does depend on t, then it is possible that the family R_t, $t \geqq 0$, has a weak limit S as $t \to \infty$, i.e., $R_t(x) = R(x + t)/R(t) \to S(x)$ weakly on $(0, \infty)$ for $t \to \infty$. It is not difficult to see that this implies that $S(x + y) = S(x)S(y)$ for $x, y > 0$ and hence that S is exponential. (Compare Feller (1966) **2** VIII, 8 Lemma 1.)

Suppose now we allow a scale transformation. Let there exist a positive function a such that $F_t(a(t)x)$ has a weak limit G as $t \to \infty$. It will be shown that

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the possible limit types in this situation are

$$\Gamma_\alpha(x) = 1 - (1 + x)^{-\alpha} \qquad \text{for} \quad x \geqq 0$$
$$\Pi(x) = 1 - e^{-x} \qquad \text{for} \quad x \geqq 0\,,$$

where α is a positive constant. All limit distributions vanish for $x < 0$. We remark that these limit distributions bear some resemblance to the limit distributions Φ_α and Λ in extreme value theory (see Section 2). This resemblance is due to the fact that both theories have a common basis, Karamata's theory of regular variation. In Section 5, we shall show how Karamata's theorems translate into a remarkable moment theorem: convergence of a positive residual life time moment as $t \to \infty$ is equivalent to weak convergence of the (properly normed) residual life time distributions F_t. For an exponential limit the relation between weak convergence and convergence of the first moments of F_t has been investigated by Meilijson (1972).

If we allow as norming functions for F_t both a scale transformation and a shift, then in addition to the limit distributions above, discrete limit distributions appear. The new limit types are:

$$\Gamma_{\gamma,\alpha}(x) = 1 - \exp -\gamma[1 + \alpha \log(1 + x)] \qquad \text{for} \quad x \geqq 0$$
$$\Pi_\gamma(x) = 1 - \exp -\gamma[1 + x] \qquad \text{for} \quad x \geqq 0\,,$$

with $\alpha, \gamma > 0$ and $[a]$ the integer part of a.

The limit types constitute a two parameter family. Indeed, set $p = \gamma > 0$, $pc = \alpha^{-1} > 0$ and

$$\Pi_{p,c}(x) = \Gamma_{\gamma,\alpha}(cx) \left(= 1 - \exp -p\left[1 + \frac{\log 1 + cx}{cp}\right] \text{ for } x \geqq 0\right).$$

On letting p or c, or p and c tend to zero we obtain the distributions

$$\Pi_{0,c}(x) = \Gamma_{c^{-1}}(cx)$$
$$\Pi_{p,0}(x) = \Pi_p\left(\frac{x}{p}\right)$$
$$\Pi_{0,0}(x) = \Pi(x)\,.$$

In this paper, we consider the general situation (allowing both a scale transformation and a shift). We shall derive the possible limit distributions (Section 1) and their domains of attraction (Sections 2 and 3). The Sections 4 and 5 refer to continuous limit distributions. In Section 4 we give approximation theorems for finite values of t. In Section 5 we shall prove that convergence of one moment entails weak convergence of the residual life times.

1. The limit distributions. Let X be a random variable with distribution tail $R(x)$, which we assume to be strictly positive. Assume moreover that for each $t > 0$ there exist a scale transformation $a(t) > 0$ and a shift $b(t)$ such that for

$t \to \infty$ we have weak convergence of the distribution tails

$$P\left(\frac{X - b(t)}{a(t)} > x \mid X > t\right) \to S(x) \qquad \text{weakly for} \quad t \to \infty\,. \tag{2}$$

The limit function S is obviously non-increasing and satisfies $0 \leqq S(x) \leqq 1$ for all x. In this section we shall prove that if $1 - S$ is a nondegenerate distribution function, then it is the type of one of the limit distributions mentioned in the introduction.

Analytically we can express (2) as

$$\min\left(1\,,\ \frac{R(b(t) + xa(t))}{R(t)}\right) \to S(x) \qquad \text{weakly for} \quad t \to \infty\,. \tag{3}$$

Following standard procedures, such as for instance Gnedenko's proof of Theorem 3 in (1943), we first show that S satisfies a functional equation (4) and then solve this equation. A complicating factor in our situation is that taking the minimum in (3) blots out convergence on one half line.

Lemma 1. *If* (2) *holds and* $1 - S$ *is a nondegenerate distribution function, then for each continuity point* y *of* S *for which* $0 < S(y) < 1$ *there exist constants* $A(y) \geqq 1$ *and* $B(y)$ *such that*

$$S(x) \cdot S(y) = S(B(y) + xA(y)) \tag{4}$$

for all x *for which* $S(x) < 1$.

Proof. Let y be a continuity point of S such that $0 < S(y) < 1$. Without loss of generality we assume $y = 0$. We know that $R(b(t) + xa(t))/R(t) \to S(x)$ weakly on the interval $I \equiv \{x \mid S(x) < 1\}$ for $t \to \infty$. Substituting $b(t)$ for t we obtain the identity

$$\frac{R(b(b(t)) + xa(b(t)))}{R(b(t))} \cdot \frac{R(b(t))}{R(t)} = \frac{R\left(b(t) + \left(\frac{b(b(t)) - b(t)}{a(t)} + x\,\frac{a(b(t))}{a(t)}\right)a(t)\right)}{R(t)}\,. \tag{5}$$

If $t \to \infty$, then $b(t) \to \infty$ (since $R(b(t))/R(t) \to S(0) < 1$) and hence the left-hand side converges weakly to $S(x) \cdot S(0)$ on I.

We shall prove that there exist constants $A(0)$ and $B(0)$ such that

$$a(b(t))/a(t) \to A(0) \qquad \text{and} \qquad (b(b(t)) - b(t))/a(t) \to B(0) \qquad \text{for} \quad t \to \infty\,. \tag{6}$$

Indeed, let (A, B) be a limit point for $t \to \infty$. Then $0 \leqq A,\ B \leqq \infty$ since $R(b(t))/R(t) \to S(0) < 1$ implies $B \geqq 0$. Moreover, A and B are finite, else (5) would yield $S(x) \cdot S(0) = 0$ for all $x \in I$. If $A > 0$, then (5) gives

$$S(x) \cdot S(0) = S(B + xA) \tag{7}$$

for all $x \in I$, using the right continuity of S. In particular, for $x = 0$ we find

$0 < S(B) = S(0)^2 < S(0) < 1$. Thus S has two continuity points 0 and B at which it assumes distinct values. This implies that A and B are uniquely determined by (7). If $A = 0$, then (5) implies $S(B+) \leqq S(x) \cdot S(0) \leqq S(B-)$ for all $x \in I$. In particular, B is the upper endpoint of the distribution $1 - S$.

Thus there is at most one limit point (A, B). This proves (6). It only remains to show that $A(0) \geqq 1$.

Let q satisfy $S(0) < q < 1$. Since $R(b(t))/R(t) \to S(0)$ there exists b_0 such that $R(b(t))/R(t) \leqq q$ for $t \geqq b_0$. Now define $b_{n+1} \equiv b(b_n)$ inductively for $n = 0, 1, 2, \cdots$. Then (b_n) is increasing (since $b(t) > t$ for $t \geqq b_0$). Assume $\lim b_n = b$ is finite. Then $R(b_{n+1})/R(b_n) \leqq q$ would imply $R(b - 0)/R(b - 0) \leqq q < 1$. Contradiction. Hence $b_n \to \infty$.

From (6) we see that $a(b_{n+1})/a(b_n) \to A(0)$. Assume $A(0) < 1$. Then $a(b_n)$ decreases exponentially to zero and since $b_{n+1} - b_n = O(a(b_n))$ by (6), the sequence (b_n) has a finite limit. Contradiction.

COROLLARY. *$S(x) > 0$ for all x.*

PROOF. Define $B_{n+1} = B(0) + B_n A(0)$ and $B_0 = 0$ (with 0 a continuity point of S and $0 < S(0) < 1$ as above). Then $B_n \to \infty$ since $B(0) > 0$ and $A(0) \geqq 1$, and $S(B_n) = S(0)^{n+1} > 0$.

REMARK. The principal argument in Lemma 1 is a form of the convergence of types theorem. We here give another version, which is proved by similar arguments as Lemma 1, and will be referred to occasionally in the remainder of the paper.

Let R be a distribution tail and suppose that $c_n R(b_n + xa_n)$ and $c_n R(b_n^* + xa_n^*)$ converge respectively to $S(x)$ and $S^*(x)$ say for $x > x_0$, with c_n, a_n and a_n^* positive and b_n, b_n^* real. If S and S^* are strictly positive on (x_0, ∞) and tend to zero for $x \to \infty$, then $a_n^*/a_n \to A > 0$, $(b_n^* - b_n)/a_n \to B$ and $S^*(x) = S(B + xA)$.

The functional equation (4) does not yet completely determine the possible limit distribution $G = 1 - S$. Let $S_0(x) = e^{-x}$ for $x \geqq 1$ and $= 1$ for $x < 1$. Then S_0 satisfies (4) with $A(y) = 1$, $B(y) = y$ although $1 - S$ is not of the type of any limit distribution $\Pi_{p,c}$! The next lemma shows why.

LEMMA 2. *Let H be an unbounded non-increasing function on an interval (x_0, ∞) such that*

$$H(x) = S(x) \qquad \text{if} \quad S(x) < 1 \,.$$

If for some pair $(A(y), B(y))$ in Lemma 1 the function H satisfies

$$H(x)H(y) = H(B(y) + xA(y)) \qquad \text{for} \quad x > x_0 \tag{8}$$

then

$$H(x) = S(x) \qquad \text{if} \quad H(x) < 1 \,.$$

PROOF. We may assume $y = 0$. Then $x_1 \equiv \inf\{x \mid S(x) < 1\}$ is negative, and we may assume x_1 to be finite since else there is nothing to prove. It suffices to show that each left neighborhood $(x_1 - \varepsilon, x_1)$ contains a point x_2 such that $H(x_2) \geqq 1$.

If $x > x_1$ then $H(x) < 1$ and (8) implies $B(0) + xA(0) > 0$. Hence $B(0) + x_2A(0) > x_1$ for some $x_2 \in (x_1 - \varepsilon, x_1)$. We may also assume x_2 and $B(0) + x_2A(0)$ to be continuity points of H.

Using (6) we see that the right-hand side of (5) converges to $H(B(0) + x_2A(0))$. The second factor on the left-hand side converges to $H(0)$. Hence the first factor converges, and the limit is equal to $H(x_2)$ by (8). Finally, $x_2 < x_1$, together with (3) implies $H(x_2) \geqq 1$.

We are now ready to prove that the weak limit distributions of the residual life times are exactly those mentioned in the introduction.

THEOREM 1. *Let F be a distribution function such that $F(x) < 1$ for all x. Let the normalized residual life time distributions $F_t(b(t) + xa(t))$ with F_t as defined in* (1) *converge weakly to a nondegenerate distribution function $G(x)$ for $t \to \infty$. Then G is of type Π, Π_γ, Γ_α or $\Gamma_{\alpha,\gamma}$ defined in the introduction.*

PROOF. We use the notation $R(x) = 1 - F(x)$, $S(x) = 1 - G(x)$ as before and denote by Y the set of continuity points of S for which $S(x) < 1$.

Thus (4) holds for $y \in Y$ and $S(x) < 1$.

We consider two cases.

Case 1. $A(y) = 1$ for all $y \in Y$. Define $\varphi(x) = \log S(x)$ when $S(x) < 1$. Then (4) becomes

$$\varphi(x) + \varphi(y) = \varphi(B(y) + x) . \tag{9}$$

We fix $y \in Y$. Then $\varphi(y) < 0$, hence $B(y) > 0$, and (9) gives

$$\varphi(x) = cx + \varphi_0(x) \tag{10}$$

with $c = \varphi(y)/B(y) < 0$ and with φ_0 periodic modulo $B(y)$. Note that $c = \lim_{x\to\infty} \varphi(x)/x$ is independent of y, and hence so is φ_0. We again distinguish two cases:

(a) φ_0 is constant: $\varphi(x) = c(x + d)$.

Define $H(x) \equiv e^{c(x+d)}$. Then $H(x) = S(x)$ when $S(x) < 1$, hence $S(x) = \min(1, H(x))$ by Lemma 2 and G is of type Π.

(b) φ_0 is not constant and has minimal period $p > 0$.

The function φ has the properties: $\varphi(x + p) - \varphi(x) = cp$ by (10) and $\varphi(y) \in \{cp, 2cp, 3cp, \cdots\}$ for all $y \in Y$ since φ_0 is periodic modulo $B(y) = \varphi(y)/c$. It follows that $\varphi(x) = cp[(x + d)/p]$. As above, we set $H(x) = \exp cp[(x + d)/p]$ for all $x \in \mathbb{R}$ and Lemma 2 yields that G is of type Π_γ with $\gamma = -cp > 0$.

Case 2. $A(y) > 1$ for some $y \in Y$. Define $\varphi(x) = \log S(x)$ when $S(x) < 1$. Suppose $y_1, y_2 \in Y$. We define $\alpha_i(x) \equiv B(y_i) + xA(y_i)$ for $i = 1, 2$. Then (4) yields $\varphi(\alpha_i(x)) = \varphi(x) + \varphi(y_i)$ for $i = 1, 2$ and hence $\varphi(\alpha_1(\alpha_2(x)) = \varphi(\alpha_2(\alpha_1(x)))$. Since φ is nonincreasing and unbounded we obtain $\alpha_1(\alpha_2) = \alpha_2(\alpha_1)$, i.e.,

$$B(y_1) + A(y_1)B(y_2) = B(y_2) + A(y_2)B(y_1) .$$

In particular $A(y) > 1$ for all $y \in Y$ and we may write

$$B(y) + xA(y) = x_0 + (x - x_0)A(y)$$

where $x_0 = B(y)/(1 - A(y))$ does not depend on y. We may also assume $x_0 = 0$. Then (4) becomes

$$\varphi(x) + \varphi(y) = \varphi(xA(y)) .$$

This implies that $x \cdot A(y) > x$ for $y \in Y$ and $S(x) < 1$. Hence $x > 0$ if $S(x) < 1$ and we may write $x = e^{\xi}$ so that (4) becomes

$$\psi(\xi) + \psi(\eta) = \psi(\xi + \alpha(\eta))$$

for $\xi > \xi_1$. This is relation (9) once more. There are two cases:

(a) $\psi(\xi) = c(\xi + d)$, $H(x) = e^{c(\xi+d)} = (e^d x)^c$ for $x > 0$ and G is of type Γ_α with $\alpha = -c > 0$.

(b) $\psi(\xi) = cp[(\xi + d)/p]$, $H(x) = \exp cp[(\xi + d)/p] = \exp cp[(\log e^d x)/p]$ for $x > 0$ and G is of type $\Gamma_{\gamma,\alpha}$ with $\gamma = -cp$, $\alpha = p^{-1}$.

It only remains to check that each of these limit functions $G = 1 - S$ satisfies

$$\min\left(1\,,\ \frac{S(B(t) + xA(t))}{S(t)}\right) = S(x)$$

for $t > 0$. □

REMARK. If we relax the condition that $F(x) < 1$ for all x, we could still consider the class of limit tails S obtained in (2) by letting t tend to the upper endpoint of the distribution F.

Using the same arguments we now obtain the tails $S(x) = e^{-x}$ for $x > 0$ and $S(x) = (1 - x)^{\lambda}$ for $x \in (0, 1)$ with $\lambda > 0$ together with their discrete counterparts. In addition we can obtain any limit distribution which is concentrated in two points. (This corresponds to the case $A(y) \equiv 0$. As an instance let F be a discrete distribution with atoms of size p_n in x_n where $p_{n+1}/p_n \to p \in (0, 1)$, $x_n \uparrow c$ and $(x_{n+1} - x_n)/(x_n - x_{n-1}) \to 0$.)

Since this aspect of the theory has little practical significance we shall not develop it further.

THEOREM 2. *Let F be a distribution function such that $F(x) < 1$ for all x. Let $F_t(a(t)x)$ with F_t as defined in* (1) *converge weakly to a nondegenerate distribution function $G(x)$ for $t \to \infty$. Then $G(x) = \Pi(ax)$ or $G(x) = \Gamma_\alpha(ax)$ for some $a > 0$, $\alpha > 0$, with Π and Γ_α defined in the introduction.*

PROOF. In relations (2) and (3) we now have $b(t) = t$. Hence relation (6) becomes

$$(b(b(t) + ya(t)) - b(t))/a(t) \to B(y)$$

(we may no longer assume $y = 0$!). Thus $B(y) = y$ and φ_0, defined in (10), being periodic modulo $B(y)$ for all $y \in Y$, is constant. One easily sees that Π and Π_α do indeed occur as limit distributions.

2. The domain of r.l.t. attraction of Γ_α and of Π. For each of the limit distributions G derived in Theorem 1 we define the domain of residual life time attraction $D_r(G)$ to consist of all distribution functions F for which there exist normalizing constants $a(t) > 0$ and $b(t)$ such that $F_t(b(t) + xa(t)) \to G(x)$ weakly for $t \to \infty$. In particular $F \in D_r(G)$ implies that $F(x) < 1$ for all x.

The concept of "domain of attraction" is well known in extreme value theory. There the limit distributions are usually denoted by

$$\Phi_\alpha(x) = e^{-(x^{-\alpha})} \qquad \text{for } x > 0$$
$$\Lambda(x) = e^{-e^{-x}}$$
$$\Psi_\alpha(x) = e^{-|x|^\alpha} \qquad \text{for } x < 0$$

with $\alpha > 0$, and if G is one of these distributions we write $F \in D(G)$ if there exist normalizing constants $a_n > 0$ and b_n such that $F(a_n x + b_n)^n \to G(x)$ weakly for $n \to \infty$. (The maxima of n independent random variables, each distributed according to F, converge, properly normed, in distribution to a random variable with distribution G.) Gnedenko (1943) determined these domains of attraction and since then there has been a substantial list of publications on this subject.

In this section we shall prove that the domains of r.l.t. attraction for continuous limit distributions are closely related to the domains of attraction for the extreme value limit distributions.

Let D_0 denote the set of all distribution functions F such that $F(x) < 1$ for all x. Then with $D_r(G)$, $D(G)$, Λ and Φ_α as above and Π and Γ_α as defined in the introduction, we have

THEOREM 3. $D_r(\Pi) = D(\Lambda) \cap D_0$.

THEOREM 4. $D_r(\Gamma_\alpha) = D(\Phi_\alpha)$ *for all* $\alpha > 0$.

The proof of these two theorems depends on a continuation principle: if the limit function satisfies a functional equation on a half line, then it will satisfy the equation on the whole line. It would be very interesting, however, to have a proof based on probabilistic arguments.

We first prove a simple lemma.

LEMMA 3. *Suppose* $F = 1 - R \in D_r(G)$ *with* G *continuous. Then* $R(x - 0) \div R(x + 0) \to 1$ *for* $x \to \infty$.

PROOF. For any $q < 1$ there exists y such that $q < 1 - G(y) < 1$. Thus $b(x) + ya(x) > x$ and $R(b(x) + ya(x)) > q \cdot R(x)$ for $x \geqq x_0$ by (3). Hence certainly $R(x + 0) > q \cdot R(x - 0)$ for $x \geqq x_0$.

PROOF OF THEOREMS 3 AND 4. We shall only prove Theorem 3. The proof of Theorem 4 is similar.

Suppose $F = 1 - R \in D(\Lambda)$ and $R(x) > 0$ for all x. By Gnedenko (1943), Theorem 6, we have $n \cdot R(b_n + xa_n) \to e^{-x}$ weakly on $\mathbb{R}$. Define $a(t) \equiv a_n$ and

$b(t) \equiv b_n$ for all t for which $(n+1)^{-1} \leqq R(t) < n^{-1}$. Then obviously $R(b(t) + xa(t))/R(t) \to e^{-x}$ which proves $F \in D_r(\Pi)$.

Now suppose $F \in D_r(\Pi)$. Then $R(b(t) + xa(t))/(t) \to e^{-x}$ for $x > 0$ for $t \to \infty$, and since $R(t+0) \sim R(t-0)$ by Lemma 3, we may choose $a_n > 0$ and b_n so that

$$nR(b_n + xa_n) \to e^{-x} \tag{11}$$

on $x > 0$. It remains to prove that (11) holds on $\mathbb{R}$.

Let (11) hold on (c, ∞) with c minimal. We have

$$2nR(b_{2n} + xa_{2n}) = 2 \cdot nR\left(b_n + \left(\frac{b_{2n} - b_n}{a_n} + x\frac{a_{2n}}{a_n}\right)a_n\right)$$

and by the remark following Lemma 1 we obtain for $n \to \infty$

$$e^{-x} = 2e^{-(B+xA)}$$

whence $A = 1$ and $B = \log 2$. Convergence on the right-hand side holds for $x + \log 2 > c$, hence convergence on the left-hand side holds for $x > c - \log 2$. In first instance this is only true for the subsequence of even numbers, but since $2n + 1 \sim 2n$ it holds for all n with the norming constants $a_{2n+1}^* = a_{2n}^* = a_{2n}$ and $b_{2n+1}^* = b_{2n}^* = b_{2n}$. Then $a_n^*/a_n \to 1$ and $(b_n^* - b_n)/a_n \to 0$ (again by the remark following Lemma 1) and hence convergence holds for the original norming constants as well.

This proves $c = c - \log 2$, i.e., $c = -\infty$ and (11) holds on $\mathbb{R}$.

COROLLARY 1. *$F \in D_r(\Gamma_\alpha)$ if and only if*

$$F_t(xt) \to \Gamma_\alpha(x) \qquad \textit{for all } x \textit{ for } t \to \infty .$$

PROOF. By Gnedenko (1943), Theorem 4, we have $F = 1 - R \in D(\Phi_\alpha)$ if and only if

$$R(t + tx)/R(t) \to (1+x)^{-\alpha} \qquad \text{for all } x > 0 .$$

COROLLARY 2. *$F \in D_r(\Pi)$ if and only if $F(x) < 1$ for all x and*

$$\lim_{t\to\infty} P\left\{\frac{X-t}{a(t)} \leqq x \mid X > t\right\} = \Pi(x) \qquad \textit{for all } x$$

with $a(t) = \int_t^\infty (1 - F(s))\, ds/(1 - F(t)) = E(X - t \mid X > t)$.

PROOF. $F = 1 - R \in D(\Lambda)$ if and only if (de Haan (1970) Theorem 2.5.1)

$$R(t + xa(t))/R(t) \to e^{-x}$$

for all x with $a(t)$ as in the statement of the corollary.

This settles the question raised by Meilijson (1972) whether $F \in D_r(\Pi)$ entails existence of the first moment. See de Haan (1970), Corollary 2.5.3.

3. The domains of r.l.t. attraction of the discrete limit distributions. Let $F = 1 - R$ be a discrete distribution function, continuous from the right, whose discontinuity points form an unbounded increasing sequence $t_0, t_1, t_2, \cdots$ such that

$$(12\text{a}) \qquad (t_{n+1} - t_n)/(t_n - t_{n-1}) \to e^{pc} \geqq 1$$

$$(12\text{b}) \qquad R(t_{n+1})/R(t_n) \to e^{-p} < 1 \, .$$

Then $F \in D_r(\Pi_{p,c})$. See introduction for definition of $\Pi_{p,c}(x)$.

Indeed, set $a_n = t_{n+1} - t_n$ and $b_n = t_{n+1}$. Then the quotient $H_n(x) = R(b_n + xa_n)/R(t_n)$ takes on the values $R(t_{n+1})/R(t_n)$, $R(t_{n+2})/R(t_n)$, $\cdots$ between the successive discontinuity points 0, a_{n+1}/a_n, $(a_{n+1} + a_{n+2})/a_n$, $\cdots$ and hence converges weakly to a function which has discontinuity points 0, e^{pc}, $e^{pc} + e^{2pc}$, $\cdots$ and takes on the values $e^{-p}, e^{-2p}, \cdots$ in between. Thus $\min(1, H_n(x))$ converges weakly to $1 - \Pi_{p,c}(x)$ and since F is discrete this proves that F lies in the domain of r.l.t. attraction of $\Pi_{p,c}$.

Obviously, if the distribution F_1 is tail equivalent to a distribution $F_2 \in D_r(\Pi_{p,c})$, then $F_1 \in D_r(\Pi_{p,c})$. Recall that two distribution functions F_i, for which $F_i(x) < 1$ for all x, are tail equivalent if $1 - F_1(x) \sim 1 - F_2(x)$ for $x \to \infty$.

THEOREM 5. *Suppose $p > 0$ and $c \geqq 0$. The distribution function F lies in the domain of residual life time attraction of $\Pi_{p,c}$ if and only if F is tail equivalent to a discrete distribution fuction F_0 which satisfies* (12a) *and* (12b).

PROOF. One part of the theorem has been proved above.

Now suppose $F = 1 - R \in D_r(\Pi_{p,c})$. Let $G = 1 - S$ be a translate of $\Pi_{p,c}$ such that $G(0) = 1 - e^{-p}$ and such that 0 is a continuity point of G. By (3)

$$R(b(t))/R(t) \to S(0) = e^{-p} < 1$$

and as in the proof of Lemma 1 we define a sequence $b_{n+1} = b(b_n)$ such that $b_n \uparrow \infty$.

We shall now prove that F is tail equivalent to a distribution F_0 which only takes the values $F(b_n)$, $n = 1, 2, \cdots$.

Suppose not. Then, since $R(b_{n+1})/R(b_n) \to e^{-p}$, there exist a sequence $s_k \to \infty$ and integers $n(k) \to \infty$ such that $R(s_k)/R(b_{n(k)}) \to q$ with $e^{-p} < q < 1$. Now $R(b_k' + xa_k')/R(b_{n(k)})$ and $R(b_k'' + xa_k'')/R(s_k)$ (with $b_k' = b(b_{n(k)})$, $a_k' = a(b_{n(k)})$, $b_k'' = b(s_k)$ and $a_k'' = a(s_k)$) both converge weakly to $S(x)$ on $x > 0$ for $k \to \infty$. By the remark following Lemma 1 we have $S(x) = S(B + xA)/q$ and since S only takes the values $1, e^{-p}, e^{-2p}, \cdots$ this implies $q = e^{kp}$ for some integer k. Contradiction.

F_0 clearly is a discrete distribution function. Let $t_0, t_1, \cdots$ be its discontinuity points such that $F_0(t) = F(b_n)$ for $t_n \leqq t < t_{n+1}$. Then $R_0(t_{n+1})/R_0(t_n) = R(b_{n+1})/R(b_n) \to e^{-p}$, which proves (12b).

By the remark following Lemma 1 and the identity

$$\frac{R_0(t_{n+1} + xa_{n+1})}{R_0(t_n)} \frac{R_0(t_n)}{R_0(t_{n-1})} = \frac{R_0(t_n + ((t_{n+1} - t_n)/a_n + xa_{n+1}/a_n)a_n)}{R_0(t_{n-1})} \, .$$

We obtain for $n \to \infty$ the functional equation

$$S(x) \cdot S(0) = S(B + xA)$$

with $B > 0$ and $A \geqq 1$, and hence

$$\frac{t_{n+1} - t_n}{t_n - t_{n-1}} = \frac{t_{n+1} - t_n}{a_n} \cdot \frac{a_{n-1}}{t_n - t_{n-1}} \cdot \frac{a_n}{a_{n-1}} \to A$$

which proves (12a).

4. Approximation for finite t. In extreme value theory, there exist convenient sufficient conditions, due to R. von Mises (1936), for a distribution function to belong to the domain of attraction of some limit distribution:

$$F \in D(\Phi_\alpha) \qquad \text{if} \quad \lim_{t\to\infty} \frac{tF'(t)}{1 - F(t)} = \alpha$$

$$F \in D(\Lambda) \qquad \text{if} \quad \lim_{t\to\infty} \frac{d}{dt} \frac{1 - F(t)}{F'(t)} = 0 .$$

By Theorems 2 and 4 these conditions are also sufficient for a distribution function to belong to the domain of residual life time attraction of a continuous limit distribution. For residual life times, one has in addition some simple inequalities which give upper and lower bounds for the normed residual life time distributions for large values of t.

THEOREM 6. *Let F be a distribution function which has a positive density F' for $t \geqq t_0$ and let α_1 and α_2 be positive real numbers such that*

$$\alpha_1 \leqq \frac{tF'(t)}{1 - F(t)} \leqq \alpha_2 \qquad \textit{for } t \geqq t_0 .$$

If X is a random variable with distribution F, then

$$\Gamma_{\alpha_1}(x) \leqq P\left\{\frac{X - t}{t} \leqq x \mid X > t\right\} \leqq \Gamma_{\alpha_2}(x)$$

for all x for $t \geqq t_0$.

PROOF. Integration between t and $(1 + x)t$ with $x > 0$ gives

$$\alpha_1 \int_t^{(1+x)t} \frac{du}{u} \leqq \int_t^{(1+x)t} \frac{F'(u)}{1 - F(u)}\, du \leqq \alpha_2 \int_t^{(1+x)t} \frac{du}{u} ,$$

and the monotonic transformation $y \to 1 - e^{-y}$ yields the stated result. □

For the next theorem, it is more convenient to use the notation $\Pi_{0,c}$. For $c \geqq 0$ the distribution functions $\Pi_{0,c}$ have been defined in the introduction. For $c < 0$ we define the distribution function

$$\Pi_{0,c}(x) = 1 - (1 + cx)^{-1/c} \qquad \text{if} \quad 0 \leqq x \leqq |c|^{-1} .$$

THEOREM 7. *Let F be a distribution function which has a positive, differentiable* *density F' for $t \geqq t_0$. Let c_1 and c_2 be real numbers such that*

$$c_1 \leqq \frac{d}{dt}\left(\frac{1-F(t)}{F'(t)}\right) \leqq c_2 \qquad \text{for } t \geqq t_0 .$$

If X is a random variable with distribution F, then for $t \geqq t_0$ and all x

$$\Pi_{0,c_2}(x) \leqq P\left\{\frac{X-t}{a(t)} \leqq x \mid X > t\right\} \leqq \Pi_{0,c_1}(x)$$

where $a(t) = (1 - F(t))/F'(t)$.

Proof. For $t \geqq t_0$ and $x \geqq 0$ integration between t and $t + xa(t)$ gives $c_1 xa(t) \leqq a(t + xa(t)) - a(t) \leqq c_2 xa(t)$ or equivalently

$$1 + c_1 x \leqq \frac{a(t + xa(t))}{a(t)} \leqq 1 + c_2 x . \tag{13}$$

By taking the logarithmic derivative with respect to x one readily checks that

$$P\left\{\frac{X-t}{a(t)} > x \mid X > t\right\} = \frac{1 - F(t + xa(t))}{1 - F(t)} = \exp - \int_0^x \frac{a(t)}{a(t + sa(t))}\, ds$$

and the theorem follows from the inequality (13) above.

Corollary. *If c and ε are nonnegative numbers such that*

$$\left|c - \frac{d}{dt}\left(\frac{1-F(t)}{F'(t)}\right)\right| \leqq \varepsilon \qquad \text{for } t \geqq t_0$$

then

$$\left|\Pi_{0,c}(x) - P\left\{\frac{X-t}{a(t)} \leqq x \mid X > t\right\}\right| \leqq \varepsilon \qquad \text{for all } x \text{ for } t \geqq t_0 .$$

Proof. It suffices to prove that

$$|\Pi_{0,c}(x) - \Pi_{0,c_0}(x)| \leqq |c - c_0|$$

for $c \geqq 0$ and all real c_0 and x. This is trivial for $|c - c_0| \geqq 1$ and follows from Lemma 4 below for $|c - c_0| < 1$ and $c \geqq 0$.

Lemma 4. $0 \leqq -(\partial/\partial c)\Pi_{0,c}(x) < 1$ *for $c > -1$ if $x \neq 0$ and $cx \neq -1$.*

Proof. We may assume (x, c) to lie in the region $R = \{c > -1,\ x > 0$ and $cx > -1\}$ since $\Pi_{0,c}(x)$ is constant on both components of $\{c > -1\}\backslash R$.

Observe that $1 - \Pi_{0,c}(x) = \exp -\psi(x, c)$ where

$$\begin{aligned}\psi(x, c) &= c^{-1} \log (1 + cx) \qquad && \text{for } c \neq 0,\ cx > -1\\ &= x && \text{if } c = 0 .\end{aligned}$$

Hence $(\partial/\partial c)\Pi_{0,c}(x) = e^{-\psi} \cdot (\partial/\partial c)\psi$. Since $\psi > 0$ on R it suffices to prove that $0 < -(\partial/\partial c)\psi < e^{\psi}$ on R.

Set $u = \log (1 + cx) = c \cdot \psi$. Then

$$-\frac{\partial}{\partial c}\psi(x, c) = \frac{\log (1 + cx)}{c^2} - \frac{x}{c(1 + cx)} = \psi^2 \cdot \frac{e^{-u} - 1 + u}{u^2} .$$

Obviously, $\sigma(u) = u^{-2}(e^{-u} - 1 + u) = \int_0^1 \int_0^t e^{-us}\,ds\,dt$ is positive and decreasing in u. Hence $c > -1$ (and therefore $u > -\psi$) implies $-(\partial/\partial c)\psi \leqq \psi^2 \cdot \sigma(-\psi) < e^{\psi}$.

5. Equivalence of weak convergence and moment convergence. In this section we state the conditions for the domains of attraction in an alternative form using certain moments of the distributions.

THEOREM 8. *Suppose X is a real-valued random variable with distribution function F and $F(x) < 1$ for all real x.*

(a) *Let $0 < \xi < \alpha$. We have $F \in D_r(\Gamma_\alpha)$*, i.e.,

$$\lim_{t\to\infty} P\left\{\frac{X}{t} \leqq x \mid X > t\right\} = \Gamma_\alpha(x-1)$$

for all $x > 0$ if and only if $\int_0^\infty y^\xi\,dF(y)$ is finite and

$$c = \lim_{t\to\infty} E\left(\left(\frac{X}{t}\right)^\xi \,\middle|\, X > t\right)$$

exists and is finite. Then $c = \int_0^\infty x^\xi\,d\Gamma_\alpha(x-1) = (1-\xi/\alpha)^{-1}$.

(b) *We have $F \in D_r(\Pi)$*, i.e.,

$$\lim_{t\to\infty} P\left\{\frac{X-t}{a(t)} \leqq x \mid X > t\right\} = \Pi(x)$$

for all positive x with (by Corollary 2 to Theorem 3)

$$a(t) = \int_t^\infty 1 - F(s)\,ds/(1-F(t)) = E(X - t \mid X > t)$$

if and only if $\int_0^\infty x^2\,dF(x)$ converges and

$$\lim_{t\to\infty} E\left(\left(\frac{X-t}{a(t)}\right)^2 \,\middle|\, X > t\right) = \int_0^\infty x^2\,d\Pi(x) = 2\,.$$

PROOF. (a) We have

$$\frac{\int_x^\infty y^\xi\,dF(y)}{x^\xi(1-F(x))} = \xi\,\frac{\int_x^\infty y^{\xi-1}(1-F(y))\,dy}{x^\xi(1-F(x))} + 1\,.$$

The statement now follows from Karamata's theorem for regularly varying functions (see, e.g., de Haan (1970) Theorem 1.2.1 and Remark 1.2.1).

(b) By Theorem 3 above and Theorems 2.5.1 and 2.5.2 of de Haan (1970) we have $F \in D_r(\Pi)$ if and only if

$$\lim_{t\to\infty} \frac{(1-F(t))(\int_t^\infty \int_v^\infty 1 - F(s)\,ds\,dv)}{(\int_t^\infty 1 - F(s)\,ds)^2} = 1\,.$$

Partial integration yields $\int_t^\infty \int_v^\infty 1 - F(s)\,ds\,dv = \frac{1}{2}\int_t^\infty (s-t)^2\,dF(s)$ and

$$\lim_{t\to\infty} E\left(\left(\frac{X-t}{a(t)}\right)^2 \,\middle|\, X > t\right) = \lim_{t\to\infty} \frac{\int_t^\infty (s-t)^2\,dF(s)}{1-F(t)} \cdot \frac{(1-F(t))^2}{(\int_t^\infty 1 - F(s)\,ds)^2} = 2\,.$$

COROLLARY. *We thus have $F \in D_r(\Pi)$ if and only if*

$$\lim_{t\to\infty} \frac{\operatorname{Var}(X - t \mid X > t)}{(E(X - t \mid X > t))^2} = 1 .$$

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF AMSTERDAM
ROETERSSTRAAT 15
AMSTERDAM-C
HOLLAND

ERASMUS UNIVERSITY
BURGEMEESTER OUDLAAN 50
ROTTERDAM-3016
HOLLAND