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# STRONG AND WEAK LIMIT POINTS OF A NORMALIZED RANDOM WALK[1]

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Let $S_n = \sum_1^n X_i$ be a random walk. A point $b$ is called a strong limit point of $n^{-\alpha}S_n$ if there exists a nonrandom sequence $n_k \to \infty$ such that $n_k^{-\alpha}S_{n_k} \to b$ w.p. 1. The possible structures for the set of strong limit points of $n^{-\alpha}S_n$ are determined. We also give a sufficient condition for $n^{-1}S_n$ to be dense in $\mathbb{R}$. In particular $n^{-1}S_n$ is dense in $\mathbb{R}$ when $E|X_1| = \infty$ and $n^{-1}S_n$ has a finite strong limit point.

**1. Introduction and notation.** In [4] the set of accumulation points of $n^{-\alpha}S_n$ for a random walk $S_n$ was studied. In this paper $S_n$ always stands for $\sum_1^n X_i$, where $X_1, X_2, \cdots$ is a sequence of independent, identically distributed random variables with common distribution function $F$. The random walk is then the sequence of partial sums $S_1, S_2, \cdots$, and the random set of accumulation points of $n^{-\alpha}S_n$ is

$$A(S_n, \alpha) = \bigcap_m \overline{\{n^{-\alpha}S_n : n \geqq m\}} . \tag{1.1}$$

The bar in the right hand side of (1.1) denotes closure in the extended real line $\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ and throughout "closure" and "closed" are taken in the topology of $\bar{\mathbb{R}}$. It is easy to show ([4] Theorem 1) that there exists a closed nonrandom set $B(\alpha) = B(F, \alpha)$, depending on $F$ and $\alpha$ only, such that

$$A(S_n, n^{\alpha}) = B(F, \alpha) \quad \text{w.p. } 1. \tag{1.2}$$

Several theorems about the possible structure of $B(\alpha)$ were derived in [4]. The present paper is centered around

THEOREM A ([4] Corollary 3 and Theorem 7). *If $B(1)$ contains more than one point, then it must contain $+\infty$ and $-\infty$. For any closed set $C \subset \bar{\mathbb{R}}$ containing $+\infty$ and $-\infty$ there exists a distribution function $F$ such that $B(F, 1) = C$.*

By (1.2) $b \in B(\alpha)$ if and only if there exists w.p. 1 a random sequence $n_k \to \infty$ such that

$$n_k^{-\alpha}S_{n_k} \to b . \tag{1.3}$$

Here we introduce also strong limit points. We call $b$ a *strong limit point* of $n^{-\alpha}S_n$ if there exists a *nonrandom* sequence $n_n \to \infty$ such that

$$n_k^{-\alpha}S_{n_k} \to b \quad \text{w.p. } 1. \tag{1.4}$$

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We denote by $B_s(\alpha) = B_s(F, \alpha)$ the set of strong limit points of $n^{-\alpha}S_n$. Clearly $B_s$ is closed in $\bar{\mathbb{R}}$ and it follows easily from (1.2) (see also (2.3) of [4]) that $B_s(F, \alpha) \subset B(F, \alpha)$. From the concentration function inequalities ([2] see also Lemma 5 and Remark 3 in Section 3) it follows easily that if (1.4) holds for some $\alpha \leqq \frac{1}{2}$ and $|b| < \infty$, then one must have $\sigma^2(X_1) = 0$, i.e. $X_1$ must be constant w.p. 1, and then by (1.4) this constant as well as $b$ must equal zero. Thus $B_s(\alpha)$ is uninteresting for $\alpha \leqq \frac{1}{2}$. For $\alpha > \frac{1}{2}$ we have

THEOREM 1. *If* $\alpha > \frac{1}{2}$, $\alpha \neq 1$ *and* $b \in B_s(\alpha)$ *for some* $0 < |b| < \infty$, *then* $\operatorname{sgn}(b)[0, \infty] \subset B_s(\alpha)$.

Thus if $\alpha > \frac{1}{2}$, $\alpha \neq 1$ the finite strong limit points of $n^{-\alpha}S_n$ (if any) either fill up the whole line $\bar{\mathbb{R}}$, or one of the half lines $[-\infty, 0]$ or $[0, \infty]$, or consist of $\{0\}$ only. Each of these possibilities can occur (see Section 4). When $\alpha = 1$ no such restrictions on $B_s$ apply since we prove

THEOREM 2. *If $D$ is any closed set in $\bar{\mathbb{R}}$, there exists a distribution function $F$ such that $B_s(F, 1) = D$.*

The question arises what happens to the set of accumulation points of $n^{-1}S_n$ which are not strong accumulation points when one forces $B_s(F, 1)$ to have a given structure. Perhaps the most striking aspect of theorem A is that $B(F, 1)$ does not have to be connected. At first sight one expects that if $-\infty < b_1 < b_2 < \infty$, $b_i \in B(F, 1)$ then all points in $[b_1, b_2]$ should be accumulation points of $n^{-1}S_n$. By theorem A this is not necessarily the case, but in [4] it was stated as a problem to find a n.a.s.c. for $B(F, 1) = \bar{\mathbb{R}}$. We do not have a n.a.s.c. here, but we do show that if $n^{-1}S_n$ has a finite strong limit point and $E|X_1| = \infty$, then $B(F, 1) = \bar{\mathbb{R}}$. This is a consequence of the stronger

THEOREM 3. *If*

$$E|X_1| = \infty \tag{1.5}$$

*and*

$$\limsup_{n\to\infty} P\left\{\left|\frac{S_n}{n}\right| \leqq a\right\} > 0 \tag{1.6}$$

*for some $a < \infty$ then $B(F, 1) = \bar{\mathbb{R}}$.*

This theorem has the following analogue for $\frac{1}{2} < \alpha < 1$.

THEOREM 4. *If* $\frac{1}{2} < \alpha < 1$ *and*

$$E(X_1^+)^{1/\alpha} = E(X_1^-)^{1/\alpha} = \infty\,, \tag{1.7}$$

*and for some $a < \infty$*

$$\limsup_{n\to\infty} P\left\{\left|\frac{S_n}{n^\alpha}\right| \leqq a\right\} > 0\,, \tag{1.8}$$

*then $B(F, \alpha) = \bar{\mathbb{R}}$.*

We shall not prove Theorem 4, because its proof is rather lengthy. We note that (1.7) cannot be replaced by $E|X_1|^{1/\alpha} = \infty$, as shown by Example 4 in Section 4. For $\alpha > 1$ the conclusion of Theorem 4 does not hold even under (1.7) as demonstrated by Example 2. Finally we note that $E|X_1|^{1/\alpha} < \infty$ implies $n^{-\alpha}(S_n - nc) \to 0$ w.p. 1 where $c = EX_1$ for $\frac{1}{2} < \alpha < 1$, respectively, $c = 0$ for $\alpha > 1$, see [5] page 243.

In Section 4 we give some more examples to illustrate the possibilities for $B(\alpha)$, $\alpha \neq 1$, as well as some conjectures.

**2. Proof of Theorem 3.** In this section it is very convenient to assume that the distribution $F$ is absolutely continuous. This causes no harm for we can always convolve $F$ with the normal distribution $N\{dx\} = (2\pi)^{-\frac{1}{2}}e^{-x^2/2}\,dx$ having mean 0; then $F * N$ is absolutely continuous and, as one may easily see from the probabilistic meaning of convolution and the Strong Law of Large Numbers, $B(F * N, \alpha) = B(F, \alpha)$ for $\alpha > \frac{1}{2}$.

Define the following quantities:

$$q(t) = P\{|X_1| > t\} = F(-t) + 1 - F(t)$$

$$\rho(t) = \frac{1}{t} \int_{-t}^{t} x^2 F\{dx\}\,, \qquad \mu(t) = \int_{-t}^{t} xF\{dx\}\,.$$

Note that $tq(t) + \rho(t) = 2t^{-1} \int_0^t xq(x)\,dx$. As customary ${}^\circ F$ denotes the distribution of the symmetrized random variable ${}^\circ X = X_1 - X_2$.

LEMMA 1. *If* (1.6) *holds then we can find* $t_k \uparrow \infty$ *such that for all* $a$ *sufficiently large*

$$\inf_{k\geq 1} P\left\{\left|\frac{S_{t_k}}{t_k}\right| \leqq a\right\} > 0 \tag{2.1}$$

*and*

$$\sup_{k\geq 1} \{t_k q(t_k) + \rho(t_k)\} < \infty \tag{2.2}$$

$(S_t = S_{[t]})$. *If for some* $t_k \uparrow \infty$ *we have*

$$|\mu(t_k)| + t_k q(t_k) + \rho(t_k) = O(1)\,,$$

*then* $\limsup P\{|S_{t_k}/t_k| \leqq a\} > 0$ *for some* $a > 0$.

PROOF. Suppose (1.6) holds. By considering subsequences we may clearly suppose (2.1) is true. From a concentration function inequality ([2] Theorem 3.1) we have

$$P\left\{\left|\frac{S_n}{n}\right| \leqq a\right\} \leqq Q(S_n; an) \leqq can^{\frac{1}{2}}\{\textstyle\int_{-\lambda}^{\lambda} y^2\, {}^\circ F\{dy\} + \lambda^2\, {}^\circ q(\lambda)\}^{-\frac{1}{2}}$$

for any $0 < \lambda < 2an$ and a universal constant $c$. Assume $a > 1$. On taking $\lambda = t_k = n$ and writing $\beta$ for the left hand side of (2.1) we get for all $k$

$$\frac{2}{t_k} \int_0^{t_k} x\, {}^\circ q(x)\,dx = \frac{1}{t_k} \int_{-t_k}^{+t_k} x^2\, {}^\circ F(dx) + t_k\, {}^\circ q(t_k) \leqq \left(\frac{ca}{\beta}\right)^2.$$

Now (2.2) follows from the symmetrization inequality $2P\{|{}^\circ X_1| > x\} \geqq P\{|X_1| > x + |m|\}$, of ([3] page 149) where $m$ is any median for $X_1$, and a change of variables.

To prove the second assertion of the lemma let $\{t_k\}$ be such that for some number $\delta_0 > 0$ and all $k$

$$|\mu_k| + \rho(t_k) + t_k q(t_k) \leqq \delta_0 < \infty$$

where $\mu_k = \mu(t_k)$. Define $X_j' = X_j$ when $|X_j| \leqq t_k$, and $X_j' = 0$ when $|X_j| > t_k$. Then for $a \geqq 2\delta_0 > 2 \sup_k |\mu_k|$

$$\begin{aligned} P\left\{\left|\frac{S_{t_k}}{t_k}\right| \geqq a\right\} &\leqq P\left\{\left|\frac{S'_{t_k}}{t_k} - \mu_k\right| \geqq \frac{a}{2}\right\} + P\{X_j \neq X_j' \text{ for some } j \leqq t_k\} \\ &\leqq \frac{4 \operatorname{Var}(S'_{t_k}/t_k)}{a^2} + 1 - [1 - q(t_k)]^{t_k} \\ &\leqq \frac{4}{a^2}\rho(t_k) + 1 - \left[1 - \frac{\delta_0}{t_k}\right]^{t_k} \\ &\leqq \frac{4\delta_0}{a^2} + 1 - e^{-\delta_0} + \varepsilon_k , \end{aligned}$$

where $\varepsilon_k \to 0$. Consequently

$$\limsup_{k\to\infty} P\left\{\left|\frac{S_{t_k}}{t_k}\right| \leqq a\right\} \geqq e^{-\delta_0} - \frac{4\delta_0}{a^2} > 0$$

for all $a$ sufficiently large.

REMARK 1. The following generalization of Lemma 1 is also true: If $\limsup P\{|S_{t_k}/t_k{}^\alpha| \leqq A\} > 0$ for some $A$ and $\alpha > \frac{1}{2}$, then

$$\sup_k [t_k{}^{1-\alpha}\rho(t_k{}^\alpha) + t_k q(t_k{}^\alpha)] < \infty .$$

If

$$|t_k{}^{1-\alpha}\mu(t_k{}^\alpha)| + t_k{}^{1-\alpha}\rho(t_k{}^\alpha) + t_k q(t_k{}^\alpha) = O(1)$$

then $\limsup P\{|S_{t_k}/t_k{}^\alpha| \leqq A\} > 0$. We omit the proof but it is analogous to that of Lemma 5 below.

Now assume (1.5) and (1.6) hold and let $t_k \uparrow \infty$ such that (2.2) holds. For $\eta > 0$ introduce the quantities

(2.3) $$\begin{aligned} \lambda_k(\eta) &= \max\{\lambda : \lambda \leqq t_k, \textstyle\int_{\lambda \leqq |x| \leqq t_k} |x| F\{dx\} = \eta\} , \\ \alpha_k(\eta) &= \max\{\rho(t) : \lambda_k(\eta) \leqq t \leqq t_k\} . \end{aligned}$$

By (1.5) for any $\eta > 0$ $\lambda_k(\eta)$ is well defined for all $k$ sufficiently large and $\lambda_k(\eta) \to \infty$, $k \to \infty$, for each fixed $\eta$. From the preceding lemma one of the following three cases must prevail.

*Case* I. There exists $t_k \uparrow \infty$ and a constant $\delta_0 < \infty$ such that

(2.4) $$|\mu(t_k)| + \rho(t_k) + t_k q(t_k) \leqq \delta_0 \qquad \text{for all } k$$

and

$$\text{(2.5)} \qquad \text{for every} \quad \eta > 0 , \quad \alpha(\eta) \equiv \sup_{k \geq 1} \alpha_k(\eta) < \infty .$$

*Case* II. There exists $t_k \uparrow \infty$ such that (2.4) holds and there is an $\eta_0 > 0$ such that

$$\text{(2.6)} \qquad \alpha_k(\eta_0) \to \infty .$$

*Case* III. There exists $t_k \uparrow \infty$ such that (2.1) and (2.2) hold but $|\mu(t_k)| \to \infty$.

In all three cases we show that at least $0 \in B(F, 1)$. This immediately gives the desired conclusion: $B(F, 1) = R$. For if a random walk $\{S_n\}$ with distribution $F$ satisfies (1.5)—(1.6), then, as the reader may easily verify, so does the random walk $\{S_n - nb\}$ with distribution $F_b\{dx\} \equiv F\{dx + b\}$ for any $b$. Hence if (1.5)—(1.6) implies $0 \in B(F, 1)$ then also $0 \in B(F_b, 1)$ or, equivalently, $b \in B(F, 1)$ for every $b$.

*Case* I. We are going to prove that under (2.4)—(2.5) $\{S_n\}$ is persistent, i.e., $P\{\liminf |S_n| = 0\} = 1$, and, *a fortiori*, $P\{\liminf |S_n/n| = 0\} = 1$ and thus $0 \in B(F, 1)$. According to Ornstein's recurrence criterion ([6] Theorem 4.1 or [7] Theorem 2) we need only to verify

$$\int_0^1 \operatorname{Re}\left(\frac{1}{1 - \varphi(\theta)}\right) d\theta = \infty$$

where

$$\varphi(\theta) = \int_{-\infty}^{\infty} e^{ix\theta} F\{dx\} \quad \text{is the ch.f. of} \quad F .$$

Put $\delta(x) = |\mu(x)| + \rho(x) + xq(x)$. Then for $\theta > 0$

$$\begin{aligned} |1 - \varphi(\theta)| &\leqq \left|\int_{-1/\theta}^{1/\theta} i\theta x F\{dx\}\right| + \int_{-1/\theta}^{1/\theta} |e^{ix\theta} - 1 - ix\theta| F\{dx\} + 2 \int_{|x| \geq 1/\theta} F\{dx\} \\ &\leqq \theta|\mu(1/\theta)| + \tfrac{1}{2}\theta^2 \int_{-1/\theta}^{1/\theta} x^2 F\{dx\} + 2q(1/\theta) \leqq 2\theta\delta(1/\theta) . \end{aligned}$$

Also

$$\operatorname{Re}(1 - \varphi(\theta)) \geqq \int_{-1/\theta}^{1/\theta} [1 - \cos x\theta] F\{dx\} \geqq \tfrac{1}{5}\theta^2 \int_{-1/\theta}^{1/\theta} x^2 F\{dx\} = \tfrac{1}{5}\theta\rho(1/\theta) .$$

Hence

$$\text{(2.7)} \qquad \begin{aligned} \int_0^1 \operatorname{Re}\left(\frac{1}{1 - \varphi(\theta)}\right) d\theta &= \int_0^1 \frac{\operatorname{Re}(1 - \varphi(\theta))}{|1 - \varphi(\theta)|^2} d\theta \\ &\geqq \frac{1}{20} \int_0^1 \frac{\rho(1/\theta)}{\theta\delta^2(1/\theta)} d\theta = \frac{1}{20} \int_1^\infty \frac{\rho(x)}{x\delta^2(x)} dx . \end{aligned}$$

From (2.3) and (2.5) if $\lambda_k(\eta) = \lambda_k \leqq x \leqq t_k$, then

$$\rho(x) \leqq \alpha_k(\eta) \leqq \alpha(\eta)$$

and

$$\text{(2.8)} \qquad |\mu(x)| = \left|\left(\int_{-t_k}^{t_k} - \int_{x \leq |y| \leq t_k}\right) y F\{dy\}\right| \leqq |\mu(t_k)| + \eta ,$$

and

$$\text{(2.9)} \qquad xq(x) = x \int_{|y| \geq t_k} F\{dy\} + x \int_{x \leq |y| \leq t_k} F\{dy\} \leqq t_k q(t_k) + \eta .$$

Consequently by (2.4)

$$\delta(x) \leqq 2\eta + \alpha(\eta) + \delta_0 = C(\eta) \qquad \text{for} \quad \lambda_k \leqq x \leqq t_k\,, \qquad k \geqq k_0\,. \tag{2.10}$$

Let us prechoose $\eta \geqq 1 + \delta_0 = 1 + \sup_{k\geq 1} \delta(t_k)$ (see (2.4)). Then, noting that the region $|y| \leqq x$, $\lambda_k \leqq x \leqq t_k$ contains $\lambda_k \leqq |y| \leqq t_k$, $|y| \leqq x \leqq t_k$, we have from (2.3)—(2.4)

$$\begin{aligned} \int_{\lambda_k}^{t_k} \frac{\rho(x)}{x}\, dx &= \int_{\lambda_k}^{t_k} \left(\int_{|y|\leq x} \frac{y^2}{x^2} F\{dy\}\right) dx \geqq \int_{\lambda_k \leq |y| \leq t_k} y^2 \left(\frac{1}{|y|} - \frac{1}{t_k}\right) F\{dy\} \\ &= \eta - \frac{1}{t_k} \int_{\lambda_k \leq |y| \leq t_k} y^2 F\{dy\} \geqq 1 + \delta_0 - \rho(t_k) \geqq 1\,. \end{aligned} \tag{2.11}$$

Now let us thin out the sequence $\{t_k\}$ so that the intervals $[\lambda_k(\eta), t_k]$ become disjoint; this can be done because $\lambda_k(\eta) \to \infty$. Then using (2.10) and (2.11) in (2.7) we get

$$\int_0^1 \operatorname{Re}\left(\frac{1}{1 - \varphi(\theta)}\right) d\theta \geqq \frac{1}{20} \sum_{k=k_0}^{\infty} \int_{\lambda_k}^{t_k} \frac{\rho(x)}{x\delta^2(x)}\, dx \geqq \frac{1}{20C^2(\eta)} \sum_{k=k_1}^{\infty} 1 = \infty\,.$$

So much for Case I.

*Case* II. Here we will prove $B(F, 1) = \bar{\mathbb{R}}$.

LEMMA 2. *Let $T > 0$ be any large number. If* (2.4) *and* (2.6) *hold, then we can find $r_n \uparrow \infty$ and a proper infinitely divisible law $G$ such that*

$$\lim_{n\to\infty} r_n q(r_n) = q_0 < \infty\,,$$

*and*

$$\lim_{n\to\infty} P\left\{\frac{S_{r_n}}{r_n} \in I \,\middle|\, \max_{i\leq r_n} |x_i| \leqq r_n\right\} = G(I) \tag{2.12}$$

*for every continuity interval $I$ of $G$. $G$ has characteristic function $\nu$ given by*

$$\log \nu(\theta) = i\mu\theta + \int_{[-1,+1]} \frac{e^{ix\theta} - 1 - ix\theta}{x^2} M(dx) \tag{2.13}$$

(*the integrand* $\equiv -\frac{1}{2}\theta^2$ *at* $x = 0$) *for some $\mu$ and canonical measure $M$* (for definition see Feller [3] page 560) *satisfying*

$$|\mu| \leqq \delta_0 + \eta_0\,, \qquad \sigma^2 \equiv M\{[-1, +1]\} \geqq |\mu| + T\,. \tag{2.14}$$

LEMMA 3. *Let $G$ be a distribution with characteristic function as in* (2.13). *Put*

$$\Lambda_k = \operatorname{supp} G^{k*}$$
$$\sigma^2 = M\{[-1, 1]\}\,.$$

*If $M\{[0, 1]\} > 0$ then for every $\varepsilon > 0$ and $a > \mu - \sigma^2$, there exists $k \geqq 1$ with*

$$(k(a - \varepsilon), k(a + \varepsilon)) \cap \Lambda_k \neq \emptyset\,. \tag{2.15}$$

Before proving these lemmas let us use them to show $B(F, 1) = \bar{\mathbb{R}}$. Let $T > 0$ be arbitrary but fixed and choose $\{r_n\}$ as in Lemma 2. By (2.14) $M \neq 0$ and there is no loss of generality in assuming $M\{[0, 1]\} > 0$. Let $a > -T$, $\varepsilon > 0$, $I = [a - \varepsilon, a + \varepsilon]$. Then

$$P\left\{\frac{S_n}{n} \in I \text{ i.o.}\right\}$$
$$= \lim_{m\to\infty} P\left\{\frac{S_n}{n} \in I \text{ for some } n \geqq m\right\}$$
$$\geqq \limsup_{n\to\infty} P\left\{\frac{S_{kr_n}}{kr_n} \in I \,\middle|\, \max_{i\leqq kr_n} |X_i| \leqq r_n\right\} P\{\max_{i\leqq kr_n} |X_i| \leqq r_n\}\,.$$

Now $S_{kr_n}/kr_n$ conditioned on $\{|X_i| \leqq r_n, i = 1, \cdots, kr_n\}$ is distributed as a sum $V_1 + \cdots + V_k$ of $k$ independent random variables where each $V_i$ is distributed as $k^{-1}(X_1 + \cdots + X_{r_n})/r_n$ conditioned on $\{|X_i| \leqq r_n, i = 1, \cdots, r_n\}$. Hence by (2.12)

$$\liminf_{n\to\infty} P\left\{\frac{S_{kr_n}}{kr_n} \in I \,\middle|\, \max_{i\leqq kr_n} |X_i| \leqq r_n\right\} \geqq G^{k*}\{kI^0\}\,,$$

$kI^0 = (k(a - \varepsilon), k(a + \varepsilon))$. By $a > -T \geqq \mu - \sigma^2$, (2.14) and Lemma 3 we may choose $k$ so that $G^{k*}\{kI^0\} > 0$. Also, by Lemma 2, $q(r_n) \sim q_0/r_n$ as $n \to \infty$ for some $q_0 < \infty$ so

$$P\{\max_{i\leqq kr_n} |X_i| \leqq r_n\} = [1 - q(r_n)]^{kr_n} \to e^{-q_0 k}\,, \qquad n \to \infty\,.$$

Putting these facts together we have for any $\varepsilon > 0$

$$P\left\{\frac{S_n}{n} \in [a - \varepsilon, a + \varepsilon] \text{ i.o.}\right\} \geqq G^{k*}\{kI^0\}e^{-q_0 k} > 0\,,$$

and hence[2] every $a > -T$ is in $B(F, 1)$. Since $T > 0$ is arbitrary and since $B(F, 1)$ is closed in $\bar{\mathbb{R}}$ we conclude $B(F, 1) = \bar{\mathbb{R}}$.

Proof of Lemma 2. By (2.6) there exists $t_k'$ with $\lambda_k(\eta_0) \leqq t_k' \leqq t_k$ and $\limsup_{k\to\infty} \rho(t_k') = \infty$, consequently we can choose sequences $\{k_n\}$ and $\{r_n\}$ such that

$$\lambda_{k_n}(\eta_0) \leqq r_n \leqq t_{k_n} \quad \text{and} \tag{2.16}$$
$$T + \eta_0 + \delta_0 \leqq \lim_{n\to\infty} \rho(r_n) \equiv \sigma^2 < \infty\,.$$

(Since $\rho$ is continuous we could even have $\rho(r_n) = \sigma^2$ for all $n$.) From (2.16), (2.8), (2.9) and (2.4) we have

$$|\mu(r_n)| \leqq \delta_0 + \eta_0 \leqq \sigma^2 - T \qquad \text{and} \qquad r_n q(r_n) \leqq \delta_0 + \eta_0\,.$$

So by selecting a subsequence of $\{r_n\}$ if necessary we can assume

$$\lim_{n\to\infty} \mu(r_n) = \mu\,, \qquad |\mu| \leqq \sigma^2 - T \tag{2.17}$$
$$\lim_{n\to\infty} r_n q(r_n) = q_0 < \infty\,.$$

[2] Compare the proof of theorem 1 in [4].

Now define

$$F_n\{dx\} = P\left\{\frac{X_k}{r_n} \in dx \,\middle|\, |X_k| \leqq r_n\right\} = \frac{F\{r_n dx\}}{1 - q(r_n)} \qquad \text{on} \quad [-1, 1]$$

$$= 0 \qquad \text{on} \quad \mathbb{R}\backslash[-1, 1]$$

$$M_n^+(x) = r_n[1 - F_n(x)]\,, \qquad M_n^-(x) = r_n F_n(-x) \qquad \text{for} \quad x > 0\,,$$

$$M_n\{dx\} = r_n x^2 F_n\{dx\}\,, \qquad \beta_{k,n} = E\left(\frac{X_k}{r_n} \,\middle|\, |X_k| \leqq r_n\right) = \frac{1}{r_n}\frac{\mu(r_n)}{1 - q(r_n)}\,,$$

$$b_n = \textstyle\sum_1^{r_n} \beta_{k,n} = \dfrac{\mu(r_n)}{1 - q(r_n)}\,, \qquad B_n = \textstyle\sum_1^{r_n} \beta_{k,n}^2 = \dfrac{1}{r_n}\left(\dfrac{\mu(r_n)}{1 - q(r_n)}\right)^2.$$

Note that $F_n^{r_n*}\{dx\} = P\{S_{r_n}/r_n \in dx \mid \max_{i \leq r_n} |X_i| \leqq r_n\}$. From (2.17), (2.16) and $q(r_n) \to 0$ we have

$$\lim b_n = \lim_{n\to\infty} \frac{\mu(r_n)}{1 - q(r_n)} = \mu$$

$$\lim B_n = \lim \frac{1}{r_n}\left(\frac{\mu(r_n)}{1 - q(r_n)}\right)^2 = 0$$

$$\lim M_n\{R\} = \lim M_n\{[-1, 1]\} = \lim \rho(r_n) = \sigma^2\,.$$

Next using (2.9), (2.4) and (2.16) gives us, for $0 < x \leqq 1$,

$$M_n^{\pm}(x) \leqq (1 - q(r_n))^{-1} \frac{1}{r_n x^2} \int_{r_n x \leq |y| \leq r_n} y^2 F\{dy\}$$

$$= O\left(\frac{1}{x^2}\,\rho(r_n)\right) = O\left(\frac{\sigma^2}{x^2}\right).$$

Thus for each $0 < x \leqq 1$, $M_n^+(x)$ and $M_n^-(x)$ are bounded in $n$, since these functions are non-increasing it follows that we can find non-increasing functions $M^+$, $M^-$ on $(0, 1]$ so that along a subsequence of $\{r_n\}$, denoted again by $\{r_n\}$, we have

$$\lim M_n^{\pm}(x) = M^{\pm}(x)$$

at all continuity points of $M^{\pm}$. We may now conclude from ([3] Theorem XVII. 7, page 585) that $(S_{r_n}/r_n) - b_n$ conditioned on $\max_{i \leq r_n} (X_i) \leqq r_n$ converges in law to an infinitely divisible distribution whose canonical measure $M$ has support in $[-1, 1]$ and satisfies (2.14). This clearly implies (2.12) since $b_n \to \mu$ and $|\mu| < \infty$.

REMARK 2. A proof similar to that of Lemma 2 shows that if $\rho(r_n) + r_n q(r_n) = O(1)$ as $n \to \infty$ but $\mu(r_n) \to \infty$. Then, along a subsequence of $\{r_n\}$, the law of $(S_{r_n}/r_n) - b_n$, $b_n = E(X_i \mid |X_i| \leqq r_n) = \mu(r_n)/(1 - q(r_n))$, conditioned on $\{\max_{i \leq r_n} |X_i| \leqq r_n\}$ converges properly to an infinitely divisible law whose mean is 0 and whose canonical measure $M$ is concentrated on $[-1, 1]$ with $M\{[-1, 1]\} = \sigma^2 = \lim \rho(r_n)$. (For the proof note that it suffices to show $\liminf r_n q(r_n x) < \infty$ and $\lim B_n = 0$. See ([3] pages 584–585), notation as before. But

$$r_n q(r_n x) \leqq \frac{1}{x^2}\rho(r_n) + r_n q(r_n) = O(1) \qquad \text{and}$$

$$\begin{aligned} B_n = \frac{1}{r_n}\left(\frac{\mu(r_n)}{1-q(r_n)}\right)^2 &\leqq O\left(\frac{\mu^2(A)}{r_n}\right) + O\left(\frac{1}{r_n}\left|\int_{A\leqq x\leqq r_n} xF\{dx\}\right|^2\right) \\ &\leqq O\left(\frac{1}{r_n}\right) + P\{|x| \geqq A\}\frac{1}{r_n}\int_{A\leqq|x|\leqq r_n} x^2F\{dx\} \\ &= O\left(\frac{1}{r_n}\right) + q(A)\rho(r_n) = O\left(\frac{1}{r_n}\right) + O(q(A)) . \end{aligned}$$

That is

$$\limsup B_n \leqq Cq(A)$$

for every $A > 0$, $C$ independent of $A$. But then we must have $\lim_{n\to\infty} B_n = 0$, since $q(A) \to 0$, $A \to \infty$.

We shall need this remark in Case III.

Proof of Lemma 3. If the canonical measure $M$ in (2.13) has an atom at the origin, say $\delta = M\{0\} > 0$, then $G$ is the convolution of some distribution $H$ and the $N(0, \delta)$ distribution and therefore has an everywhere strictly positive density. In this case (2.15) holds with $k = 1$ so we may assume $M\{0\} = 0$, $M\{(0, 1]\} > 0$. Put for $h > 0$ $b_h = \int_{[h,1]} M\{dx\}/x$ and let $V = V_h$ and $W = W_h$ be independent random variables with

$$\log Ee^{i\theta W} = \int_{x\in[-1,h)} [e^{i\theta x} - 1 - i\theta x]\frac{M\{dx\}}{x^2},$$

$$\log Ee^{i\theta V} = \int_{x\in[h,1]} (e^{i\theta x} - 1)\frac{M\{dx\}}{x^2}.$$

Also let $Y$ have distribution $G$. Since

$$Ee^{i\theta Y} = e^{i\theta(\mu - b_h)}Ee^{i\theta W}Ee^{i\theta V}$$

$\Lambda_1 = \text{supp}\,(Y) = \mu - b_h \oplus \text{supp}\,(W) \oplus \text{supp}\,(V)$. (Here we write supp $(Y)$ for the support of the distribution $G$ of $Y$, also $A \oplus B = \{a + b : a \in A, b \in B\}$.) Now $V$ has a compound Poisson distribution: $V$ has the distribution of

$$Z_1 + \cdots + Z_\tau$$

where $\tau, Z_1, Z_2, \cdots,$ are independent, $\tau$ has a Poisson distribution and

$$P\{Z_i \in A\} = \int_{A\cap[h,1]} \frac{M\{dx\}}{x^2}\Big/\int_{[h,1]}\frac{M\{dx\}}{x^2}.$$

It follows that if $d \in \text{supp}\,(M)$, $d \geqq h$, then $jd \in \text{supp}\,(V)$, $j = 1, 2, \cdots$, so, for any $j = 0, 1, \cdots$, $w_0 \in \text{supp}\,(W)$ we have $\mu - b_h + w_0 + jd \in \text{supp}\,(Y) = \Lambda_1$. This fact and $\Lambda_k = \Lambda_1 \oplus \cdots \oplus \Lambda_1$ ($k$ summands) gives us

$$\text{(2.18)} \qquad k(\mu - b_h + w_0) + jd \in \Lambda_k\,, \qquad j = 0, 1, \cdots, k = 1, 2, \cdots .$$

Now if $M\{[-1, 0)\} > 0$, then supp $(W)$ is unbounded below, see ([3] page 571(c)) and we can pick $h$, $w_0$ so that $M$ has a point of increase $d \geqq h$ and

$$\mu - b_h + w_0 < a\,. \tag{2.19}$$

If $M\{[-1, 0]\} = 0$ then

$$\sigma^2 = M\{(0, 1]\} < \lim_{h\downarrow 0} b_h = \lim_{h\downarrow 0} \int_{[h,1]} \frac{M\{dx\}}{x}$$

and $W_h \to 0$ in probability since

$$\lim_{h\downarrow 0} Ee^{i\theta W_h} = \lim_{h\downarrow 0} \exp \int_{(0,h)} [e^{i\theta x} - 1 - i\theta x] \frac{M\{dx\}}{x^2} = 1\,.$$

Thus for small $h$, $w_0$ can be taken small and we can again pick $h > 0$ so that (2.19) is satisfied (recall $\mu - \sigma^2 < a$). For $2k\varepsilon > d$ we now see that (2.19) guarantees there is at least one point of the form (2.18) in $(k(a - \varepsilon), k(a + \varepsilon))$ and (2.15) is established.

*Case* III. Here also we will show $B(F, 1) = \bar{\mathbb{R}}$. In view of (2.1) and (2.2) we may suppose

$$\begin{aligned} &t_k q(t_k) \to q_0 < \infty \\ &\rho(t_k) \to \rho_0 < \infty \qquad \text{as } k \to \infty \end{aligned} \tag{2.20}$$

and

$$P\left\{\left|\frac{S_{t_k}}{t_k}\right| \leqq a\right\} \geqq p_0 > 0 \qquad \text{for all } k\,. \tag{2.21}$$

Let $\{X_1^{(k)}, X_2^{(k)}, \cdots, Y_1^{(k)}, Y_2^{(k)}, \cdots\}$ be independent random variables such that

$$P\{X_j^{(k)} \in dx\} = P\{X_1 \in dx \mid |X_1| \leqq t_k\}$$
$$P\{Y_j^{(k)} \in dx\} = P\{X_1 \in dx \mid |X_1| > t_k\}\,.$$

Put $W_0^{(k)} = V_0^{(k)} = 0$, $V_m^{(k)} = X_1^{(k)} + \cdots + X_m^{(k)}$, $W_m^{(k)} = Y_1^{(k)} + \cdots + Y_m^{(k)}$ for $m \geqq 1$ and $\alpha_m^{(k)} = \#\{j : j \leqq m, |X_j| > t_k\}$. A simple calculation shows

$$P\{S_m \in I\} = \textstyle\sum_{j=0}^m P\{V_{m-j}^{(k)} + W_j^{(k)} \in I\} P\{\alpha_m^{(k)} = j\}\,. \tag{2.22}$$

Put $b_k = EX_1^{(k)} = \mu(t_k)/(1 - q(t_k))$. According to Remark 2 we can assume the $t_k$ have been so chosen that $(S_{t_k}/t_k) - b_k$ conditioned on $\{\max_{j\leqq t_k} |X_j| \leqq t_k\}$ converges in law to a proper distribution $G$. Moreover

$$\frac{1}{t_k} |V_{t_k}^{(k)} - V_{r_k}^{(k)} - (t_k - r_k)b_k| \to 0$$

in probability by Chebyshev's inequality, whenever $r_k/t_k \to 1$, so that

$$\lim_{k\to\infty} P\left\{\frac{1}{r_k} V_{r_k}^{(k)} - b_k \in dx\right\} = G\{dx\} \tag{2.23}$$

whenever $r_k/t_k \to 1$.

LEMMA 4. (a) *In* (2.20) $q_0 = \lim_{k\to\infty} t_k q(t_k) > 0$.

(b) *There is an integer* $j_0 \geqq 1$, *a real number* $h$, *and a subsequence of* $\{t_k\}$, *denoted* *again by* $\{t_k\}$, *such that for every* $\varepsilon > 0$

$$\text{(2.24)} \qquad \liminf P\left\{\frac{1}{t_k} W_{j_0}^{(k)} + b_k \in [h-\varepsilon, h+\varepsilon]\right\} = \gamma(\varepsilon) > 0 .$$

Before proving the lemma let us use it to show $B(F, 1) = \bar{\mathbb{R}}$. Let $g$ be a point of increase for $G$, $t$ an arbitrary real number and $\varepsilon > 0$. Put $I_1 = [g-\varepsilon, g+\varepsilon]$, $I_2 = [h-\varepsilon, h+\varepsilon]$, $h$ as in (2.24), and $m_k = (t_k + j_0)/(1 - t/b_k)$. Since $|b_k| \to \infty$ (recall $|\mu(t_k)| \to \infty$ in Case III) we have

$$\left(1 - \frac{j_0}{m_k}\right) I_1 + \frac{t_k}{m_k} I_2 + \left(1 - \frac{j_0 + t_k}{m_k}\right) b_k$$
$$\subset [h+g+t-4\varepsilon, h+g+t+4\varepsilon] \equiv J_\varepsilon$$

for all $k$ sufficiently large. So by (2.23) and (2.24)

$$\tfrac{1}{4} G\{(g-\varepsilon, g+\varepsilon)\}\gamma(\varepsilon) \leqq P\left\{\frac{V_{m_k - j_0}^{(k)}}{m_k - j_0} - b_k \in I_1\right\} P\left\{\frac{W_{j_0}^{(k)}}{t_k} + b_k \in I_2\right\}$$
$$\leqq P\left\{\frac{1}{m_k}(V_{m_k - j_0}^{(k)} + W_{j_0}^{(k)}) \in J_\varepsilon\right\}$$

for all $k$ large. From $m_k/t_k \to 1$, (2.20) and part (a) of the lemma

$$P\{\alpha_{m_k}^{(k)} = j_0\} = \binom{m_k}{j_0}(q(t_k))^{j_0}(1 - q(t_k))^{m_k - j_0} \to e^{-q_0}\frac{q_0^{j_0}}{j_0!} = C_0 > 0 .$$

Applying these estimates to (2.22) gives for all large enough $k$

$$P\left\{\frac{1}{m_k} S_{m_k} \in J_\varepsilon\right\} \geqq P\{\alpha_{m_k}^{(k)} = j_0\} P\left\{\frac{1}{m_k}(V_{m_k - j_0}^{(k)} + W_{j_0}^{(k)}) \in J_\varepsilon\right\}$$
$$\geqq \tfrac{1}{8} C_0 \gamma(\varepsilon) G\{(g-\varepsilon, g+\varepsilon)\} > 0 .$$

In other words $\limsup_{n\to\infty} P\{S_n/n \in [h+g+t-4\varepsilon, h+g+t+4\varepsilon]\} > 0$ for every $\varepsilon > 0$; consequently[3] $h+g+t \in B(F, 1)$. But then $B(F, 1) = \bar{\mathbb{R}}$ since $t$ is arbitrary.

PROOF OF LEMMA 4. From (2.23) and $|b_k| \to \infty$ it is clear that $P\{(1/t_k)|V_{t_k}^{(k)}| \leqq \varepsilon\} \to 0$ for every $\varepsilon > 0$, hence

$$P\left\{\left|\frac{V_{t_k}^{(k)}}{t_k}\right| \leqq a\right\} \leqq \tfrac{1}{4} p_0$$

for all $k$ sufficiently large. Next as noted above $P\{\alpha_{t_k}^{(k)} = j\} \to e^{-q_0} q_0^{\,j}/j!$ so there is a $j_1 \geqq 1$ such that

$$P\{\alpha_{t_k}^{(k)} > j_1\} \leqq \tfrac{1}{4} p_0$$

for all $k$ large. In (2.22) these bounds along with (2.21) give for $k$ sufficiently

[3] Compare the proof of Theorem 1 in [4].

large

$$p_0 \leqq \sum_{j=0}^{t_k} P\left\{\frac{1}{t_k}|V_{t_k-j}^{(k)} + W_j^{(k)}| \leqq a\right\} P\{\alpha_{t_k}^{(k)} = j\}$$

$$\leqq \frac{p_0}{4} + P\{1 < \alpha_{t_k}^{(k)} \leqq j_1\} \max_{1 \leqq j \leqq j_1} P\left\{\frac{1}{t_k}|V_{t_k-j}^{(k)} + W_j^{(k)}| \leqq a\right\} + \frac{p_0}{4}.$$

Hence, by going over to a subsequence, we must have for some $j_0 \in [1, j_1]$

$$\liminf_{k\to\infty} P\{1 \leqq \alpha_{t_k}^{(k)} \leqq j_1\} P\left\{\frac{1}{t_k}|V_{t_k-j_0}^{(k)} + W_{j_0}^{(k)}| \leqq a\right\} \geqq \frac{p_0}{2} > 0.$$

From this we can see firstly that $q_0 \neq 0$, since $\lim P\{1 \leqq \alpha_{t_k}^{(k)} \leqq j_1\} \leqq j_1 q_0 \max\{1, q_0^{j_1-1}\}$. Secondly for all $k$ sufficiently large we must have

$$P\left\{\frac{1}{t_k}|V_{t_k-j_0}^{(k)} + W_{j_0}^{(k)}| \leqq a\right\} \geqq \tfrac{1}{2}p_0.$$

From (2.23) we can find a $d$ so that for all $k$

$$P\left\{\left|\frac{1}{t_k} V_{t_k-j_0}^{(k)} - b_k\right| \leqq d\right\} \geqq 1 - \tfrac{1}{4}p_0.$$

Combining these last two inequalities and using $P(A \cap D) \geqq P(A) + P(D) - 1$ yields

$$P\left\{\left|\frac{1}{t_k} W_{j_0}^{(k)} + b_k\right| \leqq a + d\right\} \geqq \tfrac{1}{4}p_0 > 0 \tag{2.25}$$

for all $k$ sufficiently large. Now let us select a subsequence of $\{t_k\}$, still to be denoted by $\{t_k\}$, so that

$$\lim P\left\{\frac{1}{t_k} W_{j_0}^{(k)} + b_k \leqq x\right\} = f(x)$$

exists at all but a countable set of $x$. We get (2.24) by choosing for $h$ any point of increase of $f$ in $[-a - d, a + d]$ which exists by (2.25). This completes the proof of the lemma and of Theorem 3.

**3. Proof of Theorems 1 and 2.** First we derive a criterion for $b \in B_s(F, \alpha)$. We use the notation of Section 2.

LEMMA 5. *If for a sequence of constants $\{c_k\}$ there is a sequence of nonrandom numbers $\{t_k\}$ such that $t_k \uparrow \infty$*

$$\left|\frac{S_{t_k}}{t_k^{\alpha}} - c_k\right| \to_P 0 \qquad \text{as } k \to \infty \tag{3.1}$$

*then*

$$|c_k - t_k^{1-\alpha}\mu(t_k^{\alpha})| \to 0 \qquad \text{and} \tag{3.2}$$

$$\frac{1}{t_k^{2\alpha-1}} \int_0^{t_k^{\alpha}} xq(x)\,dx \to 0 \qquad \text{as } k \to \infty. \tag{3.3}$$

*Conversely if* (3.3) *holds for some $t_k \uparrow \infty$ then* (3.1) *holds with $c_k = t_k^{1-\alpha}\mu(t_k^{\alpha})$.*

COROLLARY. *For a finite $b$ to be in $B_s(F, \alpha)$ it is necessary and sufficient that* (3.3) *hold for some sequence $t_k \uparrow \infty$ with $t_k^{1-\alpha}\mu(t_k^\alpha) \to b$. (Recall that* (3.1) *implies* a.s. *convergence of $t_k^{-\alpha}S_{t_k} - c_k$ to 0 along a subsequence.) For $b = \pm\infty$ to be in $B_s(F, \alpha)$* (3.3) *with $t_k^{1-\alpha}\mu(t_k^\alpha) \to b$ is sufficient.*

REMARK 3. For $0 \leqq \alpha \leqq \frac{1}{2}$ $\liminf t^{1-2\alpha} \int_0^t xq(x)\,dx > 0$ unless $F$ is degenerate. Hence $B_s(F, \alpha) \cap \mathbb{R} = \emptyset$ when $0 \leqq \alpha \leqq \frac{1}{2}$ and $F$ is not concentrated at the origin.

REMARK 4. To see that (3.3) is not necessary for $\infty \in B_s(F, \alpha)$ let $F\{dx\} = C_1 x^{-2}\,dx$ for $x \geqq 1$, $C_1 > 0$, and $F\{dx\} = 0$, $x < 1$. Then $EX_1 = +\infty$ so $\mu(t) \to \infty$ and $S_n/n \to +\infty$ a.s. Thus $B_s(F, 1) = B(F, 1) = \{+\infty\}$. However $\lim_{t\to\infty} (1/t) \int_0^t xq(x)\,dx = C_1 > 0$. We do not have a good criterion for $+\infty \in B_s(F, \alpha)$.

PROOF OF LEMMA 5. Suppose first that (3.3) holds. An integration by parts shows that (3.3) is equivalent to

$$(3.4) \qquad \lim_{k\to\infty} t_k q(t_k^\alpha) = 0 \qquad \text{and} \qquad \lim_{k\to\infty} t_k^{1-\alpha}\rho(t_k^\alpha) = 0\,.$$

Define $X_i' = X_i$ if $|X_i| \leqq t_k^\alpha$ and $X_i' = 0$ for $|X_i| > t_k^\alpha$. Put $S_t' = X_1' + \cdots + X_t'$, let $c_k = t_k^{1-\alpha}\mu(t_k^\alpha)$. Then $E(S'_{t_k}/t_k^\alpha) = c_k$ and

$$\operatorname{Var}\left(\frac{S'_{t_k}}{t_k^\alpha}\right) = t_k^{1-2\alpha} \operatorname{Var}(X_1') \leqq t_k^{1-2\alpha} \int_{-t_k^\alpha}^{t_k^\alpha} x^2 F\{dx\} = t_k^{1-\alpha}\rho(t_k^\alpha)\,.$$

Hence

$$P\left\{\left|\frac{S_{t_k}}{t_k^\alpha} - c_k\right| > \varepsilon\right\} \leqq P\left\{\left|\frac{S'_{t_k}}{t_k^\alpha} - c_k\right| > \varepsilon\right\} + t_k P\{X_1 \neq X_1'\}$$
$$\leqq \frac{1}{\varepsilon^2} t_k^{1-\alpha}\rho(t_k^\alpha) + t_k q(t_k^\alpha)\,.$$

From this inequality it follows that if (3.4) holds then (3.1) holds with $c_k = t_k^{1-\alpha}\mu(t_k^\alpha)$.

Now suppose (3.1) is true. Let ${}^\circ X_i$, ${}^\circ S_n = {}^\circ X_1 + \cdots + {}^\circ X_n$ denote the symmetrized random variables. Then by ([3] page 149)

$$2P\{|S_{t_k} - t_k^\alpha c_k| > \varepsilon t_k^\alpha\} \geqq P\{|{}^\circ S_{t_k}| > 2\varepsilon t_k^\alpha\}$$
$$\geqq \tfrac{1}{2}[1 - \exp[-t_k\,{}^\circ q(2\varepsilon t_k^\alpha)]]$$

where ${}^\circ q(t) = P\{|{}^\circ X_1| > t\}$. From this we see that (3.1) implies

$$(3.5) \qquad t_k\,{}^\circ q(y t_k^\alpha) \to 0 \qquad \text{for all } y > 0$$

and

$$\frac{{}^\circ S_{t_k}}{t_k^\alpha} \to_P 0$$

as $k \to \infty$. If $\Phi_*$ denotes the characteristic function of ${}^\circ X_1$ then

$$\Phi_*^{t_k}\left(\frac{\theta}{t_k^\alpha}\right) \to 1 \qquad k \to \infty\,.$$

Since $0 \leqq \Phi_*(\theta) \leqq 1$ for all $\theta$

$$1 - \Phi_*^{t_k}\left(\frac{\theta}{t_k^\alpha}\right) = \sum_{j=0}^{t_k-1} \Phi_*^j\left(\frac{\theta}{t_k^\alpha}\right)\left(1 - \Phi_*\left(\frac{\theta}{t_k^\alpha}\right)\right)$$
$$\geqq t_k \Phi_*^{t_k}\left(\frac{\theta}{t_k^\alpha}\right)\left(1 - \Phi_*\left(\frac{\theta}{t_k^\alpha}\right)\right),$$

and thus

$$\lim_{k\to\infty} t_k\left[1 - \Phi_*\left(\frac{\theta}{t_k^\alpha}\right)\right] = 0 \qquad \text{for all } \theta . \tag{3.6}$$

But

$$t_k\left[1 - \Phi_*\left(\frac{\theta}{t_k^\alpha}\right)\right] \geqq t_k \int_{|x|\leqq|\theta|^{-1}t_k^\alpha}\left[1 - \cos\left(\frac{\theta x}{t_k^\alpha}\right)\right] {}^\circ F\{dx\}$$
$$\geqq C\theta^2 t_k^{1-2\alpha} \int_{|x|\leqq y t_k^\alpha} x^2 \, {}^\circ F\{dx\}$$
$$= C[t_k^{1-2\alpha} 2\theta^2 \int_0^{y t_k^\alpha} x \, {}^\circ q(x)\, dx - t_k \, {}^\circ q(y t_k^\alpha)] ,$$

where $y = |\theta|^{-1}$ and $C > 0$ is independent of $t_k$. Therefore by (3.5) and (3.6)

$$\lim_{k\to\infty} t_k^{1-2\alpha} \int_0^{y t_k^\alpha} x \, {}^\circ q(x)\, dx = 0 \qquad \text{for all } y > 0 . \tag{3.7}$$

In view of Remark 3 we may assume $\alpha > \frac{1}{2}$ now. Moreover, if $m$ is any median for $X_1$, then ([3] page 149),

$${}^\circ q(x) = P\{|{}^\circ X_1| > x\} \geqq \tfrac{1}{2}P\{|X_1| > x + |m|\} = \tfrac{1}{2}q(x + |m|) .$$

Using this inequality in (3.7) and a change of variables gives (3.3).

We have now proved that (3.1) implies (3.3) and that (3.3) implies (3.1) with $c_k = t_k^{1-\alpha}\mu(t_k^\alpha)$. Hence if $\{c_k\}$ is any sequence of constants for which (3.1) holds then (3.2) must necessarily also hold.

PROOF OF THEOREM 1. Assume $\alpha \neq 1$ and suppose

$$\frac{S_{t_k}}{t_k^\alpha} \to_P b , \qquad k \to \infty \tag{3.8}$$

where $0 < |b| < \infty$. If $r$ is an integer, $r \geqq 1$, (3.8) implies $S_{rt_k}/(rt_k)^\alpha \to r^{1-\alpha}b$ in probability as one may easily verify. We want to establish this for any real $r > 0$. Let $m_k \uparrow \infty$ so that

$$\frac{m_k}{t_k} \to r > 0 , \qquad k \to \infty .$$

By Lemma 5, (3.8) implies

$$t_k q(t_k^\alpha) \to 0 ,$$
$$t_k^{1-2\alpha} \int_{[-t_k^\alpha, t_k^\alpha]} x^2 F\{dx\} \to 0 ,$$
$$t_k^{1-\alpha}\mu(t_k^\alpha) \to b \qquad \text{as } k \to \infty .$$

Write $c_k = E(X_1 \,|\, |X_1| \leqq t_k^\alpha) = \mu(t_k^\alpha)/[1 - q(t_k^\alpha)]$ and let $\varepsilon > 0$. For all $k$

sufficiently large $|m_k^{1-\alpha}c_k - r^{1-\alpha}b| < \varepsilon/2$ and

$$P\left\{\left|\frac{S_{m_k}}{m_k^{\alpha}} - r^{1-\alpha}b\right| > \varepsilon\right\} \leqq P\left\{\left|\frac{S_{m_k}}{m_k^{\alpha}} - m_k^{1-\alpha}c_k\right| > \frac{\varepsilon}{2}\,\middle|\, |X_i| \leqq t_k^{\alpha},\, i \leqq m_k\right\}$$
$$+ P\{|X_i| > t_k^{\alpha} \text{ for some } i \leqq m_k\}$$
$$\leqq \frac{4}{\varepsilon^2} m_k^{1-2\alpha} \operatorname{Var}(X_1 \,|\, |X_1| \leqq t_k^{\alpha}) + m_k q(t_k^{\alpha})$$
$$= O(t_k^{1-2\alpha} \textstyle\int_{[-t_k^{\alpha}, t_k^{\alpha}]} x^2 F\{dx\} + t_k q(t_k^{\alpha})) \to 0 \qquad k \to \infty .$$

Hence

$$\frac{S_{m_k}}{m_k^{\alpha}} \to_P r^{1-\alpha} b , \qquad k \to \infty ,$$

and $r^{1-\alpha}b \in B_s(F, \alpha)$. Since this is true for any $r > 0$ it follows that

$$\operatorname{sign}(b)[0, \infty] \subset B_s(F, \alpha) .$$

PROOF OF THEOREM 2. $D$ is the given closed set in $\bar{\mathbb{R}}$ and we want to construct a distribution $F$ so that $B_s(F, 1) = D$. Select a sequence $\{c_k\}_{k=0}^{\infty} \supset \mathbb{R}$ so that $c_0 = 0$, and

$$\bigcap_{n=1}^{\infty} \overline{\{c_n, c_{n+1}, \cdots\}} = D . \tag{3.9}$$

Assume also that $c_k \neq c_{k-1}$, $k \geqq 1$. Next set

$$b_k = 8 + \max\{|c_1 - c_0|, |c_2 - c_1|, \cdots, |c_k - c_{k-1}|\} \tag{3.10}$$

and then choose $\{a_k\}$ so that

$$a_k \geqq 1 , \qquad a_{k+1} \geqq k^4 a_k b_{k+1}^2 , \qquad \frac{b_{k+1}}{a_{k+1}} \leqq \frac{b_k}{a_k} \qquad \text{and} \tag{3.11}$$
$$\sum_{k=1}^{\infty} \frac{2b_k}{a_k} = 1 .$$

Note that $\{a_k\}$ and $\{b_k\}$ are non-decreasing and $a_k \uparrow \infty$. Define the distribution $F$ of a random variable $X$ by

$$\begin{aligned} P\{X = a_k\} &= (2b_k + c_k - c_{k-1})/2a_k , \\ P\{X = -a_k\} &= (2b_k + c_{k-1} - c_k)/2a_k , \qquad k \geqq 1 . \end{aligned} \tag{3.12}$$

$F$ is a genuine probability distribution by (3.10) and (3.11); moreover

$$\mu(t) = \textstyle\int_{[-t,t]} xF\{dx\} = \sum_{j=1}^{k} (c_j - c_{j-1}) = c_k \tag{3.13}$$

for $a_k \leqq t < a_{k+1}$. Hence by the corollary to Lemma 5, ($\alpha = 1$) and by (3.9)

$$\begin{aligned} B_s^f(F, 1) &\subset (-\infty, \infty) \textstyle\bigcap_{T>0} \overline{\{\mu(t) : t > T\}} \\ &= (-\infty, \infty) \textstyle\bigcap_{n \geqq 1} \overline{\{c_k : k \geqq n\}} = D^f \end{aligned} \tag{3.14}$$

where for any set $B \subset \bar{\mathbb{R}}$, $B^f = \{\text{finite points of } B\} = B \cap (-\infty, \infty)$.

Let $t_k = (a_k a_{k+1})^{\frac{1}{2}} \in (a_k, a_{k+1})$. If we can show

$$\frac{1}{t_k} \textstyle\int_{[-t_k, t_k]} x^2 F\{dx\} + t_k q(t_k) \to 0 , \qquad k \to \infty ,$$

then by Lemma 5

$$(3.15)\qquad \left|\frac{S_{t_k}}{t_k} - \mu(t_k)\right| \to_P 0\,, \qquad k \to \infty\,,$$

and consequently by (3.9), (3.13) and (3.14),

$$(3.16)\qquad D \subset B_s(F, 1) \quad \text{and} \quad D^f = B_s^f(F, 1)\,.$$

First by (3.12)

$$\frac{1}{t_k} \int_{[-t_k, t_k]} x^2 F\{dx\} = 2(a_k a_{k+1})^{-\frac{1}{2}} \sum_{j=1}^{k} a_j b_j \leqq \frac{2}{a_k} \sum_{j=1}^{k-1} a_j b_j + 2\left(\frac{a_k b_{k+1}^2}{a_{k+1}}\right)^{\frac{1}{2}}.$$

By (3.11)

$$(3.17)\qquad \left(\frac{a_k b_{k+1}^2}{a_{k+1}}\right)^{\frac{1}{2}} \leqq \frac{1}{k^2} \to 0\,.$$

Also

$$(3.18)\qquad \sum_{j=1}^{\infty} \frac{a_j b_j}{a_{j+1}} \leqq \sum_{j=1}^{\infty} \frac{a_j b_{j+1}}{a_{j+1}} \leqq \sum_{j=1}^{\infty} \frac{1}{j^4} < \infty\,,$$

so by Kronecker's Lemma

$$(3.19)\qquad \frac{1}{a_k} \sum_{j=1}^{k-1} a_j b_j \to 0\,, \qquad k \to \infty\,.$$

It follows from (3.17) and (3.19) that

$$\frac{1}{t_k} \int_{[-t_k, t_k]} x^2 F\{dx\} \to 0\,.$$

It remains to prove $t_k q(t_k) \to 0$. But

$$t_k q(t_k) = 2(a_k a_{k+1})^{\frac{1}{2}} \sum_{j=k+1}^{\infty} \frac{b_j}{a_j}$$
$$\leqq 2\left(\frac{a_k b_{k+1}^2}{a_{k+1}}\right)^{\frac{1}{2}} + 2 \sum_{j=k+2}^{\infty} \frac{a_{j-1} b_j}{a_j} \to 0\,, \qquad k \to \infty\,,$$

by (3.17) and (3.18). This proves (3.16).

Now we must show $B_s(F, 1) = D$. Suppose first that $D$ is compact in $\mathbb{R}$, i.e., $D$ is closed and bounded, and hence by (3.13), (3.9) and (3.10)

$$(3.20)\qquad \sup_{t>0} |\mu(t)| < \infty\,,$$

$$(3.21)\qquad \sup_{k \geqq 1} b_k \coloneqq b < \infty\,.$$

Now (3.21) implies

$$(3.22)\qquad \sup_{t>0} \left[\frac{1}{t} \int_{[-t,t]} x^2 F\{dx\} + t q(t)\right] < \infty\,.$$

To see this note that

$$\sup_{t>0} t q(t) = \sup_k \sup_{a_k \leqq t < a_{k+1}} t q(t) = \sup_k a_{k+1} \sum_{j=k+1}^{\infty} \frac{b_j}{a_j}$$
$$= \sup_k (b_k + O(1)) < \infty\,,$$

where the $O(1)$ is from (3.18), and

$$\sup_t \frac{1}{t} \int_{[-t,t]} x^2 F\{dx\} = \sup_k \sup_{a_k \le t < a_{k+1}} \frac{1}{t} \sum_{i=1}^k a_j b_j$$
$$= \sup_k \left\{ b_k + \frac{1}{a_k} \sum_{j=1}^{k-1} a_j b_j \right\} = \sup_k \{b_k + O(1)\} < \infty$$

by (3.19). It follows from (3.20), (3.21) and the second assertion of Lemma 1 in Section 2 that

$$\sup_k P\left\{\left|\frac{S_{n_k}}{n_k}\right| \leqq a\right\} = \delta(a) > 0 \tag{3.23}$$

for *any* sequence $n_k \uparrow \infty$ and all $a > 0$ sufficiently large. On the other hand if

$$\frac{S_{n_k}}{n_k} \to +\infty \quad \text{or} \quad -\infty \quad \text{a.s.}$$

then

$$\lim_{k\to\infty} P\left\{\left|\frac{S_{n_k}}{n_k}\right| > a\right\} = 1$$

for every $a$, contradicting (3.23). Hence in this case $B_s(F, 1)$ is a bounded closed set and our construction leads to

$$B_s(F, 1) = B_s^f(F, 1) = D^f = D .$$

Next suppose $+\infty \in D$ *and* $-\infty \in D$. Then $D = B_s(F, 1)$ is immediate from (3.16).

Finally, for the last case, suppose $+\infty \in D$ but $-\infty \notin D$. Again by (3.16) $+\infty \in B_s(F, 1)$. To show that $-\infty \notin B_s(F, 1)$ put

$$N_k^{\pm}(t) = \text{number of} \quad i \leqq t \quad \text{with} \quad X_i = \pm a_k , \quad N_k(t) = N_k^+(t) + N_k^-(t) ,$$
$$U_k(t) = \sum_{i \le t} X_i I[|X_i| = a_k] ,$$
$$V_k(t) = \sum_{i \le t} X_i I[|X_i| < a_k] = \sum_{l \le k-1} U_l(t) ,$$

and

$$d = \inf\{x : x \in D\} .$$

By assumption $d > -\infty$, and therefore we may assume

$$-\infty < d - 1 \leqq c_k \leqq k^2 . \tag{3.24}$$

For any integer $t \in [a_k, a_{k+1})$ we have the identity

$$S_t = V_k(t) + U_k(t) + U_{k+1}(t) + \sum_{l \ge k+2} U_l(t) .$$

Thus, we will have $t^{-1}S_t \geqq d - 2$ as soon as

$$\left|\frac{V_k(t)}{t} - c_{k-1}\right| \leqq 1 , \tag{3.25}$$

$$N_l(t) = 0 \qquad \text{for all} \quad l \geqq k + 2 \tag{3.26}$$

and

$$(3.27)\qquad \frac{tb_k}{a_k} \leqq N_k(t) \leqq \frac{3tb_k}{a_k}, \qquad \frac{1}{t} U_k(t) \geqq \tfrac{1}{2}(c_k - c_{k-1}),$$

$$\frac{tb_{k+1}}{a_{k+1}} \leqq N_{k+1}(t) \leqq \frac{3tb_{k+1}}{a_{k+1}}, \qquad \frac{1}{t} U_{k+1}(t) \geqq \tfrac{1}{2}(c_{k+1} - c_k).$$

Now, as $k \to \infty$

$$(3.28)\qquad P\{(3.26) \text{ fails}\} \leqq tP\{|X_1| > a_{k+1}\} \leqq a_{k+1} \sum_{l \geqq k+2} \frac{2b_l}{a_l} \leqq 2 \sum_{l \geqq k+1} l^{-4} \to 0.$$

Moreover, $N_k(t)$ has a binomial distribution with parameters $t$, $2a_k^{-1}b_k$. Thus

$$EN_k(t) = \frac{2tb_k}{a_k}, \qquad \operatorname{Var}(N_k(t)) \leqq \frac{2tb_k}{a_k}$$

and by Chebyshev's inequality and (3.10)

$$P\left\{\frac{tb_k}{a_k} \leqq N_k(t) \leqq \frac{3tb_k}{a_k}\right\} \geqq 1 - \frac{2a_k}{tb_k} \geqq 1 - \frac{2}{b_k} \geqq \tfrac{3}{4}.$$

Similarly

$$(3.29)\qquad P\left\{\frac{tb_k}{a_k} \leqq N_k(t) \leqq \frac{3tb_k}{a_k}, \frac{tb_{k+1}}{a_{k+1}} \leqq N_{k+1}(t) \leqq \frac{3tb_{k+1}}{a_{k+1}}\right\} \geqq \tfrac{1}{2}.$$

In addition when the values of $N_k(t)$ and $N_{k+1}(t)$ are given, say $m$ and $n$, then $N_k^+(t)$ and $N_{k+1}^+(t)$ are independent, and they have binomial distributions with parameters $m$, $p_k = (2b_k + c_k - c_{k-1})/4b_k \geqq \frac{1}{4}$, respectively $n$, $p_{k+1}$. Therefore, since $N_k^- = N_k - N_k^+$,

$$(3.30)\qquad P\left\{N_k^+(t) - N_k^-(t) \geqq \frac{c_k - c_{k-1}}{2b_k} N_k(t) \,\middle|\, N_k(t) = m,\ N_{k+1}(t) = n\right\}$$

$$= P\{N_k^+(t) \geqq p_k m \mid N_k(t) = m,\ N_{k+1}(t) = n\}$$

$$= \sum_{j \geqq p_k m} \binom{m}{j} p_k{}^j (1 - p_k)^{m-j} \geqq \delta > 0$$

for some $\delta > 0$, independent of $k$, $m$, $n$ and $t$. If $N_k(t) = m \in [tb_k/a_k, 3tb_k/a_k]$ and the event in (3.30) occurs, then clearly

$$U_k(t) = a_k\{N_k^+(t) - N_k^-(t)\} \geqq \tfrac{1}{2}t(c_k - c_{k-1}).$$

Therefore, from (3.30)

$$P\left\{\frac{1}{t} U_k(t) \geqq \tfrac{1}{2}(c_k - c_{k-1}) \,\middle|\, \frac{tb_k}{a_k} \leqq N_k(t) \leqq \frac{3tb_k}{a_k}, N_{k+1}(t)\right\} \geqq \delta,$$

and slightly more generally

$$(3.31)\qquad P\left\{\frac{1}{t} U_k(t) \geqq \tfrac{1}{2}(c_k - c_{k-1}), \frac{1}{t} U_{k+1}(t) \geqq \tfrac{1}{2}(c_{k+1} - c_k) \,\middle|\, \frac{tb_k}{a_k} \leqq N_k(t) \leqq \frac{3tb_k}{a_k}, \frac{tb_{k+1}}{a_{k+1}} \leqq N_{k+1}(t) \leqq \frac{3tb_{k+1}}{a_{k+1}}\right\} \geqq \delta^2.$$

(3.30) together with (3.31) shows that

$$(3.32) \qquad P\{(3.27) \text{ occurs}\} \geqq \tfrac{1}{2}\delta^2 .$$

Lastly, given that (3.27) occurs and that $N_l(t) = 0$ for $l \geqq k+2$ we know that

$$(3.33) \qquad \sum_{l \geqq k} N_l(t) \leqq 3t\left(\frac{b_k}{a_k} + \frac{b_{k+1}}{a_{k+1}}\right) \leqq \frac{6}{k^4}\, t .$$

Given the conditions (3.26)—(3.27) and given

$$\sum_{l \geqq k} N_l(t) = r ,$$

the conditional distribution of $V_k(t)$ is that of the sum of $t - r$ independent random variables, each with the conditional distribution of $X_1$, given $|X_1| \leqq a_{k-1}$. Thus under these conditions the conditional expectation of $t^{-1}V_k(t)$ is

$$\frac{t-r}{t}\{1 - q(a_{k-1})\}^{-1}\mu(a_{k-1}) = \frac{t-r}{t}\{1 - q(a_{k-1})\}^{-1}c_{k-1}$$

$$\geqq c_{k-1} - 2\left\{\frac{r}{t} + q(a_{k-1})\right\}|c_{k-1}| \geqq c_{k-1} - \tfrac{1}{2}$$

(for $k$ large; see (3.11), (3.24), (3.33)), and the conditional variance of $t^{-1}V_k(t)$ is at most

$$\frac{1}{t}\{1 - q(a_{k-1})\}^{-1} \sum_{l \leqq k-1} 2a_l b_l = O\left(\frac{1}{a_k} \sum_{l=1}^{k-1} a_l b_l\right) = o(1)$$

(see (3.19)). An application of Chebyshev's inequality together with (3.32) and (3.28) now shows

$$P\left\{\frac{S_t}{t} \geqq d - 2\right\} \geqq P\{(3.25),\ (3.26) \text{ and } (3.27) \text{ hold}\} \geqq \tfrac{1}{4}\delta^2 > 0$$

for all sufficiently large $t$. Thus $-\infty \notin B_s(F, 1)$ as we wanted to show. The case where $-\infty \in D$ but $+\infty \in D$ is treated by interchanging positive and negative.

## 4. Examples, miscellaneous remarks and problems.

*Examples related to Theorem* 1.

$B_s(F, \alpha) = \emptyset$. Take $X_1$ symmetric stable with exponent $1/\alpha$. Then $S_n/n^\alpha$ has the same distribution as $X_1$ for all $n$ and therefore $S_{n_k}/n_k{}^\alpha \to_P b$ is impossible (see also, Example 3). Clearly $B(F, \alpha) = \bar{\mathbb{R}}$ for these examples.

$B_s(F, \alpha) = \{0\}$. Take $X_1$ symmetric and such that $(\log n/n)^\alpha S_n$ has a stable limit distribution with exponent $1/\alpha$. Then $S_n/n^\alpha \to_P 0$. (e.g., $P\{X_1 = \pm k\} \sim c/k^{1+1/\alpha}\log k$. In this example $B(\alpha) = \bar{\mathbb{R}}$ for $\alpha > \frac{1}{2}$. This follows from Theorems 3 and 4, Corollary 2 of [4] and the estimate $P\{|n^{-\alpha}S_n - b| \leqq \varepsilon\} \geqq c(\log n)^{-1}$.)

$B_s(F, \alpha) = [0, \infty]$. For $\frac{1}{2} < \alpha < 1$ see Example 1. For $\alpha = 1$ see Theorem 2 and for $\alpha > 1$ see Example 2.

$B_s(F, \alpha) = \bar{\mathbb{R}}$. For $\frac{1}{2} < \alpha < 1$ see Example 1a. For $\alpha = 1$ see Theorem 2 and for $\alpha > 1$ see Example 2a.

*Example related to Theorem* 3. Example 3 shows that (1.6) is not necessary for $B(F, 1) = \bar{\mathbb{R}}$.

*Examples related to Theorem* 4. Example 4 shows that we cannot replace (1.7) by $E|X_1|^{1/\alpha} = \infty$ for $\frac{1}{2} < \alpha < 1$. Example 2 shows that for $\alpha > 1$ not even (1.7) and (1.8) together guarantee $B(F, \alpha) = \bar{\mathbb{R}}$.

Lastly Example 5 has for $\alpha > 1$, $B(F, \alpha) = [0, \infty]$ but $B_s(F, \alpha) = \{+\infty\}$. Thus, the examples can be summarized in the following table

TABLE 1

| | $\alpha$ | $B(F, \alpha)$ | $B_s(F, \alpha)$ |
|---|---|---|---|
| As above | $\frac{1}{2} < \alpha$ | $\bar{\mathbb{R}}$ | $\emptyset$ |
| As above | $\frac{1}{2} < \alpha$ | $\bar{\mathbb{R}}$ | $\{0\}$ |
| Example 1 | $\frac{1}{2} < \alpha < 1$ | $\bar{\mathbb{R}}$ | $[0, \infty]$ |
| Example 1a | $\frac{1}{2} < \alpha < 1$ | $\bar{\mathbb{R}}$ | $\bar{\mathbb{R}}$ |
| Example 2 | $1 < \alpha$ | $[0, \infty]$ | $[0, \infty]$ |
| Example 2a | $1 < \alpha$ | $\bar{\mathbb{R}}$ | $\bar{\mathbb{R}}$ |
| Example 3 | $\alpha \leqq 1$ | $\bar{\mathbb{R}}$ | $\emptyset$ |
| Example 4 | $\frac{1}{2} < \alpha < 1$ | $[0, \infty]$ | $\{0\}$ |
| Example 5 | $1 < \alpha$ | $[0, \infty]$ | $\{+\infty\}$ |

The method of Theorem 7 in [4] can be used to construct an $F$ with $B(F, \alpha) = \{-\infty, 0, +\infty\}$ for $\alpha > \frac{1}{2}$, but we do not know if $B(F, \alpha) = \{0, \infty\}$ is possible for $\alpha \neq 1$. The table above and Theorem 1 suggest the following;

CONJECTURE. If $\alpha \neq 1$ and $b \in B(F, \alpha)$ for some $0 < |b| < \infty$ then

$$\operatorname{sign}(b)[0, \infty] \subset B(F, \alpha) .$$

As a further conjecture and problem we mention

CONJECTURE. If $\frac{1}{2} < \alpha < 1$ and $E(X_1^+)^{1/\alpha} = E(X_1^-)^{1/\alpha} = \infty$ and $B(F, \alpha)$ contains at least two points then $\{-\infty, +\infty\} \subset B(F, \alpha)$. (Note that for $\alpha = 1$ the truth of this statement follows from [4] Theorem 6 and Corollary 3.)

PROBLEM. Find a necessary and sufficient condition for $+\infty$ or $-\infty$ to be a strong accumulation point. In particular when is the condition (3.1) with $c_k \to \infty$ necessary for $+\infty \in B_s(F, \alpha)$. (Note that [1] gives a n.a.s.c. for $+\infty \in B(F, 1)$.)

EXAMPLE 1. For $\frac{1}{2} < \alpha < 1$ we construct an $F$ for which $B_s(F, \alpha) = [0, \infty]$ (and $B(F, \alpha) = \bar{\mathbb{R}}$). Pick $a_k \geqq 1$, increasing so rapidly that

$$(4.1) \qquad (a_k a_{k+1})^{1/2\alpha} \sum_{j=k+1}^{\infty} \frac{1}{a_j} \to 0 , \qquad k \to \infty ,$$

and

$$(4.2) \qquad (a_k a_{k+1})^{1/2\alpha - 1} \sum_{j=1}^{k} a_j \to 0 , \qquad k \to \infty .$$

This can be done because $1/2\alpha - 1 < 0$; at the same time we can make

$$\sum_1^{\infty} \frac{1}{a_k} = 1 .$$

Now put $t_k^\alpha = (a_k a_{k+1})^{\frac{1}{2}} \in (a_k, a_{k+1})$ and

$$a_k(p_k + q_k) = 1 , \qquad k \geqq 1$$
$$a_k(p_k - q_k) = t_k^{\alpha-1} - t_{k-1}^{\alpha-1} , \qquad k \geqq 2,\ a_1(p_1 - q_1) = t_1^{\alpha-1} .$$

Note that $p_k$, $q_k \geqq 0$ because $|t_k^{\alpha-1} - t_{k-1}^{\alpha-1}| \leqq t_k^{\alpha-1} \leqq 1$. Let $F$ be the distribution which assigns mass $p_k$ to $a_k$ and $q_k$ to $-a_k$ i.e.

$$P\{X_1 = a_k\} = p_k , \qquad P\{X_1 = -a_k\} = q_k , \qquad k = 1, 2, \cdots .$$

From the preceding it follows that $F$ is a genuine probability distribution, and from (4.1), (4.2) that

$$t_k \sum_{k+1}^{\infty} \frac{1}{a_j} + t_k^{1-2\alpha} \sum_{j=1}^{k} a_j \to 0 , \qquad k \to \infty .$$

Thus (3.4), or equivalently (3.3) holds. In addition

$$t_k^{1-\alpha}\mu(t_k^\alpha) = t_k^{1-\alpha} \sum_{j=1}^{k} a_j(p_j - q_j) = 1 ,$$

so that by Lemma 5 $1 \in B_s(F, \alpha)$. By Theorem 1 we then have $[0, \infty] \subset B_s(F, \alpha)$. Moreover for $a_k \leqq t < a_{k+1}$, $k \geqq 2$

$$\mu(t) = \sum_1^k a_j(p_j - q_j) = t_k^{\alpha-1} > 0 , \tag{4.3}$$

so that again by Lemma 5 no point of $(-\infty, 0)$ lies in $B_s(F, \alpha)$. Finally we note that $p_k - q_k = o(p_k + q_k)$ so that

$$\frac{p_k}{q_k} \to 1 . \tag{4.4}$$

Essentially the same argument as used in Theorem 2 to prove that $-\infty \notin B_s(F, 1)$ when $-\infty \notin D$ now shows that $-\infty \notin B_s(F, \alpha)$. Thus $B_s(F, \alpha) = [0, \infty]$ in this example. Finally we note that (4.4) and

$$E|X_1|^{1/\alpha} \geqq E|X_1| = \sum_1^\infty (p_k + q_k)a_k = \infty$$

show that

$$E(X_1^+)^{1/\alpha} = E(X_1^-)^{1/\alpha} = \infty .$$

Thus, by Theorem 4, $B(F, \alpha) = \bar{\mathbb{R}}$ in this example.

EXAMPLE 1a. A minor modification of the above example yields an $F$ with $B_s(F, \alpha) = \bar{\mathbb{R}}$ for $\frac{1}{2} < \alpha < 1$. With the notation of Example 1 again put $a_k(p_k + q_k) = 1$, $k \geqq 1$. However, we change $a_k(p_k - q_k)$ as follows: We pick a sequence $k_0 = 1 < k_i < k_{i+1} < \infty$ of indices and keep

$$a_k(p_k - q_k) = t_k^{\alpha-1} - t_{k-1}^{\alpha-1}$$

for

$$k_{2j} < k < k_{2j+1} , \qquad j = 0, 1, \cdots .$$

However, we put

$$a_k(p_k - q_k) = -(t_k^{\alpha-1} - t_{k-1}^{\alpha-1})$$

for

$$k_{2j+1} < k < k_{2j+2} , \qquad j = 0, 1, \cdots$$

Also $a_1(p_1 - q_1) = t_1^{\alpha-1}$ and for $l = 1, 2, \cdots$

$$a_{k_l}(p_{k_l} - q_{k_l}) = (-1)^l\{t_{k_l}^{\alpha-1} + t_{k_l-1}^{\alpha-1}\}.$$

Since we left $a_k(p_k + q_k)$ unchanged, (3.3) is still valid. However $\mu(t_k{}^\alpha)$ has been changed. For $k < k_1$ we still have (with $t_0 = 0$)

$$\mu(t_k{}^\alpha) = \sum_{j=1}^{k} a_j(p_j - q_j) = \sum_{j=1}^{k} (t_j{}^{\alpha-1} - t_{j-1}^{\alpha-1}) = t_k{}^{\alpha-1}.$$

However,

$$\mu(t_{k_1}^\alpha) = \sum_{j=1}^{k_1} a_j(p_j - q_j) = \sum_{j=1}^{k_1-1} (t_j{}^{\alpha-1} - t_{j-1}^{\alpha-1}) - t_{k_1}^{\alpha-1} - t_{k_1-1}^{\alpha-1}$$
$$= -t_{k_1}^{\alpha-1}$$

and then

$$\mu(t_k{}^\alpha) = -t_k{}^{\alpha-1}, \qquad k_1 < k < k_2, \qquad \mu(t_{k_2}) = +t_{k_2}^{\alpha-1}$$

and in general

$$\mu(t_k{}^\alpha) = +t_k{}^{\alpha-1} \qquad \text{for} \quad k_{2j} < k < k_{2j+1}$$
$$\mu(t_k{}^\alpha) = -t_k{}^{\alpha-1} \qquad \text{for} \quad k_{2j+1} < k < k_{2j+2}.$$

Thus, by Lemma 5, $+1$ and $-1$ belong to $B_s(F, \alpha)$ and then by Theorem 1, $B_s(F, \alpha) = \bar{\mathbb{R}}$.

EXAMPLE 2. For $\alpha > 1$ we construct an $F$ with

$$E(X_1{}^+)^{1/\alpha} = E(X_1{}^-)^{1/\alpha} = \infty$$
$$\limsup P\left\{\left|\frac{S_n}{n^\alpha}\right| < a\right\} > 0$$
$$B_s(F, \alpha) = B(F, \alpha) = [0, \infty].$$

Let

$$a_k = e^{k^2}, \qquad b_k = \exp[(1 - 1/\alpha)(k + \delta)^2]$$
$$p_k = c_0\frac{b_k}{a_k}, \qquad q_k = c_0 e^{-k^2/\alpha}, \qquad\qquad k = 1, 2, \cdots,$$

where $c_0 > 0$ is chosen such that

$$\sum_1^\infty (p_k + q_k) = 1$$

and $0 < \delta < 1/\alpha$. Take

$$P\{X_1 = +a_k\} = p_k, \qquad P\{X_1 = -a_k\} = q_k,$$

and, finally write

$$t_k = e^{(1/\alpha)(k+\delta)^2}.$$

Note that

$$p_k = c_0 \exp\left[-k^2 + \left(1 - \frac{1}{\alpha}\right)(k + \delta)^2\right]$$
$$= c_0 \exp\left[-\frac{k^2}{\alpha} + 2\delta\left(1 - \frac{1}{\alpha}\right)k + \left(1 - \frac{1}{\alpha}\right)\delta^2\right]$$
$$= d_0 \exp\left[-\frac{k^2}{\alpha} + 2\delta\left(1 - \frac{1}{\alpha}\right)k\right] = \frac{d_0}{c_0} q_k \exp\left[2\delta\left(1 - \frac{1}{\alpha}\right)k\right]$$

where $d_0 = c_0 \exp[(1 - 1/\alpha)\delta^2]$. In particular

$$\frac{p_{k+1}}{p_k} \to 0 , \qquad \frac{p_k}{q_k} \to \infty \qquad \text{and} \qquad \frac{p_k a_k}{p_{k-1}a_{k-1}} = \frac{b_k}{b_{k-1}} \to \infty .$$

It is easy to check from this that

$$EX_1^+ \geqq E(X_1^+)^{1/\alpha} = \infty \qquad \text{and} \qquad EX_1^- \geqq E(X_1^-)^{1/\alpha} = \infty , \tag{4.5}$$

and for $a_k \leqq t < a_{k+1}$

$$\begin{aligned} m_+(t) &\equiv \textstyle\int_0^t [1 - F(x)]\, dx \geqq m_+(a_k) \\ &\sim \textstyle\sum_{j=1}^k (a_j - a_{j-1}) \sum_{l=j}^\infty \frac{c_0 b_l}{a_l} \sim c_0 b_k , \end{aligned} \tag{4.6}$$

and consequently

$$\int_{-\infty}^0 \frac{|x|\, dF(x)}{m_+(x)} = \sum_1^\infty \frac{a_k q_k}{m_+(a_k)} < \infty . \tag{4.7}$$

It follows from (4.5) and (4.7) and Corollary 1 of [1] that

$$\frac{S_n}{n} \to +\infty \quad \text{w.p. 1.}$$

Thus $S_k \geqq 0$ eventually and

$$B_s(F, \alpha) \subset B(F, \alpha) \subset [0, \infty] . \tag{4.8}$$

To show that we have equality in (4.8) we note that $a_k < t_k^\alpha < a_{k+1}$,

$$\mu(t_k^\alpha) = \textstyle\sum_{j=1}^k (p_j - q_j)a_j \sim p_k a_k = c_0 b_k = c_0 t_k^{\alpha-1}$$

so that

$$t_k^{1-\alpha}\mu(t_k^\alpha) \to c_0 . \tag{4.9}$$

In addition, with $a_0 = 0$,

$$\begin{aligned} \textstyle\int_0^{t_k^\alpha} xq(x)\, dx &= \textstyle\sum_{j=1}^k \frac{1}{2}(a_j^2 - a_{j-1}^2) \sum_{l \geqq j} (p_l + q_l) \\ &\quad + \textstyle\frac{1}{2}(t_k^{2\alpha} - a_k^2) \sum_{l \geqq k+1} (p_l + q_l) \sim \frac{1}{2}a_k^2 p_k + \frac{1}{2}t_k^{2\alpha} p_{k+1} \\ &= o(t_k^{2\alpha-1}) \quad (\text{recall } \delta < 1/\alpha) . \end{aligned}$$

It follows from (4.9), (4.10) and Lemma 5 that $c_0 \in B_s(F, \alpha)$ and then from Theorem 1 that $[0, \infty] \subset B_s(F, \alpha)$. Thus all the sets in (4.8) are the same, and $\limsup P\{|S_n/n^\alpha| < a\} > 0$ also follows from $c_0 \in B_s(F, \alpha)$.

EXAMPLE 2a. Again a minor modification of the last example yields an $F$ with $B_s(F, \alpha) = \bar{\mathbb{R}}$ for $\alpha > 1$. Again we take a sequence of indices $1 = k_0 < k_i < k_{i+1}$, but now take $p_k$, $q_k$ as in Example 2 for

$$k_{2j} \leqq k < k_{2j+1} , \qquad j = 0, 1, \cdots ,$$

and

$$p_k = c_0 e^{-k^2/\alpha} , \qquad q_k = c_0 \frac{b_k}{a_k}$$

when $k_{2j+1} \leqq k < k_{2j+2}$ for some $j \geqq 0$. Thus for $k_{2j+1} \leqq k < k_{2j+2}$ the definitions of $p_k$ and $q_k$ have been interchanged. This does not affect (4.10), but now

$$t_k^{1-\alpha}\mu(t_k^{\alpha}) \to -c_0$$

when $k \to \infty$ such that $k_{2j+1} \leqq k < k_{2j+2}$ for some $j$ whereas (4.9) remains valid if $k \to \infty$ with $k_{2j} \leqq k < k_{2j+1}$. As in Example 1a we find $B_s(F, \alpha) = \bar{\mathbb{R}}$.

EXAMPLE 3. (1.6) is not necessary for $B(F, 1) = \bar{\mathbb{R}}$. Note that $B_s(F, \alpha) = \emptyset$ for $\alpha \leqq 1$ in this example (by (4.11)). Let $F$ be the symmetric discrete distribution given by

$$P(X_1 = \pm k) = C\frac{(\log k)^2}{k^2} \qquad k \geqq 2$$

for some $C > 0$. Then as shown in ([4] page 1182) $B(F, \alpha) = \bar{\mathbb{R}}$ for $0 < \alpha \leqq 1$. Also, by ([4] page 1182) for $C^* = \pi C$ and any $\varepsilon > 0$

$$\begin{aligned} P\left\{\frac{|S_n|}{n^{\alpha}} < a\right\} &= P\left\{\frac{|S_n|}{C^*n(\log n)^2} \leqq \frac{a}{C^*n^{1-\alpha}(\log n)^2}\right\} \\ &= P\left\{\frac{|S_n|}{C^*n(\log n)^2} \leqq \varepsilon\right\} \to \frac{1}{\pi}\int_{-\varepsilon}^{\varepsilon}\frac{1}{1+x^2}\,dx \end{aligned}$$

as $n \to \infty$. Since $\varepsilon > 0$ is arbitrary

$$\lim P\left\{\left|\frac{S_n}{n^{\alpha}}\right| \leqq a\right\} = 0 \quad \text{for all} \quad a > 0\,. \tag{4.11}$$

EXAMPLE 4. This shows that (1.8) and $E|X_1|^{1/\alpha} = \infty$ do not guarantee $B(F, \alpha) = \bar{\mathbb{R}}$ if $\frac{1}{2} < \alpha < 1$. This example has $B(F, \alpha) = [0, \infty]$ instead, and $B_s(F, \alpha) = \{0\}$.

Let $X_1$ have the characteristic function

$$\varphi(\theta) = \exp\int_2^{\infty}[e^{i\theta y} - 1 - i\theta y]\frac{dy}{y^{1+1/\alpha}\log y}\,, \qquad \tfrac{1}{2} < \alpha < 1\,. \tag{4.12}$$

Standard arguments show that the exponent of (4.12) behaves as

$$\frac{|\theta|^{1/\alpha}}{\log 1/|\theta|}b_{\alpha} \quad \text{when} \quad \theta \downarrow 0\,, \tag{4.13}$$

where

$$b_{\alpha} = \int_0^{\infty}(e^{iu} - 1 - iu)\frac{du}{u^{1+1/\alpha}} = \exp\left[-\frac{\pi}{2\alpha}i\right]\frac{\alpha^2}{1-\alpha}\Gamma\left(2 - \frac{1}{\alpha}\right).$$

It follows that

$$\left(\frac{\log n}{n}\right)^{\alpha} S_n$$

converges in law to an asymmetric stable distribution with characteristic function

$$\exp\frac{\Gamma(2 - 1/\alpha)}{1/\alpha - 1}|\theta|^{1/\alpha}\left\{\cos\frac{\pi}{2\alpha} - i\,\mathrm{sign}\,(\theta)\sin\frac{\pi}{2\alpha}\right\}.$$

In particular $n^{-\alpha}S_n \to_P 0$, (1.8) holds and $B_s(F, \alpha) = \{0\}$. By pages 540–545 of

[3] this also implies

$$P\{X_1 \geqq x\} \sim \frac{C}{x^{1/\alpha}\log x},$$
$$P\{X_1 \leqq -x\} = o\left(\frac{1}{x^{1/\alpha}\log x}\right), \qquad x \to \infty,$$

for some $C > 0$. (Actually $P\{X_1 \leqq -x\}$ decreases exponentially in $x$ by (4.14) below.) Thus $E(X_1^+)^{1/\alpha} = \infty$. But also for $b - 2\varepsilon > 0$

$$\begin{aligned} P&\left\{\frac{S_n}{n^\alpha} \in (b - 2\varepsilon, b + 2\varepsilon)\right\} \\ &\geqq \sum_{i=1}^n P\left\{\frac{X_i}{n^\alpha} \in (b-\varepsilon, b+\varepsilon), \left|\frac{S_n - X_i}{n^\alpha}\right| \leqq \varepsilon\right\} \\ &\quad - \sum_{1 \leqq i < j \leqq n} P\left\{\frac{X_i}{n^\alpha} \in (b-\varepsilon, b+\varepsilon), \frac{X_j}{n^\alpha} \in (b-\varepsilon, b+\varepsilon)\right\} \\ &\sim nP\{n^\alpha(b-\varepsilon) \leqq X_1 \leqq n^\alpha(b+\varepsilon)\} \sim \frac{C}{\log n}\left\{\frac{1}{(b-\varepsilon)^{1/\alpha}} - \frac{1}{(b+\varepsilon)^{1/\alpha}}\right\}. \end{aligned}$$

Consequently

$$\sum \frac{1}{n} P\left\{\frac{S_n}{n^\alpha} \in (b - 2\varepsilon, b + 2\varepsilon)\right\} = \infty$$

for all $b > 0$, $0 < 2\varepsilon < b$. It follows from this and Corollary 2 in [4] that $[0, \infty] \subset B(F, \alpha)$. To show that $B(F, \alpha)$ contains no points on the negative axis, we note that (4.12) implies

$$Ee^{-\lambda X_1} = \exp \int_2^\infty [e^{-\lambda y} - 1 + \lambda y] \frac{dy}{y^{1+1/\alpha}\log y}, \qquad \lambda > 0, \tag{4.14}$$

and as in (4.13), the exponent in (4.14) behaves as

$$\frac{\lambda^{1/\alpha}}{\log 1/\lambda}|b_\alpha| \qquad \text{when} \quad \lambda \uparrow 0.$$

This of course implies $E(X_1^-)^{1/\alpha} < \infty$, but more importantly, for $c > 0$, $0 < \lambda < \lambda_0$

$$\begin{aligned} P\left\{\frac{S_n}{n^\alpha} \leqq -c\right\} &\leqq e^{-\lambda c n^\alpha}(Ee^{-\lambda X_1})^n \\ &\leqq \exp\left\{-\lambda c n^\alpha + 2|b_\alpha| n \frac{\lambda^{1/\alpha}}{\log 1/\lambda}\right\}. \end{aligned}$$

If we take

$$\lambda = \frac{d}{n^\alpha(\log n)^{\alpha/(\alpha-1)}}$$

for $0 < d \leqq d_0(c)$ we obtain for large $n$

$$P\left\{\frac{S_n}{n^\alpha} \leqq -c\right\} \leqq \exp\left\{-\frac{d}{2}c(\log n)^{\alpha/(1-\alpha)}\right\}.$$

Since $\frac{1}{2} < \alpha < 1$ it follows for every $c > 0$ that

$$\sum P\left\{\frac{S_n}{n^\alpha} \leqq -c\right\} < \infty ,$$

and therefore

$$\liminf_{n\to\infty} \frac{S_n}{n^\alpha} \leqq 0 \quad \text{w.p. } 1. \tag{4.15}$$

Of course (4.15) proves $B(F, \alpha) \subset [0, \infty]$ as desired.

EXAMPLE 5. This example has $B(F, \alpha) = [0, \infty]$ and $B_s(F, \alpha) = \{+\infty\}$ for $\alpha > 1$. Take $\alpha > 1$ and

$$F(dx) = C_1 \frac{(\log \log x)^\beta}{x^{1+1/\alpha}} dx , \qquad x \geqq 10 , \tag{4.16}$$

$$F(dx) = \frac{C_2 dx}{|x|^{1+1/\alpha} \log |x| \log \log |x|} , \qquad x \leqq -10 ,$$

for some $0 < \beta < (\alpha - 1)/\alpha$, $C_1$, $C_2 > 0$ for which $\int_{-\infty}^{+\infty} F(dx) = 1$. It is easy to check from (4.16) and Corollary 1 of [1] that

$$\frac{S_n}{n} \to +\infty \quad \text{w.p. } 1.$$

so that $S_n \geqq 0$ eventually and $B(F, \alpha) \subset [0, \infty]$. It follows from pages 540–545 in [3] that

$$n^{-\alpha}(\log \log n)^{-\alpha\beta} S_n$$

converges in law to a stable distribution with exponent $1/\alpha < 1$ concentrated on $(0, \infty)$. Thus $n^{-\alpha} S_n \to_P \infty$ and $B_s(F, \alpha) = \{+\infty\}$. However, to show that $B(F, \alpha) = [0, \infty]$ we need the following estimate for $b > 0$, $0 < 2\varepsilon < b$ and some fixed $d > 0$:

$$\begin{aligned} &P\{n^{-\alpha} S_n \in (b - \varepsilon, b + \varepsilon)\} \\ &\quad \geqq \textstyle\sum_{i=1}^n P\{n^{-\alpha} X_i \in (b - \varepsilon, b + \varepsilon),\, n^{-\alpha}|X_j| \leqq d(\log \log n)^{-\alpha\beta/(\alpha-1)} \\ &\qquad\qquad \text{for } 1 \leqq j \leqq n, j \neq i,\, n^{-\alpha}|\textstyle\sum_{j\neq i, 1\leqq j\leqq n} X_j| \leqq \varepsilon\} \\ &\quad = nP\{n^{-\alpha} X_1 \in (b - \varepsilon, b + \varepsilon)\}[P\{|X_1| < dn^\alpha(\log \log n)^{-\alpha\beta/(\alpha-1)}\}]^{n-1} \\ &\qquad P\{|S_{n-1}| \leqq \varepsilon n^\alpha \,|\, |X_j| \leqq dn^\alpha(\log \log n)^{-\alpha\beta/(\alpha-1)}, j \leqq n - 1\} . \end{aligned} \tag{4.17}$$

It is easy to see from (4.16) that the product of the first two probability factors in the last member of (4.1) is at least

$$\begin{aligned} &K_1(b, \varepsilon) \frac{(\log \log n)^\beta}{n}\left[1 - K_2 \frac{(\log \log n)^{\alpha\beta/(\alpha-1)}}{d^{1/\alpha} n}\right]^{n-1} \\ &\quad \geqq K_1(b, \varepsilon) \frac{(\log \log n)^\beta}{n} \exp - \frac{2K_2}{d^{1/\alpha}} (\log \log n)^{\alpha\beta/(\alpha-1)} \geqq \frac{K_1(b, \varepsilon)}{n(\log n)^{\frac{1}{2}}} \end{aligned} \tag{4.18}$$

for some $K_1$, $K_2$ independent of $d$ and all large $n$ (recall that $\alpha\beta/(\alpha - 1) < 1$).

(4.16) also yields

$$E\{|X_1| \mid |X_1| \leqq dn^\alpha(\log\log n)^{-\alpha\beta/(\alpha-1)}\} \leqq K_3 d^{1-1/\alpha} n^{\alpha-1}$$

for some $K_3$ independent of $d$, so that for $d \leqq (\varepsilon/2K_3)^{\alpha/(\alpha-1)}$

$$(4.19) \quad \begin{aligned} &P\{|S_{n-1}| \leqq \varepsilon n^\alpha \mid |X_j| \leqq dn^\alpha(\log\log n)^{-\alpha\beta/(\alpha-1)}, j \leqq n-1\} \\ &\geqq 1 - (\varepsilon n^\alpha)^{-1} E\{|S_{n-1}| \mid |X_j| \leqq dn^\alpha(\log\log n)^{-\alpha\beta/(\alpha-1)}, j \leqq n-1\} \geqq \tfrac{1}{2} \end{aligned}$$

(4.17)—(4.19) yield

$$\sum \frac{1}{n} P\{n^{-\alpha}S_n \in (b-2\varepsilon, b+2\varepsilon)\} \geqq \tfrac{1}{2}K_1(b, \varepsilon) \sum \frac{1}{n(\log n)^{\frac{1}{2}}} = \infty$$

so that by Corollary 2 of [4], $b \in B(F, \alpha)$. Since this holds for all $b > 0$ we have indeed $B(F, \alpha) = [0, \infty]$.

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